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Hankel Operators and Their Applications

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Preface

The purpose of this book is to describe the theory of Hankel operators, one of the most important classes of operators on spaces of analytic functions. Hankel operators can be defined as operators having infinite Hankel matrices (i.e., matrices with entries depending only on the sum of the coordinates) with respect to some orthonormal basis. Finite matrices with this property were introduced by Hankel, who found interesting algebraic properties of their determinants. One of the first results on infinite Hankel matrices was obtained by Kronecker, who characterized Hankel matrices of finite rank as those whose entries are Taylor coefficients of rational functions. Since then Hankel operators (or matrices) have found numerous applications in classical problems of analysis, such as moment problems, orthogonal polynomials, etc.

Hankel operators admit various useful realizations, such as operators on spaces of analytic functions, integral operators on function spaces on $(0, \infty)$, operators on sequence spaces. In 1957 Nehari described the bounded Hankel operators on the sequence space ℓ^2. This description turned out to be very important and started the contemporary period of the study of Hankel operators.

We begin the book with introductory Chapter 1, which defines Hankel operators and presents their basic properties. We consider different realizations of Hankel operators and important connections of Hankel operators with the spaces BMO and VMO, Sz.-Nagy–Foais functional model, reproducing kernels of the Hardy class H^2, moment problems, and Carleson imbedding operators.

It turns out that for the needs of applications it is also important to consider vectorial Hankel operators, i.e., Hankel operators on spaces of vector functions. We introduce vectorial Hankel operators in Chapter 2, to be used later in the book in control theory, approximation theory, and Wiener–Hopf factorizations (Chapters 11, 13, and 14).

In Chapter 3 we introduce another very important class of operators on spaces of analytic functions, the class of Toeplitz operators. They can be defined as operators having infinite matrices with entries depending only on the difference of the coordinates. Though Hankel and Toeplitz operators have quite different properties, Hankel operators play an important role in the study of Toeplitz operators, and vice versa. We also study in Chapter 3 vectorial Toeplitz operators.

In Chapter 4 we analyze the singular values of Hankel operators. The main result of the chapter is the Adamyan–Arov–Krein theorem, which shows that the nth singular value of a Hankel operator is the distance to the set of Hankel operators of rank at most n.

Chapter 5 deals with parametrization of solutions of the Nehari problem. In other words, we parametrize the symbols of a Hankel (or a vectorial Hankel) operator that belong to the ball in L^∞ of a given radius.

In Chapter 6 we describe the Hankel operators that belong to Schatten–von Neumann classes $\boldsymbol{S}_p$ as those whose symbols belong to certain Besov classes. We consider various applications of this description. In particular we obtain sharp results on rational approximation in the norm of BMO.

In this book we study many different applications of Hankel operators (in approximation theory, prediction theory, interpolation problems, control theory, etc). In Chapters 8 and 9 we use Hankel operators to study regularity conditions for stationary processes.

Chapter 10 is an introduction to the spectral theory of Hankel operators. We continue the analysis of the spectral problems of Hankel operators in Chapter 12, where we give a complete description of the spectral properties of the self-adjoint Hankel operators. It turns out that not only are Hankel operators used in control theory, but also the theory of Hankel operators can benefit from methods of control theory. In particular, in Chapter 12 the results on spectral properties of self-adjoint Hankel operators are based on balanced linear systems with continuous time and discrete time, a notion borrowed from control theory.

Chapter 11 is devoted to applications of Hankel operators in control theory. We consider linear systems with discrete time and continuous time, the problems of robust stabilization, model reduction, and model matching.

In Chapter 13 we study hereditary properties of maximizing vectors of vectorial Hankel operators. In other words, for a broad class of function spaces X we prove that if the symbol belongs to X, then all maximizing vectors belong to the same space X. We give several applications of this result. In particular, we use it to obtain hereditary properties of Wiener–Hopf factorizations.

In Chapter 14 vectorial Hankel operators are used in the theory of approximation by analytic matrix and operator functions. We introduce the important notion of superoptimal approximation and prove the uniqueness of a superoptimal approximation under certain mild conditions on the matrix function. We obtain certain special factorizations and prove inequalities between the singular values of the corresponding Hankel operators and superoptimal singular values of their symbols. This beautiful theory has been developed during the last decade; it demonstrates the importance of vectorial Hankel operators in noncommutative analysis.

One of the most beautiful applications of Hankel operators is given in Chapter 15. The last chapter gives a solution to the famous problem of whether a polynomially bounded operator on Hilbert space must be similar to a contraction. This problem remained open for a long time. In particular, it was one of the problems in a famous paper by Paul Halmos called "Ten problems in Hilbert space". Recently it has been solved in the negative with the help of vectorial Hankel operators.

In this book we discuss only classical Hankel operators (i.e., operators with Hankel matrices or, in other words, Hankel operators on the Hardy class H^2). For the last 20 years many interesting results have been obtained about various generalizations of Hankel operators (commutators of multiplications and Calderón–Zygmund operators, paracommutators, Hankel operators on Bergman spaces, Hankel operators on function spaces on the polydisk, on the unit ball in $\mathbb{C}^n$, on classical domains, etc). However, it is physically impossible to cover such generalizations in one book, and we restrict ourselves to classical Hankel operators.

Even under this constraint it is hardly possible to cover all aspects of Hankel operators and their applications (for example, this book does not include applications of Hankel operators in noncommutative geometry, perturbation theory, or asymptotics of Toeplitz determinants). Each chapter ends with Concluding Remarks, where the reader can find references to some results not included here.

Theorems, lemmas, and corollaries (as well as displayed formulas) are numbered lexicographically. Within the same chapter Theorem 3.5 means the fifth item of Section 3. To refer to a result from a different chapter, we use three numbers: Lemma 5.5.4 means the fourth item of Section 5 in Chapter 5. Reference to §6 means Section 6 within the same chapter. Reference to §4.3 means Section 3 in Chapter 4. Displayed formulas have an independent numeration.

For convenience we add two appendices in which the reader can find necessary information on operator theory and function spaces. Reference to Appendix 2.5 means Section 5 of Appendix 2.

I would like to express my deep gratitude to my colleagues with whom I discussed many aspects of Hankel operators and their applications. I am especially grateful to A.B. Aleksandrov, R.B. Alexeev, E.M. Dyn'kin, A.Y. Kheifets, S.V. Khrushchëv, S.V. Kislyakov, A.V. Megretskii, N.K. Nikol'skii, S.R. Treil, V.I. Vasyunin, A.L. Volberg, and N.J. Young.

Okemos, Michigan VLADIMIR V. PELLER
May, 2002

Notation

Symbols $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the set of integers, real numbers, and complex numbers.

$\mathbb{R}_+ \stackrel{\text{def}}{=} \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{Z}_+ \stackrel{\text{def}}{=} \{n \in \mathbb{Z} : n \geq 0\}$.

$\mathbb{T} \stackrel{\text{def}}{=} \{\zeta \in \mathbb{C} : |\zeta| = 1\}$, $\mathbb{D} \stackrel{\text{def}}{=} \{\zeta \in \mathbb{C} : |\zeta| < 1\}$.

z denotes the identity map of $\mathbb{C}$ (or a subset of $\mathbb{C}$) onto itself.

$\boldsymbol{m}$ is normalized Lebesgue measure on $\mathbb{T}$.

$\boldsymbol{m}_2$ is planar Lebesgue measure.

$\hat{\varphi}(n)$ is the nth Fourier coefficients of φ.

L^p means the L^p space of functions on $\mathbb{T}$ with respect to $\boldsymbol{m}$, unless otherwise specified.

$\mathcal{B}(X, Y)$ for Banach spaces X and Y is the space of bounded linear operators from X to Y, $\mathcal{B}(X) \stackrel{\text{def}}{=} \mathcal{B}(X, X)$.

H^p is the Hardy class of functions analytic in $\mathbb{D}$ (see Appendix 2).

If X is a Banach space, $L^p(X)$ and $H^p(X)$ denote the X-valued L^p and H^p spaces.

If E is a subset of $\mathbb{R}$ or $\mathbb{T}$, $L^p(E)$ denotes the L^p-space of functions on E; this should not lead to a confusion with the previous notation.

$H^p(\mathbb{C}_+)$ ($H^p(\mathbb{C}_-)$, $H^p(\mathbb{C}^+)$, or $H^p(\mathbb{C}_-)$) denotes the Hardy class of functions analytic in the upper (lower, right, or left) half-plane.

If E is a subset of $\mathbb{R}$ or $\mathbb{T}$ and X is a Banach space, we use the notation $L^p(E, X)$ for the L^p-space of X-valued functions on E.

$\mathcal{F}$ is the Fourier transform, $(\mathcal{F}f)(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} f(s)e^{-2\pi \mathrm{i}st}ds$, $f \in L^1(\mathbb{R})$.

$\operatorname{span} A$ means the closed linear span of a subset E in a Banach space.

$\mathbf{1}$ is the constant function identically equal to 1.

$\mathbb{O}$ is the zero in a vector space.

$\boldsymbol{\mathcal{P}}$ is the space of trigonometric polynomials of the form $\sum c_j z^j$.

$\boldsymbol{\mathcal{P}}_+$ is the space of analytic polynomials of the form $\sum\limits_{j \ge 0} c_j z^j$.

$\boldsymbol{\mathcal{P}}_-$ is the space of antianalytic polynomials of the form $\sum\limits_{j < 0} c_j z^j$.

If X is a space of functions on $\mathbb{T}$ such that $X \subset L^1$ (or X is a space of distributions on $\mathbb{T}$), we put

$$X_+ \stackrel{\text{def}}{=} \{f \in X : \ \hat{f}(j) = 0 \text{ for } j < 0\}$$

and

$$X_- \stackrel{\text{def}}{=} \{f \in X : \ \hat{f}(j) = 0 \text{ for } j \ge 0\}.$$

$\boldsymbol{S}_p$ is the Scahtten–von Neumann class (see Appendix 1).

$\boldsymbol{\mathcal{C}} = \boldsymbol{S}_\infty$ is the space of compact operators on Hilbert space.

$\ell^2_{\mathbb{Z}}$ is the space of two-sided sequences $\{x_n\}_{n \in \mathbb{Z}}$ such that $\sum\limits_{n \in \mathbb{Z}} |x_n|^2 < \infty$.

All Hilbert spaces are assumed separable.

Unless otherwise stated, an operator means a bounded linear operator.

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1

An Introduction to Hankel Operators

In this introductory chapter we define the Hankel operators and study their basic properties. We introduce in §1 the class of Hankel operators as operators with matrices of the form $\{\alpha_{j+k}\}_{j,k\geq 0}$ and consider different realizations of such operators. One of the most important realization is the Hankel operators H_φ from the Hardy class H^2 to $H^2_- \stackrel{\text{def}}{=} L^2 \ominus H^2$. We prove the fundamental Nehari theorem, which describes the bounded Hankel operators, and we discuss the problem of finding symbols of minimal norm. We introduce the important Hilbert matrix, prove its boundedness, and estimate its norm. Then we study Hankel operators with unimodular symbols. We conclude §1 with the study of commutators of multiplication operators with the Riesz projection on L^2 and reduce the study of such commutators to the study of Hankel operators.

In §2 we study relationships between Hankel operators and functions of model operators in the Sz.-Nagy–Foias function model. We present the approach of N.K. Nikol'skii to prove the Sarason commutant lifting theorem. This approach is based on the Nehari theorem for Hankel operators. We also give some related results. Namely, we describe the Hankel operators with nontrivial kernels, the partially isometric Hankel operators, and the Hankel operators with closed ranges.

One of the earliest results on Hankel operators is Kronecker's theorem that describes the Hankel operators of finite rank as the Hankel operators with rational symbols. We give different proofs of Kronecker's theorem in §3 and we give explicit formulas for $H_\varphi f$ when φ is rational.

In §4 we consider classical interpolation problems, the Nevanlinna–Pick interpolation problem and the Carathéodory–Fejér interpolation problem. We reduce these interpolation problems to Hankel operators and solve them with the help of Hankel operators.

In §5 we find the essential norm of a Hankel operator and we describe the compact Hankel operators. Then we study the problem of the approximation of a given Hankel operator by compact Hankel operators. We show that a best approximation always exists but it is never unique unless our Hankel operator is compact. Then we introduce the notion of local distance and we give a formula for the essential norm of Hankel operators in terms of local distance. Next, we use this result to evaluate the essential norm of Hankel operators with piecewise continuous symbols. We conclude §5 with a factorization theorem for functions in $H^\infty + C$.

In §6 we show that to verify the boundedness or the compactness of a Hankel operator H_φ, it suffices to consider the norms of $H_\varphi k_\zeta$, where the k_ζ, $|\zeta| < 1$, are the normalized reproducing kernels of H^2.

Section 7 is devoted to relationships between Hankel operators and moment problems. We prove the Hamburger theorem, which characterizes the positive definite Hankel matrices in terms of moments of positive measures on $\mathbb{R}$. Then we proceed to the study of bounded positive definite Hankel matrices in terms of moment sequences. After that we consider arbitrary bounded Hankel matrices and we characterize their entries as moment sequences of so-called Carleson measures on the unit disk. We also obtain similar results for compact Hankel operators.

The final section of this chapter deals with integral operators on $L^2(\mathbb{R}_+)$ with kernels depending on the sum of the coordinates. Such operators look like continual analogs of Hankel operators. However, we show that one can also reduce their study to the study of Hankel operators. We obtain in §8 boundedness, compactness, and finite rank criteria for such integral operators.

1. Bounded Hankel Operators

Hankel operators can be defined in many different ways; they admit different realizations. Such a variety of realizations is important in applications, since in a concrete case we can choose a realization that is most suitable for the problem under consideration. We begin with the matricial definition of Hankel operators.

An infinite matrix is called a *Hankel matrix* if it has the form

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \cdots \\ \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\alpha = \{\alpha_j\}_{j\ge0}$ is a sequence of complex numbers. In other words, the Hankel matrices are the matrices whose entries depend only on the sum of the coordinates.

If $\alpha \in \ell^2$, we can consider the operator $\Gamma : \ell^2 \to \ell^2$ with matrix $\{\alpha_{j+k}\}_{j,k\ge0}$ that is defined on the dense subset of finitely supported sequences. In other words, if $a = \{a_n\}_{n\ge0}$ is a finitely supported sequence, then $\Gamma a = b \in \ell^2$, where $b = \{b_k\}_{k\ge0}$ is defined by

$$b_k = \sum_{j\ge0} \alpha_{j+k} a_j, \quad k \ge 0.$$

We call such operators *Hankel operators.* We say that Γ is a bounded operator if it extends to a bounded operator on ℓ^2.

The Nehari Theorem

The following theorem, which characterizes the bounded Hankel operators on ℓ^2, is due to Nehari.

Theorem 1.1. *The Hankel operator Γ with matrix $\{\alpha_{j+k}\}_{j,k\ge0}$ is bounded on ℓ^2 if and only if there exists a function ψ in L^∞ on the unit circle $\mathbb{T}$ such that*

$$\alpha_m = \hat{\psi}(m), \quad m \ge 0. \tag{1.1}$$

In this case

$$\|\Gamma\| = \inf\{\|\psi\|_\infty : \ \hat{\psi}(n) = \alpha_n, \ n \ge 0\}.$$

Recall that $\hat{\psi}(n)$ is the nth Fourier coefficient of ψ.

Proof. Let $\psi \in L^\infty$ and $\alpha_m = \hat{\psi}(m), \ m \ge 0$. Given $a = \{a_n\}_{n\ge0}$ and $b = \{b_k\}_{k\ge0}$ finitely supported sequences in ℓ^2, we have

$$(\Gamma a, b) = \sum_{j,k\ge0} \alpha_{j+k} a_j \bar{b}_k. \tag{1.2}$$

Let

$$f = \sum_{j\ge0} a_j z^j, \quad g = \sum_{k\ge0} \bar{b}_k z^k.$$

Then f and g are polynomials in the Hardy class H^2. Put $q = fg$. It follows from (1.2) that

$$\begin{aligned}(\Gamma a, b) &= \sum_{j,k\geq 0} \hat{\psi}(j+k)a_j\bar{b}_k = \sum_{m\geq 0}\hat{\psi}(m)\sum_{j=0}^{m} a_j\bar{b}_{m-j}\\ &= \sum_{m\geq 0}\hat{\psi}(m)\hat{q}(m) = \int_{\mathbb{T}}\psi(\zeta)q(\bar{\zeta})d\boldsymbol{m}(\zeta).\end{aligned}$$

Therefore

$$|(\Gamma a, b)| \leq \|\psi\|_\infty\|q\|_{H^1} \leq \|\psi\|_\infty\|f\|_{H^2}\|g\|_{H^2} = \|\psi\|_{L^\infty}\|a\|_{\ell^2}\|b\|_{\ell^2}.$$

Conversely, suppose that Γ is bounded on ℓ^2. Let $\mathcal{L}$ be the linear functional defined on the set of polynomials in H^1 by

$$\mathcal{L}q = \sum_{n\geq 0}\alpha_n\hat{q}(n). \tag{1.3}$$

Let us show that $\mathcal{L}$ extends by continuity to a continuous functional on H^1 and its norm $\|\mathcal{L}\|$ on H^1 satisfies

$$\|\mathcal{L}\| \leq \|\Gamma_\alpha\|. \tag{1.4}$$

By the Hahn–Banach theorem this will imply the existence of ψ in L^∞ that satisfies (1.1) and

$$\|\psi\|_\infty \leq \|\Gamma_\alpha\|.$$

Assume first that $\alpha \in \ell^1$. In this case the functional $\mathcal{L}$ defined by (1.3) is obviously continuous on H^1. Let us prove (1.4). Let $q \in H^1$ and $\|q\|_1 \leq 1$. Then q admits a representation $q = fg$, where $f, g \in H^2$ and $\|f\|_2 \leq 1$, $\|g\|_2 \leq 1$. We have

$$\begin{aligned}\mathcal{L}q &= \sum_{m\geq 0}\alpha_m\hat{q}(m) = \sum_{m\geq 0}\alpha_m\sum_{j=0}^{m}\hat{f}(j)\hat{g}(m-j)\\ &= \sum_{j,k\geq 0}\alpha_{j+k}\hat{f}(j)\hat{g}(k) = (\Gamma a, b),\end{aligned}$$

where $a = \{a_j\}_{j\geq 0}$, $a_j = \hat{f}(j)$ and $b = \{b_k\}_{k\geq 0}$, $b_k = \overline{\hat{g}(k)}$. Therefore

$$|\mathcal{L}q| \leq \|\Gamma\|\cdot\|f\|_2\|g\|_2 \leq \|\Gamma\|$$

which proves (1.4) for $\alpha \in \ell^1$.

Assume now that α is an arbitrary sequence for which Γ is bounded. Let $0 < r < 1$. Consider the sequence $\alpha^{(r)}$ defined by

$$\alpha_j^{(r)} = r^j\alpha_j, \quad j \geq 0,$$

and let Γ_r be the Hankel operator with matrix $\left\{\alpha_{j+k}^{(r)}\right\}_{j,k\geq 0}$. It is easy to see that $\Gamma_r = D_r\Gamma D_r$, where D_r is multiplication by $\{r^j\}_{j\geq 0}$ on ℓ^2. Since obviously $\|D_r\| \le 1$, it follows that the operators Γ_r are bounded and

$$\|\Gamma_r\| \le \|\Gamma\|, \quad 0 < r < 1.$$

Clearly, $\alpha^{(r)} \in \ell^1$ and so we have already proved that

$$\|\mathcal{L}_r\|_{H^1 \to \mathbb{C}} \le \|\Gamma_r\| \le \|\Gamma\|,$$

where

$$\mathcal{L}_r q \stackrel{\text{def}}{=} \sum_{n \ge 0} \alpha_n^{(r)} \hat{q}(n), \quad q \in H^1.$$

It is easy to see now that the functionals $\mathcal{L}_r$ being uniformly bounded converge strongly to $\mathcal{L}$ (i.e., $\mathcal{L}_r\psi \to \mathcal{L}\psi$ for any $\psi \in H^1$), which proves that $\mathcal{L}$ is continuous and satisfies (1.4). ■

Theorem 1.1 reduces the problem of whether a sequence α determines a bounded operator on ℓ^2 to the question of the existence of an extension of α to the sequence of Fourier coefficients of a bounded function. By virtue of the work of C. Fefferman on the space BMO of functions of bounded mean oscillation (see Appendix 2.5 for definitions) it has become possible to determine whether Γ is bounded in terms of the sequence α itself.

Recall that it follows from Fefferman's results that a function φ on the unit circle belongs to the space BMO if and only if φ admits a representation

$$\varphi = \xi + \tilde{\eta}, \quad \xi, \eta \in L^\infty,$$

where $\tilde{\eta}$ is the harmonic conjugate of η (see Appendix 2.5). Recall also that $BMOA$ is the space of BMO functions analytic in $\mathbb{D}$, i.e.,

$$BMOA = BMO \cap H^1.$$

Theorem 1.2. *The operator Γ with matrix $\{\alpha_{j+k}\}_{j,k\ge 0}$ is bounded on ℓ^2 if and only if the function*

$$\varphi = \sum_{m \ge 0} \alpha_m z^m \tag{1.5}$$

belongs to $BMOA$.

Proof. If Γ is bounded, then by Theorem 1.1 there exists a function ψ in L^∞ such that $\varphi = \mathbb{P}_+\psi$ (recall that $\mathbb{P}_+\psi \stackrel{\text{def}}{=} \sum_{m\ge 0} \hat{\psi}(m) z^m$). The fact that $\varphi \in BMO$ follows from the identity

$$\mathbb{P}_+\psi = \frac{1}{2}(\psi + \mathrm{i}\tilde{\psi} + \hat{\psi}(0))$$

(see Appendix 2.1).

Conversely, let $\varphi \in H^1$ and $\varphi = \xi + \tilde{\eta}$, $\xi, \eta \in L^\infty$. Then

$$\varphi = \mathbb{P}_+\varphi = \mathbb{P}_+\xi + \mathbb{P}_+\tilde{\eta} = \mathbb{P}_+(\xi - \mathrm{i}\eta + \mathrm{i}\hat{\eta}(0))$$

(see Appendix 2.1). ■

Clearly, Γ is a bounded operator if the function φ defined by (1.5) is bounded. However, the operator Γ can be bounded even with an unbounded φ. Let us consider an important example of such a Hankel matrix.

The Hilbert Matrix

Let $\alpha_n = \frac{1}{n+1}$, $n \geq 0$. The corresponding Hankel matrix

$$\Gamma = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots \\ 1/2 & 1/3 & 1/4 & & \ddots \\ 1/3 & 1/4 & & \ddots & \\ 1/4 & & \ddots & & \\ \vdots & \ddots & & & \end{pmatrix}$$

is called the *Hilbert matrix*. Clearly, the function

$$\sum_{n\geq 0} \frac{1}{n+1} z^n$$

is unbounded in $\mathbb{D}$. However, Γ is bounded. Indeed, consider the function ψ on $\mathbb{T}$ defined by

$$\psi(e^{\mathrm{i}t}) = \mathrm{i}e^{-\mathrm{i}t}(\pi - t), \quad t \in [0, 2\pi).$$

It is easy to see that

$$\hat{\psi}(n) = \frac{1}{n+1}, \quad n \geq 0,$$

and $\|\psi\|_{L^\infty} = \pi$. It follows from Theorem 1.1 that Γ is bounded and $\|\Gamma\| \leq \pi$. In fact, $\|\Gamma\| = \pi$ (see §1.5, where we shall prove a stronger result).

Hankel Operators on the Hardy Class

We are going to consider another realization of Hankel operators. Namely, we define Hankel operators on the Hardy class H^2 of functions on the unit circle. This realization is extremely important in applications.

Let φ be a function in the space L^2 on the unit circle. We define the Hankel operator $H_\varphi : H^2 \to H^2_- \stackrel{\text{def}}{=} L^2 \ominus H^2$ on the dense subset of polynomials in H^2 by

$$H_\varphi f = \mathbb{P}_- \varphi f, \tag{1.6}$$

where $\mathbb{P}_-$ is the orthogonal projection from L^2 onto H^2_- (clearly, $\mathbb{P}_- = I - \mathbb{P}_+$). The function φ is called *a symbol* of the Hankel operator H_φ (as we shall see below a Hankel operator has many different symbols).

If we consider the standard orthonormal bases $\{z^k\}_{k\geq 0}$ in H^2 and $\{\bar{z}^j\}_{j\geq 1}$ in H^2_-, then H_φ has Hankel matrix $\{\hat{\varphi}(-j-k)\}_{j\geq 1, k\geq 0}$ in these bases: $(H_\varphi z^k, \bar{z}^j) = \hat{\varphi}(-j-k)$.

Theorem 1.1 admits the following reformulation.

Theorem 1.3. *Let $\varphi \in L^2$. The following statements are equivalent:*
(i) H_φ *is bounded on* H^2*;*
(ii) *there exists a function* ψ *in* L^∞ *such that*

$$\hat{\psi}(m) = \hat{\varphi}(m), \quad m < 0; \tag{1.7}$$

(iii) $\mathbb{P}_-\varphi \in BMO$.
If one of the conditions (i)–(iii) *is satisfied, then*

$$\|H_\varphi\| = \inf\{\|\psi\|_{L^\infty} : \hat{\psi}(m) = \hat{\varphi}(m),\ m < 0\}. \tag{1.8}$$

Equality (1.7) is equivalent to the fact that $H_\varphi = H_\psi$. Thus (ii) means that H_φ is bounded if and only if it has a bounded symbol. Thus the operators H_φ with $\varphi \in L^\infty$ exhaust the class of bounded Hankel operators. If $\varphi \in L^\infty$, then (1.8) can be rewritten in the following way:

$$\|H_\varphi\| = \inf\{\|\varphi - f\|_\infty : f \in H^\infty\}. \tag{1.9}$$

In other words, the norm of H_φ is the distance from φ to H^∞. That is why the problem of approximation of an L^∞ function by bounded analytic functions is called the *Nehari problem.*

Symbols of Minimal Norm

Let $\varphi \in L^\infty$. Consider the infimum on the right-hand side of (1.9). It is attained for any $\varphi \in L^\infty$. Indeed, let $\{f_n\}_{n\ge1}$ be a sequence of functions in H^∞ such that

$$\lim_{n\to\infty} \|\varphi - f_n\|_\infty = \|H_\varphi\|.$$

Clearly, $\{f_n\}_{n\ge1}$ is a bounded sequence. So there is a subsequence $\{f_{n_k}\}$ that converges to a function f in H^∞ in the weak topology induced by L^1. Then

$$\|\varphi - f\|_\infty \le \lim_{n\to\infty} \|\varphi - f_n\|_\infty = \|H_\varphi\|.$$

On the other hand, it follows from Theorem 1.3 that

$$\|\varphi - f\|_\infty \ge \|H_\varphi\|.$$

Such a function f is called *a best approximation of* φ *by analytic functions in the* L^∞*-norm.*

Thus we have proved that for any bounded Hankel operator there exists a symbol of minimal norm (equal to the norm of the operator). A natural question arises of whether such a symbol of minimal norm is unique. We shall see soon that it is not the case in general. However, in the case when the Hankel operator attains its norm on the unit ball of H^2, i.e.,

$$\|H_\varphi g\|_2 = \|H_\varphi\| \cdot \|g\|_2$$

for some nonzero $g \in H^2$, we do have uniqueness as the following result shows.

Theorem 1.4. *Let φ be a function in L^∞ such that H_φ attains its norm on the unit ball of H^2. Then there exists a unique function f in H^∞ such that*

$$\|\varphi - f\|_\infty = \operatorname{dist}_{L^\infty}(\varphi, H^\infty).$$

Moreover, $\varphi - f$ has constant modulus almost everywhere on $\mathbb{T}$ and admits a representation

$$\varphi - f = \|H_\varphi\| \cdot \bar{z}\bar{\vartheta}\frac{\bar{h}}{h}, \tag{1.10}$$

where h is an outer function in H^2 and ϑ is an inner function.

Proof. Without loss of generality we can assume that $\|H_\varphi\| = 1$. Let g be a function in H^2 such that $1 = \|g\|_2 = \|H_\varphi g\|_2$. Let f be a best approximation of φ, i.e., $f \in H^\infty$ and $\|\varphi - f\|_\infty = 1$. We have

$$1 = \|H_\varphi g\|_2 = \|\mathbb{P}_-(\varphi - f)g\|_2 \le \|(\varphi - f)g\|_2 \le \|g\|_2 = 1.$$

Therefore the inequalities in the above chain are in fact equalities. The fact that

$$\|\mathbb{P}_-(\varphi - f)g\|_2 = \|(\varphi - f)\|_2$$

means that $(\varphi - f)g \in H^2_-$ and so

$$H_\varphi g = H_{\varphi - f}\, g = (\varphi - f)g. \tag{1.11}$$

The function g being in H^2 is nonzero almost everywhere on $\mathbb{T}$ (see Appendix 2.1), and so

$$f = \varphi - \frac{H_\varphi g}{g}.$$

Hence, f is determined uniquely by H_φ (the ratio $(H_\varphi g)/g$ does not depend on the choice of g).

Since $\|\varphi - f\|_\infty = 1$, the equality

$$\|(\varphi - f)g\|_2 = \|g\|_2$$

means that $|\varphi(\zeta) - f(\zeta)| = 1$ a.e. on the set $\{\zeta : g(\zeta) \ne 0\}$, which is of full measure because $g \in H^2$. Thus $\varphi - f$ has modulus one almost everywhere on $\mathbb{T}$.

Consider the functions g and $\bar{z}\overline{H_\varphi g}$ in H^2. It follows from (1.11) that they have the same moduli. Therefore they admit factorizations

$$g = \vartheta_1 h, \quad \bar{z}\overline{H_\varphi g} = \vartheta_2 h,$$

where h is an outer function in H^2, and ϑ_1 and ϑ_2 are inner functions. Consequently,

$$\varphi - f = \frac{H_\varphi g}{g} = \frac{\bar{z}\bar{\vartheta}_2\bar{h}}{\vartheta_1 h} = \bar{z}\bar{\vartheta}_1\bar{\vartheta}_2\frac{\bar{h}}{h},$$

which proves (1.10) with $\vartheta = \vartheta_1\vartheta_2$. ∎

Remark. Under the hypotheses of Theorem 1.4 the best approximant f can be expressed in the form

$$f = \varphi - \frac{H_\varphi g}{g}, \tag{1.12}$$

where g is a maximizing function (i.e., $\|H_\varphi g\|_2 = \|H_\varphi\| \cdot \|g\|_2,\ g \neq \mathbb{O}$). The function on the right-hand side does not depend on the choice of g.

Corollary 1.5. *Suppose that φ satisfies the hypotheses of Theorem 1.4 and the Fourier coefficients $\hat{\varphi}(n)$ are real for $n < 0$. Let f be the unique best uniform approximation of φ by bounded analytic functions. Then all Fourier coefficients of $\varphi - f$ are real.*

Proof. By replacing φ with $(\varphi(z) + \overline{\varphi(\bar{z})})/2$, we may assume that all Fourier coefficients of φ are real. It is easy to see that if f is a best uniform approximation of φ, then $\overline{f(\bar{z})}$ is also a best approximation. It follows now from Theorem 1.4 that $f(z) = \overline{f(\bar{z})}$. ■

Corollary 1.6. *If H_φ is a compact Hankel operator, then the conclusion of Theorem 1.4 holds.*

Indeed any compact operator attains its norm. ■

The following example shows that in general a function in L^∞ can have many best approximations by analytic functions.

Example. Let Ω be the Jordan domain defined by

$$\Omega = \left\{ \zeta \in \mathbb{C} : |\zeta| + \frac{|1-\zeta|}{2} < 1 \right\}.$$

It is easy to see that $1 \in \partial\Omega$. Let ω be a conformal map of $\mathbb{D}$ onto Ω such that $\omega(1) = 1$ (since Ω has smooth boundary, the conformal map extends to a homeomorphism of $\mathbb{T}$ onto $\partial\Omega$; see Goluzin [1], Ch. X, §1). Let B be a Blaschke product such that 1 is an accumulation point of the zeros of B. Then $\varphi = \bar{B}\omega$ has different best approximations by analytic functions.

Let us first show that $\operatorname{dist}_{L^\infty}(\varphi, H^\infty) = 1$. Indeed, $\|\varphi\|_{L^\infty} \le 1$ and if $f \in H^\infty$, then

$$\|\varphi - f\|_\infty = \|\omega - Bf\|_\infty = \sup_{\zeta \in \mathbb{D}} |\omega(\zeta) - B(\zeta)f(\zeta)| \ge 1,$$

since $|\omega(\zeta_n)| \to 1$, where the ζ_n are the zeros of B. Let now $f = (1-\omega)/2$. Then $f \in H^\infty$ and

$$\begin{aligned} \|\varphi - f\|_\infty &= \|\omega - Bf\|_\infty = \sup_{\zeta \in \mathbb{D}} \left| \omega(\zeta) - \frac{1}{2}B(\zeta)(1-\omega(\zeta)) \right| \\ &\le \sup_{\zeta \in \mathbb{D}} \left(|\omega(\zeta)| + \frac{1}{2}|1-\omega(\zeta)| \right) \le 1, \end{aligned}$$

since $\omega(\zeta) \in \Omega$. Therefore $\mathbb{O}$ and $(1-\omega)/2$ are different best approximants to φ.

It will be shown in §5.1 that in the case when there are at least two best approximations to φ there exists a best approximation g such that $\varphi-g$ has constant modulus on $\mathbb{T}$ and a parametrization of all best approximations will be given. We shall also see in §5.1 that there are functions φ in L^∞ for which there exists only one best approximation f in H^∞ but $\varphi-f$ does not have constant modulus on $\mathbb{T}$.

Unimodular Symbols

The following theorem gives a simple sufficient condition for a Hankel operator to have a *unimodular symbol* (i.e., symbol whose modulus is equal to one almost everywhere).

Theorem 1.7. *Let H_φ be a Hankel operator such that $\|H_\varphi\| < 1$. Then there exists a function u such that $|u(\zeta)| = 1$ for almost all ζ in $\mathbb{T}$, $H_u = H_\varphi$, and* $\operatorname{dist}_{L^\infty}(\bar{z}u, H^\infty) = 1$.

Proof. Without loss of generality we may assume that $\|\varphi\|_\infty < 1$. Let $t \ge 0$. Consider the function $\varphi_t = \varphi + t$. Clearly,

$$\operatorname{dist}_{L^\infty}(\varphi_0, zH^\infty) \le \|\varphi\|_\infty < 1 \quad \text{and} \quad \lim_{t\to\infty} \operatorname{dist}_{L^\infty}(\varphi_t, zH^\infty) = \infty.$$

Hence, there exists $s > 0$ such that $\operatorname{dist}_{L^\infty}(\varphi_s, zH^\infty) = 1$. Consider the Hankel operator $H_{\bar{z}\varphi_s}$. Clearly,

$$\|H_{\bar{z}\varphi_s}\| = \operatorname{dist}_{L^\infty}(\bar{z}\varphi_s, H^\infty) = \operatorname{dist}_{L^\infty}(\varphi_s, zH^\infty) = 1.$$

On the other hand,

$$\|H_{\bar{z}\varphi_s}\|_{\mathrm{e}} \le \|H_{\bar{z}\varphi_s - s\bar{z}}\| = \|H_{\bar{z}\varphi}\| \le \|\varphi\|_\infty < 1.$$

By Theorem 1.4, there exists a unique function $f \in H^\infty$ such that $\|\bar{z}\varphi_s - f\|_\infty = 1$ and $|\bar{z}\varphi_s - f| = 1$ almost everywhere on $\mathbb{T}$. Put $u = \varphi_s - zf = \varphi + sz - zf$. Clearly, u is unimodular, $H_u = H_\varphi$, and $\operatorname{dist}_{L^\infty}(\bar{z}u, H^\infty) = 1$. ■

Remark. It follows easily from Corollary 1.5 and from the construction given in the proof of Theorem 1.7 that if under the hypotheses of Theorem 1.7 the Fourier coefficients $\hat{\varphi}(n)$ are real for $n < 0$, then there exists a unimodular function u with real Fourier coefficients such that $H_u = H_\varphi$ and $\operatorname{dist}_{L^\infty}(\bar{z}u, H^\infty) = 1$.

A Commutation Relation

We show here that the Hankel operators can be characterized as the operators that satisfy a certain commutation relation. We also establish an important property of the kernel of a Hankel operator.

Theorem 1.8. *Let R be a bounded operator from H^2 to H^2_-. Then R is a Hankel operator if and only if it satisfies the following commutation* *relation:*

$$\mathbb{P}_-\mathcal{S}R = RS, \tag{1.13}$$

where S is the shift operator on H^2 and $\mathcal{S}$ is bilateral shift on L^2.

Proof. Let $R = H_\varphi$, $\varphi \in L^\infty$. Then

$$\mathbb{P}_-\mathcal{S}Rf = \mathbb{P}_- zH_\varphi f = \mathbb{P}_- z\mathbb{P}\varphi f = \mathbb{P}_- z\varphi f = H_\varphi zf.$$

Suppose now that R satisfies (1.13). Let $n \geq 1$, $k \geq 1$. We have

$$\begin{aligned}(Rz^n, \bar{z}^k) &= (RSz^{n-1}, \bar{z}^k) = (\mathbb{P}_-\mathcal{S}Rz^{n-1}, \bar{z}^k)\\ &= (\mathcal{S}Rz^{n-1}, \bar{z}^k) = (Rz^{n-1}, \bar{z}^{k+1}).\end{aligned}$$

Therefore R has Hankel matrix in the bases $\{z^n\}_{n\geq 0}$ of H^2 and $\{\bar{z}^k\}_{k\geq 1}$ of H^2_-. It follows from Theorems 1.1 and 1.3 that $R = H_\varphi$ for some φ in L^∞. ■

Corollary 1.9. *Let R be a Hankel operator on H^2. Then* $\operatorname{Ker} R$ *is an invariant subspace of S.*

The result follows immediately from (1.13). ■

Note that the fact that the kernel of a Hankel operator is S-invariant also follows directly from the definition of a Hankel operator on H^2.

Commutators of Multiplication Operators and the Riesz Projection

We consider here operators that play an important role in harmonic analysis. For a function φ in L^2 let M_φ be the multiplication operator defined on the dense subset of trigonometric polynomials $\boldsymbol{\mathcal{P}}$ in L^2 by $M_\varphi f = \varphi f$, $f \in \boldsymbol{\mathcal{P}}$. Consider now the commutator $\mathcal{C}_\varphi \stackrel{\text{def}}{=} [M_\varphi, \mathbb{P}_+]$ defined on $\boldsymbol{\mathcal{P}}$ by

$$\mathcal{C}_\varphi = [M_\varphi, \mathbb{P}_+]f = \varphi\mathbb{P}_+ f - \mathbb{P}_+\varphi f.$$

We are going to obtain a necessary and sufficient condition for $\mathcal{C}_\varphi$ to be bounded on L^2.

Let us obtain a more explicit formula for $\mathcal{C}_\varphi f$. First we find a formula for $\mathbb{P}_+ f$. Suppose first that f is a smooth function (e.g., a trigonometric polynomial). We claim that

$$(\mathbb{P}_+ f)(\zeta) = \text{p.v.}\int_{\mathbb{T}} \frac{f(\tau)}{1-\bar{\tau}\zeta}d\boldsymbol{m}(\tau) + \frac{1}{2}f(\zeta), \tag{1.14}$$

where

$$\text{p.v.}\int_{\mathbb{T}} \frac{f(\tau)}{1-\bar{\tau}\zeta}d\boldsymbol{m}(\tau) \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0}\int_{|\tau-\zeta|>\varepsilon} \frac{f(\tau)}{1-\bar{\tau}\zeta}d\boldsymbol{m}(\tau)$$

("p.v." stands for "principal value").

Indeed,

$$\text{p.v.}\int_{\mathbb{T}} \frac{f(\tau)}{1-\bar{\tau}\zeta}d\boldsymbol{m}(\tau) = \int_{\mathbb{T}} \frac{f(\tau)-f(\zeta)}{1-\bar{\tau}\zeta}d\boldsymbol{m}(\tau) + f(\zeta)\,\text{p.v.}\int_{\mathbb{T}} \frac{1}{1-\bar{\tau}\zeta}d\boldsymbol{m}(\tau).$$

Clearly,

$$\begin{aligned}\text{p.v.}\int_{\mathbb{T}}\frac{1}{1-\bar{\tau}\zeta}\boldsymbol{dm}(\tau) &= \text{p.v.}\int_{\mathbb{T}}\frac{1}{1-\bar{\tau}}\boldsymbol{dm}(\tau)\\ &= \int_{\operatorname{Im}\tau>0}\left(\frac{1}{1-\bar{\tau}}+\frac{1}{1-\tau}\right)\boldsymbol{dm}(\tau)\\ &= 2\int_{\operatorname{Im}\tau>0}\operatorname{Re}\frac{1}{1-\tau}\boldsymbol{dm}(\tau).\end{aligned}$$

It is easy to see that for $\tau\in\mathbb{T}\setminus\{1\}$ we have $\operatorname{Re}\frac{1}{1-\tau}=\frac{1}{2}$, and so

$$\text{p.v.}\int_{\mathbb{T}}\frac{1}{1-\bar{\tau}\zeta}\boldsymbol{dm}(\tau)=\frac{1}{2}.$$

On the other hand, it is easy to verify that for $f=z^n$ we have

$$\int_{\mathbb{T}}\frac{f(\tau)-f(\zeta)}{1-\bar{\tau}\zeta}\boldsymbol{dm}(\tau)=\begin{cases}0, & n\geq 0,\\ -\zeta^n, & n<0.\end{cases}$$

Therefore

$$\text{p.v.}\int_{\mathbb{T}}\frac{f(\tau)}{1-\bar{\tau}\zeta}\boldsymbol{dm}(\tau)=-(\mathbb{P}_-f)(\zeta)+\frac{1}{2}f(\zeta)=(\mathbb{P}_+f)(\zeta)-\frac{1}{2}f(\zeta),$$

which implies (1.14).

In fact, for any function f in L^2 the integral

$$\text{p.v.}\int_{\mathbb{T}}\frac{f(\tau)}{1-\bar{\tau}\zeta}\boldsymbol{dm}(\tau)$$

exists for almost all $\zeta\in\mathbb{T}$ and determines a bounded linear operator on L^2 (see Stein [2], Ch. II, §4). Hence, (1.14) holds for any function f in L^2 and for almost all $\zeta\in\mathbb{T}$.

Obviously, (1.14) implies the following formula for $\mathcal{C}_\varphi$:

$$\mathcal{C}_\varphi f=\text{p.v.}\int_{\mathbb{T}}\frac{\varphi(\zeta)-\varphi(\tau)}{1-\bar{\tau}\zeta}f(\tau)\boldsymbol{dm}(\tau),\quad f\in\boldsymbol{\mathcal{P}}.\tag{1.15}$$

The following theorem gives a boundedness criterion for $\mathcal{C}_\varphi$.

Theorem 1.10. *Let $\varphi\in L^2$. The commutator $\mathcal{C}_\varphi$ is bounded on L^2 if and only if $\varphi\in BMO$.*

Proof. Put $f_+\stackrel{\text{def}}{=}\mathbb{P}_+f$, $f_-\stackrel{\text{def}}{=}\mathbb{P}_-f$, $\varphi_+\stackrel{\text{def}}{=}\mathbb{P}_+\varphi$, and $\varphi_-\stackrel{\text{def}}{=}\mathbb{P}_-\varphi$. We have

$$\begin{aligned}\mathcal{C}_\varphi f &= \varphi\mathbb{P}_+f-\mathbb{P}_+\varphi f=\varphi_+f_++\varphi_-f_+-\varphi_+f_+-\mathbb{P}_+\varphi_-f_+-\mathbb{P}_+\varphi_+f_-\\ &= \mathbb{P}_-\varphi_-f_+-\mathbb{P}_+\varphi_+f_-=H_{\varphi_-}f_+-(H_{\overline{\varphi_+}})^*f_-.\end{aligned}$$

It is easy to see now that $\mathcal{C}_\varphi$ is bounded if and only if both Hankel operators H_{φ_-} and $H_{\overline{\varphi_+}}$ are bounded. The result now follows from Theorem 1.3. ■

2. Hankel Operators and Compressed Shift

We begin this section with the study of the commutant of compressions of the shift operator on H^2 to its coinvariant subspaces. We prove Sarason's theorem that describes this commutant in terms of a functional calculus. We use an approach suggested by N.K. Nikol'skii that is based on Hankel operators and the Nehari theorem. Then we establish an important formula that relates functions of such a compression with Hankel operators. We also describe the kernel and the closed range of Hankel operators, the partially isometric Hankel operators, and the Hankel operators with closed ranges.

Let ϑ be an inner function. Put

$$K_\vartheta = H^2 \ominus \vartheta H^2.$$

By Beurling's theorem (see Appendix 2.2) any nontrivial invariant subspace of the backward shift operator S^* on H^2 coincides with K_ϑ for some inner function ϑ. Denote by $S_{[\vartheta]}$ the compression of the shift operator S to K_ϑ:

$$S_{[\vartheta]} f = P_\vartheta z f, \quad f \in K_\theta, \tag{2.1}$$

where P_ϑ is the orthogonal projection from H^2 onto K_ϑ. It is easy to see that

$$S^*_{[\vartheta]} = S^* | K_\vartheta.$$

Since multiplication by ϑ is a unitary operator on L^2, it follows that the orthogonal projection onto ϑH^2 is given by

$$f \mapsto \vartheta \mathbb{P}_+ \bar{\vartheta} f, \quad f \in L^2.$$

Therefore

$$P_\vartheta f = f - \vartheta \mathbb{P}_+ \bar{\vartheta} f = \vartheta \mathbb{P}_- \bar{\vartheta} f, \quad f \in H^2. \tag{2.2}$$

Note that $S_{[\vartheta]}$ is the so-called model operator in the Sz.-Nagy–Foias functional model. Any contraction T for which $\lim\limits_{n\to\infty} T^{*n} = \mathbb{O}$ in the strong operator topology and

$$\operatorname{rank}(I - T^*T) = \operatorname{rank}(I - TT^*) = 1$$

is unitarily equivalent to $S_{[\vartheta]}$ for some inner function ϑ (see Sz.-Nagy and Foias [1] and N.K. Nikol'skii [2]).

The operator $S_{[\vartheta]}$ admits an H^∞ functional calculus. Indeed, given $\varphi \in H^\infty$, we define the operator $\varphi(S_{[\vartheta]})$ by

$$\varphi(S_{[\vartheta]}) f = P_\vartheta \, \varphi f, \quad f \in K_\vartheta. \tag{2.3}$$

Clearly, this functional calculus is linear. Let us show that it is multiplicative. Let $\varphi,\ \psi \in H^\infty$. We have

$$((\varphi\psi)(S_{[\vartheta]})) f = P_\vartheta \, \varphi\psi f = P_\vartheta \, \varphi P_\vartheta \, \psi f + P_\vartheta \, \varphi \, Q_\vartheta \, \psi f, \tag{2.4}$$

where Q_ϑ is the orthogonal projection from H^2 onto ϑH^2. Since $Q_\vartheta\,\psi H^2 \in \vartheta H^2$, it follows that $\varphi\,Q_\vartheta\,\psi f \in \vartheta H^2$ and so $P_\vartheta\,\varphi\,Q_\vartheta\,\psi f = \mathbb{O}$. Therefore it follows from (2.4) that

$$(\varphi\psi)(S_{[\vartheta]}) = \varphi(S_{[\vartheta]})\psi(S_{[\vartheta]}).$$

It follows from (2.4) that $\varphi(S_{[\vartheta]})$ commutes with $S_{[\vartheta]}$, $\varphi \in H^\infty$, and

$$\|\varphi(S_{[\vartheta]})\| \le \|\varphi\|_{H^\infty}$$

(von Neumann's inequality).

The following theorem, which is due to Sarason, describes the commutant of $S_{[\vartheta]}$. It is a special case of the commutant lifting theorem due to Sz.-Nagy and Foias (see Appendix 1.5).

Theorem 2.1. *Let T be an operator that commutes with $S_{[\vartheta]}$. Then there exists a function φ in H^∞ such that $T = \varphi(S_{[\vartheta]})$ and $\|T\| = \|\varphi\|_{H^\infty}$.*

To prove the theorem, we need a lemma that relates operators in the commutant of $S_{[\vartheta]}$ with Hankel operators.

Lemma 2.2. *Let T be an operator on K_ϑ. Consider the operator $A : H^2 \to H^2_-$ defined by*

$$Af = \bar{\vartheta}\,TP_\vartheta\,f. \tag{2.5}$$

Then T commutes with $S_{[\vartheta]}$ if and only if A is a Hankel operator.

Proof. A is a Hankel operator if and only if

$$\mathbb{P}_- zAf = Azf, \quad f \in H^2, \tag{2.6}$$

(see (1.13)), which means

$$\mathbb{P}_- z\bar{\vartheta}TP_\vartheta f = \bar{\vartheta}TP_\vartheta zf, \quad f \in H^2,$$

which in turn is equivalent to

$$\vartheta\mathbb{P}_-\bar{\vartheta}zTP_\vartheta f = TP_\vartheta zf, \quad f \in H^2. \tag{2.7}$$

We have by (2.2)

$$\vartheta\mathbb{P}_-\bar{\vartheta}zTP_\vartheta f = P_\vartheta zTP_\vartheta f = S_{[\vartheta]}TP_\vartheta f.$$

Since obviously the left-hand side and the right-hand side of (2.7) are zero for $f \in \vartheta H^2$, it follows from (2.1) that (2.6) is equivalent to the equality

$$S_{[\vartheta]}Tf = TS_{[\vartheta]}f, \quad f \in K_\vartheta. \quad \blacksquare$$

Proof of Theorem 2.1. By Lemma 2.2, the operator A defined by (2.5) is a Hankel operator. By the Nehari theorem, there exists a function ψ in L^∞ such that $\|\psi\|_\infty = \|A\|$ and $H_\psi = A$, i.e.,

$$\mathbb{P}_-\psi f = \bar{\vartheta}\,TP_\vartheta f, \quad f \in H^2.$$

It follows that $\mathbb{P}_-\psi f = \mathbb{O}$ for any $f \in \vartheta H^2$. That means that $H_{\psi\vartheta} = \mathbb{O}$. Put $\varphi = \psi\vartheta$. Clearly, $\varphi \in H^\infty$ and $\psi = \bar{\vartheta}\varphi$. We have

$$\bar{\vartheta}\,Tf = \mathbb{P}_-\bar{\vartheta}\varphi f, \quad f \in K_\vartheta,$$

so

$$Tf = \vartheta\,\mathbb{P}_-\bar{\vartheta}\varphi f = P_\vartheta\,\varphi f = \varphi(S_{[\vartheta]})f, \quad f \in K_\vartheta. \tag{2.8}$$

Obviously,

$$\|T\| = \|A\| = \|\psi\|_\infty = \|\varphi\|_\infty,$$

which completes the proof. ■

Remark. Formula (2.8) implies the following remarkable relation between Hankel operators and functions of model operators. Let ϑ be an inner function, $\varphi \in H^\infty$, then

$$\varphi(S_{[\vartheta]}) = M_\vartheta H_{\bar{\vartheta}\varphi}|K_\vartheta, \tag{2.9}$$

where M_ϑ is multiplication by ϑ. This formula shows that $\varphi(S_{[\vartheta]})$ has the same metric properties as $H_{\bar{\vartheta}\varphi}$ (e.g., compactness, nuclearity, etc.; we shall discuss that later).

Formula (2.9) relates the Hankel operators of the form $H_{\bar{\vartheta}\varphi}$ with functions of model operators. The following theorem describes the Hankel operators of that form.

Theorem 2.3. *Let $\psi \in L^\infty$. The following are equivalent:*
(i) *H_ψ has nontrivial kernel;*
(ii) Range *H_ψ is not dense in H^2_-;*
(iii) *$\psi = \bar{\vartheta}\varphi$ for some inner function ϑ and some function φ in H^∞.*

Proof. (i)⇒(iii). Suppose that $\operatorname{Ker} H_\psi \ne \{\mathbb{O}\}$ and $H_\psi \ne \mathbb{O}$. Then $\operatorname{Ker} H_\psi$ is a nontrivial invariant subspace of the shift operator S (see Corollary 1.9). By Beurling's theorem (see Appendix 2.2), $\operatorname{Ker} H_\psi = \vartheta H^2$ for some inner function ϑ. So $\mathbb{P}_-\psi\vartheta f = \mathbb{O}$ for any f in H^2. Therefore $H_{\psi\vartheta} = \mathbb{O}$ and so $\varphi = \psi\vartheta \in H^\infty$. Hence, $\psi = \bar{\vartheta}\varphi$.

(iii)⇒(ii). For $g = \bar{z}\bar{\vartheta}$ we have

$$(H_{\bar{\vartheta}\varphi}f, g) = (\bar{\vartheta}\varphi f, \bar{z}\bar{\vartheta}) = (f, \bar{z}\bar{\varphi}) = 0$$

for any $f \in H^2$, and so $g \perp \operatorname{Range} H_\psi$.

(ii)⇒(i). Suppose that $g \in H^2_-$, $g \perp \operatorname{Range} H_\psi$, and $g \ne \mathbb{O}$. Then $g = \bar{z}\bar{h}$, where $h \in H^2$. Then for $f \in H^2$ we have

$$0 = (H_\psi f, g) = (\psi f, \bar{z}\bar{h}) = (f, \bar{\psi}\bar{z}\bar{h}),$$

and so $\bar{\psi}\bar{z}\bar{h} \in H^2_-$, i.e., $\psi h \in H^2$, which means that $h \in \operatorname{Ker} H_\psi$. ■

Let us now describe the kernel and the closure of the range of Hankel operators of the form $H_{\bar{\vartheta}\varphi}$ when $\varphi \in H^\infty$ and ϑ is inner. Recall that two H^p functions are called *coprime* if they have no common nonconstant inner divisors.

Theorem 2.4. *Let $\varphi \in H^\infty$ and let ϑ be an inner function such that φ and ϑ are coprime. Then*

$$\operatorname{Ker} H_{\bar{\vartheta}\varphi} = \vartheta H^2 \quad \textit{and} \quad \operatorname{clos}\operatorname{Range} H_{\bar{\vartheta}\varphi} = H^2_- \ominus \bar{\vartheta} H^2_-.$$

Proof. Obviously, if $f \in \vartheta H^2$, then $H_{\bar\vartheta\varphi} f = \mathbb{O}$. Suppose that $f \in H^2$ and $H_{\bar\vartheta\varphi} f = \mathbb{O}$. Then $\bar\vartheta\varphi f \in H^2$, and so $\varphi f \in \vartheta H^2$. Since φ and ϑ are coprime, it follows that f is divisible by ϑ, i.e., $f \in \vartheta H^2$.

It follows immediately from (2.9) that $\operatorname{Range} H_{\bar\vartheta\varphi} \subset H^2_- \ominus \bar\vartheta H^2_-$. Let us show that $\operatorname{Range} H_{\bar\vartheta\varphi}$ is dense in $H^2_- \ominus \bar\vartheta H^2_-$. Suppose that $h \in H^2_-$ and $h \perp H_{\bar\vartheta\varphi} g$ for every $g \in H^2$. Then

$$0 = (\mathbb{P}_- \bar\vartheta\varphi g, h) = (\varphi g, \vartheta h) = \int_{\mathbb{T}} g\varphi\bar\vartheta\bar h \, d\boldsymbol{m}, \quad g \in H^2.$$

Hence, $\bar\varphi\vartheta h \in H^1_-$. Since φ and ϑ are coprime, it follows that $\vartheta h \in H^2_-$, i.e., $h \in \bar\vartheta H^2_-$. ■

The following result shows that the Hankel operators involved in (2.9) form a dense subset in the space of bounded Hankel operators.

Theorem 2.5. *The set*

$$\{\bar\vartheta\varphi : \ \vartheta \text{ is inner}, \ \varphi \in H^\infty\}$$

is dense in L^∞.

Proof. Let $E \subset \mathbb{T}$, $0 < \boldsymbol{m}E < 1$. Consider the function u in L^∞ defined by

$$u(\zeta) = \begin{cases} -1, & \zeta \in E, \\ 1, & \zeta \notin E. \end{cases}$$

Let $f_n = 1 + \exp n(u + \mathrm{i}\tilde u)$. Then $f_n \in H^\infty$ and $|f_n|$ is separated away from zero on $\mathbb{T}$. Indeed, $|f_n(\zeta)| \ge 1 - e^{-n}$ for $\zeta \in E$ and $|f_n(\zeta)| \ge e^n - 1$ for $\zeta \notin E$. So f_n admits a factorization $f_n = \vartheta_n h_n$, where ϑ_n is an inner function and h_n is an outer function invertible in H^∞. The result now follows from the obvious fact that $f_n^{-1} = \bar\vartheta_n h_n^{-1}$ converges uniformly to the characteristic function of E. ■

Let us now describe the Hankel operators that are partial isometries. Recall that an operator V from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$ is called a *partial isometry* if the restriction of V to $(\operatorname{Ker} V)^\perp$ maps isometrically $(\operatorname{Ker} V)^\perp$ onto $\operatorname{Range} V$. The subspace $(\operatorname{Ker} V)^\perp$ is called the *initial space* of the partial isometry V while the subspace $\operatorname{Range} V$ is called the *final space* of the partial isometry V.

Theorem 2.6. *A Hankel operator from* H^2 *to* H^2_- *is a partial isometry if and only if it has the form* $H_{\bar\vartheta}$ *for an inner function* ϑ. *The initial space of* $H_{\bar\vartheta}$ *is* K_ϑ *while the final space of* $H_{\bar\vartheta}$ *is* $H^2_- \ominus \bar\vartheta H^2_-$.

Proof. Let us first show that $H_{\bar\vartheta}$ is a partial isometry. If $f \in \vartheta H^2$, then obviously, $H_{\bar\vartheta} f = \mathbb{O}$. On the other hand, if $f \perp \vartheta H^2$, then $\bar\vartheta f \in H^2_-$, and so $\|H_{\bar\vartheta} f\|_2 = \|f\|_2$. It remains to observe that

$$H_{\bar\vartheta}(H^2 \ominus \vartheta H^2) = \bar\vartheta H^2 \ominus H^2 = H^2_- \ominus \bar\vartheta H^2_-.$$

Consider now an arbitrary partially isometric Hankel operator. Since obviously, $\|H_\psi z^n\| \to 0$ for any $\psi \in L^\infty$, a Hankel operator cannot be an isometry. Thus a partially isometric Hankel operator must have nontrivial kernel. By Theorem 2.3, it has the form $H_{\bar{\vartheta}\varphi}$, where ϑ is an inner function and $\varphi \in H^\infty$. Since $\|H_{\bar{\vartheta}\varphi}\| \le 1$, we may assume that $\|\varphi\|_\infty \le 1$. We may also assume that ϑ and φ are coprime. By Theorem 2.4, $\operatorname{Ker} H_{\bar{\vartheta}\varphi} = \vartheta H^2$. Since $H_{\bar{\vartheta}\varphi}$ is a partial isometry, it follows that multiplication by φ is an isometric map of K_ϑ into itself. In particular, if $f \in K_\vartheta$, we have

$$\int_{\mathbb{T}} |\varphi f|^2 d\boldsymbol{m} = \int_{\mathbb{T}} |f|^2 d\boldsymbol{m}.$$

Since $|f| > 0$ almost everywhere on $\mathbb{T}$, it follows that $|\varphi| = 1$ almost everywhere, i.e., φ is an inner function. Let us show that φ is constant.

Suppose that f is a nonzero function in K_ϑ. Then $\varphi^n f \in K_\vartheta$ for any $n \in \mathbb{Z}_+$, i.e.,

$$(\varphi^n f, \vartheta g) = (\varphi^n \bar{\vartheta} f, g) = 0$$

for any $g \in H^2$. Hence, $\varphi^n \bar{\vartheta} f \in H^2_-$, and so $\bar{\varphi}^n \vartheta \bar{z} \bar{f} \in H^2$, $n \in \mathbb{Z}_+$. Put $h = \vartheta \bar{z} \bar{f} \in H^2$. Then $\bar{\varphi}^n h \in H^2$ for any $n \in \mathbb{Z}_+$, i.e., h is divisible by φ^n, which is impossible unless φ is constant. This completes the proof. ■

We conclude this section with a description of the Hankel operators with closed ranges. First we determine when functions of model operators $S_{[\vartheta]}$ are invertible.

We say that H^∞ functions g_1 and g_2 satisfy the *corona condition* if there exists $\delta > 0$ such that

$$|g_1(\zeta)| + |g_2(\zeta)| \ge \delta, \quad \zeta \in \mathbb{D}.$$

By the Carleson corona theorem (see Appendix 2.1), g_1 and g_2 satisfy the corona condition if and only if there exist functions ψ_1 and ψ_2 in H^∞ such that

$$\psi_1(\zeta) g_1(\zeta) + \psi_2(\zeta) g_2(\zeta) = 1, \quad \zeta \in \mathbb{D}.$$

Theorem 2.7. *Let ϑ be an inner function and let $\varphi \in H^\infty$. The operator $\varphi(S_{[\vartheta]})$ is invertible if and only if φ and ϑ satisfy the corona condition.*

Proof. Suppose that φ and ϑ satisfy the corona condition, and u and v are functions in H^∞ such that $\varphi u + \vartheta v = \mathbf{1}$. Then

$$\varphi(S_{[\vartheta]}) u(S_{[\vartheta]}) + \vartheta(S_{[\vartheta]}) v(S_{[\vartheta]}) = I,$$

and since obviously, $\vartheta(S_{[\vartheta]}) = \mathbb{O}$, it follows that

$$\varphi(S_{[\vartheta]}) u(S_{[\vartheta]}) = u(S_{[\vartheta]}) \varphi(S_{[\vartheta]}) = I,$$

i.e., $\varphi(S_{[\vartheta]})$ is invertible.

Assume now that $\varphi(S_{[\vartheta]})$ is invertible, and φ and ϑ do not satisfy the corona condition. Then there exists a sequence $\{\zeta_n\}_{n \ge 0}$ of points in $\mathbb{D}$ such that

$$\lim_{n \to \infty} (|\varphi(\zeta_n)| + |\vartheta(\zeta_n)|) = 0.$$

Define the H^∞ functions φ_n and ϑ_n by

$$\varphi - \varphi(\zeta_n) = b_n \varphi_n \quad \text{and} \quad \vartheta - \vartheta(\zeta_n) = b_n \vartheta_n,$$

where the elementary Blaschke factor b_n is defined by

$$b_n(z) \stackrel{\text{def}}{=} (z-\zeta_n)(1-\bar{\zeta}_n z)^{-1}.$$

Consider the normalized reproducing kernels k_n defined by

$$k_n(z) \stackrel{\text{def}}{=} (1-|\zeta_n|^2)^{1/2}(1-\bar{\zeta}_n z)^{-1}$$

(see §6 for more details). Clearly, $\|k_n\|_2 = 1$. Finally, we define the functions f_n in K_ϑ by

$$f_n = P_\vartheta k_n \vartheta_n.$$

Let us show that

$$\liminf_{n\to\infty} \|f_n\|_2 > 0, \quad \text{while} \quad \lim_{n\to\infty} \left\|\varphi(S_{[\vartheta]})f_n\right\|_2 = 0,$$

which will complete the proof.

We have

$$\begin{aligned}
\|f_n\|_2 &= \left\|\mathbb{P}_-\bar{\vartheta}k_n\vartheta_n\right\|_2 = \left\|\mathbb{P}_-\overline{(\vartheta-\vartheta(\zeta_n))}k_n\vartheta_n\right\|_2 \\
&= \left\|\mathbb{P}_-\bar{b}_n\bar{\vartheta}_n k_n\vartheta_n\right\|_2 = \left\|\mathbb{P}_-\bar{b}_n|\vartheta_n|^2 k_n\right\|_2 \\
&\ge \left\|\mathbb{P}_-\bar{b}_n k_n\right\|_2 - \left\|\mathbb{P}_-\bar{b}_n(|\vartheta_n|^2-1)k_n\right\|_2.
\end{aligned}$$

It is easy to see that

$$\mathbb{P}_-\bar{b}_n k_n = (1-|\zeta_n|^2)^{1/2}\mathbb{P}_-\frac{1}{z-\zeta_n} = (1-|\zeta_n|^2)^{1/2}\frac{1}{z-\zeta_n},$$

and so $\|\mathbb{P}_-\bar{b}_n k_n\|_2 = 1$. On the other hand,

$$\begin{aligned}
\left\|\mathbb{P}_-\bar{b}_n(|\vartheta_n|^2-1)k_n\right\|_2 &\le \left\|(|\vartheta_n|^2-1)k_n\right\|_2 = \left\|(|\vartheta-\vartheta(\zeta_n)|^2-1)k_n\right\|_2 \\
&\le \left\||\vartheta-\vartheta(\zeta_n)|^2-1\right\|_\infty \le 3|\vartheta(\zeta_n)| \to 0, \quad n\to\infty,
\end{aligned}$$

and so the norms of f_n are separated away from 0.

Finally, we have

$$\begin{aligned}
\left\|\varphi(S_{[\vartheta]})f_n\right\|_2 &= \left\|\mathbb{P}_-\bar{\vartheta}\varphi\vartheta\mathbb{P}_-\bar{\vartheta}k_n\vartheta_n\right\|_2 = \left\|\mathbb{P}_-\varphi\bar{\vartheta}k_n\vartheta_n\right\|_2 \\
&= \left\|\mathbb{P}_-\left(\overline{(\vartheta-\vartheta(\zeta_n))}\varphi k_n\vartheta_n + \overline{\vartheta(\zeta_n)}\varphi\vartheta_n\right)\right\|_2 \\
&= \left\|\mathbb{P}_-\overline{(\vartheta-\vartheta(\zeta_n))}\varphi k_n\vartheta_n\right\|_2 = \left\|\mathbb{P}_-\bar{b}_n\varphi k_n|\vartheta_n|^2\right\|_2 \\
&\le \left\|\mathbb{P}_-\bar{b}_n\varphi k_n\right\|_2 + \left\|\mathbb{P}_-\bar{b}_n\varphi k_n(|\vartheta_n|^2-1)\right\|_2.
\end{aligned}$$

Since $\lim\limits_{n\to\infty} \||\vartheta_n|^2-1\|_\infty = 0$, it follows that

$$\lim_{n\to\infty} \left\|\mathbb{P}_-\bar{b}_n\varphi k_n(|\vartheta_n|^2-1)\right\|_2 = 0.$$

On the other hand,

$$\begin{aligned} \left\|\mathbb{P}_-\bar{b}_n\varphi k_n\right\|_2 &\le \left\|\mathbb{P}_-\bar{b}_n(\varphi-\varphi(\zeta_n))k_n\right\|_2 + |\varphi(\zeta_n)|\cdot\|\mathbb{P}_-\bar{b}_nk_n\|_2 \\ &= \left\|\mathbb{P}_-\bar{b}_n\varphi_nb_nk_n\right\|_2 + |\varphi(\zeta_n)|\cdot\|\bar{b}_nk_n\|_2 \\ &= |\varphi(\zeta_n)| \to 0 \quad \text{as} \quad n\to\infty. \end{aligned}$$

This completes the proof. ■

Now we are in a position to obtain a description of the Hankel operators with closed ranges.

Theorem 2.8. *A Hankel operator H_ψ has closed range in H^2_- if and only if $\psi = \bar{\vartheta}\varphi$, where $\varphi \in H^\infty$, ϑ is an inner function such that φ and ϑ satisfy the corona condition.*

Proof. Since $\|H_\psi z^n\|_2 \to 0$ as $n \to \infty$, $\operatorname{Range} H_\psi \ne H^2_-$. Thus by Theorem 2.3, we may assume that $\psi = \bar{\vartheta}\varphi$, where ϑ is inner and $\varphi \in H^\infty$. We may also assume that ϑ and φ are coprime.

By Theorem 2.7, it remains to prove that $\varphi(S_{[\vartheta]})$ is invertible if and only if $H_{\bar{\vartheta}\varphi}$ has closed range in H^2_-. If $\varphi(S_{[\vartheta]})$ is invertible, it follows immediately from (2.9) that $H_{\bar{\vartheta}\varphi}$ has closed range.

Suppose now that $H_{\bar{\vartheta}\varphi}$ has closed range. By Theorem 2.4, $H_{\bar{\vartheta}\varphi}$ is a one-to-one map of K_ϑ onto $H^2_- \ominus \bar{\vartheta}H^2_-$. Clearly, multiplication by ϑ maps $H^2_- \ominus \bar{\vartheta}H^2_-$ onto K_ϑ. It follows now from (2.9) that $\varphi(S_{[\vartheta]})$ is invertible. ■

3. Hankel Operators of Finite Rank

One of the first results about Hankel matrices was a theorem of Kronecker that describes the Hankel matrices of finite rank.

Let $r = p/q$ be a rational function where p and q are polynomials. If p/q is in its lowest terms, the degree of r is, by definition,

$$\deg r = \max\{\deg p, \deg q\},$$

where $\deg p$ and $\deg q$ are the degrees of the polynomials p and q. It is easy to see that $\deg r$ is the sum of the multiplicities of the poles of r (including a possible pole at infinity).

We are going to describe the Hankel matrices of finite rank without any assumption on the boundedness of the matrix.

We identify sequences of complex numbers with the corresponding formal power series. If $a = \{a_j\}_{j\ge 0}$ is a sequence of complex numbers, we associate with it the formal power series

$$a(z) = \sum_{j\ge 0} a_j z^j.$$

The space of formal power series forms an algebra with respect to the following multiplication:

$$(ab)(z) = \sum_{m\ge 0}\left(\sum_{j=0}^{m} a_j b_{m-j}\right) z^m, \quad a = \sum_{j\ge 0} a_j z^j, \quad b = \sum_{j\ge 0} b_j z^j.$$

It is easy to see that $a(z) = \sum\limits_{j\ge 0} a_j z^j$ is invertible in the space of formal power series if and only if $a_0 \ne 0$.

Consider the shift operator S and the backward shift operator S^* defined on the space of formal power series in the following way:

$$(Sa)(z) = za(z), \quad S^* \sum_{j\ge 0} a_j z^j = \sum_{j\ge 0} a_{j+1} z^j.$$

Let $\alpha = \{\alpha_j\}_{j\ge 0}$ be a sequence of complex numbers that we identify with the corresponding formal power series:

$$\alpha(z) = \sum_{j\ge 0} \alpha_j z^j. \tag{3.1}$$

Theorem 3.1. *Let Γ_α be a Hankel matrix $\{\alpha_{j+k}\}_{j,k\ge 0}$. Then Γ has finite rank if and only if the power series* (3.1) *determines a rational function. In this case*

$$\operatorname{rank}\Gamma_\alpha = \deg z\alpha(z).$$

Proof. Suppose that $\operatorname{rank}\Gamma_\alpha = n$. Then the first $n+1$ rows are linearly dependent. That means that there exists a nontrivial family $\{c_j\}_{0\le j\le n}$ of complex numbers (i.e., not all the c_j are equal to zero) such that

$$c_0\alpha + c_1 S^*\alpha + \cdots + c_n S^{*n}\alpha = \mathbb{O}. \tag{3.2}$$

It is easy to see that

$$S^n S^{*k}\alpha = S^{n-k}\alpha - S^{n-k}\sum_{j=0}^{k-1}\alpha_j z^j, \quad k \le n. \tag{3.3}$$

It follows easily from (3.2) and (3.3) that

$$\mathbb{O} = S^n \sum_{k=0}^{n} c_k S^{*k}\alpha = \sum_{k=0}^{n} c_k S^n S^{*k}\alpha = \sum_{k=0}^{n} c_k S^{n-k}\alpha - p, \tag{3.4}$$

where p has the form

$$p(z) = \sum_{j=0}^{n-1} p_j z^j.$$

Put

$$q(z) = \sum_{j=0}^{n} c_{n-j} z^j. \tag{3.5}$$

Then p and q are polynomials, and it follows from (3.4) that $q\alpha = p$. Note that $c_n \neq 0$, since otherwise the rank of Γ would be less than n. Thus we can divide by q within the space of formal power series. Hence, $\alpha(z) = (p/q)(z)$ is a rational function. Clearly,

$$\deg z\alpha(z) \leq \max(\deg zp(z), \deg q(z)) = n.$$

Conversely, suppose that $\alpha(z) = (p/q)(z)$, where p and q are polynomials such that $\deg p \leq n-1$ and $\deg q \leq n$. Consider the complex numbers c_j defined by (3.5). We have

$$\sum_{j=0}^{n} c_j S^{n-j}\alpha = p.$$

Therefore

$$(S^*)^n \sum_{j=0}^{n} c_j S^{n-j}\alpha = \sum_{j=0}^{n} c_j (S^*)^j \alpha = \mathbb{O},$$

which means that the first $n+1$ rows of Γ are linearly dependent. Let $m \leq n$ be the largest number for which $c_m \neq 0$. Then $(S^*)^m\alpha$ is a linear combination of the $(S^*)^j\alpha$ with $j \leq m-1$:

$$(S^*)^m\alpha = \sum_{j=0}^{m-1} d_j (S^*)^j \alpha, \quad d_j \in \mathbb{C}.$$

Let us show by induction that any row of Γ is a linear combination of the first m rows. Let $k > m$. We have

$$(S^*)^k\alpha = (S^*)^{k-m}(S^*)^m\alpha = \sum_{j=0}^{m-1} d_j (S^*)^{k-m+j}\alpha. \tag{3.6}$$

Since $k-m+j < k$ for $0 \leq j \leq m-1$, by the inductive hypothesis each of the terms on the right-hand side of (3.6) is a linear combination of the first m rows. Therefore $\operatorname{rank}\Gamma \leq m$ which completes the proof. ■

Let us reformulate Kronecker's theorem for Hankel operators H_φ on H^2.

Corollary 3.2. *Let $\varphi \in L^\infty$. The Hankel operator H_φ has finite rank if and only if $\mathbb{P}_-\varphi$ is a rational function. In this case*

$$\operatorname{rank} H_\varphi = \deg \mathbb{P}_-\varphi. \tag{3.7}$$

It is easy to derive this result from Theorem 3.1. However, here we give a different proof, which is also instructive.

Another proof. As we have already noticed, $\operatorname{Ker} H_\varphi$ is an invariant subspace of the shift operator (Corollary 1.9). By Beurling's theorem, $\operatorname{Ker} H_\varphi = \vartheta H^2$, where ϑ is an inner function.

Clearly, H_φ has finite rank if and only if $\dim(H^2 \ominus \vartheta H^2) < \infty$ and the latter coincides with $\operatorname{rank} H_\varphi$. So $\operatorname{rank} H_\varphi < \infty$ if and only if ϑ is a finite Blaschke product, and $\operatorname{rank} H_\varphi = \deg \vartheta$ (see Appendix 2.2).

It follows that $\operatorname{rank} H_\varphi < \infty$ if and only if $H_{\vartheta\varphi} = \mathbb{O}$, i.e., $\vartheta\varphi \in H^\infty$ for a finite Blaschke product ϑ.

It is easy to see that $\mathbb{P}_-\varphi$ is a rational function of degree n if and only if there exists a Blaschke product ϑ of degree n such that $\vartheta\varphi \in H^\infty$ (the zeros of the Blaschke product are the poles of $\mathbb{P}_-\varphi$ counted with multiplicities). This proves that $\operatorname{rank} H_\varphi < \infty$ if and only if $\mathbb{P}_-\varphi$ is a rational function and that (3.7) holds. ∎

Corollary 3.3. *H_φ has finite rank if and only if there exists a finite Blaschke product B such that $B\varphi \in H^\infty$.*

Note, however, that the first proof of Kronecker's theorem is universal; it works for any Hankel matrices without any additional assumption, while the second proof assumes that the Hankel operator is bounded on H^2.

Remark. Let H_φ be a finite rank operator. In this case $H_\varphi f$ can be computed explicitly. Indeed, $\mathbb{P}_-\varphi$ is a rational function. Consider first the case when

$$(\mathbb{P}_-\varphi)(z) = \frac{1}{z-\lambda}, \quad \lambda \in \mathbb{D}.$$

We have

$$H_\varphi f = \mathbb{P}_- \frac{f}{z-\lambda} = \mathbb{P}_- \left(\frac{f - f(\lambda)}{z-\lambda} + \frac{f(\lambda)}{z-\lambda} \right) = \frac{f(\lambda)}{z-\lambda}, \quad f \in H^2. \tag{3.8}$$

If

$$(\mathbb{P}_-\varphi)(z) = \frac{1}{(z-\lambda)^k}, \quad \lambda \in \mathbb{D}, \quad k \ge 1,$$

then

$$\begin{aligned} \mathbb{P}_- \frac{f}{(z-\lambda)^k} &= \mathbb{P}_- \frac{f - f(\lambda) - f'(\lambda)(z-\lambda) - \cdots - \frac{f^{(k-1)}(\lambda)}{(k-1)!}(z-\lambda)^{k-1}}{(z-\lambda)^k} \\ &+ \sum_{j=0}^{k-1} \frac{1}{j!} \frac{f^{(j)}(\lambda)}{(z-\lambda)^{k-j}} = \sum_{j=0}^{k-1} \frac{1}{j!} \frac{f^{(j)}(\lambda)}{(z-\lambda)^{k-j}} \\ &= \frac{1}{(k-1)!} \frac{\partial^{k-1}\left(\frac{f(\zeta)}{z-\zeta}\right)}{\partial \zeta^{k-1}}(z,\lambda), \quad f \in H^2. \end{aligned}$$

Therefore in the general case

$$(\mathbb{P}_-\varphi)(z) = \sum_{j=1}^{n} \sum_{k=1}^{m_j} \frac{c_{jk}}{(z-\lambda_j)^k}$$

we obtain

$$(H_\varphi f)(z) = \sum_{j=1}^{n} \sum_{k=1}^{m_j} \frac{c_{jk}}{(k-1)!} \frac{\partial^{k-1}\left(\frac{f(\zeta)}{z-\zeta}\right)}{\partial \zeta^{k-1}}(z,\lambda_j), \quad f \in H^2.$$

4. Interpolation Problems

We consider here certain classical interpolation problems and discuss their connections with Hankel operators.

Nevanlinna–Pick Interpolation Problem

Let ζ_j, $1 \le j \le n$, be distinct points in the unit disc and let w_j, $1 \le j \le n$, be complex numbers such that $|w_j| < 1$, $1 \le j \le n$. The problem is to find out under which conditions there exists a function f in H^∞ such that

$$f(\zeta_j) = w_j, \quad 1 \le j \le n; \quad \|f\|_{H^\infty} \le 1. \tag{4.1}$$

If the problem (4.1) has a solution, then we can ask a question how to describe all solutions of the problem (4.1) and find a solution of minimal norm. We are going to reduce the problem (4.1) to a problem about Hankel operators.

Let g be an arbitrary function in H^∞ for which $g(\zeta_j) = w_j$, $1 \le j \le n$. Such a function can easily be constructed. For example, we can put

$$g = \sum_{j=1}^{n} \frac{w_j}{B_j(\zeta_j)} B_j,$$

where B is the Blaschke product with zeros ζ_j,

$$B = \prod_{j=1}^{n} b_j, \quad b_j(z) = \frac{z - \zeta_j}{1 - \bar{\zeta}_j z},$$

and

$$B_j = \frac{B}{b_j}.$$

Then a function f in H^∞ interpolates the values w_j at ζ_j if and only if f has the form $f = g - Bh$, $h \in H^\infty$. Hence the problem (4.1) is solvable if and only if

$$\inf\{\|g - Bh\|_\infty : h \in H^\infty\} = \inf\{\|\bar{B}g - h\|_\infty : h \in H^\infty\} = \|H_{\bar{B}g}\| \le 1$$

by the Nehari theorem.

Later (see §7.1) we shall discuss properties of the solution of minimal norm (it is unique since $H_{\bar{B}h}$ has finite rank (see Corollaries 1.6 and 3.3)). We shall also describe the set of solutions in the case it contains at least two different ones (see §5.2).

Let us reformulate the condition $\|H_{\bar{B}g}\| \le 1$ in terms of the interpolation data.

Theorem 4.1. *The interpolation problem* (4.1) *is solvable if and only if the matrix*

$$\left\{ \frac{1 - \bar{w}_j w_k}{1 - \bar{\zeta}_j \zeta_k} \right\}_{1 \le j,k \le n}$$

is positive semi-definite.

Proof. It is easy to see that $H_{\bar{B}g}\big|BH^2 = \mathbb{O}$, so $\|H_{\bar{B}g}\| = \big\|H_{\bar{B}g}\big|K_B\big\|$. Consider the following basis in the (n-dimensional space K_B (see §2)):

$$f_j(z) = \frac{B(z)}{z-\zeta_j}, \quad 1 \le j \le n.$$

Clearly, $\|H_{\bar{B}g}\| \le 1$ if and only if

$$H_{\bar{B}g}^* H_{\bar{B}g} \le I. \tag{4.2}$$

Inequality (4.2) means that for any complex numbers c_j, $1 \le j \le n$,

$$\sum_{j,k=1}^{n} c_j \bar{c}_k (H_{\bar{B}g} f_j, H_{\bar{B}g} f_k) \le \sum_{j,k=1}^{n} c_j \bar{c}_k (f_j, f_k). \tag{4.3}$$

We have

$$H_{\bar{B}g} f_j = \mathbb{P}_- \frac{g}{z-\zeta_j} = \frac{g(\zeta_j)}{z-\zeta_j} = \frac{w_j}{z-\zeta_j}$$

(see (3.8)). Therefore

$$(H_{\bar{B}g} f_j, H_{\bar{B}g} f_k) = w_j \bar{w}_k \left(\frac{1}{z-\zeta_j}, \frac{1}{z-\zeta_k} \right) = w_j \bar{w}_k \frac{1}{1-\bar{\zeta}_k \zeta_j}.$$

Clearly,

$$(f_j, f_k) = \left(\frac{B}{z-\zeta_j}, \frac{B}{z-\zeta_k} \right) = \left(\frac{1}{z-\zeta_j}, \frac{1}{z-\zeta_k} \right) = \frac{1}{1-\bar{\zeta}_k \zeta_j}.$$

Thus (4.3) is equivalent to the following inequality:

$$\sum_{j,k=1}^{n} c_j \bar{c}_k \frac{1 - w_j \bar{w}_k}{1 - \zeta_j \bar{\zeta}_k} \ge 0$$

for any complex numbers c_j, $1 \le j \le n$. This completes the proof. ■

In a similar way one could consider Nevanlinna–Pick problem with an infinite data set. Indeed, we can reduce the case of infinitely many interpolating points to the case of finitely many points by considering a limit process.

It is also possible to consider multiple interpolation problem where not only the values of f at the ζ_j are prescribed but also its derivatives up to certain orders. We shall consider here a very special case of multiple interpolation problem.

Carathéodory–Fejér Interpolation Problem

Let $c_0, c_1, \cdots, c_n$ be complex numbers. The problem is to determine whether there exists a function f in H^∞ such that

$$\hat{f}(j) = c_j, \quad 0 \le j \le n; \quad \|f\|_{H^\infty} \le 1. \tag{4.4}$$

Let $g = \sum_{j=0}^{n} c_j z^j$. Clearly, a function f in H^∞ satisfies $\hat{f}(j) = c_j$, $0 \le j \le n$, if and only if $f = g + z^{n+1}h$, $h \in H^\infty$. So (4.4) is solvable if and only if

$$\inf\{\|g + z^{n+1}h\|_\infty : \ h \in H^\infty\} = \|H_{\bar{z}^{n+1}g}\| \le 1.$$

Theorem 4.2. *The interpolation problem* (4.4) *is solvable if and only if the operator norm of the* $(n+1) \times (n+1)$ *matrix*

$$\begin{pmatrix} c_n & c_{n-1} & \dots & c_0 \\ c_{n-1} & c_{n-2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_0 & 0 & \dots & 0 \end{pmatrix}$$

is less than or equal to one.

Proof. This is an immediate consequence of the Nehari theorem. ∎

5. Compactness of Hankel Operators

In this section we shall establish the compactness criterion for Hankel operators which is due to Hartman. Moreover we shall compute the essential norm of a Hankel operator and study the problem of approximation of a Hankel operator by compact Hankel operators. We shall also establish some useful properties of the algebra $H^\infty + C$ and compute the norm of the Hilbert matrix.

The Essential Norm

We begin with the computation of the essential norm for a Hankel operator. Recall that the *essential norm* $\|T\|_{\mathrm{e}}$ of an operator T from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$ is, by definition,

$$\|T\|_{\mathrm{e}} = \inf\{\|T - K\| : \ K \text{ is compact}\}. \tag{5.1}$$

To compute the essential norm of a Hankel operator, we have to introduce the space $H^\infty + C$.

Definition. The space $H^\infty + C$ is the set of functions φ in L^∞ such that φ admits a representation $\varphi = f + g$, where $f \in H^\infty$ and $g \in C(\mathbb{T})$.

Theorem 5.1. *The set* $H^\infty + C$ *is a closed subalgebra of* L^∞.

To prove the theorem, we need the following elementary lemma, where as usual $C_A = H^\infty \cap C(\mathbb{T})$; see Appendix 2.1.

Lemma 5.2. *Let* $\varphi \in C(\mathbb{T})$. *Then*

$$\operatorname{dist}_{L^\infty}(\varphi, H^\infty) = \operatorname{dist}_{L^\infty}(\varphi, C_A). \tag{5.2}$$

Proof. The inequality $\operatorname{dist}(\varphi, H^\infty) \le \operatorname{dist}_{L^\infty}(\varphi, C_A)$ is trivial. Let us prove the opposite one. For $f \in L^\infty$ we consider its harmonic extension to the unit disk and keep the same notation for it. Put $f_r(\zeta) = f(r\zeta)$, $\zeta \in \mathbb{D}$. Let $\varphi \in C(\mathbb{T})$, $h \in H^\infty$. We have

$$\begin{aligned}\|\varphi - h\|_\infty &\ge \lim_{r\to 1} \|(\varphi - h)_r\|_\infty \ge \lim_{r\to 1}(\|\varphi - h_r\|_\infty - \|\varphi - \varphi_r\|_\infty)\\ &= \lim_{r\to 1}\|\varphi - h_r\|_\infty \ge \operatorname{dist}_{L^\infty}(\varphi, C_A),\end{aligned}$$

since $\|\varphi - \varphi_r\|_\infty \to 0$ for continuous φ. ■

Proof of Theorem 5.1. Equality (5.2) means exactly that the natural imbedding of $C(\mathbb{T})/C_A$ in L^∞/H^∞ is isometric and so $C(\mathbb{T})/C_A$ can be considered as a closed subspace of L^∞/H^∞. Let $\rho : L^\infty \to L^\infty/H^\infty$ be the natural quotient map. It follows that $H^\infty + C = \rho^{-1}(C(\mathbb{T})/C_A)$ is closed in L^∞.

This implies that

$$H^\infty + C = \operatorname{clos}_{L^\infty}\left(\bigcup_{n\ge 0} \bar z^n H^\infty\right). \tag{5.3}$$

It is easy to see that if f and g belong to the right-hand side of (5.3), then so is fg. Hence, $H^\infty + C$ is an algebra. ■

Now we are going to compute the essential norm of a Hankel operator.

Theorem 5.3. *Let $\varphi \in L^\infty$. Then*

$$\|H_\varphi\|_{\mathrm{e}} = \operatorname{dist}_{L^\infty}(\varphi, H^\infty + C).$$

Lemma 5.4. *Let K be a compact operator from H^2 to H^2_-. Then*

$$\lim_{n\to\infty} \|KS^n\| = 0.$$

Proof of Lemma 5.4. Since any compact operator can be approximated by finite rank operators, it is sufficient to prove the assertion for rank one operators K. Let $Kf = (f, \xi)\eta$, $\xi \in H^2$, $\eta \in H^2_-$. We have $KS^n f = (f, S^{*n}\xi)\eta$ and so

$$\|KS^n\| = \|S^{*n}\xi\|_2\|\eta\|_2 \to 0. \quad ■$$

Proof of Theorem 5.3. By Corollary 3.2, H_f is compact for any trigonometric polynomial f. Therefore H_f is compact for any f in $C(\mathbb{T})$. Consequently,

$$\operatorname{dist}_{L^\infty}(\varphi, H^\infty + C) = \inf_{f\in C(\mathbb{T})} \|H_\varphi - H_f\| \ge \|H_\varphi\|_{\mathrm{e}}.$$

On the other hand, for any compact operator K from H^2 to H^2_- we have

$$\begin{aligned}\|H_\varphi - K\| &\ge \|(H_\varphi - K)S^n\| \ge \|H_\varphi S^n\| - \|KS^n\|\\ &= \|H_{z^n\varphi}\| - \|KS^n\| = \operatorname{dist}_{L^\infty}(\varphi, \bar z^n H^\infty) - \|KS^n\|\\ &\ge \operatorname{dist}_{L^\infty}(\varphi, H^\infty + C) - \|KS^n\|.\end{aligned}$$

Therefore in view of Lemma 5.4,

$$\|H_\varphi\|_e \ge \operatorname{dist}_{L^\infty}(\varphi, H^\infty + C). \quad \blacksquare$$

Compactness Criterion

Now we can easily obtain the Hartman compactness criterion.

Theorem 5.5. *Let $\varphi \in L^\infty$. The following statements are equivalent:*
(i) *H_φ is compact;*
(ii) *$\varphi \in H^\infty + C$;*
(iii) *there exists a function ψ in $C(\mathbb{T})$ such that $H_\varphi = H_\psi$.*

Proof. Obviously, (ii)$\Leftrightarrow$(iii). Suppose that $\varphi \in H^\infty + C$. Then $\|H_\varphi\|_e = 0$ by Theorem 5.3, which means that H_φ is compact. Thus (ii)$\Rightarrow$(i).

Let us show that (i)$\Rightarrow$(ii). If H_φ is compact, then by Theorem 5.3, $\operatorname{dist}_{L^\infty}(\varphi, H^\infty + C) = 0$, which together with Theorem 5.5 yields $\varphi \in H^\infty + C$. $\blacksquare$

Corollary 5.6. *Let $\varphi \in L^\infty$. Then*

$$\|H_\varphi\|_e = \inf\{\|H_\varphi - H_\psi\| : \ H_\psi \text{ is compact}\}. \tag{5.4}$$

In other words, to compute the essential norm of a Hankel operator we can consider on the right-hand side of (5.1) only compact Hankel operators.

Corollary 5.7. *Let $\varphi \in H^\infty + C$. Then for any $\varepsilon > 0$ there exists a function ψ in $C(\mathbb{T})$ such that $H_\psi = H_\varphi$ and $\|\psi\|_\infty \le \|H_\varphi\| + \varepsilon$.*

Proof. Without loss of generality we can assume that $\varphi \in C(\mathbb{T})$. By Theorem 5.3, $\|H_\varphi\| = \operatorname{dist}_{L^\infty}(\varphi, H^\infty)$. On the other hand, by Lemma 5.2, $\operatorname{dist}_{L^\infty}(\varphi, H^\infty) = \operatorname{dist}_{L^\infty}(\varphi, C_A)$. This means that for any $\varepsilon > 0$ there exists a function $h \in C_A$ such that $\|\varphi - h\| \le \|H_\varphi\| + \varepsilon$. Thus $\psi = \varphi - h$ does the job. $\blacksquare$

Remark. However, it is not always possible for a compact Hankel operator to find a continuous symbol whose L^∞-norm is equal to the norm of the operator. Indeed, let α be a real-valued function in $C(\mathbb{T})$ such that $\tilde{\alpha} \notin C(\mathbb{T})$, where $\tilde{\alpha}$ is the harmonic conjugate of α. Put $\varphi = \bar{z}e^{\mathrm{i}\tilde{\alpha}}$. Then $\varphi = \bar{z}e^{\alpha + \mathrm{i}\tilde{\alpha}}e^{-\alpha}$. Clearly, $e^{\alpha + i\tilde{\alpha}} \in H^\infty$ and $e^{-\alpha} \in C(\mathbb{T})$. It follows from Theorem 5.1 that $\varphi \in H^\infty + C$ and so H_φ is compact. Let us show that $\|H_\varphi\| = 1$. Put

$$h = \exp \frac{1}{2}(-\alpha - \mathrm{i}\tilde{\alpha}).$$

Then $h \in H^2$ and $\|H_\varphi h\|_2 = \|\bar{z}\bar{h}\|_2 = \|h\|_2$. Hence, $\|H_\varphi\| = \|\varphi\|_\infty = 1$. By Corollary 1.6, $\|\varphi + f\|_\infty > 1$ for any nonzero f in H^∞. It is also clear that $\varphi \notin C(\mathbb{T})$. This proves the result. $\blacksquare$

In §1 we have given a boundedness criterion for a Hankel operator H_φ in terms of $\mathbb{P}_-\varphi$. That criterion involves the condition $\mathbb{P}_-\varphi \in BMO$. We can give a similar compactness criterion if we replace BMO by the space VMO of functions of vanishing mean oscillation (see Appendix 2.5).

Theorem 5.8. *Let $\varphi \in L^2$. Then H_φ is compact if and only if $\mathbb{P}_-\varphi \in VMO$.*

The result can be derived from Theorem 5.5 in the same way as it has been done in Theorem 1.2 if we use the following description of VMO:

$$VMO = \{\xi + \tilde{\eta} : \ \xi, \eta \in C(\mathbb{T})\}$$

(see Appendix 2.5).

Theorem 5.8 allows us to obtain a compactness criterion for the commutators $\mathcal{C}_\varphi$ defined in §1 (see (1.15)). The following theorem can be deduced from Theorem 5.8 in exactly the same way as Theorem 1.10 has been deduced in §1 from Theorem 1.3.

Theorem 5.9. *Let $\varphi \in L^2$. The commutator $\mathcal{C}_\varphi$ is a compact operator on L^2 if and only if $\varphi \in VMO$.*

The Hartman theorem also allows us to obtain a compactness criterion for functions $\varphi(S_{[\vartheta]})$ of the model operator $S_{[\vartheta]}$ (see §2).

Theorem 5.10. *Let ϑ be an inner function and $S_{[\vartheta]}$ the compressed shift on K_ϑ. If $\varphi \in H^\infty$, then the following statements are equivalent:*
(i) $\varphi(S_{[\vartheta]})$ *is compact;*
(ii) $\bar{\vartheta}\varphi \in H^\infty + C$;
(iii) $\mathbb{P}_-\bar{\vartheta}\varphi \in VMO$.

The result follows directly from Theorems 5.5 and 5.8, and from formula (2.9).

Consider now the partial case of this situation when $\vartheta = B$ is an interpolating Blaschke product (see Appendix 2.1). In this case K_B has an unconditional basis that consists of eigenvectors of $S_{[B]}$. Namely, let

$$f_j(z) = (1 - |\zeta_j|^2)^{1/2} \frac{B}{z - \zeta_j}, \tag{5.5}$$

where the ζ_j, $j \geq 1$, are the zeros of B. Then it is easy to see that $S_{[B]}f_j = \zeta_j f_j$ and $\varphi(S_{[B]})f_j = \varphi(\zeta_j)f_j$ for any $\varphi \in H^\infty$. Since $\{f_j\}_{j\geq 1}$ is an unconditional basis, there is an invertible operator V on K_ϑ such that $\{Vf_j\}_{j\geq 1}$ is an orthogonal basis of K_ϑ (see N.K. Nikol'skii [2], Lect. VI, §3). Therefore $\varphi(S_{[B]})$ is compact if and only if $\varphi(\zeta_j) \to 0$ as $j \to \infty$. This together with Theorem 5.10 implies the following assertion.

Theorem 5.11. *Let $\varphi \in H^\infty$ and let B be an interpolating Blaschke product with zeros $\{\zeta_j\}_{j\geq 1}$. The following statements are equivalent:*
(i) $\varphi(S_{[B]})$ *is compact;*
(ii) $\bar{B}\varphi \in H^\infty + C$;
(iii) $\mathbb{P}_-\bar{B}\varphi \in VMO$;
(iv) $\varphi(\zeta_j) \to 0$ *as* $j \to \infty$.

The above facts about the bases of eigenvectors of $S_{[B]}$ can be found in N.K.Nikol'skii [2], Lect. III, §7.

Approximation by Compact Hankel Operators

In connection with (5.4) a natural question arises of whether the infimum in (5.4) is attained. This is equivalent to the existence of a best approximation from $H^\infty + C$ for any bounded function. This question turns out to be more difficult than in the case of best approximation from H^∞ since the compactness argument does not work here. Another question arising in this connection is whether such a best approximation is unique (if it exists). The following theorem answers these questions.

Theorem 5.12. *Let $\varphi \in L^\infty \setminus (H^\infty + C)$. Then there exist infinitely many different functions f in $H^\infty + C$ such that*

$$\|\varphi - f\|_\infty = \operatorname{dist}_{L^\infty}(\varphi, H^\infty + C).$$

By Theorems 1.3 and 5.5, Theorem 5.12 is a consequence of the following theorem.

Theorem 5.13. *Let Γ be a noncompact Hankel operator on H^2. Then there exist infinitely many different compact Hankel operators Υ such that*

$$\|\Gamma - \Upsilon\| = \|\Gamma\|_{\mathrm{e}}.$$

We establish a more general result. First we need the following lemma.

Lemma 5.14. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $T : \mathcal{H}_1 \to \mathcal{H}_2$ a bounded operator. If $\{A_n\}_{n\ge1}$ is a sequence of operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that*

$$A_n \to \mathbb{O} \quad \textit{and} \quad A_n^* \to \mathbb{O}$$

in the strong operator topology, then for any $\varepsilon > 0$ there exists a positive integer N such that

$$\|T + A_N\| \le \varepsilon + \max(\|T\|, \|T\|_{\mathrm{e}} + \|A_N\|).$$

Proof of Lemma 5.14. Clearly, without loss of generality we can assume that $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. Put

$$\rho_n = \max(\|T\|, \|T\|_{\mathrm{e}} + \|A_n\|).$$

Suppose that for any n there exists $x_n \in \mathcal{H}$, $\|x_n\| = 1$, such that

$$\|(T + A_n)x_n\| > \varepsilon + \rho_n. \tag{5.6}$$

By passing to a subsequence, if needed, we may assume that $x_n \to x \in \mathcal{H}$ in the weak topology. Let $y_n = x_n - x$. Then

$$(T + A_n)x_n = Tx + Ty_n + A_n y_n + A_n x. \tag{5.7}$$

Let $\mathcal{K}_n$ be a sequence of finite-dimensional subspaces of $\mathcal{H}$ such that $\mathcal{K}_n \subset \mathcal{K}_{n+1}$ and $\bigcup\limits_{n\ge1} \mathcal{K}_n$ is dense in $\mathcal{H}$. Denote by P_n the orthogonal projection onto $\mathcal{K}_n$ and by Q_n the orthogonal projection onto $\mathcal{K}_n^\perp$. Then $Q_n \to \mathbb{O}$ in the strong operator topology. Therefore for any compact operator K on $\mathcal{H}$ we have $\|Q_n K\| \to 0$. Since

$$\|T - K\| \ge \|Q_n(T - K)\| \ge \|Q_n T\| - \|Q_n K\|,$$

it follows that $\|Q_nT\| \to \|T\|_{\mathrm{e}}$.

Let δ be a positive number that we specify later. Fix a positive integer M such that $\|Q_MTx\| < \delta$, $\|Q_Mx\| < \delta$, and $\|Q_MT\| < \|T\|_{\mathrm{e}} + \delta$. Since $Ty_n \to 0$ weakly, it follows that $\|Q_MTy_n\| < \delta$ for large n. Since P_M is compact and $A_n^* \to \mathbb{O}$ in the strong operator topology, it follows that $\|P_MA_n\| \to 0$ and so $\|P_MA_ny_n\| < \delta$ for large n.

We have from (5.7)

$$(T+A_n)x_n = P_MTx+Q_M(Ty_n+A_ny_n)+A_nx+Q_MTx+P_M(Ty_n+A_ny_n).$$

Since $A_n \to \mathbb{O}$ in the strong operator topology, we have $\|A_nx\| < \delta$ for large n. Thus for large n

$$\|(T+A_n)x_n\| \le \|P_MTx + Q_M(Ty_n + A_ny_n)\| + 4\delta. \tag{5.8}$$

Since P_MTx is orthogonal to $Q_M(Ty_n + A_ny_n)$, we obtain

$$\begin{aligned}
\|P_MTx + Q_M(Ty_n + A_ny_n)\|^2 &= \|P_MTx\|^2 + \|Q_M(Ty_n + A_ny_n)\|^2 \\
&\le \|T\|^2\|x\|^2 + (\|Q_MT\| + \|A_n\|)^2\|y_n\|^2 \\
&\le \rho_n^2\|x\|^2 + (\|T\|_{\mathrm{e}} + \delta + \|A_n\|)^2\|y_n\|^2 \\
&\le (\rho_n + \delta)^2(\|x\|^2 + \|y_n\|^2).
\end{aligned}$$

It follows from the fact that $y_n \to 0$ weakly that $\|P_My_n\| < \delta$ for large n. Hence,

$$\begin{aligned}
\|x\|^2 + \|y_n\|^2 &= \|P_Mx\|^2 + \|Q_Mx\|^2 + \|P_My_n\|^2 + \|Q_My_n\|^2 \\
&\le \|P_Mx\|^2 + \|Q_My_n\|^2 + 2\delta^2 = \|P_Mx + Q_My_n\|^2 + 2\delta^2 \\
&= \|x - Q_Mx + y_n - P_My_n\|^2 + 2\delta^2 \\
&\le \left(\|x_n\| + \|Q_Mx\| + \|P_My_n\|\right)^2 + 2\delta^2 \le (1+2\delta)^2 + 2\delta^2.
\end{aligned}$$

Thus we have

$$\|P_MTx + Q_M(Ty_n + A_ny_n)\|^2 \le (\rho_n + \delta)^2\left((1+2\delta)^2 + 2\delta^2\right).$$

This together with (5.8) yields

$$\|(T+A_n)x_n\| \le (\rho_n + \delta)\left((1+2\delta)^2 + 2\delta^2\right)^{1/2} + 4\delta$$

for large n. Now we can choose δ so small that the right-hand side of the last inequality is less than $\rho_n + \varepsilon$, which contradicts (5.6). ■

Corollary 5.15. *Under the hypotheses of Lemma 5.14, for any $\varepsilon > 0$ there exists a positive integer N such that*

$$\|T + \beta A_N\| \le \varepsilon + \max(\|T\|, \|T\|_{\mathrm{e}} + \beta\|A_N\|)$$

for any $\beta \in [0,1]$.

Proof. Suppose that for any n there exists $\beta_n \in [0,1]$ such that

$$\|T + \beta_n A_n\| > \varepsilon + \max(\|T\|, \|T\|_e + \beta\|A_n\|).$$

But $\beta_n A_n \to \mathbb{O}$, $(\beta_n A_n)^* \to \mathbb{O}$ in the strong operator topology, which contradicts Lemma 5.14. ■

Theorem 5.16. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, $T : \mathcal{H}_1 \to \mathcal{H}_2$ a noncompact bounded operator. Let $\{T_n\}_{n\ge1}$ be a sequence of compact operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that $T_n \to T$ and $T_n^* \to T^*$ in the strong operator topology. Then there exist sequences $\{a_n\}_{n\ge1}$ and $\{b_n\}_{n\ge1}$ of nonnegative real numbers such that*

$$\sum_{n\ge1} a_n = \sum_{n\ge1} b_n = 1$$

and

$$\|T - K_1\| = \|T - K_2\| = \|T\|_e, \tag{5.9}$$

where $K_1 = \sum\limits_{n\ge1} a_n T_n$ and $K_2 = \sum\limits_{n\ge1} b_n T_n$. Moreover, $K_1 \ne K_2$.

Proof. Let us first find $\{a_n\}_{n\ge1}$. Let $A_n = T - T_n$. Clearly, $\lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty} A_n^* = \mathbb{O}$ in the strong operator topology.

Let us show that there exist an increasing sequence $\{n_k\}_{k\ge1}$ of positive integers and a sequence $\{\alpha_k\}_{k\ge1}$ of positive real numbers such that

$$\sum_{k=1}^{\infty} \alpha_k = 1, \tag{5.10}$$

$$\left\|\sum_{k=1}^{m} \alpha_k A_{n_k}\right\| = \|T\|_e - \varepsilon_m, \tag{5.11}$$

for any $m \ge 1$, where $\varepsilon_m = 3^{-m}\|T\|_e$.

We proceed by induction. Put $n_1 = 1$ and choose $\alpha_1 > 0$ so that $\|\alpha_1 A_1\| = \|T\|_e - \varepsilon_1$. Clearly, $\alpha_1 < 1$. Suppose that we have chosen $n_1, \cdots, n_m$ and $\alpha_1, \cdots, \alpha_m$ such that $\sum\limits_{k=1}^{m} \alpha_k < 1$ and (5.11) holds. We can choose $n_{m+1} > n_m$ such that

$$\|R + \beta A_{n_{m+1}}\| \le \varepsilon_{m+1} + \max\left\{\|R\|, \|R\|_e + \beta\|A_{n_{m+1}}\|\right\}, \tag{5.12}$$

for any β in [0,1], where $R = \sum\limits_{k=1}^{m} \alpha_k A_{n_k}$. This is clearly possible by Corollary 5.15.

Consider

$$\mu(\alpha) = \|R + \alpha A_{n_m}\|, \quad \alpha \ge 0.$$

Clearly, $\mu(\alpha) \to \infty$ as $\alpha \to \infty$ and $\mu(0) = \left\|\sum_{k=1}^{m} \alpha_k A_{n_k}\right\| = \|T\|_{\mathrm{e}} - \varepsilon_m$ by the inductive hypothesis. Therefore there exists $\alpha_{m+1} > 0$ such that $\mu(\alpha_{m+1}) = \|T\|_{\mathrm{e}} - \varepsilon_{m+1}$.

Let us show that $\sum_{k=1}^{m+1} \alpha_k < 1$. We have

$$\|T\|_{\mathrm{e}} - \varepsilon_{m+1} = \mu(\alpha_{m+1}) = \left\|\left(\sum_{k=1}^{m+1} \alpha_k\right) T - \sum_{k=1}^{m+1} \alpha_k T_{n_k}\right\| \geq \left(\sum_{k=1}^{m+1} \alpha_k\right) \|T\|_{\mathrm{e}},$$

so $\sum_{k=1}^{m+1} \alpha_k < 1$.

Put $\beta = \alpha_{m+1}$ in (5.12). Suppose that the first term under the maximum on the right-hand side of (5.12) is greater than the second. Then

$$\left\|\sum_{k=1}^{m+1} \alpha_k A_{n_k}\right\| \leq \varepsilon_{m+1} + \left\|\sum_{k=1}^{m} \alpha_k A_{n_k}\right\|.$$

Therefore by (5.11)

$$\|T\|_{\mathrm{e}} - \varepsilon_{m+1} \leq \varepsilon_{m+1} + \|T\|_{\mathrm{e}} - \varepsilon_m,$$

which implies $\varepsilon_m \leq 2\varepsilon_{m+1}$, which contradicts the definition of ε_m. Consequently, (5.12) with $\beta = \alpha_{m+1}$ turns into

$$\begin{aligned} \left\|\sum_{k=1}^{m+1} \alpha_k A_{n_k}\right\| &\leq \varepsilon_{m+1} + \left\|\sum_{k=1}^{m} \alpha_k A_{n_k}\right\|_{\mathrm{e}} + \alpha_{m+1}\|A_{n_{m+1}}\| \\ &= \varepsilon_{m+1} + \left(\sum_{k=1}^{m} \alpha_k\right) \|T\|_{\mathrm{e}} + \alpha_{m+1}\|A_{n_{m+1}}\|. \end{aligned} \tag{5.13}$$

Let $m \to \infty$, it follows from (5.11) and (5.13) that

$$\|T\|_{\mathrm{e}} = \left\|\sum_{k=1}^{\infty} \alpha_k A_{n_k}\right\| \leq \left(\sum_{k=1}^{\infty} \alpha_k\right) \|T\|_{\mathrm{e}},$$

which implies

$$\sum_{k=1}^{\infty} \alpha_k \geq 1,$$

and since we have already proved that $\sum_{k=1}^{m} \alpha_k < 1$, the sequences $\{\alpha_k\}$ and $\{n_k\}$ satisfy (5.11) and (5.13).

We can define the sequence $\{a_n\}_{n\geq 1}$ in the following way: $a_{n_k} = \alpha_k$ and $a_n = 0$ if n does not coincide with any n_k. We have

$$K_1 = \sum_{n=1}^{\infty} a_n T_n = \sum_{k=1}^{\infty} \alpha_k T_{n_k} = T - \sum_{k=1}^{\infty} \alpha_k A_{n_k},$$

$$\|T - K_1\| = \left\| \sum_{k=1}^{\infty} \alpha_k A_{n_k} \right\| = \|T\|_{\mathrm{e}}.$$

It remains to construct $\{b_n\}_{n\ge1}$. Let $\tilde T = T - K_1$, $\tilde T_n = T_n - K_1$. Then $\tilde T_n \to \tilde T$ and $\tilde T_n^* \to \tilde T^*$ in the strong operator topology. Let $\mathcal{O}$ be a convex neighborhood of $\tilde T$ in the strong operator topology whose closure does not contain $\mathbb{O}$. Since $\tilde T_n \to \tilde T$ strongly, we can assume that $\tilde T_n \in \mathcal{O}$ for all n. As we have just proved, there exists a sequence $\{b_n\}_{n\ge1}$ of nonnegative numbers such that $\sum\limits_{n=1}^{\infty} b_n = 1$ and

$$\|\tilde T - \tilde K\| = \|\tilde T\|_{\mathrm{e}} = \|T\|_{\mathrm{e}},$$

where $\tilde K = \sum\limits_{n=1}^{\infty} b_n \tilde T_n = \left(\sum\limits_{n=1}^{\infty} b_n T_n\right) - K_1$. Thus $K_2 = K_1 + \tilde K = \sum\limits_{n=1}^{\infty} b_n T_n$ satisfies (5.9). Since $\tilde K$ is an infinite convex combination of the $\tilde T_n$, we have $\tilde K \subset \mathcal{O}$. Thus $\tilde K \ne \mathbb{O}$ and $K_1 \ne K_2$. ■

Now we are in a position to deduce Theorem 5.13 from Theorem 5.16.

Proof of Theorem 5.13. Let $\Gamma = H_\varphi$. Let $\varphi_n = \varphi * K_n$ be the Césaro means of the partial sums of the Fourier series of φ (see Appendix 2.1). Then H_{φ_n} is compact. Let us show that $H_{\varphi_n} \to H_\varphi$ in the strong operator topology. Let $f \in H^2$. We have

$$\begin{aligned}\|(H_\varphi - H_{\varphi_n})f\|_2^2 &= \|\mathbb{P}_-(\varphi - \varphi_n)f\|^2 \le \|(\varphi - \varphi_n)f\|_2^2 \\ &= \int_{\mathbb{T}} |\varphi(\zeta) - \varphi_n(\zeta)|^2 |f(\zeta)|^2 d\boldsymbol{m}(\zeta) \to 0,\end{aligned}$$

since $\|\varphi_n\|_\infty \le \|\varphi\|_\infty$ and $\varphi_n(\zeta) \to \varphi(\zeta)$ a.e. on $\mathbb{T}$ (see Appendix 2.1). Using the fact that $H_{\varphi_n}^* g = \mathbb{P}_+ \bar\varphi g$, $g \in H^2_-$, one can prove in the same way that $H_{\varphi_n}^* \to H_\varphi^*$ strongly.

It follows from Theorem 5.16 that there are distinct compact Hankel operators Υ_0 and Υ_1 (infinite convex combinations of H_{φ_n}) such that

$$\|\Gamma - \Upsilon_0\| = \|\Gamma - \Upsilon_1\| = \|\Gamma\|_{\mathrm{e}}.$$

It remains to observe that if Υ_1 and Υ_2 are best compact Hankel approximations to Γ, then so is $\Upsilon_s = (1-s)\Upsilon_0 + s\Upsilon_1$ with $s \in [0,1]$. ■

The Local Distance

Let f and g be functions in L^∞, $\lambda \in \mathbb{T}$. The *local distance* $\operatorname{dist}_\lambda(f,g)$ between f and g at λ is by definition

$$\operatorname{dist}_\lambda(f,g) = \lim_{\varepsilon\to0} \|(f-g)\,|\{\zeta \in \mathbb{T} : \ |\zeta - \lambda| < \varepsilon\}\,\|_\infty.$$

If K is a subset of L^∞, the local distance from f to K is defined by

$$\operatorname{dist}_\lambda(f,K) = \inf\{\operatorname{dist}_\lambda(f,h) : \ h \in K\}.$$

It is easy to see that the function $\lambda \mapsto \operatorname{dist}_\lambda(f, K)$ is upper semicontinuous and so it attains a maximum on $\mathbb{T}$.

Theorem 5.17. *Let $\varphi \in L^\infty$. Then*

$$\|H_\varphi\|_{\mathrm{e}} = \max_{\lambda \in \mathbb{T}} \operatorname{dist}_\lambda(\varphi, H^\infty).$$

Proof. Clearly, by Theorem 5.3, we have to show that

$$\operatorname{dist}_{L^\infty}(\varphi, H^\infty + C) = \max_{\lambda \in \mathbb{T}} \operatorname{dist}_\lambda(\varphi, H^\infty).$$

It is evident that $\operatorname{dist}_\lambda(\varphi, H^\infty) \le \operatorname{dist}_{L^\infty}(\varphi, H^\infty + C)$ for any $\lambda \in \mathbb{T}$. Let us show that

$$\operatorname{dist}_{L^\infty}(\varphi, H^\infty + C) \le \max_{\lambda \in \mathbb{T}} \operatorname{dist}_\lambda(\varphi, H^\infty). \tag{5.14}$$

Let M be the right-hand side of (5.14) and let $\varepsilon > 0$. For each λ we can find h_λ in H^∞ such that

$$\operatorname{dist}_\lambda(\varphi, h_\lambda) < M + \varepsilon.$$

Since the function $\zeta \mapsto \operatorname{dist}_\zeta(\varphi, h_\lambda)$ is upper semi-continuous, it follows that for any λ there is an open interval I_λ such that $\operatorname{dist}_\zeta(\varphi, h_\lambda) < M + 2\varepsilon$ for $\zeta \in I_\lambda$. We can choose a finite cover $I_{\lambda_1}, \cdots, I_{\lambda_n}$ of $\mathbb{T}$. Let $w_j \in C(\mathbb{T})$, $1 \le j \le n$, be a partition of unity subordinate to the cover $\{I_{\lambda_j}\}$, i.e., $\operatorname{supp} w_j \subset I_{\lambda_j}$, $w_j \ge 0$, and $\sum_{j=1}^n w_j = \mathbf{1}$.

Put $h = \sum_{j=1}^n w_j h_{\lambda_j}$. Since $H^\infty + C$ is an algebra, it follows that $h \in H^\infty + C$. Let $\lambda \in \mathbb{T}$. We have

$$\begin{aligned} \operatorname{dist}_\lambda(\varphi, h) &= \operatorname{dist}_\lambda\left(\sum_{j=1}^n w_j\varphi, \sum_{j=1}^n w_j h_{\lambda_j}\right) \\ &\le \sum_{j=1}^n \operatorname{dist}_\lambda(w_j\varphi, w_j h_{\lambda_j}) = \sum_{j=1}^n \operatorname{dist}_\lambda(w_j(\lambda)\varphi, w_j(\lambda) h_{\lambda_j}) \\ &= \sum_{j=1}^n w_j(\lambda) \operatorname{dist}_\lambda(\varphi, h_{\lambda_j}) \le \sum_{j=1}^n w_j(\lambda)(M + 2\varepsilon), \end{aligned}$$

since $\operatorname{dist}_\lambda(f, h_{\lambda_j}) \le M + 2\varepsilon$ for $\lambda \in I_{\lambda_j}$ and $w_j(\lambda) = 0$ if $\lambda \notin I_{\lambda_j}$. Therefore

$$\operatorname{dist}_\lambda(\varphi, h) \le M + 2\varepsilon,$$

which proves that $\operatorname{dist}_{L^\infty}(\varphi, H^\infty + C) \le M$. ■

Let us now compute $\|H_\varphi\|_{\mathrm{e}}$ for piecewise continuous functions φ. Denote by PC the class of piecewise continuous functions on $\mathbb{T}$. A function φ on $\mathbb{T}$ is said to belong to PC if for any $\zeta \in \mathbb{T}$ the limits

$$\varphi(\zeta^+) \stackrel{\text{def}}{=} \lim_{t \to 0+} \varphi(\zeta e^{\mathrm{i}t}), \quad \varphi(\zeta^-) \stackrel{\text{def}}{=} \lim_{t \to 0-} \varphi(\zeta e^{\mathrm{i}t})$$

exist. Since we identify functions that differ on a set of zero measure, we can assume that for $\varphi \in PC$

$$\varphi(\zeta^+) = \varphi(\zeta), \quad \zeta \in \mathbb{T}.$$

Denote by $\varkappa_\zeta(\varphi)$ the jump of φ at ζ:

$$\varkappa_\zeta(\varphi) = \varphi(\zeta) - \varphi(\zeta^-).$$

It is easy to see that for any $\varepsilon > 0$ the set

$$\{\zeta \in \mathbb{T} : \ |\varkappa_\zeta(\varphi)| > \varepsilon\}$$

is finite.

Theorem 5.18. *Let $\varphi \in PC$. Then*

$$\|H_\varphi\|_{\mathrm{e}} = \frac{1}{2}\max_{\zeta\in\mathbb{T}} |\varkappa_\zeta(\varphi)|. \tag{5.15}$$

Proof. In view of Theorem 5.17 it is sufficient to show that

$$\frac{1}{2}|\varkappa_\zeta(\varphi)| = \operatorname{dist}_\zeta(\varphi, H^\infty).$$

It is easy to see that

$$\operatorname{dist}_\zeta(\varphi, H^\infty) \le \operatorname{dist}_\zeta(\varphi, c) = \frac{1}{2}|\varkappa_\zeta(\varphi)|,$$

where c is the constant function equal to $\frac{1}{2}\left(\varphi(\zeta) + \varphi(\zeta^-)\right)$. Let us show that $\frac{1}{2}|\varkappa_\zeta(\varphi)| \le \operatorname{dist}_\zeta(\varphi, H^\infty)$. By multiplying φ by a constant and adding to it a constant function if needed, we can assume without loss of generality that $\varphi(\zeta) = 1$ and $\varphi(\zeta^-) = -1$. We have to show that $\operatorname{dist}_\zeta(\varphi, H^\infty) \ge 1$. We can also assume that $\zeta = 1$.

Suppose that there exists an f in H^∞ such that $\operatorname{dist}_1(\varphi, f) < 1$. It follows that there exists $\delta > 0$ such that

$$\operatorname{Re} f(e^{\mathrm{i}t}) > \delta, \quad 0 \le t \le \varepsilon; \quad \operatorname{Re} f(e^{\mathrm{i}t}) < -\delta, \quad -\varepsilon \le t \le 0, \tag{5.16}$$

for some $\varepsilon > 0$. Let $h(\zeta) = \operatorname{Re} f(\zeta)$, $\zeta \in \mathbb{D}$. Then

$$f(\zeta) = h(\zeta) + \mathrm{i}\tilde{h}(\zeta) + \mathrm{i}\operatorname{Im} f(0),$$

where $\tilde{h}$ is the harmonic conjugate to h (see Appendix 2.1). Then $\tilde{h}$ is bounded on $\mathbb{D}$. We have

$$\tilde{h}(re^{\mathrm{i}\vartheta}) = (h * Q_r)(e^{\mathrm{i}\vartheta}),$$

where Q_r is the conjugate Poisson kernel (see Appendix 2.1). Let us show that $|\tilde{h}(r)| \to \infty$ as $r \to 1$. It is easy to see that

$$\left|\int_{\{|t|\ge\varepsilon\}} h(e^{\mathrm{i}t})Q_r(t)dt\right| \le \text{const}\,.$$

On the other hand, it follows easily from (5.16) that

$$\left|\int_{-\varepsilon}^{\varepsilon} h(e^{\mathrm{i}t})Q_r(e^{\mathrm{i}t})dt\right| \geq \text{const } \delta \int_0^{\varepsilon} \frac{\sin t}{1-2r\cos t + r^2}dt \geq \text{const} \cdot \log \frac{1}{1-r},$$

where the constant depends on δ and ε. This contradicts the fact that $\tilde{h}$ is bounded. Thus $\operatorname{dist}_1(\varphi, H^\infty) = 1$ and the proof is completed. ■

Remark. Clearly the right-hand side of (5.15) is equal to $\operatorname{dist}_{L^\infty}(\varphi, C(\mathbb{T}))$. So for $\varphi \in PC$ we have

$$\operatorname{dist}_{L^\infty}(\varphi, C(\mathbb{T})) = \operatorname{dist}_{L^\infty}(\varphi, H^\infty + C) = \|H_\varphi\|_{\mathrm{e}}.$$

Now we are in a position to compute the norm and the essential norm of the Hilbert matrix. Let $\psi(e^{\mathrm{i}t}) = \mathrm{i}(t-\pi)$, $t \in [0, 2\pi)$. Then $\hat{\psi}(-n) = 1/n$, $n \neq 0$, and H_ψ has Hilbert matrix.

Corollary 5.19. *Let ψ be as above. Then*

$$\|H_\psi\|_{\mathrm{e}} = \|H_\psi\| = \pi.$$

Proof. We have established in §1 that $\|H_\varphi\| \leq \pi$. Since $\varphi \in PC$, we have $\|H_\psi\|_{\mathrm{e}} = \pi$ by Theorem 5.18. The result follows now from the obvious inequality $\|H_\psi\|_{\mathrm{e}} \leq \|H_\psi\|$. ■

A Factorization Theorem

We conclude this section with the following elegant property of $H^\infty + C$.

Theorem 5.20. *Let $f \in L^\infty$. Then there exist a Blaschke product B and a function φ in $H^\infty + C$ such that $f = \bar{B}\varphi$.*

Proof. By Theorem 2.4, there exist sequences $\{\vartheta_n\}_{n\geq 1}$ of inner functions and $\{g_n\}_{n\geq 1}$ of H^∞-functions such that

$$\lim_{n\to\infty} \|f - \bar{\vartheta}_n g_n\|_\infty = 0.$$

By Frostman's theorem (see Appendix 2.1), each inner function is a uniform limit of Blaschke products. Therefore there exists a sequence of Blaschke products $\{B_n\}_{n\geq 1}$ such that

$$\lim_{n\to\infty} \|f - \bar{B}_n g_n\|_\infty = 0.$$

Let us decompose each B_n as follows: $B_n = B_n^{(1)} B_n^{(2)}$, where $B_n^{(1)}$ is a finite Blaschke product and the product

$$B = \prod_{n\geq 1} B_n^{(2)}$$

converges in $\mathbb{D}$. Then

$$\|Bf - \bar{B}_n^{(1)} h_n\|_\infty \to 0,$$

where

$$h_n = g_n \prod_{m\neq n} B_m^{(2)} \in H^\infty.$$

Clearly, $\bar{B}_n^{(1)} h_n \in H^\infty + C$. Hence, $Bf \in H^\infty + C$ by Theorem 5.1. ■

6. Hankel Operators and Reproducing Kernels

In this section we show that to verify the boundedness (or compactness) of a Hankel operator, we can consider the action of the operator on the so-called reproducing kernels of H^2.

Let $\zeta \in \mathbb{D}$. Put

$$K_\zeta(z) = \frac{1}{1 - \bar{\zeta} z}.$$

Then $K_\zeta \in H^2$ and for any $f \in H^2$

$$(f, K_\zeta) = f(\zeta).$$

Because of this property the functions K_ζ are called the *reproducing kernel functions* (or simply kernel functions) of H^2. It is easy to see that

$$\|K_\zeta\|_2 = (1 - |\zeta|^2)^{-1/2}.$$

The functions k_ζ defined by

$$k_\zeta(z) = \frac{(1 - |\zeta|^2)^{1/2}}{1 - \bar{\zeta} z}, \quad \zeta \in \mathbb{D},$$

are called the *normalized kernel functions*.

Obviously, for any bounded operator A on H^2 the norms $\|Ak_\zeta\|$ are uniformly bounded. It is easy to see that $k_\zeta \to \mathbb{O}$ weakly as $|\zeta| \to 1$. Therefore $\|Ak_\zeta\| \to 0$ as $|\zeta| \to 1$ for any compact operator A on H^2. We show in this section that for Hankel operators the converse is also true.

We shall work here with the Garsia norm on BMO (see Appendix 2.5). Let P_ζ be the Poisson kernel,

$$P_\zeta(z) = \frac{1 - |\zeta|^2}{|1 - \bar{\zeta} z|^2} = |k_\zeta(z)|^2.$$

Given $\varphi \in L^2$, we denote by $\varphi_\#$ its harmonic extension to the unit disc:

$$\varphi_\#(\zeta) = \int_{\mathbb{T}} \varphi(\tau) P_\zeta(\tau) d\boldsymbol{m}(\tau), \quad \zeta \in \mathbb{D}.$$

The Garsia norm of φ is defined by

$$\|\varphi\|_{\mathrm{G}} = \left(\sup_{\zeta \in \mathbb{D}} \int_{\mathbb{T}} |\varphi(\tau) - \varphi_\#(\zeta)|^2 P_\zeta(\tau) d\boldsymbol{m}(\tau) \right)^{1/2}.$$

Then $\|\cdot\|_{\mathrm{G}}$ is a norm on BMO modulo the constants which is equivalent to the initial norm (see Appendix 2.5).

Theorem 6.1. *Let $\varphi \in L^2$. The Hankel operator H_φ is bounded on H^2 if and only if*

$$\sup_{\zeta\in\mathbb{D}} \|H_\varphi k_\zeta\|_2 < \infty. \tag{6.1}$$

Proof. Let $\psi = \overline{\mathbb{P}_-\varphi}$. We have

$$\|H_\varphi K_\zeta\| = \left\|\mathbb{P}_-\frac{\mathbb{P}_-\varphi}{1-\bar{\zeta}z}\right\| = \left\|\mathbb{P}_+\frac{\bar{z}\overline{\mathbb{P}_-\varphi}}{1-\zeta\bar{z}}\right\| = \left\|\mathbb{P}_+\frac{\psi}{z-\zeta}\right\|.$$

Clearly,

$$\mathbb{P}_+\frac{\psi}{z-\zeta} = \mathbb{P}_+\frac{\psi-\psi(\zeta)}{z-\zeta} + \psi(\zeta)\mathbb{P}_+\frac{1}{z-\zeta} = \frac{\psi-\psi(\zeta)}{z-\zeta}.$$

Therefore

$$\begin{aligned}\|H_\varphi k_\zeta\|_2 &= \left(\int_{\mathbb{T}}\frac{|\psi(\tau)-\psi(\zeta)|^2(1-|\zeta|^2)}{|\tau-\zeta|^2}d\boldsymbol{m}(\tau)\right)^{1/2}\\ &= \left(\int_{\mathbb{T}}|\psi(\tau)-\psi(\zeta)|^2P_\zeta(\tau)d\boldsymbol{m}(\tau)\right)^{1/2}.\end{aligned}$$

It follows that

$$\sup_{\zeta\in\mathbb{D}}\|H_\varphi k_\zeta\|_2 = \|\psi\|_{\mathrm{G}},$$

which proves that (6.1) is equivalent to the fact that $\mathbb{P}_-\varphi \in BMO$, which in turn means that H_φ is bounded (see Theorem 1.3). ■

To prove the compactness criterion in terms of kernel functions, we need the following characterization of VMO. Let $\varphi \in L^2$. Then $\varphi \in VMO$ if and only if

$$\lim_{|\zeta|\to 1}\int_{\mathbb{T}}|\varphi(\tau)-\varphi(\zeta)|^2P_\zeta(\tau)d\boldsymbol{m}(\tau) = 0$$

(see Appendix 2.5).

Theorem 6.2. *Let $\varphi \in L^2$. The Hankel operator H_φ is compact on H^2 if and only if*

$$\lim_{|\zeta|\to 1}\|H_\varphi k_\zeta\| = 0. \tag{6.2}$$

Proof. The same argument as in the proof of Theorem 6.1 shows that (6.2) is equivalent to the condition

$$\lim_{|\zeta|\to 1}\int_{\mathbb{T}}|\psi(\tau)-\psi_\#(\zeta)|^2P_\zeta(\tau)d\boldsymbol{m}(\tau) = 0,$$

which means that $\mathbb{P}_-\varphi \in VMO$, which in turn is equivalent to the compactness of H_φ (Theorem 5.8). ■

7. Hankel Operators and Moment Sequences

In this section we see that Hankel matrices appear naturally in connection with power moment problems. We are going to solve the Hamburger moment problem, and characterize the bounded and compact positive semi-definite Hankel matrices. We also characterize arbitrary bounded and compact Hankel matrices in terms of moment sequences of Carleson (vanishing Carleson) measures.

Let $\{\alpha_j\}_{j\ge 0}$ be a sequence of complex numbers. The classical *Hamburger moment problem* is to find a positive measure μ on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} |t|^j d\mu(t) < \infty, \quad j \ge 0, \tag{7.1}$$

and such that its moments coincide with the α_j, i.e.,

$$\alpha_j = \int_{\mathbb{R}} t^j d\mu(t), \quad j \ge 0. \tag{7.2}$$

In this case μ is called a *solution of the power moment problem with data* $\{\alpha_j\}_{j\ge 0}$.

The Hamburger theorem (Theorem 7.1 below) gives a solvability criterion in terms of positive semi-definiteness of the corresponding Hankel matrix $\{\alpha_{j+k}\}_{j,k\ge 0}$. (The matrix $\{\alpha_{j+k}\}_{j,k\ge 0}$ is called *positive semi-definite* if

$$\sum_{j,k\ge 0} \alpha_{j+k} x_j \bar{x}_k \ge 0 \tag{7.3}$$

for any finitely supported sequence $\{x_j\}_{j\ge 0}$ of complex numbers.)

Theorem 7.1. *Let $\{\alpha_j\}_{j\ge 0}$ be a sequence of complex numbers. The Hamburger moment problem with data $\{\alpha_j\}_{j\ge 0}$ is solvable if and only if the Hankel matrix $\{\alpha_{j+k}\}_{j,k\ge 0}$ is positive semi-definite.*

It is very easy to prove the "only if" part. To prove the "if" part we need some facts from the extension theory of symmetric operators. We are going to deal with (possibly unbounded) densely defined operators on Hilbert space. We review briefly some basic notions and facts and refer the reader to Birman and Solomjak [1] for a detailed presentation of the theory.

Let $\mathcal{H}$ be a Hilbert space. We consider linear transformations (operators) T whose domain $\mathcal{D}(T)$ is a dense linear manifold in $\mathcal{H}$ and whose range is a subset of $\mathcal{H}$. The operator T is called *closed* if its graph $\{x \oplus Tx : x \in \mathcal{D}(T)\}$ is closed in $\mathcal{H} \oplus \mathcal{H}$ and it is called *closable* if the closure of its graph in $\mathcal{H} \oplus \mathcal{H}$ is a graph of a linear operator. If T_1 and T_2 are densely defined linear operators, we say that T_2 is an extension of T_1 if $\mathcal{D}(T_1) \subset \mathcal{D}(T_2)$ and $T_2 x = T_1 x$ for $x \in \mathcal{D}_1$. If T is a densely defined closable linear operator, we define its adjoint operator T^* as follows. The domain $\mathcal{D}(T^*)$ consists of those vectors y in $\mathcal{H}$ for which the linear functional $x \mapsto (Tx, y)$, $x \in \mathcal{D}(T)$, extends to a bounded linear functional. Since T is closable, $\mathcal{D}(T^*)$ is dense in $\mathcal{H}$. For $y \in \mathcal{D}(T^*)$ we define $T^* y$ by the equality $(Tx, y) = (x, T^* y)$.

A densely defined linear operator A is called *symmetric* if A^* is an extension of A or, in other words,

$$(Ax, y) = (x, Ay), \quad x, y \in \mathcal{D}(A).$$

The operator A is called *self-adjoint* if $A^* = A$, i.e., A is symmetric and $\mathcal{D}(A^*) = \mathcal{D}(A)$.

With a symmetric operator operator A we can associate the *deficiency indices*

$$n_-(A) = \dim(\operatorname{Range}(A + \mathrm{i}I))^\perp, \quad n_+(A) = \dim(\operatorname{Range}(A - \mathrm{i}I))^\perp.$$

The symmetric operator A has a self-adjoint extension if and only if $n_-(A) = n_+(A)$.

If A is a *cyclic* self-adjoint operator (i.e., there exists a vector x_0 such that $A^n x_0 \in \mathcal{D}(A)$ for all $n \in \mathbb{Z}_+$ and $\operatorname{span}\{A^n x_0 : n \in \mathbb{Z}_+\} = \mathcal{H}$), then it follows from the spectral theorem that there exists a positive Borel measure μ on $\mathbb{R}$ such that A is unitary equivalent to the operator A_μ of multiplication by the independent variable on $L^2(\mu)$, i.e.,

$$\mathcal{D}(A_\mu) = \left\{ f \in L^2(\mu) : \int_{\mathbb{R}} |f(t)| d\mu(t) < \infty \right\}, \quad (A_\mu f)(t) = t f(t).$$

Moreover, if x_0 is such a vector, the unitary equivalence can be chosen to send x_0 to the constant function $\mathbf{1}$ identically equal to 1.

We are ready now to prove Theorem 7.1.

Proof of Theorem 7.1. It is very easy to prove that if there exists a measure μ satisfying (7.1) and (7.2), then the matrix $\{\alpha_{j+k}\}_{j,k\ge0}$ satisfies (7.3). Indeed,

$$\begin{aligned} \sum_{j,k\ge0} \alpha_{j+k} x_j \bar{x}_k &= \sum_{j,k\ge0} x_j \bar{x}_k \int_{\mathbb{R}} t^{j+k} d\mu(t) \\ &= \int_{\mathbb{R}} \left| \sum_{j\ge0} x_j t^j \right|^2 d\mu(t) \ge 0. \end{aligned}$$

Let us prove now the sufficiency of (7.3) for the solvability of the moment problem (7.2). Let $\mathcal{H}_0$ be the set of polynomials on $\mathbb{R}$. For $p, q \in \mathcal{H}_0$ define the following pairing:

$$\langle p, q \rangle = \sum_{j=0}^{m} \sum_{k=0}^{n} p_j \bar{q}_k \alpha_{j+k}, \tag{7.4}$$

where $p(t) = p_0 + p_1(t) + \cdots + p_m t^m$ and $q(t) = q_0 + q_1 t + \cdots + q_n t^n$. We consider separately two cases.

Case 1. The form (7.4) is nondegenerate, i.e., $\langle p, p \rangle \ne 0$ for any nonzero $p \in \mathcal{H}_0$. Let $\mathcal{H}$ be the completion of $\mathcal{H}_0$ with respect to the inner product (7.4). Then $\mathcal{H}$ is a Hilbert space.

Consider the operator A_0 with domain $\mathcal{H}_0$ defined by

$$(A_0 p)(t) = tp(t), \quad p \in \mathcal{H}_0.$$

It is easy to see that A is symmetric. We claim that A has a self-adjoint extension.

Let J_0 be the operator on $\mathcal{H}_0$ defined by $J_0 p = \bar{p}$. Clearly, J_0 is conjugate-linear and isometric. So it extends by continuity to a conjugate-linear isometry J of $\mathcal{H}$ onto itself. It is also easy to see that $J\,\mathrm{Range}(A_0 + \mathrm{i}I) = \mathrm{Range}(A_0 - \mathrm{i}I)$. Therefore $n_-(A_0) = n_+(A_0)$.

Let A be a self-adjoint extension of A_0. Clearly, $\mathrm{span}\{A^n \mathbf{1} : n \in \mathbb{Z}_+\} = \mathcal{H}$. By the spectral theorem there exist a positive Borel measure on $\mathbb{R}$ and a unitary operator U from $\mathcal{H}$ to $L^2(\mu)$ such that $U\mathbf{1} = \mathbf{1}$ and UAU^{-1} is multiplication by the independent variable on $L^2(\mu)$. Obviously, $Up = p$ for any $p \in \mathcal{H}_0$. It is easy to see that μ is a solution of our moment problem.

Case 2. There exists a nonzero polynomial p for which $\langle p, p\rangle = 0$. We can choose a polynomial p minimal degree with this property. Let $n = \deg p$. We now identify polynomials q and r if $\langle q - r, q - r\rangle = 0$. We can consider now the finite-dimensional Hilbert space $\mathcal{H}$ of equivalence classes endowed with inner product $\langle \cdot, \cdot \rangle = 0$.

Let us show that if q is a polynomial such that $\langle q, q\rangle = 0$, then $\langle Aq, q\rangle = 0$. Let $m = \deg q$.

Suppose that $q(t) = q_0 + p_1 t + \cdots + q_m t^m$ and $\langle q, q\rangle = 0$. Let Γ be the Hankel matrix $\{\alpha_{j+k}\}_{j,k \ge 0}$. Consider the matrix $\Gamma_N = \{\alpha_{j+k}\}_{0 \le j,k \le m+N}$ and the vector

$$x = \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_m \\ \mathbb{O} \\ \vdots \\ \mathbb{O} \end{pmatrix} \in \mathbb{C}^{m+1+N}.$$

Since $\langle q, q\rangle = 0$, it is easy to see that $(\Gamma_N x, x) = 0$, where $(\cdot, \cdot)$ is the standard inner product in $\mathbb{C}^{m+1+N}$. Since the matrix Γ_N is positive semi-definite, it follows that $\Gamma_N x = \mathbb{O}$ for any $N \in \mathbb{Z}_+$.

Now it is easy to see that

$$\Gamma \begin{pmatrix} \mathbb{O} \\ q_0 \\ q_1 \\ \vdots \\ q_m \\ \mathbb{O} \\ \vdots \end{pmatrix} = \mathbb{O},$$

and so $\langle Aq, Aq\rangle = 0$.

Thus we can consider the operator A as a self-adjoint operator on the n-dimensional Hilbert space $\mathcal{H}$. Now the same reasoning as in Case 1 allows us to construct a measure μ (in this case it consists of at most n atoms), which solves our moment problem. ■

Remark. It is easy to see that the deficiency indices $n_-(A_0)$ and $n_+(A_0)$ are either both equal to 0 or both equal to 1. In the case $n_-(A_0) = n_+(A_0) = 0$ there is only one solution of the moment problem, since there is only one self-adjoint extension of A_0. If $n_-(A_0) = n_+(A_0) = 1$, there are infinitely many solutions of the moment problem, since there are infinitely many self-adjoint extensions of A_0. Moreover in the last case one can obtain other solutions of the moment problem by considering so-called minimal self-adjoint extensions of A_0 in Hilbert spaces larger than $\mathcal{H}$.

Let us proceed now to positive semidefinite Hankel matrices $\Gamma_\alpha = \{\alpha_{j+k}\}_{j,k\ge 0}$ that are bounded on ℓ^2. We are going to describe such matrices in terms of solutions of the corresponding moment problems.

We are going to use the Carleson imbedding theorem, which says that a measure μ on $\mathbb{D}$ is a Carleson measure if and only if

$$\sup_I \frac{|\mu|(R_I)}{|I|} < \infty, \tag{7.5}$$

where

$$R_I = \left\{\zeta \in \mathbb{D} : \frac{\zeta}{|\zeta|} \in I \quad \text{and} \quad 1 - |\zeta| \le |I|\right\},$$

the supremum is taken over all subarcs I of $\mathbb{T}$, and $|I| \stackrel{\text{def}}{=} \boldsymbol{m}(I)$ (see Appendix 2.1).

Note that for μ supported on $(-1,1)$ condition (7.5) means that

$$|\mu|(1-t,1) \le \text{const}\cdot t, \quad |\mu|(-1,-1+t) \le \text{const}\cdot t.$$

Theorem 7.2. *Let $\Gamma = \{\alpha_{j+k}\}_{j,k\ge 0}$ be a nonnegative Hankel matrix. The following statements are equivalent:*

(i) Γ *determines a bounded operator on* ℓ^2*;*

(ii) *there exists a positive measure μ on $(-1,1)$ such that*

$$\alpha_n = \int_{-1}^{1} t^n d\mu(t)$$

and μ is a Carleson measure (as a measure on $\mathbb{D}$);

(iii) $|\alpha_n| \le \text{const}(1+n)^{-1}$.

Proof. By Hamburger's theorem there exists a measure μ on $\mathbb{R}$ that satisfies (7.1) and (7.2).

Suppose that Γ is bounded on ℓ^2. Then $\alpha_n \to 0$ as $n \to \infty$ and it is easy to see that this implies that μ is supported on $[-1,1]$. It is also easy to see that $\mu(\{1\}) = \mu(\{-1\}) = 0$.

Let $\mathcal{I}$ be the imbedding operator from H^2 to $L^2(\mu)$ which is defined on the set of polynomials by

$$\mathcal{I}f = f\big|(-1,1). \tag{7.6}$$

Let $f = \sum\limits_{j\ge 0}\hat{f}(j)z^j$ be a polynomial. It is easy to see that

$$(\mathcal{I}f,\mathcal{I}f)_{L^2(\mu)} = \Big(\Gamma_\alpha\{\hat{f}(j)\}_{j\ge 0},\{\hat{f}(j)\}_{j\ge 0}\Big)_{\ell^2}. \tag{7.7}$$

It follows that Γ is bounded on ℓ^2 if and only if $\mathcal{I}$ is a bounded operator from H^2 to $L^2(\mu)$. Thus the equivalence (i)$\Leftrightarrow$(ii) follows from the Carleson imbedding theorem.

Let us show that (ii)$\Rightarrow$(iii). We have $\mu((t,1)) \le c(1-t)$ for some constant c. Then

$$\begin{aligned}\int_0^1 t^n d\mu(t) &= \mu((0,1)) - \int_0^1 nt^{n-1}\mu((0,t))dt\\ &= \int_0^1 nt^{n-1}\mu((t,1))dt \le c\int_0^1 nt^{n-1}(1-t)dt \le \frac{c}{n+1}.\end{aligned}$$

One can prove in the same way that $\left|\int_{-1}^0 t^n d\mu(t)\right| \le \mathrm{const}(1+n)^{-1}$.

The fact that (iii) implies (i) follows easily from the boundedness of the Hilbert matrix. Indeed, let $\xi = \{\xi_j\}_{j\ge 0}$ and $\eta = \{\eta_k\}_{k\ge 0}$ be finitely supported sequences. We have

$$\begin{aligned}|(\Gamma_\alpha\xi,\eta)| &= \left|\sum_{j,k\ge 0}\alpha_{j+k}\xi_j\bar{\eta}_k\right|\\ &\le \sum_{j,k\ge 0}\frac{|\xi_j|\cdot|\eta_k|}{|j+k+1|} \le \pi\|\xi\|_{\ell^2}\|\eta\|_{\ell^2},\end{aligned}$$

since the norm of the Hilbert matrix is equal to π (see Corollary 5.19). ■

Let us characterize the compact nonnegative Hankel matrices in terms of vanishing Carleson measures (see Appendix 2.1).

Theorem 7.3. *Let $\Gamma = \{\alpha_{j+k}\}_{j,k\ge 0}$ be a nonnegative Hankel matrix. The following statements are equivalent:*

(i) Γ *determines a compact operator on* ℓ^2*;*

(ii) *there exists a positive measure* μ *on* $(-1,1)$ *such that*

$$\alpha_n = \int_{-1}^1 t^n d\mu(t)$$

and μ *is a vanishing Carleson measure (as a measure in* $\mathbb{D}$*);*

(iii) $\lim\limits_{n\to\infty}\alpha_n(1+n) = 0$.

Proof. We identify H^2 with ℓ^2 in the following natural way:

$$f \mapsto \{\hat{f}(n)\}_{n\ge 0}, \quad f \in H^2.$$

It follows from (7.7) that the operator $\mathcal{I}$ defined by (7.6) satisfies

$$\Gamma[\mu] = \mathcal{I}^*\mathcal{I},$$

and so $\Gamma[\mu]$ is compact if and only if $\mathcal{I}$ is, which proves that (i)$\Leftrightarrow$(ii).

Suppose that μ is a vanishing Carleson measure. Then for any $\varepsilon > 0$, μ admits a representation $\mu = \mu_1 + \mu_2$, where μ_1 is supported on $(-r, r)$ for some $r < 1$ and μ_2 satisfies

$$\mu_2((1-r, 1)) \le \varepsilon t, \quad \mu_2((-1, -1+r)) \le \varepsilon t.$$

The proof of the implication (ii)$\Rightarrow$(iii) in Theorem 7.1 gives the estimate

$$\left| \int_{-1}^{1} t^n d\mu_2(t) dt \right| \le \frac{2\varepsilon}{n+1}.$$

It is easy to see that

$$\int_{-r}^{r} t^n d\mu_1(t) \le \text{const } r^n.$$

This proves that (ii)$\Rightarrow$(iii).

To show that (iii)$\Rightarrow$(i) we can represent $\Gamma[\mu]$ as $\Gamma_1^{(n)} + \Gamma_2^{(n)}$, where

$$\Gamma_1 = \left\{ \alpha_{j+k}^{(1)} \right\}_{j,k \ge 0}, \quad \Gamma_2 = \Gamma - \Gamma_1,$$

and $\alpha_j^{(1)} = \alpha_j$ for $j \le n$, $\alpha_j^{(1)} = 0$ for $j > n$. Then $\Gamma_1^{(n)}$ has finite rank and the proof of the application (iii)$\Rightarrow$(i) in Theorem 7.1 shows that $\left\| \Gamma_2^{(n)} \right\| \to 0$, which completes the proof. ■

Note that (ii) in Theorem 7.3 means that

$$\lim_{t \to 0+} \frac{\mu((1-t, 1))}{t} = \lim_{t \to 0+} \frac{\mu((-1, -1+t))}{t} = 0.$$

Let now μ be a finite complex measure on $\mathbb{D}$. As in the case of measures on $[-1, 1]$ we can define the Hankel matrix $\Gamma[\mu]$ as follows:

$$\Gamma[\mu] = \left\{ \int_{\mathbb{D}} \zeta^{j+k} d\mu(\zeta) \right\}_{j,k \ge 0}.$$

As before we can identify ℓ^2 with H^2 and consider $\Gamma[\mu]$ as the matrix of an operator on H^2. It is easy to see that for any polynomials f and g in H^2

$$(\Gamma[\mu] f, g) = \int_{\mathbb{D}} f(\zeta) \overline{g(\bar{\zeta})} d\mu(\zeta). \tag{7.8}$$

Theorem 7.4. *If μ is a complex Carleson measure, then $\Gamma[\mu]$ is bounded on H^2. If $\Gamma = \{\alpha_{j+k}\}_{j,k \ge 0}$ is a bounded Hankel matrix, then there exists a complex Carleson measure μ such that*

$$\alpha_j = \int_{\mathbb{D}} \zeta^j d\mu(\zeta).$$

Proof. If μ is a Carleson measure, then it follows from (7.8) that for any polynomials f and g in H^2

$$\begin{aligned}|(\Gamma[\mu]f,g)| &= \left(\int_{\mathbb{D}}|f(\zeta)|^2 d|\mu|(\zeta)\right)^{1/2}\left(\int_{\mathbb{D}}|g(\bar{\zeta})|^2 d|\mu|(\zeta)\right)^{1/2}\\ &\le \text{const}\,\|f\|_{H^2}\|g\|_{H^2}\end{aligned}$$

by the definition of Carleson measures. So $\Gamma[\mu]$ is bounded.

To prove the converse, we need the fact that any function in BMO can be represented as the Poisson balayage of a Carleson measure (see Appendix 2.5).

Suppose that $\{\alpha_{j+k}\}_{j,k\ge 0}$ is a bounded matrix on ℓ^2. By Theorem 1.2,

$$\varphi \stackrel{\text{def}}{=} \sum_{j\ge 0}\alpha_j z^j \in BMOA.$$

Then there exists a Carleson measure μ whose Poisson balayage coincides with φ, that is,

$$\varphi(\zeta) = \int_{\mathbb{D}} P_\lambda(\zeta)d\mu(\lambda) \quad \text{for almost all} \quad \zeta\in\mathbb{T},$$

where P_λ is the Poisson kernel. It is easy to see that

$$\alpha_j = \hat{\varphi}(j) = \int_{\mathbb{D}}\zeta^j d\mu(\zeta), \quad j\ge 0. \quad \blacksquare$$

In a similar way one can prove the following compactness criterion.

Theorem 7.5. *If μ is a complex vanishing Carleson measure, then $\Gamma[\mu]$ is compact on H^2. If $\Gamma = \{\alpha_{j+k}\}_{j,k\ge 0}$ is a compact Hankel matrix, then there exists a complex vanishing Carleson measure μ such that*

$$\alpha_j = \int_{\mathbb{D}}\zeta^j d\mu(\zeta).$$

To conclude this section, we evaluate the Hankel matrices $\Gamma_{[\boldsymbol{m}^1_{-1}]}$ and $\Gamma_{[\boldsymbol{m}^1_0]}$, where $\boldsymbol{m}^1_{-1}$ is Lebesgue measure on $[-1,1]$ and $\boldsymbol{m}^1_0$ is Lebesgue measure on $[0,1]$. It is easy to see that

$$\int_{-1}^1 x^j dx = \begin{cases}\frac{2}{j+1}, & j \text{ is even,}\\ 0, & j \text{ is odd.}\end{cases}$$

Hence,

$$\Gamma_{[\boldsymbol{m}^1_{-1}]} = 2\begin{pmatrix}1 & 0 & \frac13 & 0 & \frac15 & \cdots\\ 0 & \frac13 & 0 & \frac15 & 0 & \cdots\\ \frac13 & 0 & \frac15 & 0 & \frac17 & \cdots\\ 0 & \frac15 & 0 & \frac17 & 0 & \cdots\\ \frac15 & 0 & \frac17 & 0 & \frac19 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}. \tag{7.9}$$

On the other hand,

$$\int_0^1 x^j\,dx = \frac{1}{j+1},$$

and so

$$\Gamma_{[\boldsymbol{m}_0^1]} = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots \\ 1/2 & 1/3 & 1/4 & & \ddots \\ 1/3 & 1/4 & & \ddots & \\ 1/4 & & \ddots & & \\ \vdots & \ddots & & & \end{pmatrix} \tag{7.10}$$

is the Hilbert matrix (see §1).

Corollary 7.6. *The matrices on the right-hand sides of* (7.9) *and* (7.10) *are bounded positive semi-definite matrices.*

Proof. This is an immediate consequence of Theorem (7.2). ■

8. Hankel Operators as Integral Operators on the Semi-Axis

In this section we consider integral operators on the space $L^2(\mathbb{R}_+)$ of functions on the semi-axis $\mathbb{R}_+$ with kernels depending only on the sum of the variables. We show that such operators are unitarily equivalent to Hankel operators on ℓ^2 introduced in §1 and this will allow us to reduce the study of such operators to the study of Hankel operators on ℓ^2.

For a function $k \in L^1(\mathbb{R}_+)$ consider the operator $\boldsymbol{\Gamma}_k$ on $L^2(\mathbb{R}_+)$,

$$(\boldsymbol{\Gamma}_k f)(t) = \int_0^\infty k(s+t)f(s)ds. \tag{8.1}$$

Below we shall see that $\boldsymbol{\Gamma}_k$ is a bounded operator on $L^2(\mathbb{R}_+)$.

The definition of $\boldsymbol{\Gamma}_k$ is similar to the definition of Hankel operators as operators with matrices whose entries depend only on the sum of the coordinates (see §1). We are going to define the operator $\boldsymbol{\Gamma}_k$ in a much more general case where k does not necessarily belong to $L^1(\mathbb{R}_+)$. We shall see that $\boldsymbol{\Gamma}_k$ can be bounded for certain distributions. We shall find in this section boundedness, compactness, and finite rank criteria for $\boldsymbol{\Gamma}_k$, and we establish the unitary equivalence between operators $\boldsymbol{\Gamma}_k$ and Hankel operators on ℓ^2.

We call operators $\boldsymbol{\Gamma}_k$ *Hankel operators on* $L^2(\mathbb{R}_+)$.

It will be slightly more convenient for us to deal with the following integral operators. Let $q \in L^1(\mathbb{R}_-)$. We define the integral operator G_q from

$L^2(\mathbb{R}_+)$ to $L^2(\mathbb{R}_-)$ by

$$(G_q f)(t) = \int_0^\infty q(t-s)f(s)ds, \quad t < 0. \tag{8.2}$$

Clearly,

$$(\boldsymbol{\Gamma}_k f)(t) = (G_q f)(-t), \quad k(t) = q(-t), \quad t \in \mathbb{R}_+. \tag{8.3}$$

We can assume for convenience that $q \in L^1(\mathbb{R})$ and $q(t) = 0$ for $t > 0$, since the right-hand side in (8.2) does not depend on the values of q on $\mathbb{R}_+$. Denote by P_+ and P_- the operators of multiplication by the characteristic function of $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively. We can also identify the spaces $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$ with subspaces of $L^2(\mathbb{R})$. Then P_- and P_+ are the orthogonal projections of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}_-)$ and $L^2(\mathbb{R}_+)$ respectively. It is easy to see that G_q can also be defined by

$$G_q f = P_-(q * f), \quad f \in L^2(\mathbb{R}_+),$$

where the the *convolution* $\varphi * \psi$ of functions φ and ψ on $\mathbb{R}$ is defined by

$$(\varphi * \psi)(t) = \int_{\mathbb{R}} \varphi(s)\psi(t-s)\,ds.$$

It is easy to see now that G_q is bounded for $q \in L^1(\mathbb{R})$. We have

$$\|G_q f\|_2 = \|P_-(q * f)\|_2 \le \|q * f\|_2 \le \|q\|_1 \|f\|_2;$$

the last inequality is well known and can be proved elementarily.

Digression. Distributions

To define the operators G_q and $\boldsymbol{\Gamma}_k$ in a more general context we need the notion of distributions (see Schwartz [1] for more detailed information). Let $\boldsymbol{\mathcal{D}}$ be the space of infinitely differentiable functions on $\mathbb{R}$ with compact support. The space $\boldsymbol{\mathcal{D}}$ is endowed with the natural topology of inductive limit. Without entering into detail let us mention the following. A sequence $\{f_j\}_{j\ge 0}$ converges in $\boldsymbol{\mathcal{D}}$ to f if and only if there is a compact set K in $\mathbb{R}$ such that $\operatorname{supp} f_j \subset K$ for any j and all derivatives $f_j^{(m)}$ converge uniformly on K to the derivatives $f^{(m)}$, $m \ge 0$. A linear operator T from $\boldsymbol{\mathcal{D}}$ to a locally convex space X is continuous if and only if for any sequence $\{f_j\}_{j\ge 0}$ converging to $\mathbb{O}$ in $\boldsymbol{\mathcal{D}}$, the sequence $\{Tf_j\}_{j\ge 0}$ converges in X to $\mathbb{O}$. For convenience we define the space of distributions $\boldsymbol{\mathcal{D}}'$ as the space of continuous *antilinear* functionals on $\boldsymbol{\mathcal{D}}$, i.e.,

$$< g, \lambda f > \stackrel{\text{def}}{=} g(\lambda f) = \bar{\lambda} < g, f >, \quad f \in \boldsymbol{\mathcal{D}}, \quad g \in \boldsymbol{\mathcal{D}}', \quad \lambda \in \mathbb{C};$$

$$< g, f_1 + f_2 > = < g, f_1 > + < g, f_2 >, \quad f_1, f_2 \in \boldsymbol{\mathcal{D}}, \quad g \in \boldsymbol{\mathcal{D}}'.$$

If g is a locally summable function on $\mathbb{R}$, it can be interpreted as a distribution that acts on $\boldsymbol{\mathcal{D}}$ as follows:

$$< g, f > = \int_{\mathbb{R}} g(t)\overline{f(t)}dt, \quad f \in \boldsymbol{\mathcal{D}}. \tag{8.4}$$

For $g \in \mathcal{D}'$, we define its support $\operatorname{supp} g$ as the minimal closed set F for which $< g, f >= 0$ for any $f \in \mathcal{D}$ with support in $\mathbb{R} \setminus F$.

If Ω is an open set in $\mathbb{R}$, we can consider in a similar way the space $\mathcal{D}(\Omega)$ of infinitely smooth functions with compact support in Ω and we can define the space of distributions $\mathcal{D}'(\Omega)$.

We need the important subclass of $\mathcal{D}'$, the space of tempered distributions. Let $\mathcal{S}$ be the space of infinitely differentiable functions f on $\mathbb{R}$ such that for any m, $n \in \mathbb{Z}_+$

$$\sup_{t\in\mathbb{R}} \left| f^{(m)}(t) \right| (1 + |t|)^n < \infty. \tag{8.5}$$

The topology on $\mathcal{S}$ is determined by the system of seminorms on the left-hand side of (8.5). As in the case of the space $\mathcal{D}'$ we define the space $\mathcal{S}'$ of *tempered distributions* as the space of continuous *antilinear* functionals on $\mathcal{S}$.

Note that if g is a measurable function on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \frac{|g(t)|}{(1 + |t|)^n} dt < \infty$$

for some $n \in \mathbb{Z}_+$, then g can be considered as a tempered distribution that acts on $\mathcal{S}$ as in (8.4).

Recall that the *Fourier transform* $\mathcal{F}f$ of a function f in $L^1(\mathbb{R})$ is defined by

$$(\mathcal{F}f)(s) = \int_{\mathbb{R}} e^{-2\pi i ts} f(t)dt.$$

By Plancherel's theorem $\mathcal{F}\big| L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$ extends to a unitary operator on $L^2(\mathbb{R})$ and the inverse Fourier transform $\mathcal{F}^{-1} = \mathcal{F}^*$ is defined on $L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$ by

$$(\mathcal{F}^* f)(t) = \int_{\mathbb{R}} e^{2\pi i ts} f(t)ds.$$

It is easy to see that $\mathcal{F}^*\mathcal{S} = \mathcal{S}$, which allows us to define the Fourier transform on the space of tempered distributions $\mathcal{S}'$ by

$$< \mathcal{F}g, f > \stackrel{\text{def}}{=} < g, \mathcal{F}^* f >, \quad f \in \mathcal{S}, \quad g \in \mathcal{S}'.$$

Recall one more elementary identity:

$$\mathcal{F}^*(f * g) = (\mathcal{F}^* f)(\mathcal{F}^* g), \quad f \in L^1(\mathbb{R}), \quad g \in L^2(\mathbb{R}). \tag{8.6}$$

Boundedness, Compactness, and Finite Rank Criteria

Now we are in a position to define the operators G_q and $\boldsymbol{\Gamma}_k$ in a more general situation. If $q \in L^1(\mathbb{R})$, it is an elementary exercise to see that

$$(G_q f, g) = (q, \bar{f}_\circ * g) \stackrel{\text{def}}{=} \int_{\mathbb{R}} q(s)\overline{(\bar{f}_\circ * g)(s)}ds, \quad f \in L^2(\mathbb{R}_+), \quad g \in L^2(\mathbb{R}_-), \tag{8.7}$$

where $f_\circ(s) \stackrel{\text{def}}{=} f(-s)$.

Formula (8.7) suggests the following definition of G_q for some distributions q in $\boldsymbol{\mathcal{D}}'(-\infty,0)$. Put

$$(G_q f, g) \stackrel{\text{def}}{=} < q, \bar{f}_\circ * g >, \quad f \in \boldsymbol{\mathcal{D}}(0,\infty), \quad g \in \boldsymbol{\mathcal{D}}(-\infty,0) \tag{8.8}$$

(we identify in a natural way $\boldsymbol{\mathcal{D}}(0,\infty)$ and $\boldsymbol{\mathcal{D}}(-\infty,0)$ with the subspaces of functions in $\boldsymbol{\mathcal{D}}$ with support in $(0,\infty)$ and $(-\infty,0)$, respectively). We say that G_q *is bounded on* $L^2(\mathbb{R}_+)$ if

$$|(G_q f, g)| \le \text{const} \cdot \|f\|_2 \|g\|_2.$$

In this case for $f \in \boldsymbol{\mathcal{D}}(0,\infty)$ we define $G_q f$ as the function in $L^2(\mathbb{R}_-)$ such that $(G_q f, g) =< q, \bar{f}_\circ * g >$ for $g \in \boldsymbol{\mathcal{D}}(-\infty,0)$.

We can define now the operator $\boldsymbol{\Gamma}_k$ by (8.3) in the case when the operator G_q is bounded. The identity $k(t) = q(-t),\ t \in \mathbb{R}$, means that

$$< k, f >=< q, f_\circ >, \quad f \in \boldsymbol{\mathcal{D}}.$$

The following result characterizes the bounded operators G_q.

Theorem 8.1. *Let $q \in \boldsymbol{\mathcal{D}}'(-\infty,0)$. The operator G_q defined by* (8.8) *is bounded on $L^2(\mathbb{R}_+)$ if and only if there exists a function $\varkappa \in L^\infty(\mathbb{R})$ such that $\mathcal{F}\varkappa\big|(-\infty,0) = q$. Moreover,*

$$\|G_q\| = \inf\left\{\|\varkappa\|_\infty : \varkappa \in L^\infty(\mathbb{R}),\ \mathcal{F}\varkappa\big|(-\infty,0) = q\right\}. \tag{8.9}$$

(By $\mathcal{F}\varkappa\big|(-\infty,0) = q$ we mean that $< \mathcal{F}\varkappa, g >=< q, g >$ for any $g \in \boldsymbol{\mathcal{D}}(-\infty,0)$).

To prove Theorem 8.1 we need the notion of Hankel operator on the Hardy class $H^2(\mathbb{C}_+)$ of functions in the upper half-plane (see Appendix 2.1). Recall that the Paley–Wiener theorem allows us to identify $H^2(\mathbb{C}_+)$ with the following subspace of $L^2(\mathbb{R})$:

$$H^2(\mathbb{C}_+) = \left\{f \in L^2(\mathbb{R}) :\ \mathcal{F}f\big|(-\infty,0) = \mathbb{O}\right\}.$$

Similarly, the Hardy class $H^2(\mathbb{C}_-)$ of functions analytic in the lower half-plane can be identified with the subspace of functions in $L^2(\mathbb{R})$ whose Fourier transforms vanish on $\mathbb{R}_+$. Clearly, under this identification $L^2(\mathbb{R}) = H^2(\mathbb{C}_+) \oplus H^2(\mathbb{C}_-)$.

Recall that the unitary operator $\mathcal{U}$ on $L^2(\mathbb{T})$ defined by

$$(\mathcal{U}f)(t) = \frac{1}{\sqrt{\pi}} \frac{(f \circ \omega^{-1})(t)}{t + \mathrm{i}}, \quad t \in \mathbb{R}, \tag{8.10}$$

is a unitary operator from $L^2(\mathbb{T})$ onto $L^2(\mathbb{R})$ and $\mathcal{U}(H^2) = H^2(\mathbb{C}_+)$ (see Appendix 2.1). Here ω is the conformal map ω of $\mathbb{D}$ onto $\mathbb{C}_+$ defined by

$$\omega(\zeta) = \mathrm{i}\frac{1+\zeta}{1-\zeta}, \quad \zeta \in \mathbb{D},$$

and

$$\omega^{-1}(w) = \frac{w - \mathrm{i}}{w + \mathrm{i}}, \quad w \in \mathbb{C}_+.$$

Note that ω also maps conformally $\hat{\mathbb{C}} \setminus \operatorname{clos} \mathbb{D}$ onto $\mathbb{C}_-$, where $\hat{\mathbb{C}} \stackrel{\text{def}}{=} \mathbb{C} \cup \infty$.

To prove Theorem 8.1 we need the following lemma.

Lemma 8.2. *Let q be a distribution on $(-\infty, 0)$ such that the operator G_q defined by (8.8) is bounded on $L^2(\mathbb{R}_+)$. Then the operator $\Gamma : H^2 \to H^2_-$ defined by*

$$\Gamma = \mathcal{U}^* \mathcal{F}^* G_q \mathcal{F} \mathcal{U} \tag{8.11}$$

is a Hankel operator on H^2.

Proof. In view of the commutation relation (1.13) it suffices to show that

$$(\mathbb{P}_- z \Gamma \xi, \eta) = (\Gamma z \xi, \eta) \tag{8.12}$$

for ξ in a dense subset of H^2 and η in a dense subset of H^2_-. We have

$$(\Gamma z \xi, \eta) = (G_q \mathcal{F} \mathcal{U} z \xi, \mathcal{F} \mathcal{U} \eta).$$

Put

$$F \stackrel{\text{def}}{=} \mathcal{U} \xi, \quad G \stackrel{\text{def}}{=} \mathcal{U} \eta, \quad f \stackrel{\text{def}}{=} \mathcal{F} F, \quad g \stackrel{\text{def}}{=} \mathcal{F} G.$$

We have by (8.10)

$$(\mathcal{U} z \xi)(t) = F(t) \frac{t - \mathrm{i}}{t + \mathrm{i}} = F(t) \left(1 - 2\mathrm{i} \frac{1}{t + \mathrm{i}}\right).$$

Put $\alpha(t) = 2\mathrm{i}/(t + \mathrm{i})$, $t \in \mathbb{R}$. Clearly, $\alpha \in H^2(\mathbb{C}_+)$. Let $h = \mathcal{F} \alpha \in L^2(\mathbb{R}_+)$. It follows from (8.6) that $\mathcal{F}^*(f * h) = F\alpha$. Therefore

$$(\Gamma z \xi, \eta) = (G_q f, g) + (G_q (h * f), g).$$

Clearly, $\mathcal{F}^* \bar{h}_\circ = \bar{\alpha}$ and so we can obtain in a similar way

$$(\mathbb{P}_- z \Gamma \xi, \eta) = (\Gamma \xi, \bar{z} \eta) = (G_q \mathcal{F} \mathcal{U} \xi, \mathcal{F} \mathcal{U} \bar{z} \eta) = (G_q f, g) + (G_q f, \bar{h}_\circ * g).$$

Therefore to prove (8.12) it is sufficient to show that

$$(G_q (h * f), g) = (G_q f, \bar{h}_\circ * g) \tag{8.13}$$

for any functions $f \in \boldsymbol{\mathcal{D}}(0, \infty)$, $g \in \boldsymbol{\mathcal{D}}(-\infty, 0)$ and any $h \in L^2(\mathbb{R}_+)$. In view of (8.8) equality (8.13) is equivalent to the following one:

$$\overline{(h * f)}_\circ * g = \bar{f}_\circ * (\bar{h}_\circ * g).$$

The latter follows from the following identity, which can be verified elementarily:

$$(h * f)_\circ = h_\circ * f_\circ. \quad \blacksquare$$

Before we proceed to the proof of Theorem 8.1 we define Hankel operators on the Hardy class $H^2(\mathbb{C}_+)$. Denote by $\boldsymbol{P}_+$ and $\boldsymbol{P}_-$ the orthogonal projections from $L^2(\mathbb{R}_+)$ onto $H^2(\mathbb{C}_+)$ and $H^2(\mathbb{C}_-)$. It follows from the Paley–Wiener theorem that

$$\boldsymbol{P}_+ = \mathcal{F}^* P_+ \mathcal{F}, \quad \boldsymbol{P}_- = \mathcal{F}^* P_- \mathcal{F}.$$

Let $\psi \in L^\infty(\mathbb{R})$. The *Hankel operator* $\mathcal{H}_\psi : H^2(\mathbb{C}_+) \to H^2(\mathbb{C}_-)$ is defined by

$$\mathcal{H}_\psi F = \boldsymbol{P}_- \psi F, \quad F \in H^2(\mathbb{C}_+).$$

Clearly, $\mathcal{H}_\psi$ is a bounded operator on $H^2(\mathbb{C}_+)$.

Lemma 8.3. *Let $\psi \in L^\infty(\mathbb{R})$ and $\varphi \stackrel{\text{def}}{=} \psi \circ \omega \in L^\infty(\mathbb{T})$. Then*

$$H_\varphi = \mathcal{U}^* \mathcal{H}_\psi \mathcal{U}.$$

Proof. Let $f \in H^2$, $g \in H^2_-$. We have

$$(H_\varphi f, g) = (\varphi f, g);$$

$$(\mathcal{U}^* \mathcal{H}_\psi \mathcal{U} f, g) = (\mathcal{H}_\psi \mathcal{U} f, \mathcal{U} g) = (\psi \mathcal{U} f, \mathcal{U} g) = (\mathcal{U} \varphi f, \mathcal{U} g) = (\varphi f, g)$$

by (8.10). ■

Corollary 8.4. *Let $\psi \in L^\infty(\mathbb{R})$. Then*

$$\|\mathcal{H}_\psi\| = \operatorname{dist}_{L^\infty(\mathbb{R})}(\psi, H^\infty(\mathbb{C}_+)).$$

The next result gives a compactness criterion for Hankel operators $\mathcal{H}_\psi$. The space $C(\hat{\mathbb{R}})$ is by definition the space of continuous functions on $\mathbb{R}$ which have equal limits at ∞ and $-\infty$.

Corollary 8.5. *Let $\psi \in L^\infty(\mathbb{R})$. $\mathcal{H}_\psi$ is compact if and only if $\psi \in H^\infty(\mathbb{C}_+) + C(\hat{\mathbb{R}})$.*

Clearly, Corollary 8.5 is an immediate consequence of Lemma 8.3.

Now we are in a position to prove Theorem 8.1.

Proof of Theorem 8.1. Suppose that G_q is bounded. Then by Lemma 8.2, the operator Γ defined by (8.11) is a bounded Hankel operator on H^2. So $\Gamma = H_\varphi$, where $\varphi \in L^\infty(\mathbb{T})$ and $\|\varphi\|_\infty = \|G_q\|$. It follows from (8.11) and from Lemma 8.3 that

$$\mathcal{U} H_\varphi \mathcal{U}^* = \mathcal{H}_\psi = \mathcal{F}^* G_q \mathcal{F},$$

where $\psi = \varphi \circ \omega^{-1}$. Hence, $G_q = \mathcal{F} \mathcal{H}_\psi \mathcal{F}^*$.

Put $r \stackrel{\text{def}}{=} \mathcal{F}\psi \in \boldsymbol{\mathcal{S}}'$. Let us show that $\mathcal{F} \mathcal{H}_\psi \mathcal{F}^* = G_r$. We have

$$\begin{aligned} (\mathcal{F} \mathcal{H}_\psi \mathcal{F}^* f, g) &= (\psi \mathcal{F}^* f, \mathcal{F}^* g) = < \psi, \overline{\mathcal{F}^* f} \cdot \mathcal{F}^* g > \\ &= < r, \mathcal{F}(\overline{\mathcal{F}^* f} \cdot \mathcal{F}^* g) > = < r, \bar{f}_\circ * g >, \\ & \qquad f \in \boldsymbol{\mathcal{D}}(0,\infty), \quad g \in \boldsymbol{\mathcal{D}}(-\infty, 0). \end{aligned} \tag{8.14}$$

Therefore $G_q = G_r$. Since the set of functions

$$\bar{f}_\circ * g, \quad f \in \boldsymbol{\mathcal{D}}(0,\infty), \quad g \in \boldsymbol{\mathcal{D}}(-\infty, 0),$$

is obviously dense in $\boldsymbol{\mathcal{D}}(-\infty, 0)$, it follows that

$$q = r \,\big|\, (-\infty, 0) = \mathcal{F}\psi \big| (-\infty, 0)$$

and $\|G_q\| = \|\psi\|_\infty$, which proves that $\|G_q\|$ is greater than or equal to the right-hand side of (8.9).

If $q = \mathcal{F}\psi\big|(-\infty,0)$ and $\psi \in L^\infty(\mathbb{R})$, then the same reasoning proves that

$$G_q = \mathcal{F}\mathcal{H}_\psi\mathcal{F}^*, \tag{8.15}$$

which implies that $\|G_q\| \le \|\psi\|_\infty = \|\varkappa\|_\infty$. ■

In the case of functions on $\mathbb{T}$ we can represent any bounded function φ as $\varphi = \mathbb{P}_+\varphi + \mathbb{P}_-\varphi$ and the Hankel operator H_φ depends only on $\mathbb{P}_-\varphi$. However, in the case of functions on $\mathbb{R}$ the projections $\boldsymbol{P}_+$ and $\boldsymbol{P}_-$ are not defined on $L^\infty(\mathbb{R})$. Instead we define on $L^\infty(\mathbb{R})$ the operators $\boldsymbol{P}^+$ and $\boldsymbol{P}^-$ by

$$\boldsymbol{P}^+\psi \stackrel{\text{def}}{=} (\mathbb{P}_+(\psi\circ\omega))\circ\omega^{-1}, \quad \boldsymbol{P}^-\psi \stackrel{\text{def}}{=} (\mathbb{P}_-(\psi\circ\omega))\circ\omega^{-1}, \quad \psi \in L^\infty(\mathbb{R}).$$

We shall also use the notation $\psi^+ = \boldsymbol{P}^+\psi$, $\psi^- = \boldsymbol{P}^-\psi$. Clearly, $\psi = \psi^+ + \psi^-$. Note that $\boldsymbol{P}^+$ and $\boldsymbol{P}^-$ are unbounded on $L^\infty(\mathbb{R})$.

Lemma 8.6. *Let* $\psi \in L^\infty(\mathbb{R})$. *Then* $\operatorname{supp}\mathcal{F}\psi^+ \subset [0,\infty)$ *and* $\operatorname{supp}\mathcal{F}\psi^- \subset (-\infty,0]$.

Proof. Let $g \in \boldsymbol{\mathcal{S}}$, $\operatorname{supp} g \subset \mathbb{R}_-$. Let us show that $<\mathcal{F}\psi^+, g> = 0$, which would mean that $\operatorname{supp}\mathcal{F}\psi^+ \subset \mathbb{R}_+$. The proof of the fact that $\operatorname{supp}\mathcal{F}\psi^- \subset \mathbb{R}_-$ is similar. We have

$$<\mathcal{F}\psi^+, g> = <\psi^+, \mathcal{F}^*g> = \int_{\mathbb{R}} \psi^+(t)\overline{(\mathcal{F}^*g)(t)}dt = 0$$

since $\psi^+\overline{(\mathcal{F}^*g)} \in H^1(\mathbb{C}_+)$ (see Appendix 2.1).■

We can obtain now an analog of Theorem 1.2 in terms of the space $BMO(\mathbb{R})$ of functions on $\mathbb{R}$ (see Appendix 2.5). Recall that $\psi \in BMO$ if and only if ψ can be represented as $\psi = \xi + \boldsymbol{P}^+\eta$, for some ξ and η in $L^\infty(\mathbb{R})$. As in the case of functions on the unit circle the following result follows easily from Theorem 8.1.

Corollary 8.7. *Let* $q \in \boldsymbol{\mathcal{D}}'(-\infty,0)$. *Then* G_q *is bounded on* $L^2(\mathbb{R}_+)$ *if and only if* $q = r\big|(-\infty,0)$ *for some* $r \in \boldsymbol{\mathcal{S}}'$ *such that* $\operatorname{supp} r \subset \mathbb{R}_-$ *and* $\mathcal{F}^*r \in BMO(\mathbb{R})$.

Let us now obtain the boundedness criterion for $\boldsymbol{\Gamma}_k$.

Theorem 8.8. *Let* $k \in \boldsymbol{\mathcal{D}}'(0,\infty)$. *The following statements are equivalent:*

(i) $\boldsymbol{\Gamma}_k$ *is bounded on* $L^2(\mathbb{R}_+)$;

(ii) *there exists a function* $\varkappa$ *in* $L^\infty(\mathbb{R})$ *such that* $\mathcal{F}\varkappa\big|(0,\infty) = k$;

(iii) *there exists* $r \in \boldsymbol{\mathcal{S}}'$ *such that* $k = r\big|(0,\infty)$, $\operatorname{supp} r \subset \mathbb{R}_+$, *and* $\mathcal{F}^*r \in BMO(\mathbb{R})$.

If $\boldsymbol{\Gamma}_k$ *is bounded, then*

$$\|\boldsymbol{\Gamma}_k\| = \inf\{\|\varkappa\|_\infty : \ \varkappa \in L^\infty(\mathbb{R}), \ \mathcal{F}\varkappa\big|\mathbb{R}_+ = k\}.$$

Proof. It remains to prove that (ii) implies (iii); everything else follows immediately from Theorem 8.1 and the definition of $\boldsymbol{\Gamma}_k$. Let $\varkappa \in L^\infty(\mathbb{R})$ such that $\mathcal{F}\varkappa\big|(0,\infty) = k$. Define r by $r = \mathcal{F}\boldsymbol{P}^+\varkappa$. By Lemma 8.6, $k = r\big|(0,\infty)$ and $\operatorname{supp} r \subset \mathbb{R}_+$. Clearly, $\mathcal{F}^*r = \boldsymbol{P}^+\varkappa \in BMO(\mathbb{R})$. ■

The operator $\boldsymbol{\Gamma}_k$ defined by (8.1) is a continual analog of a Hankel matrix. The next result shows that a bounded operator $\boldsymbol{\Gamma}_k$ is unitarily equivalent to a Hankel operator on ℓ^2 (i.e., an operator with Hankel matrix).

Theorem 8.9. *If $\boldsymbol{\Gamma}_k$ is a bounded operator, then the operator $\mathcal{U}^*\mathcal{F}^*\boldsymbol{\Gamma}_k\mathcal{F}\mathcal{U}$ on H^2 has Hankel matrix in the basis $\{z^n\}_{n\geq 0}$.*

Proof. Let $q(t) = k(-t)$, $t \in \mathbb{R}$. Then G_q is bounded and so by (8.15) $G_q = \mathcal{F}\mathcal{H}_\psi\mathcal{F}^*$ for some bounded function ψ on $\mathbb{R}$. Put $\varphi = \psi \circ \omega \in L^\infty(\mathbb{T})$. Let $\xi_n \stackrel{\text{def}}{=} \mathcal{U}z^n$, $n \in \mathbb{Z}_+$. We have

$$(\mathcal{U}^*\mathcal{F}^*\boldsymbol{\Gamma}_k\mathcal{F}\mathcal{U}z^n, z^m) = (\boldsymbol{\Gamma}_k\mathcal{F}\xi_n, \mathcal{F}\xi_m) = (G_q\mathcal{F}\xi_n, (\mathcal{F}\xi_m)_\circ)$$

$$= (G_q\mathcal{F}\xi_n, \mathcal{F}^*\xi_m) = (\mathcal{F}^*G_q\mathcal{F}\xi_n, (\mathcal{F}^*)^2\xi_m) = (\mathcal{H}_\psi\xi_n, (\xi_m)_\circ)\,.$$

It is easy to see that $(\xi_m)_\circ \in H^2(\mathbb{C}_-)$ and so $(\mathcal{H}_\psi\xi_n, (\xi_m)_\circ) = (\psi\xi_n, (\xi_m)_\circ)$. Therefore

$$\begin{aligned}(\mathcal{U}^*\mathcal{F}^*\boldsymbol{\Gamma}_k\mathcal{F}\mathcal{U}z^n, z^m) &= (\psi\xi_n, (\xi_m)_\circ) = -\frac{1}{\pi}\int_{\mathbb{R}}\psi(t)\frac{(t-\mathrm{i})^{n+m}}{(t+\mathrm{i})^{n+m+2}}dt \\ &= -\frac{1}{\pi}\int_{\mathbb{R}}\psi(t)\left(\frac{t-\mathrm{i}}{t+\mathrm{i}}\right)^{n+m+1}\frac{dt}{|t+\mathrm{i}|^2} \\ &= -\int_{\mathbb{T}}\varphi(\zeta)\zeta^{n+m+1}d\boldsymbol{m}(\zeta) = -\hat{\varphi}(-n-m-1),\end{aligned}$$

which proves the result. ■

Remark. Theorem 8.9 says that $\boldsymbol{\Gamma}_k$ has Hankel matrix in the orthonormal basis $\{\mathcal{F}\mathcal{U}z^n\}_{n\geq 0}$ of the space $L^2(\mathbb{R}_+)$. Using the residue theorem, one can show that

$$(\mathcal{F}\mathcal{U}z^n)(t) = -2\sqrt{\pi}\mathrm{i}L_n(4\pi t)e^{-2\pi t}, \quad t > 0,$$

where the functions L_n are the so-called *Lageurre polynomials*, $L_n(s) = \frac{1}{n!}e^s\left(\frac{d}{ds}\right)^n(e^{-s}s^n)$; see Szegö [1].

To proceed to compactness criteria, we need the space $VMO(\mathbb{R})$ (see Appendix 2.5 for the definition). Note that there are several possible definitions of $VMO(\mathbb{R})$ and ours differs from the one in Garnett [1]. The space $VMO(\mathbb{R})$ consists of functions of the form $\xi + \boldsymbol{P}^+\eta$ with ξ, $\eta \in C(\hat{\mathbb{R}})$.

It is easy to obtain from Corollary 8.5 and formula (8.15) a compactness criterion for G_q. We state instead the following compactness criterion for $\boldsymbol{\Gamma}_k$.

Theorem 8.10. *Let $k \in \boldsymbol{\mathcal{D}}'(0,\infty)$. The following statements are equivalent:*

(i) *$\boldsymbol{\Gamma}_k$ is a compact operator on $L^2(\mathbb{R}_+)$;*

(ii) *there exists a function $\varkappa$ in $C(\hat{\mathbb{R}})$ such that $\mathcal{F}\varkappa\big|(0,\infty) = k$;*

(iii) *there exists $r \in \boldsymbol{\mathcal{S}}'$ such that* $\operatorname{supp} r$ *$\subset \mathbb{R}_+$, $k = r\big|(0,\infty)$, and $\mathcal{F}^*r \in VMO(\mathbb{R})$.*

We have started this section with the operators $\boldsymbol{\Gamma}_k$ for $k \in L^1(\mathbb{R}_+)$ and observed that such operators are bounded. In fact it follows immediately from Theorem 8.10 that they are compact.

Corollary 8.11. *Let $k \in L^1(\mathbb{R}_+)$. Then $\boldsymbol{\Gamma}_k$ is compact on $L^2(\mathbb{R}_+)$.*

Proof. Since $\mathcal{F}^* k \in C(\hat{\mathbb{R}})$ by the Riemann–Lebesgue lemma, the result follows directly from statement (ii) of Theorem 8.10. ■

Let us now characterize the finite rank operators $\boldsymbol{\Gamma}_k$. It is easy to see from Lemma 8.3 that $\mathcal{H}_\psi$ has finite rank if and only if $\boldsymbol{P}^-\psi$ is a rational function and $\operatorname{rank}\mathcal{H}_\psi = \deg \boldsymbol{P}^-\psi$. This implies the following result.

Lemma 8.12. *Let $\boldsymbol{\Gamma}_k$, $k \in \boldsymbol{\mathcal{D}}'(0,\infty)$, be a bounded operator and let r be a tempered distribution such that $\operatorname{supp} r \subset \mathbb{R}_+$ and $k = r\big|(0,\infty)$. Then $\boldsymbol{\Gamma}_k$ has finite rank if and only if $\mathcal{F}^* k$ is a rational function. In this case $\operatorname{rank}\boldsymbol{\Gamma}_k = \deg \mathcal{F}^* k$.*

It is easy to see that if $\operatorname{supp} r \subset \mathbb{R}_+$, then $\mathcal{F}^* r$ is rational if and only if r admits a representation

$$r(t) = \sum_{j=1}^{n} \sum_{l=0}^{m_j-1} c_{j,l} t^l e^{\lambda_j t}, \quad t > 0,$$

where the λ_j, $1 \le j \le n$, are complex numbers satisfying $\operatorname{Re}\lambda_j < 0$. This implies the following finite rank criterion.

Theorem 8.13. *Let $\boldsymbol{\Gamma}_k$, $k \in \boldsymbol{\mathcal{D}}'(0,\infty)$, be a bounded operator and let r be a tempered distribution such that $\operatorname{supp} r \subset \mathbb{R}_+$ and $k = r|(0,\infty)$. Then $\boldsymbol{\Gamma}_k$ has finite rank if and only if*

$$r(t) = \sum_{j=1}^{n} \sum_{l=0}^{m_j-1} c_{j,l} t^l e^{\lambda_j t}, \quad t > 0, \tag{8.16}$$

where $\operatorname{Re}\lambda_j < 0$. If the operator $\boldsymbol{\Gamma}_k$ has finite rank and $c_{j,m_j-1} \ne 0$ in (8.16), then

$$\operatorname{rank}\boldsymbol{\Gamma}_k = \sum_{j=1}^{n} m_j.$$

The Carleman Operator

Consider the operator $\boldsymbol{\Gamma} \stackrel{\text{def}}{=} \boldsymbol{\Gamma}_k$ on $L^2(\mathbb{R}_+)$, where $k(t) = 1/t$, $t > 0$. It is called the *Carleman operator*. Here we interpret k as an element of $\boldsymbol{\mathcal{D}}'(0,\infty)$. Let us show that $\boldsymbol{\Gamma}$ is bounded. By Theorem 8.8 it suffices to find a function $\chi \in L^\infty(\mathbb{R})$ such that $\mathcal{F}\chi|(0,\infty) = k$. Define χ as follows:

$$\chi(t) = \begin{cases} \pi\mathrm{i}, & t > 0, \\ -\pi\mathrm{i}, & t < 0. \end{cases}$$

Let us prove that

$$< \mathcal{F}\chi, f >= \int_0^\infty \frac{\overline{f(t) - f(-t)}}{t} dt, \quad f \in \boldsymbol{\mathcal{D}}, \tag{8.17}$$

which would imply that $\mathcal{F}\chi|(0,\infty) = k$ (note that the integrand on the right-hand side belongs to L^1). Put

$$\chi_n(t) = \begin{cases} \pi\mathrm{i}, & 0 < t < n, \\ -\pi\mathrm{i}, & -n < t < 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $n > 0$. Clearly, $\chi_n \to \chi$ in the weak topology $\sigma(\boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{D}}')$. Thus it remains to prove that $< \mathcal{F}\chi_n, f >$ converges to the right-hand side of (8.17) for any $f \in \boldsymbol{\mathcal{D}}$. We have

$$(\mathcal{F}\chi_n)(t) = \pi\mathrm{i}\left(\int_0^n e^{-2\pi\mathrm{i}ts} ds - \int_{-n}^0 e^{-2\pi\mathrm{i}ts} ds\right) = \frac{1 - \cos 2\pi nt}{t}.$$

It follows that

$$\begin{aligned} < \mathcal{F}\chi_n, f > &= \int_{\mathbb{R}} \frac{\overline{f(t)}(1 - \cos 2\pi nt)}{t} dt \\ &= \int_0^\infty \frac{\overline{(f(t) - f(-t))}(1 - \cos 2\pi nt)}{t} dt, \end{aligned}$$

which tends to the right-hand side of (8.17) by the Riemann–Lebesgue lemma.

By Theorem 8.9, $\boldsymbol{\Gamma}$ has Hankel matrix in the basis $\{\mathcal{F}\xi_n\}_{n\ge 0}$. Let us find its entries. Put $\psi = -\chi$. Then it follows from (8.14) that $G_q = \mathcal{F}\mathcal{H}_\psi\mathcal{F}^*$, where $q(t) = k(-t)$, $t < 0$. By Theorem 8.9,

$$(\boldsymbol{\Gamma}\mathcal{F}\xi_n, \mathcal{F}\xi_m) = -\hat{\varphi}(-n - m - 1),$$

where $\varphi = \psi \circ \omega$. Clearly,

$$\varphi(\zeta) = \begin{cases} \pi\mathrm{i}, & \operatorname{Im}\zeta > 0, \\ -\pi\mathrm{i}, & \operatorname{Im}\zeta < 0. \end{cases} \tag{8.18}$$

It is easy to see that

$$\hat{\varphi}(j) = \begin{cases} \frac{2}{j}, & j \text{ odd}, \\ 0, & j \text{ even}. \end{cases}$$

Thus $\boldsymbol{\Gamma}$ has the following matrix in the basis $\{\mathcal{F}\xi_n\}_{n\ge 0}$:

$$2\begin{pmatrix} 1 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & \cdots \\ 0 & \frac{1}{3} & 0 & \frac{1}{5} & 0 & \cdots \\ \frac{1}{3} & 0 & \frac{1}{5} & 0 & \frac{1}{7} & \cdots \\ 0 & \frac{1}{5} & 0 & \frac{1}{7} & 0 & \cdots \\ \frac{1}{5} & 0 & \frac{1}{7} & 0 & \frac{1}{9} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Clearly, this is the matrix on the right-hand side of (7.9).

Theorem 8.14. *Let $\boldsymbol{\Gamma}$ be the Carleman operator. Then*

$$\|\boldsymbol{\Gamma}\| = \|\boldsymbol{\Gamma}\|_{\mathrm{e}} = \pi.$$

Proof. Clearly, $\|\boldsymbol{\Gamma}\| = \|H_\varphi\|$ and $\|\boldsymbol{\Gamma}\|_{\mathrm{e}} = \|H_\varphi\|_{\mathrm{e}}$, where φ is defined by (8.18). The result now follows from Theorem 5.18. ■

Concluding Remarks

Hankel matrices appeared in the paper by Hankel [1] who studied determinants of finite Hankel matrices. They were used on numerous occasions in the study of moment problems, orthogonal polynomials (see Akhiezer [1], Szegö [1]).

The modern period of the study of Hankel operators begins with the paper Nehari [1], where the bounded Hankel matrices were characterized (see Theorem 1.1). The Fefferman characterization of BMO found in Fefferman [1] (see also Fefferman and Stein [1]) shed a new light on Hankel operators. Theorem 1.4 is taken from Adamyan, Arov, and Krein [1]. Note that earlier several authors studied the problem of best approximations of continuous functions on $\mathbb{T}$ by bounded analytic functions in $\mathbb{D}$. S.Ya. Khavinson [1] proved that for a continuous function on $\mathbb{T}$ the best approximation is unique. We also mention here the paper Rogosinski and Shapiro [1] in which the authors rediscovered the results of Khavinson and constructed a function with different best approximations. The example of a function with different best approximations given in §1 is taken from J.L. Walsh [1], Ch. X, §3 (see also Sarason [1]). Commutators $\mathcal{C}_\varphi$ were studied by many authors, see e.g., Coifman, Rochberg, and Weiss [1], where the authors considered commutators in the case of several variables. We also mention here the paper Treil and Volberg [1], where a fixed point approach to the Nehari problem was given.

Theorem 2.1 is due to Sarason [1]. It is a special case of the Sz.-Nagy–Foias commutant lifting theorem (see Appendix 1.5). The approach given in §2 is due to N.K. Nikol'skii (see N.K. Nikol'skii [1]). Formula (2.9) is also due to N.K. Nikol'skii [1]. Theorem 2.5 is a special case of the Douglas–Rudin theorem (Douglas and Rudin [1]); it can be found in Douglas [2]. The description of the partially isometric Hankel operators (Theorem 2.6) is taken from Power [2]. Theorem 2.7 was obtained by Fuhrmann [1], [3]. Theorem 2.8 is due to Clark [3].

The characterization of the finite rank Hankel matrices was obtained in Kronecker [1]. The second proof is taken from Sarason [5]; it was suggested by Axler.

The Nevanlinna–Pick interpolation problem was studied independently by Nevanlinna [1] and Pick [1]. The Carathéodory–Fejér problem was studied in Carathéodory and Fejér [1].

The Hartman theorem (Theorem 5.5) was obtained in Hartman [1]. Theorem 5.1 was obtained in Sarason [1]. It is amusing that the fact that $H^\infty + C$ is closed in L^∞ follows immediately from the Hartman theorem and the fact that the set of compact operators is closed in the space of bounded linear operators. However, in this book we use the Sarason theorem to prove the Hartman theorem. The essential norm of a Hankel operator (see Theorem 5.4) was computed in Adamyan, Arov, and Krein [1]. The equivalence of (i) and (ii) in Theorem 5.10 was established in Sarason [1]. The class VMO was introduced in Sarason [4]. Theorem 5.11 is essentially due to Clark [2]. The results on approximation by compact Hankel operators were obtained in Axler, Berg, Jewell, and Shields [1]. These results give a complete solution to a problem posed in Adamyan, Arov, and Krein [5]. Another solution to the Adamyan–Arov–Krein problem (based on the notion of M-ideal) was given by Luecking [1]. Theorem 5.17 was found in Douglas and Sarason [1], while Theorem 5.18 was obtained in Bonsall and Gillespie [1]. The factorization theorem for L^∞ functions (Theorem 5.20) is due to Axler [1].

Theorems 6.1 and 6.2 were published in Bonsall [1], [2], though they were known before to some experts.

The Hamburger theorem (Theorem 7.1) was established in Hamburger [1]. The approach based on self-adjoint extensions of symmetric operators can be found in Akhiezer [1]. Our approach is taken from Sarason [6]. Theorems 7.1 and 7.2 are obtained in Widom [3]. Theorems 7.4 and 7.5 can be found in Power [2].

The equivalence of the operators $\boldsymbol{\Gamma}_k$ and the Hankel operators is well known; see Power [2] for some related results and see Devinatz [2] for similar results on Wiener–Hopf operators. The Carleman operator was studied in Carleman [1]. In connection with the material of §8 we would like to offer the following exercise.

Exercise. *Let $k(x) = e^{-x}/x$, $x > 0$. Prove that $\boldsymbol{\Gamma}_k$ is bounded on $L^2(\mathbb{R}_+)$ and is unitarily equivalent to the Hankel operator with Hilbert matrix.*

Let us mention here the following boundedness criterion for Hankel operators (see Bonsall [4], and Holland and D. Walsh [1]). For $\zeta \in \mathbb{T}$ consider the following polynomials:

$$u_{n,\zeta} \stackrel{\text{def}}{=} n^{-1/2} \sum_{j=0}^{n-1} \zeta^j z^j, \quad v_{n,\zeta} \stackrel{\text{def}}{=} n^{-1/2} \sum_{j=n}^{2n-1} \zeta^j z^j.$$

Then the following are equivalent:

(i) H_φ is bounded;

(ii) $\sup\limits_{n>0,\, \zeta\in\mathbb{T}} \|H_\varphi u_{n,\zeta}\| < \infty$;

(iii) $\sup\limits_{n>0,\, \zeta\in\mathbb{T}} \|H_\varphi v_{n,\zeta}\| < \infty$.

In Bonsall [1] it was shown that for a sequence $\{a_j\}_{j\ge0}$ of nonnegative numbers the Hankel matrix $\{\alpha_{j+k}\}_{j,k\ge0}$ is bounded if and only if

$$\sup_{n>0}\frac{1}{n}\sum_{j=0}^{\infty}\left(\sum_{k=0}^{n-1}\alpha_{j+k}\right)^2<\infty.$$

This is also equivalent to the condition

$$\sup_{n>0}\sum_{j=1}^{\infty}\left(\sum_{k=0}^{n-1}\alpha_{jn+k}\right)^2<\infty.$$

The fact that the last inequality for nonnegative α_j is equivalent to the condition $\sum\limits_{j\ge0}\alpha_j z^j\in BMO$ is an unpublished theorem of C. Fefferman; its proof is contained in Bonsall [1].

We would like to mention here a series of papers by Arocena, Cotlar, and Sadosky (see Arocena [1], Arocena and Cotlar [1], Cotlar and Sadosky [1], and Arocena, Cotlar, and Sadosky [1]). In those papers the authors develop an interesting theory of generalized Toeplitz kernels which generalizes the Nehari theory for Hankel matrices.

Let $K:\mathbb{Z}\times\mathbb{Z}:\to\mathbb{C}$ be a *Toepliz kernel*, i.e., $K(i+1,j+1)=K(i,j)$, $i,\,j\in\mathbb{Z}$, which is positive semidefinite, i.e.,

$$\sum_{i,j\in\mathbb{Z}}K(i,j)x_i\bar{x}_j\ge0$$

for any finitely supported sequence $\{x_j\}_{j\in\mathbb{Z}}$ of complex numbers. Then by the F. Riesz–Herglotz theorem (see Riesz and Sz.-Nagy [1], §53) there exists a positive Borel measure μ on $\mathbb{T}$ such that $\hat{\mu}(n)=K(j+n,j)$, $j\in\mathbb{Z}$.

Suppose now that K is a positive semidefinite *generalized Toeplitz kernel*, i.e.,

$$K(i+1,j+1)=K(i,j),\quad i,\,j\in\mathbb{Z},\quad i\ne-1,\ j\ne-1.$$

Then there are sequences $\{k_j^+\}_{j\in\mathbb{Z}}$, $\{k_j^-\}_{j\in\mathbb{Z}}$, $\{k_j^{+-}\}_{j>0}$, and $\{k_j^{-+}\}_{j<0}$ such that

$$K(i,j)=\begin{cases}k_{i-j}^+, & (i,j)\in\mathbb{Z}_+\times\mathbb{Z}_+,\\ k_{i-j}^-, & (i,j)\in\mathbb{Z}_-\times\mathbb{Z}_-,\\ k_{i-j}^{+-}, & (i,j)\in\mathbb{Z}_+\times\mathbb{Z}_-,\\ k_{i-j}^{-+}, & (i,j)\in\mathbb{Z}_-\times\mathbb{Z}_+,\end{cases}$$

and $k_j^{+-}=\overline{k_{-j}^{-+}}$, $j>0$. Here $\mathbb{Z}_-\stackrel{\text{def}}{=}\mathbb{Z}\setminus\mathbb{Z}_+$.

If $\mu=\begin{pmatrix}\mu^- & \mu^{-+}\\ \mu^{-+} & \mu^+\end{pmatrix}$ is a positive matrix Borel measure on $\mathbb{T}$, then we can put

$$k^+(j)=\mu^+(j),\quad k^-(j)=\mu^-(j),\quad j\in\mathbb{Z}_+,$$

$$k^{+-}(j)=\mu^{+-}(j),\quad k^{-+}(-j)=\mu^{-+}(-j),\quad j<0,$$

and obtain a positive semidefinite generalized Toeplitz kernel. We say that this generalized Toeplitz kernel is determined by the matrix μ. Obviously, the measures μ^+ and μ^- are uniquely determined by the generalized Toeplitz kernel while only the Fourier coefficients $\hat{\mu}^{-+}(n)$ with $n < 0$ are uniquely determined. Thus μ^{-+} is determined up to a function in the Hardy class H^1. The main result obtained in the papers by Arocena, Cotlar, and Sadosky mentioned above says that each positive semidefinite generalized Toeplitz kernel is determined by a positive matrix measure.

This theorem generalizes the Nehari theorem. Indeed, let $\{\alpha_{i+j}\}_{i\geq 0, j\geq 1}$ be a bounded Hankel matrix and let $c = \|\{\alpha_{i+j}\}_{i\geq 0, j\geq 1}\|$. Put

$$k^+(j) = k^-(j) = \begin{cases} c, & j = 0, \\ 0, & j \neq 0, \end{cases} \qquad \text{and} \quad k^{-+}(j) = \overline{k^{+-}(-j)} = \alpha_j, \quad j > 0.$$

It is easy to see that the corresponding generalized Toeplitz kernel is positive semi-definite, and so it is determined by a positive matrix measure $\mu = \begin{pmatrix} \mu^- & \mu^{-+} \\ \mu^{-+} & \mu^+ \end{pmatrix}$. Since $\mu \geq \mathbb{O}$, it follows that μ^{-+} is absolutely continuous with respect to Lebesgue measure, $d\mu = \varphi d\boldsymbol{m}$, $\varphi \in L^\infty$, $\|\varphi\|_\infty \leq c$, and $\hat{\varphi}(-j) = \alpha_j$, $j > 0$. The authors also found other interesting applications; in particular, they obtained two weight inequalities for the operator of harmonic conjugation.

Let us also mention the Axler–Chang–Sarason–Volberg theorem that characterizes the compact operators of the form $H_\psi^* H_\varphi$, φ, $\psi \in L^\infty$: the operator $H_\psi^* H_\varphi$ is compact if and only if $H^\infty[\varphi] \cap H^\infty[\psi] \subset H^\infty + C$. Here $H^\infty[\varphi]$ is the smallest closed subalgebra of L^∞ that contains φ. The part "if" was obtained in Axler, Chang, and Sarason [1], while the part "only if" was obtained in Volberg [1].

We conclude this chapter with the following list of books and survey articles that discuss Hankel operators: Ando [1], Khrushchëv and Peller [2], N.K. Nikol'skii [2], [3], [4], Partington [1], Peetre [2], Peller [23], Peller and Khrushchëv [1], Power [2], Sarason [5], and Zhu [1].

2

Vectorial Hankel Operators

In this chapter we study Hankel operators on spaces of vector functions. We prove in §2 a generalization of the Nehari theorem which describes the bounded block Hankel matrices of the form $\{\Omega_{j+k}\}_{j,k\geq 0}$, where the Ω_j are bounded linear operators from a Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{K}$. The proof is based on a more general result on completing matrix contractions. This result is obtained in §1. Namely, we obtain in §1 a necessary and sufficient condition on Hilbert space operators A, B, and C for the existence a Hilbert space operator Z such that the block matrix $\begin{pmatrix} A & B \\ C & Z \end{pmatrix}$ is a contraction (i.e., has norm at most 1). Moreover, we describe in §1 all solutions of this completion problem. Note that the results of §2 will be used in Chapter 5 to parametrize all solutions of the Nehari problem.

In §3 we give an alternative approach to the problem of boundedness of vectorial Hankel operators. This approach is based on the Sz.-Nagy–Foias commutant lifting theorem. Section 4 is devoted to the description of the compact vectorial Hankel operators. We obtain an analog of the Hartman theorem.

In §5 we introduce the important notion of a Blaschke–Potapov product, we give a description of the finite rank vectorial Hankel operators, and we obtain an explicit formula for the rank of such operators. Finally, in §6 we discuss relationships between vectorial Hankel operators and imbedding theorems.

1. Completing Matrix Contractions

This section is devoted to the following problem. Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces, A a bounded linear operator on $\mathcal{H}$, B a bounded linear operator from $\mathcal{K}$ to $\mathcal{H}$, and C a bounded linear operator from $\mathcal{H}$ to $\mathcal{K}$. The problem is to find out under which conditions there exists a bounded linear operator Z on $\mathcal{K}$ such that the operator

$$Q_Z = \begin{pmatrix} A & B \\ C & Z \end{pmatrix} \tag{1.1}$$

on $\mathcal{H} \oplus \mathcal{K}$ is a contraction, that is, $\|Q_Z\| \le 1$.

It is easy to see that if the problem is solvable, then the operators

$$\begin{pmatrix} A \\ C \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & B \end{pmatrix} \tag{1.2}$$

from $\mathcal{H}$ to $\mathcal{H} \oplus \mathcal{K}$ and from $\mathcal{H} \oplus \mathcal{K}$ to $\mathcal{H}$, respectively, are contractions. It turns out that the converse is also true.

Theorem 1.1. *Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces, $A : \mathcal{H} \to \mathcal{H}$, $B : \mathcal{K} \to \mathcal{H}$, and $C : \mathcal{H} \to \mathcal{K}$ bounded linear operators. There is an operator $Z : \mathcal{K} \to \mathcal{K}$ for which the operator Q_Z defined by* (1.1) *is a contraction on $\mathcal{H} \oplus \mathcal{K}$ if and only if*

$$\left\| \begin{pmatrix} A \\ C \end{pmatrix} \right\| \le 1 \quad \textit{and} \quad \left\| \begin{pmatrix} A & B \end{pmatrix} \right\| \le 1. \tag{1.3}$$

Another problem we shall consider here is to describe all operators Z on $\mathcal{K}$ for which Q_Z is a contraction. Clearly, the second problem is meaningful only under the condition (1.3). To state the description we need some preliminaries.

Lemma 1.2. *Let $\mathcal{H}$, $\mathcal{H}_1$, and $\mathcal{H}_2$ be Hilbert spaces, and let $T : \mathcal{H}_1 \to \mathcal{H}$ and $R : \mathcal{H}_2 \to \mathcal{H}$ be bounded linear operators. Then $TT^* \le RR^*$ if and only if there exists a contraction $Q : \mathcal{H}_1 \to \mathcal{H}_2$ such that $T = RQ$.*

Proof. Suppose that $T = RQ$ and $\|Q\| \le 1$. We have

$$\begin{aligned} (TT^*x, x) &= (RQQ^*R^*x, x) = (Q^*R^*x, Q^*R^*x) \\ &= \|Q^*R^*x\|^2 \le \|R^*x\|^2 = (RR^*x, x). \end{aligned}$$

Conversely, assume that $TT^* \le RR^*$. We define the operator L on $\operatorname{Range} R^*$ as follows:

$$LR^*x = T^*x, \quad x \in \mathcal{H}_2.$$

The inequality $TT^* \le RR^*$ implies that L is well-defined on $\operatorname{Range} R^*$ and $\|L\| \le 1$ on $\operatorname{Range} R^*$. We can extend L by continuity to $\operatorname{clos} \operatorname{Range} R^*$ and put

$$L \big| \operatorname{Ker} R = L \big| (\operatorname{Range} R^*)^{\perp} = \mathbb{O}.$$

Set $Q = L^*$. Clearly $T = RQ$. ■

For a contraction $A : \mathcal{H}_1 \to \mathcal{H}_2$ the *defect operator* D_A is defined on $\mathcal{H}_1$ by $D_A = (I - A^*A)^{1/2}$. It is also convenient besides D_A to consider other operators $\mathcal{D}_A : \mathcal{H}_1 \to \tilde{\mathcal{H}}$ such that $\mathcal{D}_A^*\mathcal{D}_A = I - A^*A$, where $\tilde{\mathcal{H}}$ is a Hilbert space. In this case $\mathcal{D}_A = VD_A$ for some isometry V defined on $\operatorname{clos}\operatorname{Range} D_A$.

Lemma 1.3. *Let $\mathcal{H}$, $\mathcal{K}$, $\tilde{\mathcal{H}}$ be Hilbert spaces, $A : \mathcal{H} \to \mathcal{H}$, $B : \mathcal{K} \to \mathcal{H}$ linear operators such that $\|A\| \le 1$. Let $\mathcal{D}_{A^*} : \mathcal{H} \to \tilde{\mathcal{H}}$ be an operator such that $\mathcal{D}_{A^*}^*\mathcal{D}_{A^*} = I - A\,A^*$. Then*

$$\left\| \begin{pmatrix} A & B \end{pmatrix} \right\| \le 1 \tag{1.4}$$

if and only if $B = \mathcal{D}_{A^*}^* K$ for a contraction $K : \mathcal{K} \to \tilde{\mathcal{H}}$.

Proof. It is easy to see that (1.4) is equivalent to the fact that

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} A^* \\ B^* \end{pmatrix} \le I_{\mathcal{H}},$$

which means that $AA^* + BB^* \le I$ or, which is the same, $BB^* \le \mathcal{D}_{A^*}^*\mathcal{D}_{A^*}$. By Lemma 1.2, this is equivalent to the fact that $B = \mathcal{D}_{A^*}^* K$ for some contraction $K : \mathcal{K} \to \tilde{\mathcal{H}}$. ■

Remark. The conclusion of the lemma is valid if $\mathcal{D}_{A^*} = D_{A^*} = (I - AA^*)^{1/2}$. It is easy to see that we can choose a contraction $K : \mathcal{K} \to \mathcal{H}$ such that $\operatorname{Range} K \subset \operatorname{clos}\operatorname{Range}(I - AA^*)$ and $B = D_{A^*}K$. Clearly, such a contraction K is unique.

Lemma 1.4. *Let $\mathcal{H}$, $\mathcal{K}$, $\tilde{\mathcal{H}}$ be Hilbert spaces and let $A : \mathcal{H} \to \mathcal{H}$, $C : \mathcal{H} \to \mathcal{K}$ be linear operators such that $\|A\| \le 1$. Let $\mathcal{D}_A : \mathcal{H} \to \tilde{\mathcal{H}}$ be an operator such that $\mathcal{D}_A^*\mathcal{D}_A = I - A^*A$. Then*

$$\left\| \begin{pmatrix} A \\ C \end{pmatrix} \right\| \le 1 \tag{1.5}$$

if and only if $C = L\mathcal{D}_A$ for some contraction $L : \tilde{\mathcal{H}} \to \mathcal{K}$.

Proof. The result follows from Lemma 1.3, since (1.5) is equivalent to the inequality

$$\left\| \begin{pmatrix} A^* & C^* \end{pmatrix} \right\| \le 1. \quad ■$$

Remark. As in Lemma 1.3 we can take $\mathcal{D}_A = D_A = (I - A^*A)^{1/2}$. Clearly, one can find a contraction $L : \mathcal{H} \to \mathcal{K}$ such that

$$L \Big| (\operatorname{Range}(I - A^*A))^\perp = \mathbb{O}$$

and $C = LD_A$. As in Lemma 1.3 it is easy to see that such a contraction L is unique.

Now we are in a position to state the description of those operators $Z : \mathcal{K} \to \mathcal{K}$ for which the operator Q_Z defined by (1.1) is a contraction. As we have already observed, the operators in (1.2) are contractions. Therefore (see the Remarks after Lemmas 1.3 and 1.4) there exist unique contractions $K : \mathcal{K} \to \mathcal{H}$ and $L : \mathcal{H} \to \mathcal{K}$ such that

$$\operatorname{Range} K \subset \operatorname{clos}\operatorname{Range}(I - AA^*), \quad B = D_{A^*}K, \tag{1.6}$$

$$L\Big|(\operatorname{Range}(I-A^*A))^\perp=\mathbb{O},\quad C=LD_A. \tag{1.7}$$

Theorem 1.5. *Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces, $A:\mathcal{H}\to\mathcal{H}$, $B:\mathcal{K}\to\mathcal{H}$, and $C:\mathcal{H}\to\mathcal{K}$ bounded linear operators satisfying* (1.3). *Let $K:\mathcal{K}\to\mathcal{H}$ and $L:\mathcal{H}\to\mathcal{K}$ be the operators satisfying* (1.6) *and* (1.7). *If $Z:\mathcal{K}\to\mathcal{K}$ is a bounded linear operator, then the operator Q_Z defined by* (1.1) *is a contraction if and only if Z admits a representation*

$$Z=-LA^*K+D_{L^*}MD_K, \tag{1.8}$$

where M is a contraction on $\mathcal{K}$.

Note that we may always assume that

$$M\big|(\operatorname{Range}D_K)^\perp=\mathbb{O}\quad\text{and}\quad\operatorname{Range}M\subset\operatorname{clos}\operatorname{Range}D_{L^*}. \tag{1.9}$$

If these two conditions are satisfied, then Z determines M uniquely, and so the contractions M satisfying (1.9) parametrize the solutions Z.

It is easy to see that Theorem 1.1 follows from Theorem 1.5. Indeed, we can always take $M=\mathbb{O}$. To prove Theorem 1.5, we need one more lemma.

Lemma 1.6. *Let A, B be as above and let $K:\mathcal{K}\to\mathcal{H}$ be an operator satisfying* (1.6). *Then the operator*

$$\mathcal{D}_{(AB)}=\begin{pmatrix} D_A & -A^*K\\ \mathbb{O} & D_K\end{pmatrix} \tag{1.10}$$

satisfies

$$\mathcal{D}^*_{(AB)}\mathcal{D}_{(AB)}=I_{\mathcal{H}\oplus\mathcal{K}}-\begin{pmatrix}A^*\\ B^*\end{pmatrix}\begin{pmatrix}A & B\end{pmatrix}.$$

Proof. We have

$$\begin{aligned}
&\begin{pmatrix} D_A & -A^*K\\ \mathbb{O} & D_K\end{pmatrix}^*\begin{pmatrix} D_A & -A^*K\\ \mathbb{O} & D_K\end{pmatrix}+\begin{pmatrix}A^*\\ B^*\end{pmatrix}\begin{pmatrix}A & B\end{pmatrix}\\
=\;&\begin{pmatrix} D_A & \mathbb{O}\\ -K^*A & D_K\end{pmatrix}\begin{pmatrix} D_A & -A^*K\\ \mathbb{O} & D_K\end{pmatrix}+\begin{pmatrix}A^*\\ B^*\end{pmatrix}\begin{pmatrix}A & B\end{pmatrix}\\
=\;&\begin{pmatrix} D_A^2 & -D_AA^*K\\ -K^*AD_A & K^*AA^*K+D_K^2\end{pmatrix}+\begin{pmatrix}A^*A & A^*B\\ B^*A & B^*B\end{pmatrix}\\
=\;&\begin{pmatrix} I_{\mathcal{H}}-A^*A & -D_AA^*K\\ -K^*AD_A & K^*K+D_K^2-K^*D_{A^*}D_{A^*}K\end{pmatrix}\\
+\;&\begin{pmatrix}A^*A & A^*B\\ B^*A & B^*B\end{pmatrix}.
\end{aligned}$$

Let us show that $D_AA^*=A^*D_{A^*}$. We have $(I-A^*A)A^*=A^*(I-AA^*)$. It follows that $\varphi(I-A^*A)A^*=A^*\varphi(I-AA^*)$ for any polynomial φ, so the same equality holds for any continuous function φ. If we take $\varphi(t)=t^{1/2}$, $t\ge 0$, we obtain $D_AA^*=A^*D_{A^*}$. Similarly, $D_{A^*}A=AD_A$.

Consequently,

$$\begin{aligned} &\mathcal{D}^*_{(AB)}\mathcal{D}_{(AB)} \quad + \quad \begin{pmatrix} A^* \\ B^* \end{pmatrix} \begin{pmatrix} A & B \end{pmatrix} \\ =& \begin{pmatrix} I_{\mathcal{H}} - A^*A & -A^*B \\ -B^*A & I - B^*B \end{pmatrix} \quad + \quad \begin{pmatrix} A^*A & A^*B \\ B^*A & B^*B \end{pmatrix} = \begin{pmatrix} I_{\mathcal{H}} & \mathbb{O} \\ \mathbb{O} & I_{\mathcal{H}} \end{pmatrix}. \quad \blacksquare \end{aligned}$$

Proof of Theorem 1.5. Suppose that $\|Q_Z\| \le 1$. By Lemma 1.2

$$\begin{pmatrix} C & Z \end{pmatrix} = \begin{pmatrix} X & Y \end{pmatrix} \mathcal{D}_{(AB)},$$

where $\begin{pmatrix} X & Y \end{pmatrix}$ is a contraction from $\mathcal{H} \oplus \mathcal{K}$ to $\mathcal{K}$ and $\mathcal{D}_{(AB)}$ is defined by (1.10). Then $C = XD_A$. Let P be the orthogonal projection from $\mathcal{H}$ onto $\operatorname{clos}\operatorname{Range}(I - A^*A)$. Put $\tilde{X} = XP$. Clearly, $C = \tilde{X}D_A$ and by the Remark after Lemma 1.4, $\tilde{X} = L$. It is easy to see that

$$\begin{pmatrix} C & Z \end{pmatrix} = \begin{pmatrix} L & Y \end{pmatrix} \mathcal{D}_{(AB)}. \tag{1.11}$$

Clearly,

$$\begin{pmatrix} L & Y \end{pmatrix} = \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} P & \mathbb{O} \\ \mathbb{O} & I_{\mathcal{H}} \end{pmatrix},$$

which proves that $\begin{pmatrix} L & Y \end{pmatrix}$ is a contraction. Then by Lemma 1.3, the operator Y admits a representation $Y = D_{L^*}M$ for a contraction M on $\mathcal{K}$. Formula (1.8) follows now immediately from (1.11).

Suppose that Z satisfies (1.8), where M is a contraction on $\mathcal{K}$. Then it is easy to see that

$$\begin{pmatrix} A & B \\ C & Z \end{pmatrix} = \begin{pmatrix} I & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & L & D_{L^*} \end{pmatrix} \begin{pmatrix} A & D_{A^*} & \mathbb{O} \\ D_A & -A^* & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & M \end{pmatrix} \begin{pmatrix} I & \mathbb{O} \\ \mathbb{O} & K \\ \mathbb{O} & D_K \end{pmatrix}. \tag{1.12}$$

The result follows from the fact that all factors on the right-hand side of (1.12) are contractions, which is a consequence of the following lemma.

Lemma 1.7. *Let T be a contraction on Hilbert space. Then the operator*

$$\begin{pmatrix} T & D_{T^*} \\ D_T & -T^* \end{pmatrix}$$

is unitary.

Proof. We have

$$\begin{aligned} &\begin{pmatrix} T & D_{T^*} \\ D_T & -T^* \end{pmatrix}^* \begin{pmatrix} T & D_{T^*} \\ D_T & -T^* \end{pmatrix} \\ =& \begin{pmatrix} T^*T + D_T^2 & T^*D_{T^*} - D_T T^* \\ D_{T^*}T - TD_T & D_{T^*}^2 + TT^* \end{pmatrix}. \end{aligned}$$

It has been shown in the proof of Lemma 1.6 that $T^*D_{T^*} = D_T T^*$ and $TD_T = D_{T^*}T$. Thus

$$\begin{pmatrix} T & D_{T^*} \\ D_T & -T^* \end{pmatrix}^* \begin{pmatrix} T & D_{T^*} \\ D_T & -T^* \end{pmatrix} = \begin{pmatrix} I & \mathbb{O} \\ \mathbb{O} & I \end{pmatrix}.$$

Similarly,

$$\begin{pmatrix} T & D_{T^*} \\ D_T & -T^* \end{pmatrix} \begin{pmatrix} T & D_{T^*} \\ D_T & -T^* \end{pmatrix}^* = \begin{pmatrix} I & \mathbb{O} \\ \mathbb{O} & I \end{pmatrix}. \quad \blacksquare$$

The following uniqueness criterion is an immediate consequence of Theorem 1.5.

Corollary 1.8. *The following statements are equivalent:*

(i) *there is only one operator Z for which $\|Q_Z\| \le 1$;*

(ii) *either $D_{L^*} = \mathbb{O}$ or $D_K = \mathbb{O}$;*

(iii) *either K is an isometry or L^* is an isometry.*

Remark. The same results are valid if A is an operator from $\mathcal{H}_1$ to $\mathcal{H}_2$, B is an operator from $\mathcal{K}_1$ to $\mathcal{H}_2$, C is an operator from $\mathcal{H}_1$ to $\mathcal{K}_2$, and Z is an operator from $\mathcal{K}_1$ to $\mathcal{K}_2$, where $\mathcal{H}_1,\ \mathcal{H}_2,\ \mathcal{K}_1,\ \mathcal{K}_2$ are Hilbert spaces. The proof given above works in this more general situation.

2. Bounded Block Hankel Matrices

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces and $\Omega = \{\Omega_j\}_{j\ge 0}$ a sequence of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. Consider the block matrix

$$\Gamma_\Omega = \{\Omega_{j+k}\}_{j,k\ge 0}. \tag{2.1}$$

Such matrices are called *block Hankel matrices*. In the case $\mathcal{H} = \mathcal{K} = \mathbb{C}$ the matrix Γ_Ω can be considered as a Hankel matrix with scalar entries. The main aim of this section is to obtain an analog of the Nehari theorem for block Hankel matrices, that is to describe the block Hankel matrices that determine bounded operators from $\ell^2(\mathcal{H})$ to $\ell^2(\mathcal{K})$. There are several different approaches to this problem. Here we use the one based on the problem of completing matrix contractions that was considered in §1. In §3 we present another approach based on the Sz.-Nagy–Foias commutant lifting theorem.

As in the scalar case we consider a realization of Hankel operators as operators on the Hardy class $H^2(\mathcal{H})$ of functions with values in a Hilbert space $\mathcal{H}$. We study elementary properties of symbols with minimal norm and maximizing vectors of vectorial Hankel operators.

Denote by $\mathcal{B}(\mathcal{H},\mathcal{K})$ the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and by $L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ the space of bounded weakly measurable $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued functions. In the case $\mathcal{H} = \mathcal{K}$ we shall use the notation

$\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H},\mathcal{H})$. If $\Phi \in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$, we define the Fourier coefficients $\hat{\Phi}(n)$, $n \in \mathbb{Z}$, of Φ as usual:

$$\hat{\Phi}(n) = \int_{\mathbb{T}} \bar{\zeta}^n \Phi(\zeta) d\boldsymbol{m}(\zeta).$$

Clearly, $\hat{\Phi}(n) \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $n \in \mathbb{Z}$.

Theorem 2.1. *Let $\{\Omega_j\}_{j\ge 0}$ be a sequence of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. The block Hankel matrix Γ_Ω determines a bounded linear operator from $\ell^2(\mathcal{H})$ to $\ell^2(\mathcal{K})$ if and only if there exists a function Φ in $L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ such that*

$$\Omega_n = \hat{\Phi}(n), \quad n \ge 0. \tag{2.2}$$

In this case

$$\|\Gamma_\Omega\| = \inf\left\{\|\Phi\|_{L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))} : \ \hat{\Phi}(n) = \Omega_n, \ n \ge 0\right\}.$$

As in the scalar case we can consider vectorial Hankel operators as operators on the Hardy class H^2 of vector-valued functions. If $\mathcal{H}$ is a separable Hilbert space, the Hardy class $H^2(\mathcal{H})$ is defined as follows:

$$H^2(\mathcal{H}) = \{F \in L^2(\mathcal{H}) : \ \hat{F}(n) = 0, \ n < 0\},$$

where $L^2(\mathcal{H})$ is the space of weakly measurable $\mathcal{H}$-valued functions F for which

$$\|F\|^2_{L^2(\mathcal{H})} = \int_{\mathbb{T}} \|F(\zeta)\|^2_{\mathcal{H}} d\boldsymbol{m}(\zeta) < \infty.$$

Put $H^2_-(\mathcal{H}) = L^2(\mathcal{H}) \ominus H^2(\mathcal{H})$. Let $\mathcal{K}$ be another separable Hilbert space and let Ψ be a function in $L^2_{\mathrm{s}}(\mathcal{B}(\mathcal{H},\mathcal{K}))$ (see Appendix 2.3), i.e.,

$$\int_{\mathbb{T}} \|\Psi(\zeta)x\|^2_{\mathcal{K}} d\boldsymbol{m}(\zeta) < \infty \quad \text{for any} \quad x \in \mathcal{H}. \tag{2.3}$$

Recall that for functions Ψ satisfying (2.3) the Fourier coefficients $\hat{\Psi}(n) \in \mathcal{B}(\mathcal{H},\mathcal{K})$ are defined by

$$\hat{\Psi}(n)x = \int_{\mathbb{T}} \bar{\zeta}^n \Psi(\zeta)x \, d\boldsymbol{m}(\zeta), \quad n \in \mathbb{Z}, \quad x \in \mathcal{H}.$$

We can also define for such functions Ψ the Hankel operator $H_\Psi : H^2(\mathcal{H}) \to H^2_-(\mathcal{K})$ on the set of polynomials in $H^2(\mathcal{H})$ by

$$H_\Psi F = \mathbb{P}_- \Psi F, \quad F \in H^2(\mathcal{H}), \tag{2.4}$$

where as in the scalar case we denote by $\mathbb{P}_+$ and $\mathbb{P}_-$ the orthogonal projections onto $H^2(\mathcal{H})$ and $H^2_-(\mathcal{K})$.

If Ψ is an operator function satisfying (2.3), we denote by $\mathbb{P}_-\Psi$ and $\mathbb{P}_+\Psi$ the functions

$$((\mathbb{P}_-\Psi)(\zeta))\, x = (\mathbb{P}_-(\Psi x))\,(\zeta), \quad \zeta \in \mathbb{T}, \ x \in \mathcal{H},$$

$$((\mathbb{P}_+\Psi)(\zeta))\, x = (\mathbb{P}_+(\Psi x))\,(\zeta), \quad \zeta \in \mathbb{T}, \ x \in \mathcal{H}.$$

We can identify in a natural way $\ell^2(\mathcal{H})$ with $H^2(\mathcal{H})$:

$$\{x_k\}_{k\ge 0} \in \ell^2(\mathcal{H}) \mapsto \sum_{k\ge 0} z^k x_k.$$

As in the scalar case it is easy to see that Theorem 2.1 is equivalent to the following statement in which

$$H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K})) \stackrel{\text{def}}{=} \left\{\Phi \in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K})) :\ \hat{\Phi}(n) = 0,\ n < 0\right\}.$$

Theorem 2.2. *Let Ψ be a $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued function that satisfies* (2.3). *Then the operator H_Ψ defined by* (2.4) *extends to a bounded operator on $H^2(\mathcal{H})$ if and only if there exists a function Φ in $L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ such that*

$$\hat{\Phi}(n) = \hat{\Psi}(n), \quad n < 0,$$

and

$$\|H_\Psi\| = \operatorname{dist}_{L^\infty}\left(\Phi, H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K})\right).$$

Proof of Theorem 2.1. Let $\Phi \in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ and $\Gamma_\Omega = \{\Omega_{j+k}\}_{j,k\ge 0}$ be the block Hankel matrix defined by (2.1) and (2.2). Let us show that Γ_Ω is a bounded operator from $\ell^2(\mathcal{H})$ to $\ell^2(\mathcal{K})$ and $\|\Gamma_\Omega\| \le \|\Phi\|_{L^\infty}$.

Put $\Psi(z) = \bar{z}\Phi(\bar{z})$. It is easy to see that Γ_Ω is bounded on $\ell^2(\mathcal{H})$ if and only if H_Ψ is bounded on $H^2(\mathcal{H})$ and $\|\Gamma_\Omega\| = \|H_\Psi\|$. We have

$$\|H_\Psi F\|_{L^2(\mathcal{K})} = \|\mathbb{P}_-\Psi F\|_{L^2(\mathcal{K})} \le \|\Psi F\|_{L^2(\mathcal{K})} \le \|\Psi\|_{L^\infty}\|F\|_{H^2(\mathcal{K})}.$$

Hence, $\|H_\Psi\| \le \|\Psi\|_{L^\infty} = \|\Phi\|_{L^\infty}$.

To show the converse, we extend the sequence $\{\Omega_j\}_{j\ge 0}$ to a two-sided sequence $\{\Omega_j\}_{j\in\mathbb{Z}}$ so that the block matrix $\Lambda = \{\Omega_{j+k}\}_{j\in\mathbb{Z},k\ge 0}$ determines a bounded operator from $\ell^2(\mathcal{H})$ to $\ell^2_{\mathbb{Z}}(\mathcal{K})$, where

$$\ell^2_{\mathbb{Z}}(\mathcal{K}) = \left\{x = \{x_j\}_{j\in\mathbb{Z}} :\ x_j \in \mathcal{K},\ \|x\|^2_{\ell^2_{\mathbb{Z}}(\mathcal{K})} = \sum_{j\in\mathbb{Z}} \|x_j\|^2_{\mathcal{K}} < \infty\right\}.$$

Moreover, we show that $\|\Lambda\| = \|\Gamma_\Omega\|$.

Let us first find Ω_{-1}. Consider the matrix $\{\Omega_{j+k}\}_{j\ge -1,k\ge 0}$ where $\Omega_{-1} = Z$ is an unknown operator:

$$\left(\begin{array}{c|cccc} Z & \Omega_0 & \Omega_1 & \Omega_2 & \cdots \\ \hline \Omega_0 & \Omega_1 & \Omega_2 & \Omega_3 & \cdots \\ \Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 & \cdots \\ \Omega_2 & \Omega_3 & \Omega_4 & \Omega_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right). \tag{2.5}$$

Let

$$A = \begin{pmatrix} \Omega_1 & \Omega_2 & \Omega_3 & \cdots \\ \Omega_2 & \Omega_3 & \Omega_4 & \cdots \\ \Omega_3 & \Omega_4 & \Omega_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad B = \begin{pmatrix} \Omega_0 \\ \Omega_1 \\ \Omega_2 \\ \vdots \end{pmatrix},$$

$$C = \begin{pmatrix} \Omega_0 & \Omega_1 & \Omega_2 & \cdots \end{pmatrix}.$$

The matrix in (2.5) can be identified with the matrix

$$\begin{pmatrix} Z & C \\ B & A \end{pmatrix}.$$

Clearly,

$$\left\| \begin{pmatrix} B & A \end{pmatrix} \right\| = \|\Gamma_\Omega\|, \qquad \left\| \begin{pmatrix} C \\ A \end{pmatrix} \right\| = \|\Gamma_\Omega\|.$$

By Theorem 1.1 and the Remark at the end of §1, there exists an operator Z such that

$$\left\| \begin{pmatrix} Z & C \\ B & A \end{pmatrix} \right\| = \|\Gamma_\Omega\|.$$

Put $\Omega_{-1} = Z$. We have

$$\|\{\Omega_{j+k}\}_{j\le -1, k\ge 0}\| = \|\Gamma_\Omega\|.$$

If we continue this process, we can find a sequence of operators $\{\Omega_j\}_{j\le -1}$ such that for any $m \in \mathbb{Z}_+$

$$\|\{\Omega_{j+k}\}_{j\le -m, k\ge 0}\| = \|\Gamma_\Omega\|.$$

Given $m \in \mathbb{Z}_+$, consider the matrix $\Lambda_m = \{T_{jk}\}_{j\in\mathbb{Z},k\ge 0}$, $T_{jk} = \Omega_{j+k}$, $j \ge -m$, and $T_{jk} = \mathbb{O}$, $j < -m$. Clearly, $\|\Lambda\| = \|\Gamma_\Omega\|$ and the sequence $\{\Lambda_m\}_{m\ge 1}$ converges to $\Lambda = \{\Omega_{j+k}\}_{j\in\mathbb{Z},k\ge 0}$ in the weak operator topology. Thus $\|\Lambda\| = \|\Gamma_\Omega\|$.

We can now consider the block matrix $Q = \{\Omega_{j+k}\}_{j,k\in\mathbb{Z}}$. As above it is easy to see that $\|\{\Omega_{j+k}\}_{j\in\mathbb{Z},k\ge -m}\| = \|\Gamma_\Omega\|$ and $\|Q\| = \|\Gamma_\Omega\|$. Thus Q determines a bounded operator from $\ell^2_{\mathbb{Z}}(\mathcal{H})$ to $\ell^2_{\mathbb{Z}}(\mathcal{K})$.

Let us identify $\ell^2_{\mathbb{Z}}(\mathcal{H})$ with $L^2(\mathcal{H})$ and $\ell^2_{\mathbb{Z}}(\mathcal{K})$ with $L^2(\mathcal{K})$ in the standard way:

$$\{x_j\}_{j\in\mathbb{Z}} \mapsto \sum_{j\in\mathbb{Z}} z^j x_j.$$

We can consider now the operator Q as an operator from $L^2(\mathcal{H})$ to $L^2(\mathcal{K})$.

Let $x \in \mathcal{H}$ and let $\boldsymbol{x}$ be the constant $\mathcal{H}$-valued function on $\mathbb{T}$ identically equal to x. Then

$$(Q\boldsymbol{x})(\zeta) = \sum_{k\in\mathbb{Z}} \zeta^k \Omega_k x$$

and $Q\boldsymbol{x} \in L^2(\mathcal{K})$. Since $\mathcal{K}$ is separable, we can define a $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued function Φ almost everywhere on $\mathbb{T}$ such that

$$\Phi(\zeta)x = \sum_{k\in\mathbb{Z}} \zeta^k \Omega_k x, \quad \text{a.e.}$$

It is easy to see that

$$(QF)(\zeta) = \Phi(\zeta)F\big(\bar{\zeta}\big), \quad \text{a.e.} \tag{2.6}$$

for any trigonometric polynomial F with coefficients in $\mathcal{H}$.

Now let M be multiplication by Φ defined on the set of trigonometric polynomials in $L^2(\mathcal{H})$ by

$$(MF)(\zeta) = \Phi(\zeta)F(\zeta).$$

It is easy to see from (2.6) that M extends to a bounded operator from $L^2(\mathcal{H})$ to $L^2(\mathcal{K})$ and its norm is equal to $\|Q\|$.

To prove that $\Phi \in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ and $\|\Phi\|_{L^\infty} \le \|Q\|$, it is sufficient to show that

$$\operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} |(\Phi(\zeta)x, y)| \le \|Q\|$$

for any $x \in \mathcal{H}$ and $y \in \mathcal{K}$ with $\|x\| \le 1$ and $\|y\| \le 1$. Put $\varphi(\zeta) = (\Phi(\zeta)x, y)$, $\zeta \in \mathbb{T}$. Clearly, multiplication by the scalar function φ defined on the set of trigonometric polynomial extends to a bounded operator on L^2. It is also obvious that $\varphi \in L^2$, and so multiplication by φ is a bounded operator from L^2 to L^1. By continuity, multiplication by φ is a bounded operator on L^2. It is easy to see now that $\|\varphi\|_{L^\infty} \le \|M\| = \|Q\|$, which completes the proof. ■

Now let $\Phi \in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$. As in the scalar case it is easy to use a compactness argument to show that there exists a function $F \in H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ such that

$$\|H_\Phi\| = \|\Phi - F\|_{L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))}.$$

In other words, $\Phi - F$ is a symbol of H_Φ of minimal norm. Suppose that H_Φ has a maximizing vector $f \in H^2(\mathcal{H})$, i.e., f is a nonzero vector function in $H^2(\mathcal{H})$ such that $\|H_\Phi f\|_{H^2_-(\mathbb{C}^m)} = \|H_\Phi\| \cdot \|f\|_{H^2(\mathbb{C}^n)}$. We obtain a result similar to Theorem 1.1.4. However, unlike the scalar case H_Φ may have infinitely many symbols of minimal norm (we will discuss this in more detail in Chapter 14).

Theorem 2.3. *Let $\Phi \in L^\infty(\mathcal{H},\mathcal{K})$ and $F \in H^\infty(\mathcal{H},\mathcal{K})$ satisfy*

$$\|\Phi - F\|_{L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))} = \|H_\Phi\| > 0.$$

Suppose that $f \in H^2(\mathcal{H})$ is a maximizing vector of H_Φ. Then $(\Phi - F)f \in H^2_-(\mathcal{K})$, $\|(\Phi - F)(\zeta)\|_{\mathcal{B}(\mathcal{H},\mathcal{K})} = \|H_\Phi\|$ for almost all $\zeta \in \mathbb{T}$, and $f(\zeta)$ is a maximizing vector of $(\Phi - F)(\zeta)$ almost everywhere on $\mathbb{T}$. If $g = \|H_\Phi\|^{-1}H_\Phi f$, then

$$\|f(\zeta)\|_{\mathcal{H}} = \|g(\zeta)\|_{\mathcal{K}} \tag{2.7}$$

for almost all $\zeta \in \mathbb{T}$.

Proof. Without loss of generality we may assume that $\|H_\Phi\| = 1$. We have

$$\|f\|_2 = \|H_\Phi f\|_2 = \|\mathbb{P}_-(\Phi - F)f\|_2 \le \|(\Phi - F)f\|_2 \le \|f\|_2.$$

Hence,

$$\|\mathbb{P}_-(\Phi - F)f\|_2 = \|(\Phi - F)f\|_2 = \|f\|_2.$$

The first equality means that $(\Phi - F)f \in H^2_-(\mathcal{K})$. Since $f(\zeta) \ne \mathbb{O}$ and $\|(\Phi - F)(\zeta)\| \le 1$ almost everywhere on $\mathbb{T}$, it follows from the second equality that $\|(\Phi-F)(\zeta)\| = 1$ and $f(\zeta)$ is a maximizing vector of $(\Phi-F)(\zeta)$ for almost all $\zeta \in \mathbb{T}$. Therefore $g(\zeta)$ is a maximizing vector of $(\Phi - F)^*(\zeta)$ for almost all $\zeta \in \mathbb{T}$. Hence,

$$\|g(\zeta)\|_{\mathcal{K}} = \|H_\Phi\|^{-1}\|(\Phi - F)^*(\zeta)\|_{\mathcal{B}(\mathcal{K},\mathcal{H})}\|f(\zeta)\|_{\mathcal{H}} = \|f(\zeta)\|_{\mathcal{H}},$$

which proves (2.7). ■

Remark. We can also consider vectorial Hankel operators on the Hardy class $H^2(\mathbb{C}_+, \mathcal{H})$ of vector-valued functions on the upper half-plane $\mathbb{C}_+$ (see §1.8) and integral operators $\boldsymbol{\Gamma}_\Xi$ on spaces of vector functions on $\mathbb{R}_+$ (see §1.8). The same reasoning as in §1.8 shows that bonded operators $\boldsymbol{\Gamma}_\Xi$ have block Hankel matrices with respect to the block bases of vector-valued Laguerre polynomials.

3. Hankel Operators and the Commutant Lifting Theorem

In this section we give an alternative approach to the boundedness problem for vectorial Hankel operators. We show that Theorem 2.1 can be deduced from the commutant lifting theorem. On the other hand, we show that in an important special case the commutant lifting theorem can be deduced from the vector version of the Nehari theorem (Theorem 2.1).

An alternative proof of Theorem 2.1. We give a proof of the nontrivial part of Theorem 2.1. Namely, we prove that if $\Gamma_\Omega = \{\Omega_{j+k}\}_{j,k\ge0}$ determines a bounded operator from $\ell^2(\mathcal{H})$ to $\ell^2(\mathcal{K})$, then there exists a function $\Phi \in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ such that $\hat{\Phi}(j) = \Omega_j$, $j \in \mathbb{Z}_+$, and $\|\Phi\|_{L^\infty} = \|\Gamma_\Omega\|$.

We denote by $S_{\mathcal{H}}$ the shift operator on $\ell^2(\mathcal{H})$:

$$S_{\mathcal{H}}(x_0, x_1, \cdots) = (\mathbb{O}, x_0, x_1, \cdots), \quad (x_0, x_1, \cdots) \in \ell^2(\mathcal{H}).$$

Similarly, we denote by $S_{\mathcal{K}}$ the shift operator on $\ell^2(\mathcal{K})$.

It is easy to see that

$$S^*_{\mathcal{K}}\Gamma_\Omega = \Gamma_\Omega S_{\mathcal{H}} \tag{3.1}$$

and both operators in (3.1) are given by the block matrix $\{\Omega_{j+k+1}\}_{j,k\ge0}$. Equality (3.1) means that the operator Γ_Ω intertwines the contractions $S^*_{\mathcal{K}}$ and $S_{\mathcal{H}}$.

Consider now the bilateral shift $\mathcal{S}_{\mathcal{H}}$ on the space $\ell^2_{\mathbb{Z}}(\mathcal{H})$ of two-sided sequences:

$$\mathcal{S}_{\mathcal{H}}\{x_j\}_{j\in\mathbb{Z}} = \{x_{j-1}\}_{j\in\mathbb{Z}}, \quad \{x_j\}_{j\in\mathbb{Z}} \in \ell^2(\mathcal{H}).$$

Similarly, we can define the bilateral shift $\mathcal{S}_{\mathcal{K}}$ on the space $\ell^2_{\mathbb{Z}}(\mathcal{K})$.

We identify in a natural way the spaces $\ell^2(\mathcal{H})$ and $\ell^2(\mathcal{K})$ with subspaces of $\ell^2_{\mathbb{Z}}(\mathcal{H})$ and $\ell^2_{\mathbb{Z}}(\mathcal{K})$. It is easy to see that under these identifications the operator $\mathcal{S}_{\mathcal{H}}$ is a minimal unitary dilation of $S_{\mathcal{H}}$ while the operator $\mathcal{S}^*_{\mathcal{K}}$ is a minimal unitary dilation of $S^*_{\mathcal{K}}$.

Now we are in a position to apply the commutant lifting theorem (see Appendix 1.5), which implies that there exists an operator $R : \ell^2(\mathcal{H}) \to \ell^2(\mathcal{K})$ such that

$$\mathcal{S}^*_{\mathcal{K}} R = R\mathcal{S}_{\mathcal{H}}, \tag{3.2}$$

$$P_{\ell^2(\mathcal{K})} R\big|\ell^2(\mathcal{H}) = \Gamma_\Omega, \tag{3.3}$$

and

$$\|R\| = \|\Gamma_\Omega\| \tag{3.4}$$

(as usual $P_{\ell^2(\mathcal{K})}$ is the orthogonal projection onto $\ell^2(\mathcal{K})$).

It follows easily from (3.2) that R has a block matrix representation in the form $\{R_{j+k}\}_{j,k\in\mathbb{Z}}$, where the R_j are bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. It is easy to see from (3.3) that $R_j = \Omega_j$, $j \in \mathbb{Z}_+$.

The rest of the proof is the same as in the proof of Theorem 2.1 given in §2. As in that proof we can find a function Φ in $L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ such that $\hat{\Phi}(j) = R_j$, $j \in \mathbb{Z}$, and $\|\Phi\|_{L^\infty} = \|R\|$. It follows now from (3.4) that $\|\Phi\|_{L^\infty} = \|\Gamma_\Omega\|$. ■

We are going to consider now an important special case of the commutant lifting theorem and deduce it from the vector version of the Nehari theorem.

Let $\mathcal{H}$ be a Hilbert space and let $\Theta \in H^\infty(\mathcal{B}(\mathcal{H}))$ be an operator inner function such that $\Theta(\zeta)$ is a unitary operator for almost all $\zeta \in \mathbb{T}$. On the space

$$K_\Theta \stackrel{\text{def}}{=} H^2(\mathcal{H}) \ominus \Theta H^2(\mathcal{H})$$

we consider the compression $S_{[\Theta]}$ of the shift operator $S_{\mathcal{H}}$ (multiplication by z) on $H^2(\mathcal{H})$ defined by

$$S_{[\Theta]} = P_\Theta S_{\mathcal{H}}\big|K_\Theta,$$

where P_Θ is the orthogonal projection onto K_Θ. In the case of scalar functions such operators have been considered in §1.2. As in the scalar case it is easy to see that

$$P_\Theta f = \Theta\mathbb{P}_-\Theta^* f, \quad f \in H^2(\mathcal{H}).$$

Recall that T is unitary equivalent to an operator $S_{[\Theta]}$ for some unitary-valued operator inner function Θ if and only if T is a C_{00}-*contraction* on Hilbert space (i.e., $\|T\| \le 1$ and $\lim\limits_{n\to\infty} T^n = \lim\limits_{n\to\infty} (T^*)^n = \mathbb{O}$ in the strong operator topology) (see Appendix 1.6).

The following theorem is a special case of the commutant lifting theorem (see Appendix 1.5). It will be deduced from the vector version of the Nehari theorem (Theorem 2.1). Recall that the case of scalar functions has been considered in §1.2 (Theorem 1.2.1).

Theorem 3.1. *Let $\mathcal{H}$ be a Hilbert space and let $\Theta \in H^\infty(\mathcal{B}(\mathcal{H}))$ be a unitary-valued operator inner function. If R commutes with $S_{[\Theta]}$, then there exists a function $\Phi \in H^\infty(\mathcal{B}(\mathcal{H}))$ such that $\|\Phi\|_{L^\infty} = \|R\|$ and*

$$Rf = P_\Theta \Phi f, \quad f \in K_\Theta.$$

The proof of Theorem 3.1 is similar to that of Theorem 1.2.1.

Proof. Define the operator $A : H^2(\mathcal{H}) \to H^2_-(\mathcal{H})$ by

$$Af = \Theta^* R P_\Theta f, \quad f \in H^2(\mathcal{H}). \tag{3.5}$$

As in Lemma 1.2.2 we prove that A is a Hankel operator. Again, as in the scalar case it is easy to see that A is Hankel if and only if

$$\mathbb{P}_- zAf = Azf, \quad f \in H^2(\mathcal{H}). \tag{3.6}$$

We have

$$\begin{aligned} \mathbb{P}_- zAf &= \mathbb{P}_- z\Theta^* R P_\Theta f = \Theta^* \Theta \mathbb{P}_- \Theta^* z R P_\Theta f \\ &= \Theta^* P_\Theta z R P_\Theta f = \Theta^* S_{[\Theta]} R P_\Theta f. \end{aligned}$$

On the other hand,

$$Azf = \Theta^* R P_\Theta z f.$$

Clearly, for $f \in \Theta H^2(\mathcal{H})$

$$\mathbb{P}_- zAf = Azf = \mathbb{O}.$$

If $f \in K_\Theta$, then bearing in mind that R and $S_{[\Theta]}$ commute, we obtain

$$\mathbb{P}_- zAf = \Theta^* S_{[\Theta]} Rf = \Theta^* R S_{[\Theta]} f = Azf,$$

which proves (3.6).

By Theorem 2.1, there exists $\Psi \in L^\infty(\mathcal{B}(\mathcal{H}))$ such that $\|\Psi\|_{L^\infty} = \|A\|$ and $A = H_\Psi$. It follows from (3.5) that $\Theta \mathbb{P}_- \Psi H^2(\mathcal{H}) \perp H^2_-(\mathcal{H})$. Hence, $\Theta \Psi H^2(\mathcal{H}) \perp H^2_-(\mathcal{H})$, and so $\Theta\Psi \in H^\infty(\mathcal{B}(\mathcal{H}))$.

Now put $\Phi \stackrel{\text{def}}{=} \Theta\Psi \in H^\infty(\mathcal{B}(\mathcal{H}))$. We have

$$Rf = \Theta \mathbb{P}_- \Psi f = \Theta \mathbb{P}_- \Theta^* \Phi f = P_\Theta \Phi f, \quad f \in K_\Theta. \tag{3.7}$$

We have $\|\Phi\|_{L^\infty} = \|\Psi\|_{L^\infty} = \|A\| \le \|R\|$. On the other hand, $\|R\| \le \|P_\Theta\| \cdot \|\Phi\|_{L^\infty} = \|\Phi\|_{L^\infty}$. ■

Remark. Formula (3.7) implies that

$$R = M_\Theta H_{\Theta^*\Phi} \big| K_\Theta, \tag{3.8}$$

where M_Θ is multiplication by Θ. By (3.5),

$$H_{\Theta^*\Phi} \big| \Theta H^2(\mathcal{H}) = H_\Psi \big| \Theta H^2(\mathcal{H}) = \mathbb{O}.$$

It follows now from (3.8) that the operators R and $H_{\Theta^*\Phi}$ have the same metric properties (compactness, singular values, etc.; see §6.9).

4. Compact Vectorial Hankel Operators

Here we obtain an analog of the Hartman theorem for vectorial Hankel operators. In §4.3 we obtain a formula for the essential norm of a vectorial Hankel operator.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Denote by $H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))+C(\boldsymbol{\mathcal{C}}(\mathcal{H},\mathcal{K}))$ the subspace of $L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$, which consists of functions F that admit a representation $F=G+H$, where $G\in H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ and H is a continuous function with values in the space $\boldsymbol{\mathcal{C}}(\mathcal{H},\mathcal{K})$ of compact operators from $\mathcal{H}$ to $\mathcal{K}$.

Theorem 4.1. *Let $\Phi\in L^\infty(\mathcal{H},\mathcal{K}))$. The following statements are equivalent:*
(i) *the Hankel operator H_Φ is compact on $H^2(\mathcal{H})$;*
(ii) $\Phi\in H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))+C(\boldsymbol{\mathcal{C}}(\mathcal{H},\mathcal{K}))$*;*
(iii) *there exists a function Ψ in $C(\boldsymbol{\mathcal{C}}(\mathcal{H},\mathcal{K}))$ such that $\hat\Phi(n)=\hat\Psi(n)$ for $n<0$.*

It is easy to see that Theorem 4.1 can be reformulated as follows.

Theorem 4.2. *Let $\{\Omega_j\}_{j\ge0}$ be a sequence of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. Then the block Hankel matrix $\{\Omega_{j+k}\}_{j,k\ge0}$ determines a compact operator from $\ell^2(\mathcal{H})$ to $\ell^2(\mathcal{K})$ if and only if there exists a function Ψ in $C(\boldsymbol{\mathcal{C}}(\mathcal{H},\mathcal{K}))$ such that $\hat\Psi(n)=\Omega_n$ for $n\ge0$.*

To prove Theorem 4.1, we need a lemma where $C_A(\mathcal{B}(\mathcal{H},\mathcal{K}))$ denotes the set of functions $\Phi\in C(\mathcal{B}(\mathcal{H},\mathcal{K}))$ for which $\hat\Phi(n)=\mathbb{O}$, $n<0$.

Lemma 4.3. *Let $\mathcal{H}$, $\mathcal{K}$ be finite-dimensional Hilbert spaces. Let $\Phi\in C(\mathcal{B}(\mathcal{H},\mathcal{K}))$. Then*

$$\operatorname{dist}_{L^\infty}\left(\Phi,H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))\right)=\operatorname{dist}_{L^\infty}\left(\Phi,C_A(\mathcal{B}(\mathcal{H},\mathcal{K}))\right).$$

The proof of the lemma is exactly the same as that of Lemma 1.5.2.

Proof of Theorem 4.1. Clearly, (ii)$\Leftrightarrow$(iii). The implication (iii)$\Rightarrow$(i) is very easy. Indeed, if Ψ is a trigonometric polynomial with coefficients in $\boldsymbol{\mathcal{C}}(\mathcal{H},\mathcal{K})$, then H_Ψ is compact since the block matrix $\{\hat\Psi(-j-k)\}_{j\ge0,k\ge1}$ corresponding to H_Ψ has finitely many nonzero entries, each of which is compact. If $\Psi\in C(\boldsymbol{\mathcal{C}}(\mathcal{H},\mathcal{K}))$, the result follows from the fact that Ψ can be approximated by trigonometric polynomials with values in $\boldsymbol{\mathcal{C}}(\mathcal{H},\mathcal{K})$.

Suppose now that H_Φ is compact. Let $\{P_n\}_{n\ge1}$ be a sequence of finite rank orthogonal projections on $\mathcal{H}$ such that $P_n\mathcal{H}\subset P_{n+1}\mathcal{H}$ and $\mathcal{H}=\operatorname{clos}\bigcup_{n\ge1}P_n\mathcal{H}$. Let $\{Q_n\}_{n\ge1}$ be a sequence of projections on $\mathcal{K}$ with the same properties.

Let $\Phi_1=Q_1\Phi P_1$, $\Phi_n=Q_n\Phi P_n-Q_{n-1}\Phi P_{n-1}$, $n\ge2$. It is easy to see that

$$H_{\Phi_1}=\mathcal{Q}_1H_\Phi\mathcal{P}_1,\quad H_{\Phi_n}=\mathcal{Q}_nH_\Phi\mathcal{P}_n-\mathcal{Q}_{n-1}H_\Phi\mathcal{P}_{n-1},\quad n\ge2,$$

where $\mathcal{P}_n$ is the orthogonal projection from $L^2(\mathcal{H})$ onto the subspaces of functions with values in $P_n\mathcal{H}$ and $\mathcal{Q}_n$ is the orthogonal projection from $L^2(\mathcal{K})$ onto the subspace of functions with values in $Q_n\mathcal{K}$.

Since H_Φ is compact, it follows that $\|H_{\Phi_n}\| \to 0$ as $n \to \infty$. Therefore it is possible to choose sequences $\{P_n\}_{n\ge1}$ and $\{Q_n\}_{n\ge1}$ such that

$$\sum_{n\ge1} \|H_{\Phi_n}\| < \infty.$$

Clearly,

$$H_\Phi = \sum_{n\ge1} H_{\Phi_n}. \tag{4.1}$$

We need the following lemma.

Lemma 4.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be finite-dimensional Hilbert spaces and let H_Φ be a compact Hankel operator from $H^2(\mathcal{H})$ to $H^2_-(\mathcal{K})$. Then for any $\varepsilon > 0$ there exists a function $\Psi \in C(\mathcal{B}(\mathcal{H},\mathcal{K}))$ such that $H_\Phi = H_\Psi$ and*

$$\|\Psi\|_{L^\infty} \le \|H_\Phi\| + \varepsilon.$$

Let us first complete the proof of Theorem 4.1 and then prove Lemma 4.4.

Consider $H_{\Phi_n}\big|H^2(P_n\mathcal{H})$ as a Hankel operator from $H^2(P_n\mathcal{H})$ to $H^2_-(Q_n\mathcal{K})$. It follows from Lemma 4.4 that for any $\varepsilon > 0$ there exists a function $\Psi_n \in C(\mathcal{B}(P_n\mathcal{H}, Q_n\mathcal{K}))$ such that $H_{\Psi_n} = \mathcal{Q}_n H_{\Phi_n}\big|H^2(P_n\mathcal{H})$ and $\|\Psi_n\|_\infty \le \|\Phi_n\|_\infty + \varepsilon 2^{-n}$. Now put

$$\Psi = \sum_{n\ge1} \Psi_n P_n.$$

Clearly, $\Psi_n P_n \in C(\boldsymbol{C}(\mathcal{H},\mathcal{K}))$ and

$$\sum_{n\ge1} \|\Psi_n P_n\|_\infty \le \sum_{n\ge1} \|\Phi_n\|_\infty + \varepsilon. \tag{4.2}$$

It follows from (4.1) that $H_\Psi = H_\Phi$, and it follows from (4.2) that $\Psi \in C(\boldsymbol{C}(\mathcal{H},\mathcal{K}))$. ■

Proof of Lemma 4.4. We can assume that $\mathcal{H} = \mathbb{C}^m$ and $\mathcal{K} = \mathbb{C}^d$. Then we can interpret Φ as a matrix function $\{\varphi_{jk}\}_{1\le j\le d, 1\le k\le m}$. It is easy to see that H_Φ is compact if and only if so are the $H_{\varphi_{jk}}$ for all j, k. By the Hartman theorem (Theorem 1.5.5) for each j, k there exists a function ξ_{jk} in $C(\mathbb{T})$ such that $H_{\xi_{jk}} = H_{\varphi_{jk}}$. Let $\Xi = \{\xi_{jk}\}_{1\le j\le d, 1\le k\le m}$. Then $\Xi \in C(\mathcal{B}(\mathbb{C}^m, \mathbb{C}^d))$ and $H_\Phi = H_\Xi$.

By Theorem 2.2

$$\|H_\Xi\| = \operatorname{dist}_{L^\infty}\left(\Xi, H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))\right).$$

Since $\Xi \in C(\mathcal{B}(\mathbb{C}^m, \mathbb{C}^d))$, it follows from Lemma 4.3 that for any $\varepsilon > 0$ there exists a function $\Upsilon \in C_A(\mathcal{B}(\mathcal{H},\mathcal{K}))$ such that $\|\Xi - \Upsilon\|_\infty \le \|H_\Phi\| + \varepsilon$. Now the function $\Psi \stackrel{\text{def}}{=} \Xi - \Upsilon$ is desired. ■

Remark. It is easy to see that we can choose sequences $\{P_n\}_{n\geq 1}$ and $\{Q_n\}_{n\geq 1}$ such that

$$\sum_{n\geq 1} \|H_{\Phi_n}\| \leq \|H_\Phi\| + \varepsilon$$

for any given positive ε. It follows that if H_Φ is a compact vectorial Hankel operator, then for any $\varepsilon > 0$ there exists a function Ψ in $C(\boldsymbol{\mathcal{C}}(\mathcal{H}, \mathcal{K}))$ such that $H_\Psi = H_\Phi$ and $\|\Psi\|_\infty \leq \|H_\Phi\| + \varepsilon$.

5. Vectorial Hankel Operators of Finite Rank

In this section we obtain an analog of Kronecker's theorem for vectorial Hankel operators. We shall show that a vectorial Hankel operator has finite rank if and only if it has a rational symbol whose coefficients have finite rank. We shall also obtain a formula for the rank of such operators. However, this formula is considerably more complicated than in the scalar case.

First we describe the invariant subspaces of the multiple shift operator (multiplication by z on $H^2(\mathcal{H})$) that have finite codimension. To this end we need the notion of a finite Blaschke–Potapov product.

Let $\mathcal{H}$ be a Hilbert space. A function B in $H^\infty(\mathcal{B}(\mathcal{H}))$ is called a *finite Blaschke–Potapov product* if it admits a factorization

$$B = UB_1B_2\cdots B_n, \tag{5.1}$$

where

$$B_j(z) = \frac{\lambda_j - z}{1 - \bar{\lambda}_j z} P_j + (I - P_j)$$

for some λ_j in $\mathbb{D}$, orthogonal projections P_j on $\mathcal{H}$, and a unitary operator U. The degree of the Blaschke–Potapov product (5.1) is defined by

$$\deg B = \sum_{j=1}^{n} \operatorname{rank} P_j.$$

Lemma 5.1. *Let L be an invariant subspace of multiplication by z on $H^2(\mathcal{H})$. Then L has finite codimension if and only if $L = BH^2(\mathcal{H})$ for a Blaschke–Potapov product B of finite degree. Moreover,*

$$\operatorname{codim} L = \deg B.$$

Proof. Denote by $S_\mathcal{H}$ multiplication by z on $H^2(\mathcal{H})$. Clearly, $L^\perp$ is an invariant subspace of $S^*_\mathcal{H}$ and $\dim L^\perp = \operatorname{codim} L$. If $\dim L^\perp < \infty$, then $S^*_\mathcal{H} \big| L^\perp$ has an eigenvalue $\bar{\lambda} \in \mathbb{D}$. Since $\operatorname{Ker}(S^*_\mathcal{H} - \bar{\lambda} I)$ consists of the functions

$$\frac{x}{1 - \bar{\lambda} z}, \quad x \in \mathcal{H},$$

it follows that

$$\operatorname{Ker}\left((S^*_{\mathcal{H}} - \bar{\lambda} I)\,\middle|\, L^{\perp}\right) = \left\{ \frac{x}{1-\bar{\lambda} z} :\ x \in \mathcal{H}_\lambda \right\},$$

where $\mathcal{H}_\lambda$ is a finite-dimensional subspace of $\mathcal{H}$. Let

$$B_1(z) = \frac{z-\lambda}{1-\bar{\lambda} z} P_{\mathcal{H}_\lambda} + (I - P_{\mathcal{H}_\lambda}),$$

where $P_{\mathcal{H}_\lambda}$ is the orthogonal projection onto $\mathcal{H}_\lambda$.

It is easy to see that

$$\operatorname{Ker}\left((S^*_{\mathcal{H}} - \bar{\lambda} I)\,\middle|\, L^{\perp}\right) = \operatorname{Ker} \mathbb{P}_+ \boldsymbol{B}_1^*,$$

where $\boldsymbol{B}_1^*$ is multiplication by B_1^*.

Clearly, $\mathbb{P}_+ \boldsymbol{B}_1^* L^{\perp}$ is an invariant subspace of $S^*_{\mathcal{H}}$ whose dimension is less than $\dim L^{\perp}$. Therefore we can repeat the above procedure finitely many times and we find that $L^{\perp} = \operatorname{Ker} \mathbb{P}_+ \boldsymbol{B}^*$, where B is a Blaschke–Potapov product of the form (5.1) and $\boldsymbol{B}^*$ is multiplication by B^*. Clearly, $\deg B < \infty$. Hence, $L = BH^2(\mathcal{H})$.

Conversely, let B be a Blaschke–Potapov product of the form (5.1). Clearly, multiplication by B_j on $H^2(\mathcal{H})$ is an isometry and the result will follow from the fact that

$$\operatorname{codim} B_j H^2(\mathcal{H}) = \operatorname{rank} P_j.$$

Let us show that

$$H^2(\mathcal{H}) \ominus B_j H^2(\mathcal{H}) = \left\{ \frac{x}{1-\bar{\lambda}_j z} :\ x \in P_j \mathcal{H} \right\}. \tag{5.2}$$

Indeed, if $x \in P_j\mathcal{H}$, $F \in H^2(\mathcal{H})$, we have

$$\left(B_j F, \frac{x}{1-\bar{\lambda}_j z} \right) = \left(F, \frac{x}{z-\lambda_j} \right) = 0,$$

since $1/(z-\lambda_j) \in H^2_-$.

Suppose now that $G \in H^2(\mathcal{H})$ and $G \perp B_j H^2(\mathcal{H})$. We have

$$(B_j F, G) = (bP_j F, G) + ((I-P_j)F, G) = (F, \bar{b} P_j G + (I-P_j)G) = 0$$

for any $F \in H^2(\mathcal{H})$, where $b(z) = (\lambda_j - z)(1-\bar{\lambda}_j z)^{-1}$. Hence, $\bar{b}P_j G + (I-P_j)G \in H^2(\mathcal{H})$. Consequently, $(I-P_j)G = \mathbb{O}$, that is, G takes values in the finite-dimensional space $P_j\mathcal{H}$ and $\bar{b}G \in H^2_-$.

Let $x \in P_j\mathcal{H}$ and let $f(z) \stackrel{\text{def}}{=} (G(z), x)$. Then $f \in H^2$ and $\bar{b} f \in H^2_-$. It follows that

$$f \in \{ g \in H^2 :\ \|H_{\bar{b}} g\|_2 = \|g\|_2 \}. \tag{5.3}$$

Since $\operatorname{rank} H_{\bar{b}} = 1$ (see Corollary 1.3.2), the space in (5.3) is one-dimensional; obviously it contains the function $1/\left(1-\bar{\lambda}_j z\right)$, and so $f = c/\left(1-\bar{\lambda}_j z\right)$ for some $c \in \mathbb{C}$. It is easy to see now that G belongs to the right-hand side of (5.2). ■

The following analog of Corollary 1.3.3 describes the vectorial Hankel operators of finite rank. As above $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces.

Theorem 5.2. *Let $\Phi \in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$. Then H_Φ has finite rank if and only if there exists a Blaschke–Potapov product B in $H^\infty(\mathcal{B}(\mathcal{H}))$ of finite degree such that $\Phi B \in H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$. In this case it is possible to choose B so that* $\deg B = \operatorname{rank} H_\Phi$.

Proof. Let $\operatorname{rank} H_\Phi < \infty$. Then $\operatorname{Ker} H_\Phi$ is an invariant subspace of $S_\mathcal{H}$ of codimension $\operatorname{rank} H_\Phi$. By Lemma 5.1, $\operatorname{Ker} H_\Phi = BH^2(\mathcal{H})$, where B is a Blaschke–Potapov product such that $\deg B = \operatorname{rank} H_\Phi$. Clearly, $H_{\Phi B} = \mathbb{O}$, and so $\Phi B \in H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$.

Conversely, if $\Phi B \in H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$, then $H_\Phi \big| BH^2(\mathcal{H}) = \mathbb{O}$, and so $\operatorname{Ker} H_\Phi$ has finite codimension. ■

Remark. It can also be shown that H_Φ has finite rank if and only if there exists a Blaschke–Potapov product B_1 in $H^\infty(\mathcal{B}(\mathcal{K}))$ of finite degree such that $B_1\Phi \in H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$. Moreover, B_1 can be chosen so that $\deg B_1 = \operatorname{rank} H_\Phi$.

Indeed, let $\Psi(z) = (\Phi(\bar{z}))^*$. Then $\Psi \in L^\infty(\mathcal{B}(\mathcal{K},\mathcal{H}))$, and it is easy to see that $\operatorname{rank} H_\Phi = \operatorname{rank} H_\Psi$. Therefore by Theorem 5.2, $\Psi B \in H^\infty(\mathcal{B}(\mathcal{K},\mathcal{H}))$ for some Blaschke–Potapov product B in $H^\infty(\mathcal{B}(\mathcal{K}))$ of finite degree. Let $B_1(z) = (B(\bar{z}))^*$. It is easy to show that B_1 is a Blaschke–Potapov product in $H^\infty(\mathcal{B}(\mathcal{K}))$ and $\deg B_1 = \deg B$. Clearly, $B_1\Phi \in H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$. ■

Theorem 5.2 gives a description of the vectorial Hankel operators of finite rank. However, it is difficult to compute the rank of such an operator by using Theorem 5.2. We conclude this section with a more explicit formula for the rank of a vectorial Hankel operator. To state the result, we need some preliminaries.

Let Λ be a finite subset of the unit disc and let Ψ be a rational function of the form

$$\Psi(\zeta) = \sum_{\lambda\in\Lambda}\sum_{n=1}^{k(\lambda)} \frac{T_{\lambda,n}}{(z-\lambda)^n}, \quad T_{\lambda,k(\lambda)} \neq \mathbb{O}, \tag{5.4}$$

where the $k(\lambda)$ are positive integers and $T_{\lambda,n} \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $\lambda \in \Lambda$, $1 \le n \le k(\lambda)$. Let us construct by induction on j the operators $T^{(j)}_{\lambda,n}$ for $0 \le j \le k(\lambda) - n$. Put

$$T^{(0)}_{\lambda,n} = T_{\lambda,n}, \quad T^{(j+1)}_{\lambda,n} = T^{(j)}_{\lambda,n} P_{\operatorname{Ker} T^{(j)}_{\lambda,k(\lambda)-j}} + T^{(j)}_{\lambda,n+1} P_{\operatorname{Range}\left(T^{(j)}_{\lambda,k(\lambda)-j}\right)^*}, \tag{5.5}$$

where P_M denotes the orthogonal projection onto M.

Theorem 5.3. *Let $\Phi \in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$. Then* $\operatorname{rank} H_\Phi < \infty$ *if and only if $\Psi = \mathbb{P}_-\Phi$ admits a representation* (5.4)*, where Λ is a finite set in $\mathbb{D}$ and the $T_{\lambda,n}$ are finite rank operators from $\mathcal{H}$ to $\mathcal{K}$, $\lambda \in \Lambda$, $1 \le n \le k(\lambda)$. In* *this case*

$$\operatorname{rank} H_\Phi = \sum_{\lambda\in\Lambda} \sum_{j=0}^{k(\lambda)-1} \operatorname{rank} T^{(j)}_{\lambda,k(\lambda)-j}, \tag{5.6}$$

where the operators $T^{(j)}_{\lambda,n}$ *are defined by* (5.5).

Proof. Let $\tilde{S}$ be the compressed shift on $H^2_-(\mathcal{K})$ defined by

$$\tilde{S}f = \mathbb{P}_- zf, \quad f \in H^2_-(\mathcal{K}).$$

Let $L = \operatorname{Range} H_\Phi$. Clearly, L is $\tilde{S}$-invariant and $\dim L = \operatorname{rank} H_\Phi$. If H_Φ has finite rank, there exist a finite set Λ in $\mathbb{D}$ and $N \in \mathbb{Z}_+$ such that

$$L \subset \operatorname{span}\{\operatorname{Ker}(\tilde{S} - \lambda I)^N : \lambda \in \Lambda\}.$$

Put

$$k(\lambda) = \min\left\{n \in \mathbb{Z}_+ : \operatorname{Ker}((\tilde{S} - \lambda I)\big|L)^n = \operatorname{Ker}((\tilde{S} - \lambda I)\big|L)^{n+1}\right\}.$$

It is easy to see that

$$\operatorname{Ker}(\tilde{S} - \lambda I)^n = \operatorname{span}\left\{\frac{x}{(z-\lambda)^m} : x \in \mathcal{K},\ 1 \le m \le n\right\}.$$

Therefore

$$L \subset L' \stackrel{\text{def}}{=} \operatorname{span}\left\{\frac{x}{(z-\lambda)^m} : \lambda \in \Lambda,\, x \in \mathcal{K},\, 1 \le m \le k(\lambda)\right\}.$$

Let us now define the operators $T_{\lambda,n}$, $\lambda \in \Lambda$, $1 \le n \le k(\lambda)$, by

$$T_{\lambda,n}x = (z-\lambda)^n \mathcal{P}_{\lambda,n}\mathbb{P}_-(\Phi x), \quad x \in \mathcal{H}, \tag{5.7}$$

where $\mathcal{P}_{\lambda,n}$ is the projection onto the subspace

$$\left\{\frac{x}{(z-\lambda)^n} : x \in \mathcal{K}\right\}$$

along the other subspaces

$$\left\{\frac{x}{(z-\mu)^m} : x \in \mathcal{K}\right\}, \quad \mu \in \Lambda, \quad 1 \le m \le k(\lambda),$$

that is,

$$\mathcal{P}_{\lambda,n}\frac{x}{(z-\lambda)^m} = \mathbb{O}$$

whenever $\lambda \ne \mu$ or $m \ne n$. In (5.7) we identify vectors in $\mathcal{K}$ with constant $\mathcal{K}$-valued functions. It is easy to see that the operators $T_{\lambda,n}$ have finite rank and $\Psi = \mathbb{P}_-\Phi$ satisfies (5.4).

Let

$$L_\lambda = L \bigcap \operatorname{Ker}(\tilde{S} - \lambda I)^{k(\lambda)}, \quad \lambda \in \Lambda.$$

Clearly,

$$\operatorname{rank} H_\Phi = \sum_{\lambda\in\Lambda} \dim L_\lambda.$$

Therefore formula (5.6) will be established if we prove that

$$\dim L_\lambda = \sum_{j=0}^{k(\lambda)-1} \operatorname{rank} T^{(j)}_{\lambda,k(\lambda)-j}, \quad \lambda \in \Lambda.$$

Assume for simplicity that $\lambda = 0$. The general case can be treated similarly; moreover it can be reduced to the case $\lambda = 0$ by a conformal change of variable.

Let $k = k(\lambda)$, $T_n = T_{0,n}$, and $T^{(j)}_n = T^{(j)}_{\lambda,n}$. We have

$$\begin{aligned} L_0 &= \left(\sum_{m=1}^{k} \mathcal{P}_{0,m}\right) H_\Phi H^2(\mathcal{H}) = \mathbb{P}_-\Psi_0 H^2(\mathcal{H}) \\ &= \operatorname{span}\left\{\sum_{m=1}^{k-n} \frac{T_{k+n}x}{(z-\lambda)^m} : \ 0 \le n < k,\, x \in \mathcal{H}\right\}, \end{aligned}$$

where

$$\Psi_0(z) = \sum_{n=1}^{k} z^{-n} T_n.$$

Let $E = \operatorname{Ker} T_k$. Then all nonzero functions in $\mathbb{P}_-\Psi_0 E^\perp$ are polynomials of z^{-1} of degree k. On the other hand, the space

$$\mathbb{P}_-\Psi_0\left(H^2(\mathcal{H}) \ominus E^\perp\right) = \mathbb{P}_-\Psi_0\left(H^2(E) \oplus zH^2(E^\perp)\right)$$

consists of polynomials of degree less than k. (Here we identify vectors in $E^\perp$ with constant $E^\perp$-valued functions.) Therefore the above subspaces are linearly independent and so

$$\dim \mathbb{P}_-\Psi_0 H^2(\mathcal{H}) = \dim \mathbb{P}_-\Psi_0 E^\perp + \dim \mathbb{P}_-\left(\Psi_0 H^2(E) + z\Psi_0 H^2(E^\perp)\right).$$

Clearly,

$$\dim \mathbb{P}_-\Psi_0 E^\perp = \operatorname{rank} T_k$$

and

$$\dim \mathbb{P}_-\left(\Psi_0 H^2(E) + z\Psi_0 H^2(E^\perp)\right) = \dim \Psi_1 H^2(\mathcal{H}),$$

where

$$\Psi_1 = \Psi P_E + \tilde{S}\Psi P_{E^\perp} = \sum_{n=1}^{k-1} z^{-n} T^{(1)}_n,$$

the operators $T^{(1)}_n = T^{(1)}_{0,n}$ being defined by (5.5).

Repeating the above procedure k times, we arrive at the formula

$$\dim L_0 = \sum_{j=0}^{k-1} \operatorname{rank} T^{(j)}_{k-j}. \quad \blacksquare$$

Suppose that $\lambda \in \mathbb{C}$ and Φ is a rational operator function of the form

$$\Phi = \sum_{n=1}^{k(\lambda)} \frac{T_{\lambda,n}}{(z-\lambda)^n} + Q_\lambda, \quad T_{\lambda,k(\lambda)} \ne \mathbb{O},$$

where Q_λ is analytic at λ. Then the *McMillan degree* $\deg_\lambda \Phi$ of Ψ at λ can be defined as

$$\deg_\lambda \Phi = \sum_{j=0}^{k(\lambda)-1} \operatorname{rank} T^{(j)}_{\lambda,k(\lambda)-j},$$

where the $T^{(j)}_{\lambda,k(\lambda)-j}$ are defined by (5.5). Similarly, one can define the McMillan degree $\deg_\infty \Phi$ of Φ at infinity. The *McMillan degree* $\deg \Phi$ of Φ is defined by

$$\deg \Phi = \sum_{\lambda \in \mathbb{C}\cup\{\infty\}} \deg_\lambda \Phi.$$

Theorem 5.3 says that $\operatorname{rank} H_\Phi$ is equal to the McMillan degree of $\mathbb{P}_-\Phi$.

6. Imbedding Theorems

In this section we find relations between vectorial Hankel operators and certain imbedding operators. In particular we shall show that for any Carleson measure μ there exists a bounded Hankel operator H_Φ, $\Phi \in L^\infty(\mathcal{B}(\mathbb{C},\mathcal{H}))$, such that

$$\|f\|_{L^2(\mu)} = \|H_\Phi f\|, \quad f \in H^2.$$

In other words, the Carleson imbedding operator $\mathcal{I}_\mu : H^2 \to L^2(\mu)$ (see Appendix 2.1) can be represented as $\mathcal{I}_\mu = V H_\Phi$ for some isometry V. This allows one to reduce the study of metric properties of $\mathcal{I}_\mu$ (such as the norm, singular values, etc.) to the corresponding properties of Hankel operators. Note that in §1.7 we have already found a relation between scalar Hankel operators and Carleson imbedding operators.

We are going to consider more general imbedding operators and relate them with certain Hankel operators. Recall that $k_\zeta(z) = (1-|\zeta|^2)(1-\bar{\zeta}z)^{-1}$ is the normalized reproducing kernel of H^2.

Theorem 6.1. *Let $\mathcal{H}$ be a Hilbert space, $\Phi \in L^2(\mathcal{H})$. The Hankel operator $H_\Phi : H^2 \to H^2_-(\mathcal{H})$ defined on the set of polynomials by*

$$H_\Phi f = \mathbb{P}_- f\Phi \tag{6.1}$$

is bounded if and only if

$$\sup_{\zeta\in\mathbb{D}} \|H_\Phi k_\zeta\| < \infty.$$

The proof is the same as that of Theorem 1.6.1. The only difference is that we have to deal with vector-valued functions. Clearly, in our case $\mathcal{B}(\mathbb{C},\mathcal{H})$-valued functions can be identified with $\mathcal{H}$-valued functions. By Theorem 2.2, H_Φ is bounded if and only if Φ determines a bounded linear functional on $H^1(\mathcal{H})$. Therefore Theorem 6.1 is equivalent to the fact that an $\mathcal{H}$-valued function Φ determines a bounded linear functional on $H^1(\mathcal{H})$ if and only if the Garsia norm

$$\sup_{\zeta\in\mathbb{D}}\int_{\mathbb{T}}\|\Psi(\zeta)-\Psi(\tau)\|_{\mathcal{H}}^2 P_\zeta(\tau)d\boldsymbol{m}(\tau)$$

of $\Psi \stackrel{\text{def}}{=} \mathbb{P}_-\Phi$ is finite, where as before we denote the harmonic extension of Ψ to $\mathbb{D}$ by the same symbol Ψ.

The proof of this Hilbert-valued version can be obtained in exactly the same way as the corresponding scalar result (see Garnett [1], Ch. VI, §2, where the scalar case is treated).

We consider here the following imbedding operators. Let ν be a positive Borel measure on a subset Λ of $\mathbb{D}$ and let $\lambda \mapsto \vartheta_\lambda$, $\lambda \in \Lambda$, be a measurable family of inner functions. As before P_{ϑ_λ} is the orthogonal projection of H^2 onto $K_{\vartheta_\lambda} = H^2 \ominus \vartheta_\lambda H^2$. Let $f \in H^2$. Define the function $\mathcal{I}f$ in the direct integral

$$\int_\Lambda \oplus K_{\vartheta_\lambda} d\nu(\lambda)$$

by

$$(\mathcal{I}f)(\lambda) = P_{\vartheta_\lambda} f, \quad \lambda \in \Lambda, \tag{6.2}$$

(see Appendix 1.4 for the definition of direct integrals). We show that $\mathcal{I} = VH_\Phi$ for a certain vectorial Hankel operator H_Φ and an isometry V. The following generalized imbedding theorem holds.

Theorem 6.2. *Let ν be a positive measure on a subset Λ of $\mathbb{D}$ and let $\{\vartheta_\lambda\}_{\lambda\in\Lambda}$ be a measurable family of inner functions. The following statements are equivalent:*

(i) *the operator $\mathcal{I}$ defined by* (6.2) *is bounded, i.e.,*

$$\int_\Lambda \|P_{\vartheta_\lambda} f\|^2 d\nu(\lambda) \le c\|f\|_2^2, \quad f \in H^2;$$

(ii)

$$\sup_{\zeta\in\mathbb{D}}\int_\Lambda (1-|\vartheta_\lambda(\zeta)|^2)d\nu(\lambda) < \infty;$$

(iii)

$$\sup_{\zeta\in\mathbb{D}}\int_\Lambda \|P_{\vartheta_\lambda} k_\zeta\|^2 d\nu(\lambda) < \infty.$$

Note that the Carleson imbedding operator can be considered as a special case of the above operators (6.2). Indeed, if

$$\vartheta_\lambda(z) = \frac{z-\lambda}{1-\bar\lambda z}, \quad d\mu(\lambda) = (1-|\lambda|^2)d\nu(\lambda),$$

then it is easy to see that

$$\int_\Lambda \|P_{\vartheta_\lambda} f\|^2 d\nu(\lambda) = \int_\Lambda |f(\lambda)|^2 d\mu(\lambda), \quad f \in H^2.$$

In particular, Theorem 6.2 implies the following well-known version of the Carleson imbedding theorem (see N.K. Nikol'skii [2], Lect. VII, §1).

Theorem 6.3. *Let μ be a positive measure on $\mathbb{D}$. Then the imbedding operator $\mathcal{I}_\mu : H^2 \to L^2(\mu)$ is bounded if and only if*

$$\sup_{\zeta \in \mathbb{D}} \int_{\mathbb{D}} \frac{1-|\zeta|^2}{|1-\bar{\zeta}\lambda|^2} d\mu(\lambda) < \infty.$$

In other words, μ is a Carleson measure if and only if

$$\sup_{\zeta \in \mathbb{D}} \|\mathcal{I}_\mu k_\zeta\|_{L^2(\mu)} < \infty.$$

To prove Theorem 6.2, we need the following result, which reduces the study of the imbedding operators (6.2) to the study of vectorial Hankel operators.

Theorem 6.4. *Let ν be a positive measure on a subset Λ of $\mathbb{D}$ and let $\{\vartheta_\lambda\}_{\lambda \in \Lambda}$ be a measurable family of inner functions. For $\mathcal{H} = L^2(\nu)$ define the $\mathcal{H}$-valued function Φ by*

$$(\Phi(\zeta)) = \overline{\vartheta_\lambda(\zeta)}, \quad \zeta \in \mathbb{T}, \ \lambda \in \Lambda.$$

Then

$$\|\mathcal{I}f\| = \|H_\Phi f\|$$

for $f \in H^2$, where $\mathcal{I}$ is defined by (6.2) *and the Hankel operator $H_\Phi : H^2 \to H^2_-(\mathcal{H})$ is defined by* (6.1).

Proof. Clearly,

$$\|P_{\vartheta_\lambda} f\| = \|\vartheta_\lambda \mathbb{P}_- \bar{\vartheta}_\lambda f\| = \|H_{\bar{\vartheta}_\lambda} f\|.$$

It follows that

$$\|\mathcal{I}f\|^2 = \int_\Lambda \|P_{\vartheta_\lambda} f\|^2 d\nu(\lambda) = \int_\Lambda \|H_{\bar{\vartheta}_\lambda} f\|^2 d\nu(\lambda) = \|H_\Phi f\|^2. \quad \blacksquare$$

Proof of Theorem 6.2. Let us show that (ii)⇔(iii). We have

$$\|P_{\vartheta_\lambda} k_\zeta\|^2 = 1 - \|\vartheta_\lambda \mathbb{P}_+ \bar{\vartheta}_\lambda k_\zeta\|^2 = 1 - \|\overline{\vartheta_\lambda(\zeta)} \vartheta k_\zeta\|^2 = 1 - |\vartheta_\lambda(\zeta)|^2,$$

which proves that (ii)⇔(iii).

The fact that (i)⇒(iii) is trivial. The implication (iii)⇒(i) is an immediate consequence of Theorems 6.1 and 6.4. ■

Remark. It can also be proved that $\mathcal{I}$ is compact if and only if

$$\lim_{|\zeta| \to 1} \int_\Lambda \|P_{\vartheta_\lambda} k_\zeta\|^2 d\nu(\lambda) = 0.$$

Concluding Remarks

Theorem 1.1 was established in Parrott [1]. Lemma 1.3 is taken from Douglas [1]. Theorem 1.5 in the special case of self-adjoint operators was proved in Krein [1]. A more general case was considered in Arlinskii and Tsekanovskii [1]. In the general case Theorem 1.5 was obtained independently in Shmul'yan and Yanovskaya [1], Arsene and Gheondea [1], and Davis, Kahan, and Weinberger [1]. The proof of Theorem 1.5 given in §1 is suggested by Vasyunin.

Theorem 2.1, which is the operator-valued version of the Nehari theorem, was obtained first in Page [1]. The approach given in §2 is due to Adamyan, Arov, and Krein [4]; see also Parrott [1]. Theorem 2.3 is a well-known analog of Theorem 1.1.4.

The approach to the operator-valued version of the Nehari theorem, given in §3 is due to Page [1]. We also mention the approach found in Sarason [1] which is based on factorizations of operator functions of class $H^1(\boldsymbol{S}_1)$. The proof Theorem 3.1 based on the operator version of the Nehari theorem was obtined by N.K. Nikol'skii [2]. Formula (3.8) is also due to N.K. Nikol'skii [2].

The compactness criterion obtained in §4 was first given in Page [1]. Theorem 5.2 is well known. Theorem 5.3 is due to Treil [3]. The results of §6 were obtained in Treil [6].

Theorem 4.1 was used in the paper Yafaev [1] to construct a function $\Phi \in H^\infty(\mathcal{B}(\mathcal{H}))$ ($\mathcal{H}$ is a Hilbert space) such that Φ is continuous on a nondegenerate closed arc $I \subset \mathbb{T}$, $\Phi(\zeta)$ is compact for $\zeta \in I$ but $\Phi(\zeta)$ is not compact for $\zeta \notin I$.

Theorem 2.2 leads naturally to the Nehari problem for operator-valued functions: for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, and $\Phi \in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ find a best approximation from $H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$, i.e., find an operator function $Q \in H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ such that

$$\|\Phi - Q\|_{L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))} = \|H_\Phi\|.$$

A more general problem, very important in control theory (see Ch. 11), is the so-called *four block problem*. Let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{K}_1$, and $\mathcal{K}_2$ be Hilbert spaces, and let $\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}$ be an operator function such that $\Phi_{ij} \in L^\infty(\mathcal{B}(\mathcal{H}_j,\mathcal{K}_i))$, $i,j = 1,2$. The four block problem is to find $Q \in H^\infty(\mathcal{B}(\mathcal{H}_1,\mathcal{K}_1))$, which minimizes

$$\left\| \begin{pmatrix} \Phi_{11} - Q & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \right\|_{L^\infty(\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{K}_1 \oplus \mathcal{K}_2))}.$$

The four block problem reduces to the Nehari problem in the case Φ_{12}, Φ_{21}, and Φ_{22} are the zero functions.

Consider now the *four block operator*

$$\not{4}_\Phi : H^2(\mathcal{H}_1) \oplus L^2(\mathcal{H}_2) \to H^2_-(\mathcal{K}_1) \oplus L^2(\mathcal{K}_2),$$

which plays the same role in the four block problem as the Hankel operators play in the Nehari problem:

$$\mathit{4}_{\Phi}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \mathbb{P}^{-}\left(\Phi\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right), \quad f_1 \in H^2(\mathcal{H}_1),\ f_2 \in L^2(\mathcal{H}_2),$$

where $\mathbb{P}^{-}$ is the orthogonal projection from $L^2(\mathcal{K}_1) \oplus L^2(\mathcal{K}_2)$ onto $H^2_-(\mathcal{K}_1) \oplus L^2(\mathcal{K}_2)$. The analog of the Nehari theorem is

$$\|\mathit{4}_{\Phi}\| = \inf_{Q \in H^{\infty}(\mathcal{H}_1, \mathcal{K}_1)} \left\| \begin{pmatrix} \Phi_{11} - Q & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \right\|_{L^{\infty}(\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{K}_1 \oplus \mathcal{K}_2))}.$$

Note that $\mathit{4}_{\Phi}$ can be compact only if Φ_{12}, Φ_{21}, and Φ_{22} are the zero functions. We refer the reader to Foias and Tannenbaum [1], [2] for the above facts and detailed information on the four block problem.

3

Toeplitz Operators

We study in this chapter another very important class of operators on spaces of analytic functions, the class of Toeplitz operators. Toeplitz operators can be defined as operators with matrices of the form $\{t_{j-k}\}_{j,k\geq 0}$.

In §1 we characterize the bounded Toeplitz operators and study elementary properties of Toeplitz operators on the Hardy class H^2. We obtain spectral inclusion theorems, and we reduce the problem of the invertibility of an arbitrary Toeplitz operator to the case of Toeplitz operators with unimodular symbols. Then we use the Nehari theorem to obtain necessary and sufficient conditions for left invertibility and right invertibility in the case of unimodular symbols. We obtain similar results for left Fredholmness and right Fredholmness.

In §2 we obtain a general invertibility criterion for Toeplitz operators. Although this criterion is not geometric, it turns out to be very useful in obtaining geometric descriptions of the spectra for various classes of Toeplitz operators (see §3).

We give geometric descriptions of spectra for several classes of Toeplitz operators in §3. We consider here the classes of Toeplitz operators with real-valued symbols, with continuous symbols, with symbols in $H^\infty + C$, and the class of Toeplitz operators with piecewise continuous symbols.

Section 4 is devoted to Toeplitz operators on spaces of vector functions. We obtain a boundedness criterion and prove general spectral inclusion theorems. Then we study invertibility conditions for Toeplitz operators with isometric-valued symbols, and we conclude §4 with a Fredholmness criterion for Toeplitz operators with matrix-valued symbols of class $H^\infty + C$.

In §5 we prove that if a Toeplitz operator with matrix-valued symbol is Fredholm, then its symbol admits a Wiener–Hopf factorization. We characterize the invertible and the Fredholm Toeplitz operators with matrix-valued symbols in terms of such Wiener–Hopf factorizations. We also find formulas for the indices of such factorizations.

We conclude this chapter with §6 in which we obtain a criterion for the left invertibility of an operator function Ω in the space of bounded analytic operator functions. We prove that Ω is left invertible if and only if the Toeplitz operator with symbol $\Omega(\bar{z})$ is left invertible.

1. Basic Properties

We define here Toeplitz operators on the Hardy class H^2 as those which have Toeplitz matrices (i.e., their entries depend only on the difference of the coordinates) in the basis $\{z^n\}_{n\geq 0}$. In this section we study basic properties of Toeplitz operators. We shall see that a special role is played by Toeplitz operators with unimodular symbols.

Definition. An operator $T : H^2 \to H^2$ defined on the set of polynomials is called a *Toeplitz operator* if there is a two-sided sequence of complex numbers $\{t_n\}_{n\in\mathbb{Z}}$ such that

$$(Tz^k, z^j) = t_{j-k}, \quad j,\, k \in \mathbb{Z}_+. \tag{1.1}$$

Theorem 1.1. *A Toeplitz operator T defined by* (1.1) *is bounded on H^2 if and only if there exists a bounded function φ on $\mathbb{T}$ whose Fourier coefficients coincide with the t_j:*

$$\hat{\varphi}(n) = t_n.$$

In this case $\|T\| = \|\varphi\|_\infty$.

Proof. Suppose that T is a bounded Toeplitz operator. For each $n \in \mathbb{Z}$ we consider the operator T_n on $L^2(\mathbb{T})$ defined by

$$T_n f = \bar{z}^n T\mathbb{P}_+ z^n f, \quad f \in L^2.$$

Clearly, T_n is bounded and $\|T_n\| \leq \|T\|$. If $j \geq -n$ and $k \geq -n$, then $(T_n z^k, z^j) = (Tz^{n+k}, z^{n+j}) = t_{j-k}$. Therefore the sequence $\{T_n\}_{n\geq 0}$ converges weakly to a bounded operator M on L^2. Obviously, $\|M\| \leq \|T\|$ and $(Mz^k, z^j) = t_{j-k}$ for any $j,\, k \in \mathbb{Z}$. It is also clear that $Mzf = zMf$, $f \in L^2$ (it suffices to verify this equality for $f = z^j$, $j \in \mathbb{Z}$).

Let $\varphi = M\mathbf{1} \in L^2$. It is easy to see that $Mf = \varphi f$, for any polynomial f. Clearly, multiplication by φ is a bounded operator from L^2 to L^1, and it follows that $Mf = \varphi f$ for any $f \in L^2$. Since M is a bounded operator on L^2, it is easy to see that $\varphi \in L^\infty$ and $\|\varphi\|_\infty = \|M\|$ (it suffices to apply M to the charcteristic functions of the sets $\{\zeta \in \mathbb{T} :\ |\varphi(\zeta)| > N\}$).

We have $\hat{\varphi}(n) = (M\mathbf{1}, z^n) = t_n$ and $Tf = \mathbb{P}_+ Mf = \mathbb{P}_+\varphi f$ for any polynomial f. Consequently, $\|Tf\|_2 \le \|\varphi\|_\infty \|f\|_2 = \|M\| \cdot \|f\|_2$. Thus $\|T\| \le \|M\| = \|\varphi\|_\infty$.

The converse follows immediately from the equality $Tf = \mathbb{P}_+\varphi f$ (which is sufficient to verify for $f = z^n$, $n \ge 0$). ■

Given $\varphi \in L^\infty$ we define the *Toeplitz operator* T_φ on H^2 by

$$T_\varphi f = \mathbb{P}_+\varphi f, \quad f \in H^2. \tag{1.2}$$

It follows from Theorem 1.1 that $\|T_\varphi\| = \|\varphi\|_\infty$. The function φ is called the *symbol* of T_φ. In contrast with Hankel operators the symbol of a Toeplitz operator is uniquely determined by the operator.

It is easy to see that

$$T_\varphi^* = T_{\bar{\varphi}}, \quad \varphi \in L^\infty. \tag{1.3}$$

Although the definition of Toeplitz operators (1.2) looks similar to the definition of Hankel operators (see (1.1.6)), the properties of Hankel operators are quite different from those of Toeplitz operators. In particular the Toeplitz operator T_φ is never compact unless $\varphi = \mathbb{O}$. Indeed, let $\varphi \in H^2$ and $T_\varphi f \ne \mathbb{O}$. Then it is easy to see that

$$\|T_\varphi z^n f\|_2 \ge \|T_\varphi f\|_2, \quad n \ge 0.$$

On the other hand, the sequence $\{z^n f\}_{n\ge 0}$ converges weakly to $\mathbb{O}$, which makes it impossible for T_φ to be compact.

However, there are important relations between Hankel and Toeplitz operators and there are useful formulas that relate Hankel operators with Toeplitz ones. We begin with the simplest one. Given $\varphi \in L^\infty$, denote by M_φ multiplication by φ on L^2:

$$M_\varphi f = \varphi f, \quad f \in L^2.$$

Then obviously

$$M_\varphi f = H_\varphi f + T_\varphi f, \quad f \in H^2. \tag{1.4}$$

Another formula that will be used frequently is the following:

$$T_{\varphi\psi} - T_\varphi T_\psi = H_{\bar{\varphi}}^* H_\psi, \quad \varphi,\ \psi \in L^\infty. \tag{1.5}$$

The proof of (1.5) is straightforward:

$$\begin{aligned} T_{\varphi\psi} f - T_\varphi T_\psi f &= \mathbb{P}_+\varphi\psi f - \mathbb{P}_+(\varphi \mathbb{P}_+ \psi f) \\ &= \mathbb{P}_+(\varphi \mathbb{P}_- \psi f) = H_{\bar{\varphi}}^* H_\psi f, \quad f \in H^2. \quad ■ \end{aligned}$$

The Toeplitz operators can also be described as the operators satisfying a certain commutation relation.

Theorem 1.2. *Let T be an operator on H^2. Then T is a Toeplitz operator if and only if*

$$S^* T S = T, \tag{1.6}$$

where S is multiplication by z on H^2.

Proof. Let $j, k \in \mathbb{Z}_+$. Clearly, (1.6) means that

$$(Tz^k, z^j) = (S^*TSz^k, z^j) = (Tz^{k+1}, z^{j+1}),$$

that is, T has Toeplitz matrix (see (1.1)). ■

It follows from Theorem 1.5.5 and from (1.5) that the functional calculus $\varphi \mapsto T_\varphi$, $\varphi \in H^\infty + C$, is multiplicative modulo the compact operators. But it is not multiplicative. The following assertion characterizes the pairs $\varphi,\ \psi \in L^\infty$ for which $T_{\varphi\psi} = T_\varphi T_\psi$.

Theorem 1.3. *Let $\varphi,\ \psi \in L^\infty$. Then*

$$T_{\varphi\psi} = T_\varphi T_\psi$$

if and only if $\psi \in H^\infty$ or $\bar\varphi \in H^\infty$.

Proof. If $\psi \in H^\infty$, then $T_{\varphi\psi} f = \mathbb{P}_+\varphi\psi f = T_\varphi \psi f = T_\varphi T_\psi f$, $f \in H^2$. If $\bar\varphi \in H^\infty$, it follows from (1.5) that $T^*_{\varphi\psi} = T_{\bar\varphi\bar\psi} = T_{\bar\psi} T_{\bar\varphi} = T^*_\psi T^*_\varphi = (T_\varphi T_\psi)^*$, so $T_{\varphi\psi} = T_\varphi T_\psi$.

Suppose now that $T_\varphi T_\psi = T_{\varphi\psi}$. It follows from (1.5) that $H^*_{\bar\varphi} H_\psi = \mathbb{O}$. Let us show that either $H_\psi = \mathbb{O}$ or $H_{\bar\varphi} = \mathbb{O}$. We have

$$(H^*_{\bar\varphi} H_\psi z^n, z^k) = (H_\psi z^n, H_{\bar\varphi} z^k) = (\mathbb{P}_- z^n \psi, \mathbb{P}_- z^k \bar\varphi) = 0, \quad n,\, k \geq 0.$$

Therefore

$$(\mathbb{P}_+ \bar z^n f, \mathbb{P}_+ \bar z^k g) = (S^{*n} f, S^{*k} g) = 0, \quad n,\, k \geq 0, \tag{1.7}$$

where $f = \bar z \overline{\mathbb{P}_- \psi} \in H^2$, $g = \bar z \overline{\mathbb{P}_- \bar\varphi} = S^* \mathbb{P}_+ \varphi \in H^2$. If $f \neq \mathbb{O}$ and $g \neq \mathbb{O}$, put

$$L_1 = \operatorname{span}\{S^{*n} f :\ n \geq 0\}, \quad L_2 = \operatorname{span}\{S^{*k} g : k \geq 0\}.$$

The subspaces L_1 and L_2 are S^*-invariant and nontrivial. So by Beurling's theorem (see Appendix 2.2) there exist nonconstant inner functions ϑ_1 and ϑ_2 such that

$$L_1 = H^2 \ominus \vartheta_1 H^2, \quad L_2 = H^2 \ominus \vartheta_2 H^2.$$

It follows from (1.7) that $L_1 \perp L_2$ and so $H^2 \ominus \vartheta_1 H^2 \subset \vartheta_2 H^2$. It is easy to see that if $\tau h \in L_1$, and τ is inner, then $h \in L_1$. Indeed, $(\tau h, \vartheta_1 \eta) = 0$ for every $\eta \in H^2$. Put $\eta = \tau\xi$, $\xi \in H^2$. Then $(\tau h, \vartheta_1 \tau \xi) = (h, \vartheta_1 \xi) = 0$ for every $\xi \in H^2$, which means that $h \in L_1$. Therefore L_1 contains an outer function that certainly cannot belong to $\vartheta_2 H^2$.

Thus either $f = \mathbb{O}$ or $g = \mathbb{O}$, which means that either $H_\psi = \mathbb{O}$ or $H_{\bar\varphi} = \mathbb{O}$. ■

One of the most important questions in studying Toeplitz operators is to find an invertibility criterion and describe the spectrum. To study such a problem we need the notion of Fredholm operators.

Recall (see Appendix 1.2 for more information) that an operator T on Hilbert space $\mathcal{H}$ is called *Fredholm* if it is invertible modulo compact operators. The *index* of a Fredholm operator T is defined by

$$\operatorname{ind} T = \dim \operatorname{Ker} T - \dim \operatorname{Ker} T^*,$$

and the *essential spectrum* $\sigma_e(A)$ of a bounded operator A is, by definition,

$$\sigma_e(A) = \{\lambda \in \mathbb{C} : \ A - \lambda I \text{ is not Fredholm}\}.$$

Clearly, $\sigma_e(A) \subset \sigma(A)$, where $\sigma(A)$ is the spectrum of A.

Obviously, the index of an invertible operator is zero. It turns out that for Fredholm Toeplitz operators the converse is also true.

Theorem 1.4. *Let φ be a nonzero function in L^∞. Then either* $\operatorname{Ker} T_\varphi = \{\mathbb{O}\}$ *or* $\operatorname{Ker} T_\varphi^* = \{\mathbb{O}\}$.

Proof. Let $f \in \operatorname{Ker} T_\varphi$ and $g \in \operatorname{Ker} T_\varphi^*$. Then $\varphi f \in H^2_-$ and $\bar\varphi g \in H^2_-$. Therefore $\varphi f \bar g \in H^1_- \stackrel{\text{def}}{=} \{\psi \in L^1 : \ \hat\psi(j) = 0,\, j \ge 0\}$ and $\bar\varphi \bar f g \in H^1_-$. Let $h = \varphi f \bar g$. Then both h and $\bar h$ belong to H^1_-, which means that $\hat h(n) = 0$ for any $n \in \mathbb{Z}$ and so $h = \mathbb{O}$. Since $\varphi \ne \mathbb{O}$, it follows that either f or g must vanish on a set of positive measure, which implies that either $f = \mathbb{O}$ or $g = \mathbb{O}$ (see Appendix 2.1). ■

Corollary 1.5. *A Toeplitz operator T_φ is invertible if and only if it is Fredholm and* $\operatorname{ind} T_\varphi = 0$.

In the case $\varphi \in H^\infty$ the operator T_φ is simply multiplication by φ. Such operators are called *analytic Toeplitz operators*. The spectrum of such operators can easily be described.

Theorem 1.6. *Let $\varphi \in H^\infty$. Then* $\sigma(T_\varphi) = \operatorname{clos} \varphi(\mathbb{D})$.

Proof. Let us show that T_φ is invertible if and only if φ is invertible in H^∞. Let $\psi \in H^\infty$ and $\varphi\psi = \mathbf{1}$. Then clearly $T_\varphi T_\psi = T_\psi T_\varphi = T_{\varphi\psi} = I$.

Suppose now that T_φ is invertible. Then $\varphi H^2 = H^2$. Consequently, there exists a function ψ in H^2 such that $\varphi\psi = \mathbf{1}$. Let $g = \varphi f$, $f \in H^2$. Then $T_\varphi^{-1} g = f = \psi g$. Since $\varphi H^2 = H^2$, it follows that $T_\varphi^{-1} g = \psi g$ for any g in H^2. Thus ψ is bounded and so φ is invertible in H^∞. ■

We are going to establish some elementary properties of the spectrum of Toeplitz operators.

Definition. Let $\varphi \in L^\infty$. The set $\mathcal{R}(\varphi)$ of those λ in $\mathbb{C}$ for which the set

$$\{\zeta \in \mathbb{T} : \ |f(\zeta) - \lambda| < \varepsilon\}$$

has positive Lebesgue measure for any $\varepsilon > 0$ is called the *essential range of* φ. In other words $\lambda \in \mathcal{R}(\varphi)$ if and only if $\varphi - \lambda$ is noninvertible in L^∞. Clearly, $\mathcal{R}(\varphi)$ is closed in $\mathbb{C}$.

Theorem 1.7. *Let $\varphi \in L^\infty$. Then $\mathcal{R}(\varphi) \subset \sigma_e(T_\varphi)$.*

Proof. It suffices to show that if T_φ is Fredholm, then φ is invertible in L^∞. By Theorem 1.4, either $\operatorname{Ker} T_\varphi = \{\mathbb{O}\}$ or $\operatorname{Ker} T_\varphi^* = \{\mathbb{O}\}$. To be definite, suppose that $\operatorname{Ker} T_\varphi = \{\mathbb{O}\}$. Then there exists $\varepsilon > 0$ such that

$$\varepsilon \|f\|_2 \le \|T_\varphi f\|_2 \le \|\varphi f\|_2, \quad f \in H^2.$$

Therefore

$$\varepsilon \|\bar z^n f\|_2 \le \|\varphi \bar z^n f\|_2.$$

Since the set $\{\bar{z}^n f: \ f \in H^2, n \geq 0\}$ is dense in L^2, it follows that $\varepsilon\|g\|_2 \leq \|\varphi g\|_2$ for any $g \in L^2$, which implies that $1/\varphi \in L^\infty$. ■

Corollary 1.8. *Let* $\varphi \in L^\infty$. *Then* $\mathcal{R}(\varphi) \subset \sigma(T_\varphi)$.

Denote by $\operatorname{conv} E$ the convex hull of a set E.

Theorem 1.9. *Let* $\varphi \in L^\infty$. *Then* $\sigma(T_\varphi) \subset \operatorname{conv}\mathcal{R}(\varphi)$.

Proof. It is sufficient to prove that if an open half-plane contains $\mathcal{R}(\varphi)$, then it also contains $\sigma(T_\varphi)$. By the linearity of the map $\varphi \mapsto T_\varphi$ we can assume that the half-plane in question is $\{\zeta \in \mathbb{C}: \ \operatorname{Re}\zeta > 0\}$. Moreover it is sufficient to show that if $\mathcal{R}(\varphi) \subset \{\zeta \in \mathbb{C}: \operatorname{Re}\zeta > 0\}$, then T_φ is invertible.

Let us show that there exist $\varepsilon > 0$ and $r < 1$ such that

$$\varepsilon\mathcal{R}(\varphi) \subset \{\zeta: \ |1 - \zeta| < r\}. \tag{1.8}$$

Indeed, the discs

$$D_R = \{\zeta: \ |R - \zeta| < R\}, \quad R > 0,$$

cover the half-plane $\{\zeta \in \mathbb{C}: \ \operatorname{Re}\zeta > 0\}$. Therefore $\mathcal{R}(\varphi)$, being compact, is contained in some D_R. Then

$$\mathcal{R}(\varphi) \subset \{\zeta: \ |R - \zeta| < R - \delta\}$$

for some $\delta > 0$. Now (1.8) holds with $\varepsilon = 1/R$, $r = 1 - \delta/R$.

It follows that $\|\varepsilon\varphi - \mathbf{1}\|_\infty < 1$, so $\|T_{\varepsilon\varphi} - I\| < 1$, which implies that $T_{\varepsilon\varphi}$ is invertible and so is T_φ. ■

In the general case the problem of the invertibility of a Toeplitz operator can be reduced to the case of a Toeplitz operator with unimodular symbol. (A function u is called *unimodular* if $|u(\zeta)| = 1$ almost everywhere on $\mathbb{T}$.)

Lemma 1.10. *Let* $\varphi \in L^\infty$. *The following statements are equivalent:*

(i) T_φ *is invertible;*

(ii) φ *is invertible in* L^∞ *and* $T_{\varphi/h}$ *is invertible, where* h *is an outer function with* $|h| = |\varphi|$;

(iii) φ *is invertible in* L^∞ *and* $T_{\varphi/|\varphi|}$ *is invertible.*

Proof. If T_φ is invertible, then φ is invertible in L^∞ by Corollary 1.8. By Theorem 1.3, $T_\varphi = T_{\varphi/h}T_h$. Since T_h is invertible, it follows that (i)⇔(ii).

To show that (i)⇔(iii), take an outer function h_1 such that $|h_1| = |\varphi|^{-1/2}$. By Theorem 1.3, $T_{\varphi/|\varphi|} = T_{\bar{h}_1\varphi h_1} = T^*_{h_1}T_\varphi T_{h_1}$. The result follows from the invertibility of T_{h_1} and $T^*_{h_1}$. ■

Theorem 1.11. *Let* u *be a unimodular function on* $\mathbb{T}$. *Then* T_u *is left invertible if and only if* $\operatorname{dist}_{L^\infty}(u, H^\infty) < 1$. T_u *is right invertible if and only if* $\operatorname{dist}_{L^\infty}(\bar{u}, H^\infty) < 1$.

Proof. It follows from (1.4) that

$$\|f\|^2 = \|H_u f\|^2 + \|T_u f\|^2, \quad f \in H^2.$$

Hence, $\|T_u f\|^2 \geq \varepsilon\|f\|^2$ for some $\varepsilon > 0$ if and only if $\|H_u\| < 1$. By Theorem 1.1.3 this is equivalent to the fact that $\operatorname{dist}(u, H^\infty) < 1$.

The second assertion of the theorem follows from the equality $T_u^* = T_{\bar{u}}$. ■

Theorem 1.12. *Let u be a unimodular function on $\mathbb{T}$. If T_u is invertible and h is a function in H^∞ such that $\|u-h\|_\infty < 1$, then h is an outer function.*

Proof. We have

$$\|I - T_{\bar{u}h}\| = \|\mathbf{1} - \bar{u}h\|_\infty = \|u-h\|_\infty.$$

Therefore $T_{\bar{u}h} = T_u^* T_h$ is invertible and so is T_h, which means by Theorem 1.6 that h is invertible in H^∞.

Theorem 1.13. *Let u be a unimodular function on $\mathbb{T}$. Then T_u is invertible if and only if there exists an outer function h such that $\|u-h\|_\infty < 1$.*

Proof. The existence of such an outer function for an invertible T_u follows from Theorems 1.11 and 1.12.

Suppose that $\|u-h\|_\infty < 1$ for an outer function h. Then h is separated away from zero on $\mathbb{T}$ and since h is outer, it follows that h is invertible in H^∞. We have $\|I - T_{\bar{u}}T_h\| < 1$, which implies that $T_{\bar{u}}T_h$ is invertible. Hence, T_u is also invertible. ■

Theorem 1.14. *Let u be a unimodular function on $\mathbb{T}$ such that T_u is left invertible. Then T_u is invertible if and only if $T_{\bar{z}u}$ is not left invertible.*

Proof. If T_u is invertible, then there is a function f in H^2 such that $T_u f = \mathbf{1}$. Hence, $T_{\bar{z}u} f = \mathbb{O}$.

Suppose now that $T_{\bar{z}u}$ is not left invertible. That means that

$$\inf\{\|T_{\bar{z}u}f\|_2 : \ \|f\|_2 = 1\} = 0$$

or, which is the same,

$$\inf\left\{\|T_u f - \widehat{T_u f}(0)\cdot\mathbf{1}\|_2 : \ \|f\|_2 = 1\right\} = 0.$$

By the hypothesis

$$\inf\{\|T_u f\|_2 : \ \|f\|_2 = 1\} > 0.$$

Hence, $\mathbf{1} \in \operatorname{clos} T_u H^2 = T_u H^2$. Let g be a function in H^2 such that $T_u g = \mathbf{1}$. From the identity

$$\mathbf{1} = T_{\bar{z}^n} T_u z^n g, \quad n \in \mathbb{Z}_+,$$

it follows by induction that $z^n \in T_u H^2$, $n \in \mathbb{Z}_+$, that is, $T_u H^2 = H^2$. ■

We are now going to obtain similar descriptions of Fredholm, left Fredholm, and right Fredholm Toeplitz operators with unimodular symbols. Recall that a bounded linear operator T on Hilbert space is called *left Fredholm* if there exists a bounded linear operator R such that $RT - I$ is compact and T is called *right Fredholm* if there is a bounded linear operator R such that $TR - I$ is compact.

Theorem 1.15. *Let u be a unimodular function on $\mathbb{T}$. Then T_u is left Fredholm if and only if $\|H_u\|_e < 1$. T_u is right Fredholm if and only if $\|H_{\bar{u}}\|_e < 1$.*

Proof. Clearly, it is sufficient to prove only the first part of the theorem. Suppose that $\|H_u\|_e = \mathrm{dist}_{L^\infty}(u, H^\infty + C) < 1$. Then there exists $n \in \mathbb{Z}_+$ such that

$$\mathrm{dist}_{L^\infty}(z^n u, H^\infty) = \mathrm{dist}_{L^\infty}(u, \bar{z}^n H^\infty) < 1.$$

By Theorem 1.11, the operator $T_{z^n u} = T_u T_{z^n}$ is left invertible. Let R be a bounded linear operator such that $RT_u T_{z^n} = I$. It follows that $RT_u T_{z^n} T_{\bar{z}^n} = T_{\bar{z}^n}$. Clearly, $I - T_{z^n} T_{\bar{z}^n}$ is compact, and so $RT_u - T_{\bar{z}^n}$ is compact. Multiplying this operator by T_{z^n} on the left, we find that $T_{z^n} RT_u - I$ is compact which proves that T_u is left Fredholm.

Suppose now that T_u is left Fredholm. Then there exists a bounded linear operator R such that $RT_u - I$ is compact. Hence, $\dim \operatorname{Ker} RT_u < \infty$. It follows that there exists $n \in \mathbb{Z}_+$ such that $\operatorname{Ker} RT_{z^n u} = \{\mathbb{O}\}$. Since $RT_{z^n u} - I$ is compact, it follows that $RT_{z^n u}$ is invertible, and so $T_{z^n u}$ is left invertible. By Theorem 1.11, $\mathrm{dist}_{L^\infty}(z^n u, H^\infty) < 1$. The result follows now from the obvious equality

$$\mathrm{dist}_{L^\infty}(u, H^\infty + C) = \mathrm{dist}_{L^\infty}(z^n u, H^\infty + C). \quad \blacksquare$$

Corollary 1.16. *Let u be a unimodular function on $\mathbb{T}$. Then T_u is Fredholm if and only if $\|H_u\|_e < 1$ and $\|H_{\bar{u}}\|_e < 1$.*

2. A General Invertibility Criterion

The following theorem due to Devinatz and Widom gives an invertibility criterion for a Toeplitz operator. Although the criterion is not geometric, we shall see later that it can be used to obtain geometric descriptions of the spectra of certain Toeplitz operators.

Theorem 2.1. *Let $\varphi \in L^\infty$. Then T_φ is invertible if and only if φ is invertible in L^∞ and $\varphi/|\varphi|$ admits a representation*

$$\varphi/|\varphi| = \exp \mathrm{i}(\tilde{\alpha} + \beta + c) \tag{2.1}$$

where α and β are real functions in L^∞, $\|\beta\|_\infty < \pi/2$, and $c \in \mathbb{R}$.

Recall that $\tilde{\alpha}$ is the harmonic conjugate of α (see Appendix 2.1).

Proof. By Lemma 1.10, we may assume that φ is unimodular. Suppose that T_φ is invertible. Then there exists an outer function h such that $\|\mathbf{1} - \bar{\varphi} h\|_\infty < 1$ (see Theorem 1.13). Hence,

$$\mathcal{R}(\bar{\varphi} h) \subset \left\{ \zeta \in \mathbb{C} : \ |\arg \zeta| < \frac{\gamma\pi}{2} \right\}$$

for some $\gamma < 1$. Therefore there exists a real-valued function β such that $\|\beta\|_\infty < \pi/2$ and $\bar{\varphi} h = |h| e^{-\mathrm{i}\beta}$. Hence,

$$\varphi = |h| \bar{h}^{-1} e^{\mathrm{i}\beta} = \exp(\widetilde{\log |h|} + \beta + c)$$

for some $c \in \mathbb{R}$. Clearly, $\alpha \stackrel{\text{def}}{=} \log|h| \in L^\infty$.

Suppose now that $\varphi = \exp \mathrm{i}(\tilde{\alpha} + \beta + c)$. Put $\varphi_1 = \exp \mathrm{i}(\beta + c)$ and $h = \exp((\alpha + \mathrm{i}\tilde{\alpha})/2)$. Then h is invertible in H^∞, $\varphi = \varphi_1 h/\bar{h}$ and so

$$T_\varphi = T_{1/\bar{h}} T_{\varphi_1} T_h.$$

The operators $T_{1/\bar{h}}$ and T_h are obviously invertible. Since $\|\beta\|_\infty < \pi/2$, it follows that $0 \notin \operatorname{conv}(\mathcal{R}(\varphi_1))$. Thus T_{φ_1} is invertible by Theorem 1.9. ∎

To obtain further results we make use of Zygmund's theorem, which asserts that $\exp \tilde{\beta} \in L^p$ whenever β is a real-valued function satisfying $\|\beta\|_\infty < \pi/2p$ (see Appendix 2.1).

Corollary 2.2. *Let u be a unimodular function for which T_u is invertible on H^2. Then there exists an outer function h such that both h and $1/h$ belong to H^p for some $p > 2$ and $u = \bar{h}/h$.*

Proof. By Theorem 2.1, $u = \exp \mathrm{i}(\tilde{\alpha} + \beta + c)$, where $c \in \mathbb{R}$, α, β are real-valued functions in L^∞ and $\|\beta\|_\infty < \pi/2$. Put

$$h_1 = \exp\left(-\frac{\alpha}{2} - \mathrm{i}\frac{\tilde{\alpha}}{2}\right), \quad h_2 = \exp\left(\frac{\tilde{\beta}}{2} - \mathrm{i}\frac{\beta}{2}\right), \quad h = e^{-\mathrm{i}c/2} h_1 h_2.$$

Then $u = \bar{h}/h$. It is also clear that h_1 is invertible in H^∞. By Zygmund's theorem h_2 and $1/h_2$ belong to H^p for some $p > 2$. ∎

Let us show that the representation of a unimodular function obtained in Corollary 2.2 is unique.

Lemma 2.3. *Let h_1 and h_2 be outer functions such that $h_1 \in H^2$ and $1/h_2 \in H^2$. If $\bar{h}_1/h_1 = \bar{h}_2/h_2$, then $h_2 = ch_1$ for some $c \in \mathbb{R}$.*

Proof. Since $h_1 h_2^{-1} \in H^1$, it follows from $h_1 h_2^{-1} = \bar{h}_1 \bar{h}_2^{-1}$ that h_1/h_2 is constant. ∎

Corollary 2.4. *Suppose that under the hypotheses of Corollary 2.2 the function u has real Fourier coefficients. Then there exists an outer function h with real Fourier coefficients such that h and $1/h$ belong to H^p for some $p > 2$ and $u = \bar{h}/h$.*

Proof. The result follows immediately from Corollary 2.2 and Lemma 2.3. ∎

Theorem 2.5. *Let $u = \bar{h}/h$, where h is an outer function in H^2. Then T_φ is invertible if and only if $|h|^2$ admits a representation*

$$|h|^2 = \exp(\xi + \tilde{\eta}), \tag{2.2}$$

where ξ and η are real functions in L^∞ and $\|\eta\|_\infty < \pi/2$.

Proof. Suppose that (2.2) holds with ξ and η as required. Then

$$\exp\left(\frac{\xi}{2} + \mathrm{i}\frac{\tilde{\xi}}{2}\right) \exp\left(\frac{\tilde{\eta}}{2} - \mathrm{i}\frac{\eta}{2}\right)$$

is an outer function whose modulus coincides with $|h|$ on $\mathbb{T}$. So

$$h = \tau \exp\left(\frac{\xi}{2} + \mathrm{i}\frac{\tilde{\xi}}{2}\right) \exp\left(\frac{\tilde{\eta}}{2} - \mathrm{i}\frac{\eta}{2}\right)$$

for some unimodular constant τ. Therefore

$$u = \bar{h}/h = \frac{\bar{\tau}}{\tau} \exp \mathrm{i}(\eta - \tilde{\xi}) = \exp \mathrm{i}(\eta - \tilde{\xi} + c),$$

where $c \in \mathbb{R}$, $e^{\mathrm{i}c} = \bar{\tau}/\tau$. Thus u can be represented as in (2.1) and so T_u is invertible by Theorem 2.1.

Suppose now that T_u is invertible. Then u admits a representation (2.1) with α, $\beta \in L^\infty$ and $\|\beta\|_\infty < \pi/2$. Put

$$h_1 = \tau \exp\left(\frac{\alpha}{2} + \mathrm{i}\frac{\tilde{\alpha}}{2}\right) \exp\left(-\frac{\tilde{\beta}}{2} + \mathrm{i}\frac{\beta}{2}\right),$$

where τ is a unimodular constant such that $\bar{\tau}/\tau = e^{\mathrm{i}c}$. Then h_1 and h_1^{-1} belong to H^2 by Zygmund's theorem and $\bar{h}_1/h_1 = \bar{h}/h$. By Lemma 2.3, $h = \gamma h_1$ for some constant γ. Then

$$|h|^2 = |\gamma|^2 \cdot |h_1|^2 = \exp(\delta + \alpha - \tilde{\beta}),$$

where $\delta = \log|\gamma|$. Clearly, $\xi \overset{\text{def}}{=} \delta + \alpha$ and $\eta = -\beta$ satisfy the requirements. ■

Remark. Positive functions that can be represented as $\exp(\xi + \tilde{\eta})$ with ξ and η as in the statement of Theorem 2.5 are said to satisfy the *Helson–Szegö condition*. We shall discuss this condition in more detail in connection with prediction theory. It follows from Zygmund's theorem that if $|h|^2$ satisfies the Helson–Szegö condition, then $h \in H^p$ and $1/h \in H^p$ for some $p > 2$.

Corollary 2.6. *Let $u = \exp \mathrm{i}(\tilde{\alpha} + \beta)$, where α and β are real-valued functions in $C(\mathbb{T})$. Then T_u is invertible and u can be represented as $u = \bar{h}/h$, where h is an outer function such that h and $1/h$ belong to H^p with any $p < \infty$. Moreover $\tilde{\alpha} + \beta = \widetilde{\log|h|^2} + c$ for some $c \in \mathbb{R}$.*

Proof. Let q be a trigonometric polynomial such that $\|\beta - q\|_\infty < \pi/2p$, where $0 < p < \infty$. Then

$$\tilde{\alpha} + \beta = \widetilde{(\alpha - \tilde{q})} + \beta - q + \hat{q}(0).$$

So by Theorem 2.1, T_u is invertible. Let $\alpha_1 = \alpha - \tilde{q}$, $\beta_1 = \beta - q$, $d = \hat{q}(0)$. Put

$$h_1 = \exp \frac{1}{2}(\alpha_1 + \mathrm{i}\tilde{\alpha}_1), \quad h_2 = \exp \frac{1}{2}(-\tilde{\beta}_1 + \mathrm{i}\beta), \quad h = e^{-d/2} h_1 h_2.$$

Then $u = \bar{h}/h$. Clearly, h_1 is invertible in H^∞ and by Zygmund's theorem h_2, $h_2^{-1} \in H^p$. It is easy to see that

$$\log|h|^2 = \alpha - \tilde{\beta} - d$$

and so

$$\widetilde{\log |h|^2} = \tilde{\alpha} + \beta + c$$

for some $c \in \mathbb{R}$. ■

Corollary 2.7. *If f and g are real functions in VMO such that* $\exp \mathrm{i} f = \exp \mathrm{i} g$, *then* $f(\zeta) - g(\zeta) = 2k\pi$, $\zeta \in \mathbb{T}$, *for some* $k \in \mathbb{Z}$.

Proof. By Corollary 2.6 there exist outer functions h_1 and h_2 such that

$$\exp \mathrm{i} f = \bar{h}_1/h_1, \quad \exp \mathrm{i} g = \bar{h}_2/h_2$$

and h_1, h_1^{-1}, h_2, $h_2^{-1} \in H^2$. By Lemma 2.3, $h_1 = \gamma h_2$, $\gamma \in \mathbb{C}$. It follows from Corollary 2.6 that

$$f - g = \widetilde{\log |h_1|^2} - \widetilde{\log |h_2|^2} + \text{const} = \text{const}. \quad ■$$

3. Spectra of Certain Toeplitz Operators

In this section we find geometric descriptions of spectra of Toeplitz operators with symbols belonging to certain classes. We begin with the class of Toeplitz operators with real symbols.

Theorem 3.1. *Let φ be a real function in L^∞. Then*

$$\sigma(T_\varphi) = \sigma_{\mathrm{e}}(T_\varphi) = [\operatorname{ess\,inf} \varphi, \operatorname{ess\,sup} \varphi].$$

Proof. It follows from Theorems 1.7 and 1.9 that

$$\sigma(T_\varphi) \subset [\operatorname{ess\,inf} \varphi, \operatorname{ess\,sup} \varphi]$$

and the endpoints of this interval belong to $\sigma(T_\varphi)$. Suppose that $\lambda \in (\operatorname{ess\,inf} \varphi, \operatorname{ess\,sup} \varphi)$ and $T_{\varphi-\lambda}$ is invertible. Then there exists a function f in H^2 such that $T_{\varphi-\lambda} f = \mathbf{1}$. We have

$$(T_{\varphi-\lambda} f, f z^n) = (\mathbf{1}, f z^n) = 0, \quad n \geq 1.$$

Therefore

$$((\varphi - \lambda) f, f z^n) = \int_{\mathbb{T}} (\varphi - \lambda) |f|^2 \bar{z}^n d\boldsymbol{m} = 0, \quad n \geq 1.$$

Since $\varphi - \lambda$ is real-valued, it follows that the Fourier coefficients of $(\varphi - \lambda)|f|^2$ vanish for $n \neq 0$. Therefore $(\varphi - \lambda)|f|^2$ is constant, which is impossible, since it changes sign on $(\operatorname{ess\,inf} \varphi, \operatorname{ess\,supp} \varphi)$. Thus $\sigma(T_\varphi) = [\operatorname{ess\,inf} \varphi, \operatorname{ess\,sup} \varphi]$. Since T_φ is self-adjoint, it is evident that $\sigma_{\mathrm{e}}(T_\varphi) = [\operatorname{ess\,inf} \varphi, \operatorname{ess\,sup} \varphi]$. ■

Let us proceed to the class of Toeplitz operators with continuous symbols and symbols of class $H^\infty + C$.

Theorem 3.2. *Let $\varphi \in H^\infty + C$. Then T_φ is Fredholm if and only if φ is invertible in $H^\infty + C$.*

Proof. Let φ, $\psi \in H^\infty + C$ and $\varphi\psi = \mathbf{1}$. Then by (1.5)

$$I - T_\varphi T_\psi = H^*_{\bar\varphi} H_\psi, \quad I - T_\psi T_\varphi = H^*_{\bar\psi} H_\varphi.$$

By the Hartman theorem both operators are compact, which implies that T_φ is Fredholm.

Suppose now that T_φ is Fredholm. Then $1/\varphi \in L^\infty$ by Theorem 1.7 and so $\varphi = u\eta$, where η is an outer function invertible in H^∞ and u is a unimodular function. Clearly, $T_u = T_\varphi T_{\eta^{-1}}$ is Fredholm. Let $n = \operatorname{ind} T_u$.

Let us show that $T_{z^n u}$ is invertible. By Corollary 1.5, it is sufficient to show that it is Fredholm with zero index. If $n \ge 0$, then $T_{z^n u} = T_u T_{z^n}$ and

$$\operatorname{ind} T_{z^n u} = \operatorname{ind} T_u + \operatorname{ind} T_{z^n} = n - n = 0. \tag{3.1}$$

If $n < 0$, then $T_{z^n u} = T_{z^n} T_u$ and again (3.1) holds. Thus $T_{z^n u}$ is invertible and so is $T_{\bar z^n \bar u} = T^*_{z^n u}$. By Theorem 1.13, there exists an outer function h in H^∞ such that

$$\|\mathbf{1} - z^n u h\|_\infty = \|\bar z^n \bar u - h\|_\infty < 1.$$

Consequently, $z^n uh$ is invertible in $H^\infty + C$, and so $\varphi = uh$ is also invertible in $H^\infty + C$. ■

It is very easy now to evaluate the spectrum of a Toeplitz operator with symbol in $C(\mathbb{T})$. Let φ be a function in $C(\mathbb{T})$ that does not vanish on $\mathbb{T}$. We define the *winding number* $\operatorname{wind}\varphi$ with respect to the origin in the following way. Consider a continuous branch of the argument $\arg_\varphi$ of the function $t \mapsto \varphi(e^{\mathrm{i}t})$, $t \in [0, 2\pi]$, i.e.,

$$\arg_\varphi \in C([0,2\pi]), \quad \frac{\varphi(e^{\mathrm{i}t})}{|\varphi(e^{\mathrm{i}t})|} = \exp(\mathrm{i}\arg_\varphi(t)), \quad t \in [0, 2\pi].$$

Then

$$\operatorname{wind}\varphi \stackrel{\text{def}}{=} \frac{1}{2\pi}\left(\arg_\varphi(2\pi) - \arg_\varphi(0)\right).$$

Theorem 3.3. *Let $\varphi \in C(\mathbb{T})$. Then $\sigma_{\mathrm{e}}(T_\varphi) = \varphi(\mathbb{T})$. For $\lambda \notin \sigma_{\mathrm{e}}(T_\varphi)$ we have*

$$\operatorname{ind}(T_\varphi - \lambda I) = -\operatorname{wind}(\varphi - \lambda).$$

Proof. By Theorem 3.2, the operator $T_\varphi - \lambda I$ is Fredholm if and only if $\varphi - \lambda$ is invertible in $H^\infty + C$. Since $\varphi \in C(\mathbb{T})$, the last condition means that $\varphi - \lambda$ does not vanish on $\mathbb{T}$, which proves that $\sigma_{\mathrm{e}}(T_\varphi) = \varphi(\mathbb{T})$.

Let $\lambda \notin \sigma_{\mathrm{e}}(T_\varphi)$. Without loss of generality we can assume that $\lambda = 0$. Since φ does not vanish on $\mathbb{T}$, the curve $t \mapsto \varphi(e^{\mathrm{i}t})$, $t \in [0, 2\pi]$, is homotopic in $\mathbb{C} \setminus \{0\}$ to the curve $t \mapsto e^{\mathrm{i}nt}$ for some integer n. Since the index depends continuously on the operator, it does not change under homotopy. It is also clear that $\operatorname{wind}\varphi = \operatorname{wind} z^n = n$. Thus

$$\operatorname{ind} T_\varphi = \operatorname{ind} T_{z^n} = -n = -\operatorname{wind} z^n = -\operatorname{wind}\varphi. \quad ■$$

Corollary 3.4. *Let $\varphi \in C(\mathbb{T})$. Then T_φ is invertible if and only if $\varphi = e^f$ for some $f \in C(\mathbb{T})$.*

To obtain an invertibility criterion for Toeplitz operators with symbols in $H^\infty + C$, we have to define the winding number of an invertible function in $H^\infty + C$.

Let $\varphi \in H^\infty + C$. Consider its harmonic extension to the unit disc, which we also denote by φ. Given $r < 1$, let $\varphi_r(\zeta) \stackrel{\text{def}}{=} \varphi(r\zeta)$, $\zeta \in \mathbb{T}$. Clearly, $\varphi_r = \varphi * P_r$, where P_r is the Poisson kernel.

Theorem 3.5. *Let $\varphi \in H^\infty + C$. Then φ is invertible in $H^\infty + C$ if and only if there exists $r_0 \in (0,1)$ such that $|\varphi|$ is separated away from zero on the annulus $\{\zeta : r_0 < |\zeta| < 1\}$. In this case* $\operatorname{wind}\varphi_r$ *is constant for $r_0 < r < 1$.*

To prove the theorem we need a lemma on the asymptotic multiplicativity of the harmonic extension of functions in $H^\infty + C$.

Lemma 3.6. *Let φ, $\psi \in H^\infty + C$. Then*

$$\lim_{r \to 1-} \|(\varphi\psi)_r - \varphi_r\psi_r\|_\infty = 0.$$

Proof. Let $\varphi = f + \alpha$, $\psi = g + \beta$, where f, $g \in H^\infty$, α, $\beta \in C(\mathbb{T})$. Since $(fg)_r = f_r g_r$, we have

$$(\varphi\psi)_r - \varphi_r\psi_r = (f\beta)_r - f_r\beta_r + (g\alpha)_r - g_r\alpha_r + (\alpha\beta)_r - \alpha_r\beta_r.$$

It is obvious that $\|(\alpha\beta)_r - \alpha_r\beta_r\|_\infty \to 0$ as $r \to 1$, since α and β can be approximated by trigonometric polynomials and so it is sufficient to consider the case $\alpha = z^n$, $\beta = z^m$. Then

$$\begin{aligned} \|(\alpha\beta)_r - \alpha_r\beta_r\|_\infty &= \|r^{|n+m|} z^{m+n} - r^{|n|} z^n r^{|m|} z^m\|_\infty \\ &= |r^{|n+m|} - r^{|n|+|m|}| \to 0. \end{aligned}$$

Let us show that $\|(f\beta)_r - f_r\beta_r\|_\infty \to 0$. The proof of the fact that $\|(g\alpha)_r - g_r\alpha_r\|_\infty \to 0$ is similar. It is also clear that we can consider only the case $\beta = \bar{z}^n$, $n > 0$, since the trigonometric polynomials are dense in $C(\mathbb{T})$ and for $\beta \in H^\infty$ the situation is trivial.

Let $f = q + z^n h$, where q is an analytic polynomial, $\deg q \le n - 1$, and $h \in H^\infty$. We have

$$\|(f\beta)_r - f_r\beta_r\|_\infty \le \|(q\bar{z}^n)_r - q_r(\bar{z}^n)_r\|_\infty + \|h_r - (z^n h)_r(\bar{z}^n)_r\|_\infty.$$

Clearly,

$$\begin{aligned} \|h_r - (z^n h)_r(\bar{z}^n)_r\|_\infty &= \|h_r - r^n z^n h_r \cdot r^n \bar{z}^n\|_\infty \\ &= \|(1 - r^{2n})h_r\|_\infty \;\to\; 0, \quad r \to 1, \end{aligned}$$

and $\|(q\bar{z}^n)_r - q_r(\bar{z}^n)_r\|_\infty \to 0$ since q and $\bar{z}^n$ are trigonometric polynomials. ∎

Proof of Theorem 3.5. Suppose that φ is invertible in $H^\infty + C$. Then $\psi = 1/\varphi \in H^\infty + C$ and by Lemma 3.6,

$$\|\mathbf{1} - \varphi_r\psi_r\|_\infty = \|(\varphi\psi)_r - \varphi_r\psi_r\|_\infty \to 0.$$

Since $\|\psi_r\|_\infty \le \|\psi\|_\infty$, it follows that $|\varphi|$ is separated away from zero on $\{\zeta :\ r_0 < |\zeta| < 1\}$ for r_0 sufficiently close to 1.

Suppose now that $|\varphi(\zeta)| > \delta > 0$ for $|\zeta| > r_0$. Then there exist a function f in H^∞ and $n \in \mathbb{Z}_+$ such that $\|\varphi - \bar{z}^n f\|_\infty = \|z^n\varphi - f\|_\infty < \varepsilon$, where $0 < \varepsilon < \delta/2$. By Lemma 3.6,

$$\|(z^n\varphi)_r - r^n z^n \varphi_r\|_\infty \to 0.$$

Consequently, $|f(\zeta)| \ge \delta - \varepsilon$ for $|\zeta| > r_1$, where $r_0 < r_1 < 1$. Let $f = \vartheta h$, where ϑ is an inner function and h is an outer function. It follows that ϑ is a finite Blaschke product and h is invertible in H^∞. Therefore f is invertible in $H^\infty + C$. We have

$$\|z^n\varphi/f - 1\|_\infty = \|(z^n\varphi - f)/f\|_\infty \le \varepsilon\|f^{-1}\|_\infty \le \varepsilon(\delta - \varepsilon)^{-1} < 1.$$

Hence, $z^n\varphi/f$ is invertible in $H^\infty + C$ and so is φ.

The fact that $\operatorname{wind}\varphi_r$ does not depend on r follows from the fact that φ_r depends on r continuously and so $\operatorname{wind}\varphi_r$ is a continuous function on $(r_0, 1)$. ■

Let φ be an invertible function in $H^\infty + C$. Its winding number $\operatorname{wind}\varphi$ can be defined as follows. Let r_0 be a number in $(0,1)$ such that

$$\inf\{|\varphi(\zeta)| :\ |\zeta| > r_0\} > 0.$$

Then $\operatorname{wind}\varphi$ is by definition $\operatorname{wind}\varphi_r$, where r is an arbitrary number in $(r_0, 1)$. The winding number is well-defined in view of Theorem 3.5.

Before we proceed to the description of the spectrum of Toeplitz operators with symbols in $H^\infty + C$, we compute the index of analytic Toeplitz operators.

Lemma 3.7. *Let $\varphi \in H^\infty$. Then*

$$\sigma_{\mathrm{e}}(T_\varphi) = \mathcal{R}(\varphi) \bigcup \{\lambda \in \mathbb{C} :\ \dim H^2 \ominus \vartheta_\lambda H^2 = \infty\}, \tag{3.2}$$

where ϑ_λ is the inner factor of $\varphi - \lambda$. If $\lambda \notin \sigma_{\mathrm{e}}(T_\varphi)$, then

$$\operatorname{ind}(T_\varphi - \lambda I) = -\deg\vartheta_\lambda = -\operatorname{wind}(\varphi - \lambda).$$

Proof. By Theorem 3.5 to prove (3.2) it is sufficient to show that if $|\varphi|$ is separated away from zero on $\mathbb{T}$, then T_φ is Fredholm if and only if the inner part ϑ of φ is a finite Blaschke product. Let $\varphi = \vartheta h$ where h is outer. Since T_h is invertible, it follows that T_φ is Fredholm if and only if so is T_ϑ. Since $\operatorname{Ker} T_\vartheta^* = H^2 \ominus \vartheta H^2$, the operator T_φ is Fredholm if and only if ϑ is a finite Blaschke product, which proves (3.2).

Clearly, if T_φ is Fredholm, then $\deg\vartheta = \dim\operatorname{Ker} T_\vartheta^* = -\operatorname{ind} T_\varphi = \operatorname{wind}\vartheta$ by Theorem 3.3. ■

Now we are in a position to describe the spectrum of Toeplitz operators with $H^\infty + C$ symbols.

Theorem 3.8. *Let $\varphi \in H^\infty + C$. Then for $\lambda \notin \sigma_e(T_\varphi)$*

$$\operatorname{ind}(T_\varphi - \lambda I) = -\operatorname{wind}(\varphi - \lambda).$$

Proof. Clearly, we can assume that $\lambda = 0$. Let $n = \operatorname{ind} T_\varphi$. Then $T_{z^n\varphi}$ is invertible. Let us show that $\operatorname{wind} z^n\varphi = n + \operatorname{wind}\varphi$. Indeed, by Lemma 3.6, $\|(z^n\varphi)_r - r^{|n|}z^n\varphi_r\|_\infty \to 0$ as $r \to 1$. Since $(z^n\varphi)_r$ and $r^{|n|}z^n\varphi_r$ are separated away from zero for r sufficiently close to 1, it follows that

$$\operatorname{wind}(z^n\varphi)_r = \operatorname{wind} r^{|n|}z^n\varphi_r = n + \operatorname{wind}\varphi_r$$

Therefore without loss of generality we can assume that $n = 0$, that is, T_φ is invertible.

Let $\varepsilon > 0$. There exist $k \in \mathbb{Z}_+$ and $f \in H^\infty$ such that $\|\varphi - \bar{z}^k f\|_\infty < \varepsilon$. If ε is sufficiently small, then $T_{\bar{z}^k f}$ is invertible and $0 = \operatorname{ind} T_{\bar{z}^k f} = \operatorname{ind} T_f + k$. By Lemma 3.7, $\operatorname{ind} T_f = -\operatorname{wind} f$. Therefore $\operatorname{ind} T_{\bar{z}^k f} = -\operatorname{wind}\bar{z}^k f$. It is easy to see that if ε is sufficiently small, then $\operatorname{wind}\varphi = \operatorname{wind}\bar{z}^k f = 0$, which completes the proof. ■

Let us now introduce the important class of *quasicontinuous functions* $QC = (H^\infty + C)\bigcup\overline{(H^\infty + C)}$. It is easy to see that QC is the maximal self-adjoint subalgebra of $H^\infty + C$.

Lemma 3.9. $QC = VMO\bigcap L^\infty$.

Proof. Let $\varphi \in QC$. Then $\varphi \in H^\infty + C$ and $\mathbb{P}_-\varphi \in \mathbb{P}_-C(\mathbb{T}) \subset VMO$. Similarly, $\mathbb{P}_+\varphi \in VMO$.

Now let $\varphi \in VMO\bigcap L^\infty$. Then $\varphi = \mathbb{P}_+ f + g$, where $f, g \in C(\mathbb{T})$ (see Appendix 2.5). Since $\varphi \in L^\infty$, it follows that $\mathbb{P}_+ f \in H^\infty$, so $\varphi \in H^\infty + C$. Similarly, $\bar{\varphi} \in H^\infty + C$. ■

The following theorem, which describes the unimodular functions in QC, will be important in what follows.

Theorem 3.10. *A unimodular function u belongs to QC if and only if it admits a factorization*

$$u = z^n \exp \mathrm{i}(\tilde{\alpha} + \beta), \tag{3.3}$$

where $n \in \mathbb{Z}$, and α and β are real functions in $C(\mathbb{T})$.

Proof. If u satisfies (3.3), then

$$u = z^n \exp(\alpha + \mathrm{i}\tilde{\alpha})\exp(-\alpha + \mathrm{i}\beta) \in H^\infty + C,$$

since $H^\infty + C$ is an algebra (see Theorem 1.5.1). Similarly, $\bar{u} \in H^\infty + C$.

Suppose now that $u \in QC$, then $\bar{u} \in QC$ and so u is invertible in $H^\infty + C$. By Theorem 3.2, T_u is Fredholm. Let $n = -\operatorname{ind} T_u$ and $v = \bar{z}^n u$. Then T_v is invertible. By Theorem 1.13, there exists an outer function h in H^∞ such that $\|v - h\|_\infty = \|1 - \bar{v}h\|_\infty < 1$. Consequently, $\bar{v}h$ has a logarithm in the Banach algebra $H^\infty + C$. Therefore there exist an f in $C(\mathbb{T})$ and a g in H^∞ such that $(\bar{v}h)^{-1} = v/h = \exp(f + g)$. Hence,

$$v = h\exp(f + g) = \exp(\mathrm{i}c + \log|h| + \mathrm{i}\widetilde{\log|h|} + f + g)$$

for some $c \in \mathbb{R}$. Since v is unimodular, it follows that $\log|h| + \operatorname{Re}(f + g) = \mathbb{O}$. Put $\alpha = \log|h| + \operatorname{Re} g = -\operatorname{Re} f \in C(T)$. Since $g \in H^\infty$, we have

$$\tilde{\alpha} = \widetilde{\log|h|} + \operatorname{Im} g - \operatorname{Im} g(0).$$

Putting now

$$\beta = \operatorname{Im} f + c + \operatorname{Im} g(0) \in C(\mathbb{T}),$$

we see that u satisfies (3.3). ■

In Chapter 7 we shall obtain similar results for many other classes of functions.

We proceed now to Toeplitz operators with piecewise continuous symbols (see §1.5, where the class PC of piecewise continuous is defined).

If $\varphi \in PC$, we can consider the set γ that consists of $\varphi(\mathbb{T})$ and the union of the straight line intervals $[\varphi(\zeta^-), \varphi(\zeta)]$, where ζ ranges over all jump points. It is easy to find a parametrization of that curve, that is, a function $\varphi_\#$ in $C(\mathbb{T})$ such that $\varphi_\#(\mathbb{T}) = \gamma$. Thus

$$\gamma = \varphi_\#(\mathbb{T}) = \varphi(\mathbb{T}) \cup \bigcup_\zeta [\varphi(\zeta^-), \varphi(\zeta)],$$

the union being taken over all jump points ζ.

Theorem 3.11. *Let $\varphi \in PC$. Then $\sigma_{\mathrm{e}}(T_\varphi) = \varphi_\#(\mathbb{T})$. If $\lambda \notin \sigma_{\mathrm{e}}(T_\varphi)$, then*

$$\operatorname{ind}(T_\varphi - \lambda I) = -\operatorname{wind}(\varphi_\# - \lambda).$$

Proof. It suffices to prove that T_φ is Fredholm if and only if $0 \notin \varphi_\#(\mathbb{T})$ and if T_φ is Fredholm, then $\operatorname{ind} T_\varphi = -\operatorname{wind} \varphi_\#$.

Assume that $0 \notin \varphi_\#(\mathbb{T})$. That means that φ does not vanish on $\mathbb{T}$ and the intervals $[\varphi(\zeta^-), \varphi(\zeta)]$ do not contain the origin. Without loss of generality we may assume that φ is continuous at 1. Clearly, $\varphi/|\varphi| \in PC$ and it is possible to find a branch of the argument α of φ (i.e., $e^{\mathrm{i}\alpha} = \varphi/|\varphi|$) such that $\alpha \in PC$, α is continuous at the continuity points of φ, except for a possible jump at 1, and

$$|\alpha(\zeta^-) - \alpha(\zeta)| < \pi, \quad \zeta \neq 1.$$

In this case

$$\operatorname{wind} \varphi_\# = \frac{1}{2\pi}\left(\lim_{t\to 2\pi-} \alpha(e^{\mathrm{i}t}) - \lim_{t\to 0+} \alpha(e^{\mathrm{i}t})\right).$$

If $\operatorname{wind}\varphi_\# = 0$, then α has no jump at 1 and so all jumps of α have moduli less than $\pi - \delta$ for some $\delta > 0$. It is easy to see that α admits a representation

$$\alpha = \xi + \eta,$$

where $\xi \in C(\mathbb{T})$ and $\|\eta\|_\infty \le (\pi - \delta)/2$. Let q be a trigonometric polynomial such that $\|\xi - q\|_\infty < \delta/4$ and $r = \tilde{q}$. We have

$$\alpha = \eta + \xi - q - \tilde{r} + \hat{q}(0).$$

Clearly, $\|\eta + \xi - q\| < \pi/2$ and so by Theorem 2.1, the operator $T_{\varphi/|\varphi|} = T_{\exp \mathrm{i}\alpha}$ is invertible.

Let $n = \operatorname{wind}\varphi_\# \neq 0$ and $0 \notin \varphi_\#(\mathbb{T})$. Then $(\bar{z}^n\varphi)_\#$ has zero winding number and as we have just proved, $T_{\bar{z}^n\varphi}$ is invertible. If $n > 0$, then

$T_{\bar{z}^n\varphi} = T_{\bar{z}^n}T_\varphi$; if $n < 0$, then $T_{\bar{z}^n\varphi} = T_\varphi T_{z^n}$. In both cases it follows that T_φ is Fredholm and

$$0 = \operatorname{ind} T_{\bar{z}^n f} = \operatorname{ind} T_{\bar{z}^n} + \operatorname{ind} T_\varphi = n + \operatorname{ind} T_\varphi,$$

which implies that $\operatorname{ind} T_\varphi = -n$.

Suppose now that $0 \in \varphi_\#(\mathbb{T})$. If $0 \in \operatorname{clos}\varphi(\mathbb{T})$, then T_φ is not Fredholm by Theorem 1.7. If $0 \notin \operatorname{clos}\varphi(\mathbb{T})$, then $0 \in [\varphi(\zeta^-), \varphi(\zeta)]$ for some jump point ζ in $\mathbb{T}$. If T_φ were Fredholm, then so would be any operator $T_{\varphi-\lambda}$ for sufficiently small λ. Moreover, $\operatorname{ind} T_{\varphi-\lambda} = \operatorname{ind} T_\varphi$ for all such λ. Let us first take a very small λ_0 such that λ_0 belongs to only one interval of the form $[\varphi(\zeta^-), \varphi(\zeta)]$. Let ε be such a small number that the disc $D_\varepsilon = \{\zeta : |\lambda_0 - \zeta| < \varepsilon\}$ does not intersect $\varphi(\mathbb{T})$ and intersects only one interval of the form $[\varphi(\zeta^-), \varphi(\zeta)]$. Then as we have already proved, if λ_1 and λ_2 are in different components of

$$D_\varepsilon \setminus [\varphi(\zeta^-), \varphi(\zeta)],$$

the operators $T_{\varphi-\lambda_1}$ and $T_{\varphi-\lambda_2}$ are Fredholm and have different indices, which contradicts the assumption that T_φ is Fredholm. ■

4. Toeplitz Operators on Spaces of Vector Functions

In this section we study vectorial Toeplitz operators (i.e., Toeplitz operators on spaces of vector functions). Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. We identify the spaces $H^2(\mathcal{H})$ and $H^2(\mathcal{K})$ with $\ell^2(\mathcal{H})$ and $\ell^2(\mathcal{K})$ in the usual way:

$$f \mapsto \{\hat{f}(j)\}_{j\ge0}. \tag{4.1}$$

We define vectorial Toeplitz operators from $H^2(\mathcal{H})$ to $H^2(\mathcal{K})$ as operators that with respect to the decompositions (4.1) have block matrices of the form

$$\{\Omega_{j-k}\}_{j,k\ge0}, \tag{4.2}$$

where $\{\Omega_j\}_{j\in\mathbb{Z}}$ is a two-sided sequence of operators from $\mathcal{H}$ to $\mathcal{K}$.

Here we obtain basic results on vectorial Toeplitz operators and characterize in the finite-dimensional case the Fredholm Toeplitz operators with symbols in $H^\infty + C$. In the next section we obtain a characterization of the invertible and Fredholm Toeplitz operators in terms of certain factorizations of their symbols.

The following theorem characterizes the bounded vectorial Toeplitz operators.

Theorem 4.1. *Let $\{\Omega_j\}_{j\in\mathbb{Z}}$ be a sequence of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. The block matrix* (4.2) *determines a bounded operator from*

$H^2(\mathcal{H})$ to $H^2(\mathcal{K})$ if and only if $\{\Omega_j\}_{j\in\mathbb{Z}}$ is the sequence of the Fourier coefficients of a function $\Phi \in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$. In this case

$$\|\{\Omega_{j-k}\}_{j,k\ge 0}\| = \|\Phi\|_{L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))}.$$

Note that, as usual, we identify operators from $H^2(\mathcal{H})$ to $H^2(\mathcal{K})$ with their block matrices with respect to the decomposition (4.1).

Proof. Denote by T the operator with block matrix (4.2). Suppose that T is a bounded Toeplitz operator from $H^2(\mathcal{H})$ to $H^2(\mathcal{K})$. For each $n \in \mathbb{Z}$ we consider the operator T_n from $L^2(\mathcal{H})$ to $L^2(\mathcal{K})$ defined by

$$T_n f = \bar{z}^n T\mathbb{P}_+ z^n f, \quad f \in L^2(\mathcal{H}).$$

Clearly, T_n is bounded and $\|T_n\| \le \|T\|$. If $j \ge -n$ and $k \ge -n$, then $(T_n z^k x, z^j y) = (Tz^{n+k}x, z^{n+j}y) = (\Omega_{j-k}x, y)$, $x \in \mathcal{H}$, $y \in \mathcal{K}$. Therefore the sequence $\{T_n\}_{n\ge 0}$ converges weakly to a bounded operator M from $L^2(\mathcal{H})$ to $L^2(\mathcal{K})$. Obviously, $\|M\| \le \|T\|$ and $(Mz^k x, z^j y) = (\Omega_{j-k}x, y)$ for any j, $k \in \mathbb{Z}$. It is also clear that $Mzf = zMf$, $f \in L^2$ (it suffices to verify this equality for $f = z^j x$, $j \in \mathbb{Z}$, $x \in \mathcal{H}$).

Let $x \in \mathcal{H}$ and let $\boldsymbol{x}$ be the constant $\mathcal{H}$-valued function identically equal to x. We have

$$(M\boldsymbol{x})(\zeta) = \sum_{j\in\mathbb{Z}} \zeta^j \Omega_{-j} x.$$

Clearly, $M\boldsymbol{x} \in L^2(\mathcal{K})$. We can define a measurable $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued function on $\mathbb{T}$ by

$$\Phi(\zeta)x = \sum_{\mathrm{j}\in\mathbb{Z}} \zeta^j \Omega_{-j} x, \quad \text{a.e.}$$

It is easy to see that

$$(Mf)(\zeta) = \Phi(\zeta)f(\zeta), \quad \text{a.e.}$$

for any trigonometric $\mathcal{H}$-valued polynomial f. In exactly the same way as in the proof of Theorem 2.2.2 it can easily be shown that $\Phi \in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ and $\|T\| = \|\Phi\|_{L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))}$.

The converse follows immediately from the equality $Tf = \mathbb{P}_+\Phi f$, $f \in H^2(\mathcal{H})$, (which is sufficient to verify for $f = z^n x$, $n \ge 0$, $x \in \mathcal{H}$). ∎

As in the scalar case an operator T_Φ can be compact only if $\Phi = \mathbb{O}$ (the proof is the same).

The scalar formula (1.5) is also valid in the vectorial case: if $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ are Hilbert spaces, $\Psi \in L^\infty(\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2))$, $\Phi \in L^\infty(\mathcal{B}(\mathcal{H}_2,\mathcal{H}_3))$, then

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi \tag{4.3}$$

(the proof is exactly the same). It is also easy to see that

$$T^*_\Phi = T_{\Phi^*}, \quad \Phi \in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K})).$$

It follows from (4.3) that

$$T_{\Phi\Psi} = T_\Phi T_\Psi$$

if $\Psi \in H^\infty(\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2))$ or $\Phi \in (H^\infty(\mathcal{B}(\mathcal{H}_2,\mathcal{H}_3)))^*$ (cf. Theorem 1.3). However, in contrast with the scalar case the converse does not hold. Indeed, let $\mathcal{L}$ be a nonzero subspace of $\mathcal{H}_2$ such that $\mathcal{L} \neq \mathcal{H}_2$. Suppose that $\operatorname{Range}\Psi(\zeta) \in \mathcal{L}$, a.e. on $\mathbb{T}$ and $\operatorname{Ker}\Phi(\zeta) \subset \mathcal{L}$, a.e. on $\mathbb{T}$. Then it is easy to see that

$$T_{\Phi\Psi} = T_\Phi T_\Psi = \mathbb{O},$$

while Ψ does not have to be in $H^\infty(\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2))$ as well as Φ does not have to be in $(H^\infty(\mathcal{B}(\mathcal{H}_2,\mathcal{H}_3)))^*$.

In the case when the corresponding Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are finite-dimensional we can identify $\mathcal{H}$ with $\mathbb{C}^n$, $\mathcal{K}$ with $\mathbb{C}^m$, and $\mathcal{B}(\mathcal{H},\mathcal{K})$ with the *space* $\mathbb{M}_{m,n}$ *of* $m \times n$ *matrices.* We shall use the notation $\mathbb{M}_n$ for $\mathbb{M}_{n,n}$.

Note that the study of spectral problems for Toeplitz operators on spaces of vector functions is much harder than in the scalar case. In particular fundamental Theorem 1.4 does not generalize to the Toeplitz operators with matricial symbols. Indeed, let $m = n = 2$. Consider the matrix function $\Phi \in L^\infty(\mathbb{M}_2)$ defined by

$$\Phi(\zeta) = \begin{pmatrix} \zeta & 0 \\ 0 & \bar{\zeta} \end{pmatrix}, \quad \zeta \in \mathbb{T}.$$

It is easy to see that T_Φ is Fredholm,

$$\operatorname{Ker} T_\Phi = \operatorname{span}\left\{\begin{pmatrix} \mathbb{O} \\ \mathbf{1} \end{pmatrix}\right\}, \quad \operatorname{Ker} T_\Phi^* = \operatorname{span}\left\{\begin{pmatrix} \mathbf{1} \\ \mathbb{O} \end{pmatrix}\right\}$$

(recall that $\mathbf{1}$ is the function identically equal to 1), and so $\operatorname{ind} T_\Phi = 0$ though T_Φ is noninvertible.

Let us obtain some general results.

Theorem 4.2. *Let $\mathcal{H}$ be a Hilbert space and let $\Phi \in L^\infty(\mathcal{B}(\mathcal{H}))$. Suppose that T_Φ is Fredholm. Then $\Phi(\zeta)$ is invertible almost everywhere and $\Phi^{-1} \in L^\infty(\mathcal{B}(\mathcal{H}))$.*

Proof. Assume first that T_Φ is left invertible. Then there exists $\varepsilon > 0$ such that

$$\varepsilon\|f\|_{H^2(\mathcal{H})} \le \|T_\Phi f\|_{H^2(\mathcal{H})} \le \|\Phi f\|_{H^2(\mathcal{H})}, \quad f \in H^2(\mathcal{H}).$$

Therefore

$$\varepsilon\|\bar{z}^n f\|_{L^2(\mathcal{H})} \le \|\Phi \bar{z}^n f\|_{L^2(\mathcal{H})}.$$

Since the set $\{\bar{z}^n f :\ f \in H^2(\mathcal{H}),\ n \ge 0\}$ is dense in $L^2(\mathcal{H})$, it follows that

$$\varepsilon\|g\|_{L^2(\mathcal{H})} \le \|\Phi g\|_{L^2(\mathcal{H})}, \quad g \in L^2(\mathcal{H}). \tag{4.4}$$

Let us show that $\|\Phi(\zeta)x\|_{\mathcal{H}} \ge \varepsilon\|x\|_{\mathcal{H}}$, $x \in \mathcal{H}$, for almost all $\zeta \in \mathbb{T}$. Indeed, otherwise there are a positive δ, $\delta < \varepsilon$, and a measurable subset Δ of $\mathbb{T}$ of positive measure such that for each $\zeta \in \Delta$ there exists a nonzero $x \in \mathcal{H}$ for which $\|\Phi(\zeta)x\|_{\mathcal{H}} \le \delta\|x\|_{\mathcal{H}}$. Let χ be the characteristic function of $[0,\delta^2]$. Clearly, $\chi((\Phi(\zeta))^*\Phi(\zeta)) \neq \mathbb{O}$ for $\zeta \in \Delta$. Denote by M the operator of multiplication by $\Phi^*\Phi$ on $L^2(\mathcal{H})$. It is easy to see that $\chi(M) \neq \mathbb{O}$. Clearly, for $f \in \operatorname{Range}\chi(M)$ we have $\|\Phi f\|_{L^2(\mathcal{H})} \le \delta\|f\|_{L^2(\mathcal{H})}$, which contradicts (4.4).

Let us get rid of the assumption that T_Φ is left invertible. Since T_Φ is left invertible on a subspace of finite codimension, it follows that T_Φ is left invertible on $z^n H^2(\mathcal{H})$ for some $n \ge 0$, which is the same as saying that $T_{z^n\Phi}$ is left invertible on $H^2(\mathcal{H})$. Therefore $\|\zeta^n\Phi(\zeta)x\|_{\mathcal{H}} \ge \varepsilon\|x\|_{\mathcal{H}}$, $x \in \mathcal{H}$, a.e. on $\mathbb{T}$, which is equivalent to the inequality $\|\Phi(\zeta)x\|_{\mathcal{H}} \ge \varepsilon\|x\|_{\mathcal{H}}$, $x \in \mathcal{H}$, a.e. on $\mathbb{T}$.

The same reasoning applied to Φ^* yields $\|\Phi^*(\zeta)x\|_{\mathcal{H}} \ge \varkappa\|x\|_{\mathcal{H}}$, $x \in \mathcal{H}$, a.e. on $\mathbb{T}$ for some $\varkappa > 0$. Consequently, $\Phi(\zeta)$ is invertible almost everywhere on $\mathbb{T}$ and $\Phi^{-1} \in L^\infty(\mathcal{B}(\mathcal{H}))$. ■

Note that in the finite-dimensional case for $\Phi \in L^\infty(\mathbb{M}_n)$ the condition $\Phi^{-1} \in L^\infty(\mathbb{M}_n)$ is equivalent to the condition $(\det\Phi)^{-1} \in L^\infty$.

As in the scalar case the Toeplitz operators with unitary-valued symbols play a special role as one can see from the following lemma.

Lemma 4.3. *Let $\mathcal{H}$ be a Hilbert space and let Φ be a function in $L^\infty(\mathcal{B}(\mathcal{H}))$ such that $\Phi^{-1} \in L^\infty(\mathcal{B}(\mathcal{H}))$. Then Φ admits a representation $\Phi = U\Psi$, where U is a unitary-valued function in $L^\infty(\mathcal{B}(\mathcal{H}))$ and Ψ is an invertible function in $H^\infty(\mathcal{B}(\mathcal{H}))$.*

Proof. Consider the subspace $\mathcal{L}$ of $L^2(\mathcal{H})$ defined by

$$\mathcal{L} \stackrel{\text{def}}{=} \{\Phi g : \ g \in H^2(\mathcal{H})\}.$$

It is easy to see that $\mathcal{L}$ is a closed subspace of $L^2(\mathcal{H})$ invariant under multiplication by z. Let us show that it is completely nonreducing (see Appendix 2.3). Indeed, suppose that f is a nonzero function and $\bar{z}^n f \in \mathcal{L}$ for $n \ge 0$. Then $f = \Phi h$ for $h \in H^2(\mathcal{H})$. We can choose $m > 0$ such that $\bar{z}^m h \notin H^2(\mathcal{H})$. Then $\bar{z}^m f \notin \mathcal{L}$, which contradicts our assumption.

Then $\mathcal{L}$ has the form $\mathcal{L} = UH^2(\mathcal{K})$, where $\mathcal{K}$ is a Hilbert space and U is a function in $L^\infty(\mathcal{B}(\mathcal{K},\mathcal{H}))$ such that $U(\zeta)$ is isometric for almost all $\zeta \in \mathbb{T}$ (see Appendix 2.3). Since Φ is invertible in $L^\infty(\mathcal{B}(\mathcal{H}))$, we have $H^2(\mathcal{H}) = \Phi^{-1}UH^2(\mathcal{K})$ and it is easy to see that $U(\zeta)$ maps $\mathcal{K}$ onto $\mathcal{H}$ for almost all $\zeta \in \mathbb{T}$. Therefore we can identify $\mathcal{K}$ with $\mathcal{H}$ after which U becomes a unitary-valued function in $L^\infty(\mathcal{B}(\mathcal{H}))$.

Put now $\Psi \stackrel{\text{def}}{=} U^*\Phi$. Clearly, $\Psi H^2(\mathcal{H}) = H^2(\mathcal{H})$, and so $\Psi \in H^\infty(\mathcal{B}(\mathcal{H}))$. The invertibility of Ψ in $H^\infty(\mathcal{B}(\mathcal{H}))$ is obvious. ■

If Φ satisfies the conclusion of Lemma 4.3, then $T_\Phi = T_U T_\Psi$, and so T_Φ is invertible (Fredholm) if and only if T_U is.

As in the scalar case the following criterion holds.

Theorem 4.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let U be a function in $L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ such that $U(\zeta)$ is isometric for almost all $\zeta \in \mathbb{T}$. Then T_U is a left invertible operator from $H^2(\mathcal{H})$ to $H^2(\mathcal{K})$ if and only if*

$$\operatorname{dist}_{L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))}\{U, H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))\} < 1.$$

Proof. As in the scalar case we have

$$\|Uf\|^2_{L^2(\mathcal{K})} = \|T_U f\|^2_{H^2(\mathcal{H})} + \|H_U f\|^2_{H^2_-(\mathcal{H})}, \quad f \in H^2(\mathcal{H}).$$

It follows that T_U is left invertible if and only if $\|H_U\| < 1$. The result now follows from Theorem 2.2.2. ■

Corollary 4.5. *Let $\mathcal{H}$ be a Hilbert space and let U be a unitary-valued function in $L^\infty(\mathcal{B}(\mathcal{H}))$. Then T_U is invertible if and only if*

$$\operatorname{dist}_{L^\infty(\mathcal{B}(\mathcal{H}))}\{U, H^\infty(\mathcal{B}(\mathcal{H}))\} < 1 \quad \text{and} \quad \operatorname{dist}_{L^\infty(\mathcal{B}(\mathcal{H}))}\{U^*, H^\infty(\mathcal{B}(\mathcal{H}))\} < 1.$$

As in §1 we can obtain now a Fredholmness criterion for Toeplitz operators with matricial unitary-valued symbols.

Theorem 4.6. *Let n be a positive integer and let U be a unitary-valued function in $L^\infty(\mathbb{M}_n)$. The operator T_U is Fredholm if and only if $\|H_U\|_{\mathrm{e}} < 1$ and $\|H_{U^*}\|_{\mathrm{e}} < 1$.*

The proof of Theorem 4.6 is exactly the same as the proof of Theorem 1.15. As in Theorem 1.15 one can also state necessary and sufficient conditions for left Fredholmness and right Fredholmness.

To conclude this section we obtain a Fredholmness criterion for Toeplitz operators with symbols in $(H^\infty + C)(\mathbb{M}_n)$ and compute their index.

Throughout the book we use the following convention. If X is a space of functions, we denote by $X(\mathbb{C}^n)$ the space of $n \times 1$ column functions whose entries belong to X. Sometimes we shall use the notation $\{f_j\}_{1 \le j \le n}$ for the column function with entries f_j, $1 \le j \le n$. Similarly, we denote by $X(\mathbb{M}_{n,m})$ (or $X(\mathbb{M}_n)$ if $m = n$) the space of $m \times n$ matrix functions with entries in X. If it does not lead to a confusion, we can write $f \in X$ instead of $f \in X(\mathbb{C}^n)$ or $\Phi \in X$ instead of $\Phi \in X(\mathbb{M}_{m,n})$.

The following theorem characterizes the Fredholm Toeplitz operators with symbols in $H^\infty + C$.

Theorem 4.7. *Let n be a positive integer and let $\Phi \in (H^\infty + C)(\mathbb{M}_n)$. The operator T_Φ is Fredholm if and only if* $\det \Phi$ *is invertible in $H^\infty + C$.*

Proof. Suppose that $\det \Phi$ is invertible in $H^\infty + C$ and $\Psi = \Phi^{-1}$. Then $\Psi \in H^\infty + C$. By (4.3),

$$I - T_\Phi T_\Psi = T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi, \quad I - T_\Psi T_\Phi = T_{\Psi\Phi} - T_\Psi T_\Phi = H^*_{\Psi^*} H_\Phi,$$

and so it follows from the Hartman theorem that both $I - T_\Phi T_\Psi$ and $I - T_\Psi T_\Phi$ are compact, which means that T_Φ is Fredholm.

Suppose now that T_Φ is Fredholm. It follows that T_Φ is left invertible on $z^m H^2(\mathbb{C}^n)$ for some $m \ge 0$, which means that the operator $T_{z^m\Phi}$ is left invertible. Clearly, it is sufficient to show that $z^m\Phi$ is invertible in $H^\infty + C$. Without loss of generality we may assume that $m = 0$.

By Theorem 4.2, $\Phi^{-1} \in L^\infty$, and so by Lemma 4.3, Φ admits a representation $\Phi = U\Psi$, where U is a unitary-valued matrix function and Ψ is invertible in H^∞. We have $T_\Phi = T_U T_\Psi$. Since T_Ψ is invertible, it follows that T_U is left invertible and it is sufficient to show that U is invertible in $H^\infty + C$.

By Theorem 4.4, there exists a function Ξ in $H^\infty(\mathbb{M}_n)$ such that $\|U - \Xi\|_{L^\infty} < 1$. Hence, $\|I - T_{U^*\Xi}\| < 1$. It follows that $T_{U^*\Xi} = T_U^* T_\Xi$ is invertible. Since T_U is Fredholm, it follows that T_Ξ is Fredholm. By Theorem 4.2, Ξ is invertible in L^∞, and so by Lemma 4.3, Ξ admits a representation $\Xi = \Theta G$, where Θ is a unitary-valued inner function and G is invertible in $H^\infty(\mathbb{M}_n)$.

It remains to show that Θ is invertible in $H^\infty + C$. Since G is invertible in H^∞, it follows that T_G is invertible, and so $T_\Theta = T_\Psi T_G^{-1}$ is Fredholm. Hence, the subspace $\Theta H^2(\mathbb{C}^n)$ of $H^2(\mathbb{C}^n)$ has finite codimension. By Lemma 2.5.1, Θ is a Blaschke–Potapov product of finite degree that implies that Θ is invertible in $H^\infty + C$. ■

The following theorem computes the index of a Fredholm Toeplitz operator with symbol in $(H^\infty + C)(\mathbb{M}_n)$ (see §3.3 for the definition of the winding number for scalar functions invertible in $H^\infty + C$).

Theorem 4.8. *Let Φ be a function in $H^\infty + C(\mathbb{M}_n)$ such that T_Φ is Fredholm. Then*

$$\operatorname{ind} T_\Phi = -\operatorname{wind}\det\Phi.$$

Proof. We have

$$\lim_{m\to\infty} \operatorname{dist}_{L^\infty(\mathbb{M}_n)}\{\Phi, \bar{z}^m H^\infty(\mathbb{M}_n)\} = 0,$$

and so for any $\varepsilon > 0$ there exists $F \in H^\infty(\mathbb{M}_n)$ such that $\|\Phi - \bar{z}^m F\|_\infty < \varepsilon$. Clearly, if ε is sufficiently small, then $T_{\bar{z}^m F}$ is Fredholm and $\operatorname{ind} T_\Phi = \operatorname{ind} T_{\bar{z}^m F}$. It is also clear that for a sufficiently small ε we have

$$\operatorname{wind}\det\Phi = \operatorname{wind}\det\bar{z}^m F.$$

Hence, it is sufficient to show that $\operatorname{ind} T_{\bar{z}^m F} = -\operatorname{wind}\det\bar{z}^m F$. Since $T_{\bar{z}^m F} = T_{\bar{z}^m I_n} T_F$, we have

$$\operatorname{ind} T_{\bar{z}^m F} = \operatorname{ind} T_{\bar{z}^m I_n} + \operatorname{ind} T_F = \operatorname{ind} T_F + mn$$

(recall that I_n stands for the $n \times n$ identity matrix). It is also easy to see that

$$\operatorname{wind}\det\bar{z}^m F = \operatorname{wind}\det F - mn.$$

Therefore it is sufficient to show that $\operatorname{ind} T_F = -\operatorname{wind}\det F$.

Since F is invertible in L^∞, it follows from Lemma 4.3 that $F = \Theta G$, where Θ is an inner function in $H^\infty(\mathbb{M}_n)$ and G is invertible in $H^\infty(\mathbb{M}_n)$. Clearly, T_G is invertible, $\operatorname{wind}\det G = 0$ (follows from Lemma 3.7). Hence, it suffices to show that $\operatorname{ind} T_\Theta = -\operatorname{wind}\det\Theta$.

Since T_Θ is Fredholm, it follows from Lemma 2.5.1 that Θ is a Blaschke–Potapov product of finite degree. Clearly, it is sufficient to consider the case when Θ has degree 1, i.e.,

$$\Theta(\zeta) = \frac{\lambda - \zeta}{1 - \bar{\lambda}\zeta} P + (I - P),$$

where $\lambda \in \mathbb{D}$ and P is an orthogonal projection on $\mathbb{C}^n$ of rank 1. Then by Lemma 2.5.1, $\operatorname{ind} T_\Theta = -1$. On the other hand, it is easy to see that

$$\det \Theta(\zeta) = \frac{\lambda - \zeta}{1 - \bar{\lambda}\zeta},$$

and so $\operatorname{wind} \det \Theta = 1$. ■

5. Wiener–Hopf Factorizations of Symbols of Fredholm Toeplitz Operators

In this section we show that the symbol of a Fredholm Toeplitz operator on $H^2(\mathbb{C}^n)$ admits a certain factorization (Wiener–Hopf factorization). This allows us to obtain a general Fredholmness criterion. With a Wiener–Hopf factorization we associate factorization indices. We obtain formulas for the dimension of the kernel of a Fredholm Toeplitz operators and for its index in terms of the factorization indices of its symbol. We also show that if the indices are arranged in the nondecreasing order, they are uniquely determined by the symbol.

Let us first characterize the invertible Toeplitz operators.

Theorem 5.1. *Let Φ be an $n\times n$ matrix function in L^∞. The following are equivalent:*

(i) *the Toeplitz operator T_Φ is invertible;*

(ii) *Φ admits a factorization $\Phi = \overline{\Psi_1}\Psi_2$, where Ψ_1 and Ψ_2 are $n \times n$ matrix functions such that both $\Psi_1(\zeta)$ and $\Psi_2(\zeta)$ are invertible for almost all $\zeta \in \mathbb{T}$; Ψ_1, Ψ_2, Ψ_1^{-1}, Ψ_2^{-1} belong to $H^2(\mathbb{M}_n)$, and the operator R defined on the set of polynomials in $H^2(\mathbb{C}^n)$ by*

$$Rf = \Psi_2^{-1}\mathbb{P}_+\overline{\Psi_1}^{-1} f \tag{5.1}$$

extends to a bounded operator on $H^2(\mathbb{C}^n)$.

Note that $\mathbb{P}_+\overline{\Psi}_1^{-1} f$ is a polynomial for any polynomial f and so R maps polynomials in $H^2(\mathbb{C}^n)$ into $H^2(\mathbb{C}^n)$.

Proof. Suppose that T_Φ is invertible. Then there exist functions $f_j \in H^2(\mathbb{C}^n)$ and $g_j \in H^2(\mathbb{C}^n)$, $1 \le j \le n$, such that

$$T_\Phi f_j = T_{\Phi^*} g_j = \begin{pmatrix} \mathbb{O} \\ \vdots \\ \mathbf{1} \\ \vdots \\ \mathbb{O} \end{pmatrix}$$

($\mathbf{1}$ is in the jth row). Consider the $n \times n$ matrix functions

$$F = \begin{pmatrix} f_1 & \cdots & f_n \end{pmatrix}, \quad G = \begin{pmatrix} g_1 & \cdots & g_n \end{pmatrix}.$$

Clearly,

$$(\mathbb{P}_+\Phi F)(\zeta) = (\mathbb{P}_+\Phi^* G)(\zeta) = I, \quad \text{a.e. on } \mathbb{T}.$$

It follows that $\overline{\Phi F} \in H^2$ and $\overline{\Phi^* G} \in H^2$.

Put $\Xi = G^*\Phi F$. Then $\Xi = G^*(\Phi F) \in \overline{H^1}$ since both G^* and ΦF are in $\overline{H^2}$. On the other hand, $\Xi = (G^*\Phi)F \in H^1$ since both $G^*\Phi = (\overline{\Phi^* G})^{\mathrm{t}}$ and F are in H^2. Therefore Ξ is a constant function.

Assume that $\det \Xi = 0$. We have $\det \Xi = \overline{\det G(\zeta)} \cdot \overline{\det \overline{\Phi(\zeta) F(\zeta)}}$ for almost all $\zeta \in \mathbb{T}$. Since the columns $g_1(\zeta), \cdots, g_n(\zeta)$ of $G(\zeta)$ are linearly independent a.e. on $\mathbb{T}$ as well as the columns $f_1(\zeta) \cdots, f_n(\zeta)$ of $F(\zeta)$, it follows that $\det \Phi(\zeta) = 0$ a.e. on $\mathbb{T}$. However, this contradicts the invertibility of T_Φ (see Theorem 4.2). Hence, $\det \Xi \neq 0$, and so Ξ is invertible.

Put $\Psi_1 \stackrel{\text{def}}{=} \overline{\Phi F}$, $\Psi_2 \stackrel{\text{def}}{=} F^{-1}$. Clearly, $\overline{\Psi_1}\Psi_2 = \Phi F F^{-1} = \Phi$. We have already shown that $\Psi_1 \in H^2$. We have

$$\Psi_1^{-1} = (\overline{\Phi F})^{-1} = \overline{\Xi^{-1} G^*} \in H^2.$$

Obviously, $\Psi_2^{-1} = F \in H^2$. We have

$$\Psi_2 = F^{-1} = \Xi^{-1} G^* \Phi = \Xi^{-1}(\Phi^* G)^* \in H^2.$$

Suppose now that Ψ_1, Ψ_2, Ψ_1^{-1}, $\Psi_2^{-1} \in H^2$ and $\Phi = \overline{\Psi_1}\Psi_2$. It remains to show that T_Φ is invertible if and only if the operator R defined by (5.1) is bounded on $H^2(\mathbb{C}^n)$.

Suppose that R is bounded. In this case it is easy to see that $\Psi_2 R f = \mathbb{P}_+ \overline{\Psi_1}^{-1} f$ for an arbitrary $f \in H^2(\mathbb{C}^n)$ and so $\mathbb{P}_+ \overline{\Psi_1}^{-1} f \in H^1(\mathbb{C}^n)$ for any $f \in H^2(\mathbb{C}^n)$. Hence, $Rf = \Psi_2^{-1} \mathbb{P}_+ \overline{\Psi_1}^{-1} f$ for any $f \in H^2(\mathbb{C}^n)$.

Let us show that T_Φ is invertible and $T_\Phi^{-1} = R$. We have for a polynomial f in $H^2(\mathbb{C}^n)$

$$\begin{aligned} T_\Phi R f &= \mathbb{P}_+ \overline{\Psi_1} \Psi_2 \Psi_2^{-1} \mathbb{P}_+ \overline{\Psi_1}^{-1} f = \mathbb{P}_+ \overline{\Psi_1} \mathbb{P}_+ \overline{\Psi_1}^{-1} f \\ &= \mathbb{P}_+ \overline{\Psi_1}\, \overline{\Psi_1}^{-1} f - \mathbb{P}_+ \overline{\Psi_1} \mathbb{P}_- \overline{\Psi_1}^{-1} f = f, \end{aligned}$$

since obviously, $\mathbb{P}_+ \overline{\Psi_1} \mathbb{P}_- \overline{\Psi_1}^{-1} f = \mathbb{O}$. On the other hand,

$$R T_\Phi f = \Psi_2^{-1} \mathbb{P}_+ \overline{\Psi_1}^{-1} \mathbb{P}_+ \overline{\Psi_1} \Psi_2 f = \Psi_2^{-1} \mathbb{P}_+ \Psi_2 f = f,$$

and so T_Φ is invertible and $T_\Phi^{-1} = R$.

Suppose now that T_Φ is invertible. We have

$$f = T_\Phi T_\Phi^{-1} f = \mathbb{P}_+ \overline{\Psi}_1 \Psi_2 T_\Phi^{-1} f = \mathbb{P}_+ \overline{\Psi}_1 \mathbb{P}_+ \Psi_2 T_\Phi^{-1} f.$$

Multiplying by $\overline{\Psi}_1^{-1}$ on the left and applying $\mathbb{P}_+$, we obtain

$$\mathbb{P}_+ \overline{\Psi}_1^{-1} f = \mathbb{P}_+ \overline{\Psi}_1^{-1} \mathbb{P}_+ \overline{\Psi}_1 \mathbb{P}_+ \Psi_2 T_\Phi^{-1} f \in (\mathbb{P}_+ L^1)(\mathbb{C}^n).$$

Therefore

$$\begin{aligned} \mathbb{P}_+ \overline{\Psi}_1^{-1} f &= \mathbb{P}_+ \overline{\Psi}_1^{-1} \overline{\Psi}_1 \mathbb{P}_+ \Psi_2 T_\Phi^{-1} f - \mathbb{P}_+ \overline{\Psi}_1^{-1} \mathbb{P}_- \overline{\Psi}_1 \mathbb{P}_+ \Psi_2 T_\Phi^{-1} f \\ &= \mathbb{P}_+ \overline{\Psi}_1^{-1} \overline{\Psi}_1 \mathbb{P}_+ \Psi_2 T_\Phi^{-1} f = \Psi_2 T_\Phi^{-1} f, \end{aligned}$$

since, obviously, $\mathbb{P}_+\overline{\Psi}_1^{-1}\mathbb{P}_-\overline{\Psi}_1\mathbb{P}_+\Psi_2 T_\Phi^{-1} f = \mathbb{O}$. It follows that

$$T_\Phi^{-1} f = \Psi_2^{-1}\mathbb{P}_+\overline{\Psi}_1^{-1} f, \quad f \in H^2(\mathbb{C}^n).$$

Hence, $\Psi_2^{-1}\mathbb{P}_+\overline{\Psi}_1^{-1} f \in H^2(\mathbb{C}^n)$ for any $f \in H^2(\mathbb{C}^n)$ and $R = T_\Phi^{-1}$ is a bounded operator on $H^2(\mathbb{C}^n)$. ■

Let us now characterize the Fredholm Toeplitz operators in terms of factorizations of their symbols.

Theorem 5.2. *Let Φ be an $n\times n$ matrix function in L^∞. The following are equivalent:*

(i) *the Toeplitz operator T_Φ is Fredholm;*

(ii) *Φ admits a factorization $\Phi = \overline{\Psi_1}\Lambda\Psi_2$, where Ψ_1 and Ψ_2 are $n\times n$ matrix functions such that both $\Psi_1(\zeta)$ and $\Psi_2(\zeta)$ are invertible for almost all $\zeta \in \mathbb{T}$; Ψ_1, Ψ_2, Ψ_1^{-1}, $\Psi_2^{-1} \in H^2(\mathbb{M}_{n,n})$; Λ is a diagonal matrix function of the form*

$$\Lambda = \begin{pmatrix} z^{k_1} & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & z^{k_2} & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & z^{k_n} \end{pmatrix}, \quad k_1, \cdots, k_n \in \mathbb{Z}; \tag{5.2}$$

and the operator B defined on the set of polynomials in $H^2(\mathbb{C}^n)$ by

$$Bf = \Psi_2^{-1}\mathbb{P}_+\overline{\Lambda}\mathbb{P}_+\overline{\Psi_1}^{-1} f \tag{5.3}$$

extends to a bounded operator on $H^2(\mathbb{C}^n)$.

Factorizations of the form $\Phi = \overline{\Psi_1}\Lambda\Psi_2$, where Ψ_1 and Ψ_2 are matrix functions invertible in $H^2(\mathbb{M}_{n,n})$ and Λ is of the form (5.2), are called *Wiener–Hopf factorizations* of Φ. With such a factorization we associate *Wiener–Hopf factorization indices* (or simply factorization indices) $k_1, k_2, \cdots, k_n$.

It is easy to see that if Φ has a Wiener–Hopf factorization, we can always find such a factorization that $k_1 \le k_2 \le \cdots \le k_n$. We show in this section that if the factorization indices are arranged in the nondecreasing order, they are uniquely determined by the function Φ itself.

To prove Theorem 5.2 we start with the following lemma.

Lemma 5.3. *Let L be a finite-dimensional subspace of $H^2(\mathbb{C}^n)$. Then there exist a closed subspace $L_\#$ of $H^2(\mathbb{C}^n)$ and analytic polynomials $p_1, \cdots, p_n$ with zeros in $\mathbb{D}$ such that $L \cap L_\# = \{\mathbb{O}\}$, $L + L_\# = H^2(\mathbb{C}^n)$, and multiplication by the matrix function P,*

$$P = \begin{pmatrix} p_1 & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & p_2 & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & p_n \end{pmatrix},$$

maps $H^2(\mathbb{C}^n)$ isomorphically onto $L_\#$.

Proof. Let $k = \dim L$. We argue by induction. Let us construct subspaces $L_0, L_1, \cdots, L_k = L_\#$ satisfying the following properties:

(1) $H^2(\mathbb{C}^n) = L_j + L$, $0 \le j \le k$;

(2) $\dim L_j \cap L = k - j$;

(3) $L_j = \{P_j f : f \in H^2(\mathbb{C}^n)\}$, where P_j is a diagonal matrix function of the form

$$P_j = \begin{pmatrix} p_{1,j} & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & p_{2,j} & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & p_{n,j} \end{pmatrix},$$

and $p_{1,j}, \cdots, p_{n,j}$ are analytic polynomials with zeros in $\mathbb{D}$.

Put $L_0 = H^2(\mathbb{C}^n)$. Suppose that we have already constructed L_j, $j < k$. Let us construct L_{j+1}. Let $g = \{g_r\}_{1 \le r \le n}$ be a nonzero element in $L_j \cap L$. Then there exist m, $1 \le m \le n$, and $\zeta_0 \in \mathbb{D}$ such that $g_m(\zeta_0) \ne 0$. Put $L_{j+1} \stackrel{\text{def}}{=} \{P_{j+1} f : f \in H^2(\mathbb{C}^n)\}$, where

$$P_{j+1} \stackrel{\text{def}}{=} \begin{pmatrix} p_{1,j} & \mathbb{O} & \cdots & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & p_{2,j} & \cdots & \mathbb{O} & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & (z-\zeta_0)p_{m,j} & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \cdots & p_{n,j} \end{pmatrix}.$$

Clearly, $L_{j+1} \subset L_j$. Let us show that $L_j = L_{j+1} + \operatorname{span}\{g\}$. Indeed, let $g = P_j \breve{g}$, $\breve{g} \in H^2(\mathbb{C}^n)$. Let $f = P_j \breve{f}$, $\breve{f} \in H^2(\mathbb{C}^n)$, be an arbitrary function in L_j. We have $f = P_j(\breve{f} + \lambda\breve{g} - \lambda\breve{g}) = P_j(\breve{f} + \lambda\breve{g}) - \lambda g$, $\lambda \in \mathbb{C}$. We can now choose λ so that the mth component of $(\breve{f} + \lambda\breve{g})(\zeta_0)$ is zero. Clearly, $P_j(\breve{f} + \lambda\breve{g}) \in L_{j+1}$, while $\lambda g \in \operatorname{span}\{g\}$. It follows that $\dim L_{j+1} \cap L = k - j - 1$.

We have

$$L_{j+1} + L = L_{j+1} + (\operatorname{span}\{g\} + L) = (L_{j+1} + \operatorname{span}\{g\}) + L = L_j + L = L$$

by the inductive hypothesis.

Doing it this way, we can construct L_k. It remains to put $L_\# \stackrel{\text{def}}{=} L_k$. ■

We have to introduce several classes of functions.

We denote by $\mathcal{R}$ the algebra of rational matrix functions $\{f_{jk}\}_{1 \le j,k \le n}$ such that the poles of the f_{jk} are outside $\mathbb{T}$. We denote by $\mathcal{R}^{-1}$ the set of invertible elements of $\mathcal{R}$.

We also consider the class $\mathcal{R} + H^2$ consisting of $n \times n$ matrix functions F that can be represented as $F = G_1 + G_2$, where $G_1 \in \mathcal{R}$ and $G_2 \in H^2(\mathbb{M}_n)$. It is easy to see that $R \cdot H^2 \subset \mathcal{R} + H^2$, i.e., $FG \in \mathcal{R} + H^2$ for any $F \in \mathcal{R}$ and $G \in H^2$. We denote by $(\mathcal{R} + H^2)^{-1}$ the set of functions invertible in $\mathcal{R} + H^2$.

Finally we need the class $\mathcal{R}\cap H^2$, which is a subalgebra of $\mathcal{R}$ that consists of matrix functions in $\mathcal{R}$ with poles outside $\operatorname{clos}\mathbb{D}$. We denote by $(\mathcal{R}\cap H^2)^{-1}$ the set of invertible elements in $\mathcal{R}\cap H^2$.

To prove Theorem 5.2 we need the following result.

Theorem 5.4. *Let Ψ be a function of class $(\mathcal{R}+H^2)^{-1}$. Then Ψ admits a representation $\Psi=\Xi_1\Lambda\overline{\Xi_2}$, where Ξ_1, Ξ_2, Λ are matrix functions such that Ξ_1, $\Xi_1^{-1}\in H^2$, Ξ_2, $\Xi_2^{-1}\in\mathcal{R}\cap H^2$, and Λ is a diagonal matrix function of the form* (5.2). *If $\Psi\in\mathcal{R}^{-1}$, then we can factorize Ψ so that Ξ_1, $\Xi_1^{-1}\in\mathcal{R}\cap H^2$.*

The proof of Theorem 5.4 is based on the following lemma.

Lemma 5.5. *Let F be an $n\times n$ matrix function in H^2 such that $\det F$ is not identically equal to 0 and $\det F(\zeta_0)=0$ for some $\zeta_0\in\mathbb{D}$. Then F admits a representation $F=F_1W$, where $F_1\in H^2$, $\det F_1(\zeta)=(\zeta-\zeta_0)^{-1}\det F(\zeta)$, and W is a matrix function of the form*

$$W(\zeta)=\begin{pmatrix} 1 & 0 & \cdots & 0 & \xi_1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0 & \xi_2 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1 & \xi_{k-1} & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & \zeta-\zeta_0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{5.4}$$

for some $k\in\mathbb{Z}$, $1\le k\le n$, and $\xi_1,\cdots,\xi_{k-1}\in\mathbb{C}$.

Proof of Lemma 5.5. Let $f_1,\cdots,f_n\in H^2(\mathbb{C}^n)$ be the columns of F. Since $\det\Phi(\zeta_0)=0$, there exist $k\le n$ and complex numbers $\xi_1,\cdots,\xi_{k-1}$ such that

$$f_k(\zeta_0)=\xi_1f_1(\zeta_0)+\cdots+\xi_{k-1}f_{k-1}(\zeta_0). \tag{5.5}$$

Consider the matrix function W defined by (5.4). It is easy to see that for $\zeta\ne\zeta_0$ the matrix $W(\zeta)$ is invertible and

$$W^{-1}(\zeta)=\begin{pmatrix} 1 & 0 & \cdots & 0 & \frac{-\xi_1}{\zeta-\zeta_0} & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0 & \frac{-\xi_2}{\zeta-\zeta_0} & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1 & \frac{-\xi_{k-1}}{\zeta-\zeta_0} & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & \frac{1}{\zeta-\zeta_0} & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Put $F_1(\zeta) \stackrel{\text{def}}{=} F(\zeta)W^{-1}(\zeta)$, $\zeta \in \mathbb{T}$. Clearly, $\det F_1(\zeta) = (\zeta - \zeta_0)^{-1} \det F(\zeta)$. Let us show that $F_1 \in H^2$. It suffices to show that the kth column of F_1 belongs to $H^2(\mathbb{C}^n)$. It is easy to see that the kth column of F_1 is equal to

$$\frac{1}{z - \zeta_0}(-\xi_1 f_1 - \cdots - \xi_{k-1} f_{k-1} + f_k) \in H^2(\mathbb{C}^n)$$

by (5.5). ■

Proof of Theorem 5.4. Obviously, there exists a scalar analytic polynomial p with zeros in $\mathbb{D}$ such that $P\Psi \in H^2$, where

$$P = \begin{pmatrix} p & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & p & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & p \end{pmatrix}.$$

Let $m \stackrel{\text{def}}{=} \deg p$. We have

$$\Psi = (P\Psi) \begin{pmatrix} \bar{z}^m & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & \bar{z}^m & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & \bar{z}^m \end{pmatrix} (z^m P^{-1}). \tag{5.6}$$

Clearly, $P\Psi \in H^2$, $(P\Psi)^{-1} \in \mathcal{R} + H^2$, and $(z^m P^{-1})^{\pm 1} \in \mathcal{R} + H^2$. Since $\Psi \in (\mathcal{R} + H^2)^{-1}$, it follows that $\det P\Psi$ has finitely many zeros in $\mathbb{D}$. Let d be the number of zeros of $\det P\Psi$ in $\mathbb{D}$ (counted with multiplicities).

For $j = 0, \cdots, d$ we construct by induction factorizations

$$\Psi = \Xi_{1,j} \Lambda_j \overline{\Xi}_{2,j}, \tag{5.7}$$

where $\Xi_{2,j} \in (\mathcal{R} \cap H^2)^{-1}$, Λ_j is a diagonal matrix function of the form (5.2), $\Xi_{1,j} \in H^2$, $\Xi_{1,j}^{-1} \in \mathcal{R} + H^2$, and $\det \Xi_{1,j}$ has $d - j$ zeros in $\mathbb{D}$. After we do that we simply put $\Xi_1 = \Xi_{1,d}$, $\Lambda = \Lambda_d$, and $\Xi_2 = \Xi_{2,d}$.

For $j = 0$ we take representation (5.6). Let $0 \le j < d$ and suppose that Ψ admits a representation of the form (5.7) satisfying the desired properties. Let

$$\Lambda_j = \begin{pmatrix} z^{s_1} & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & z^{s_2} & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & z^{s_n} \end{pmatrix}.$$

Without loss of generality we may assume that $s_1 \ge s_2 \ge \cdots \ge s_n$ (otherwise we can change the order of the basis vectors in $\mathbb{C}^n$ and write matrix representations in the new basis).

Let us apply Lemma 5.5 to $F = \Xi_{1,j}$ and obtain a representation $\Xi_{1,j} = \Xi_{1,j+1} W$, where W is given by (5.4), $\Xi_{1,j+1} \in H^2$, $\Xi_{1,j+1}^{-1} \in \mathcal{R} + H^2$, and $\det \Xi_{1,j+1}$ has $d - j - 1$ zeros in $\mathbb{D}$.

We have $\Psi = \Xi_{1,j+1} W \Lambda_j \overline{\Xi}_{2,j}$. It is easy to see that $W\Lambda_j = W_\# \Lambda_{j+1}$, where

$$W_\#(\zeta) \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & \cdots & 0 & \xi_1\bar{\zeta} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \xi_2\bar{\zeta} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \xi_{k-1}\bar{\zeta} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1-\zeta_0\bar{\zeta} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \zeta \in \mathbb{T},$$

and

$$\Lambda_{j+1} \stackrel{\text{def}}{=} \begin{pmatrix} z^{s_1} & \cdots & \mathbb{O} & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \cdots & z^{s_{k-1}} & \mathbb{O} & \mathbb{O} & \cdots & 0 \\ \mathbb{O} & \cdots & \mathbb{O} & z^{s_k+1} & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} & z^{s_{k+1}} & \cdots & \mathbb{O} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} & \mathbb{O} & \cdots & z^{s_n} \end{pmatrix}.$$

Let V be the matrix function satisfying the equation $W_\#\Lambda_{j+1} = \Lambda_{j+1}V$. Then $V = \Lambda_{j+1}^{-1} W_\# \Lambda_{j+1}$. It is easy to see that

$$V(\zeta) \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & \cdots & 0 & \xi_1\zeta^{s_k-s_1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \xi_2\zeta^{s_k-s_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \xi_{k-1}\zeta^{s_k-s_{k-1}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1-\frac{\zeta_0}{\zeta} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \zeta \in \mathbb{T}.$$

We have $\Psi = \Xi_{1,j+1}\Lambda_{j+1}\overline{\Xi}_{2,j+1}$, where $\Xi_{2,j+1} \stackrel{\text{def}}{=} \overline{V}\Xi_{2,j}$. Clearly, $\overline{V} \in \mathcal{R} \cap H^2$ and $\overline{V}^{-1} \in \mathcal{R} \cap H^2$, and so $\Xi_{2,j+1} \in (\mathcal{R} \cap H^2)^{-1}$.

It is easy to see that if $\Psi \in \mathcal{R}^{-1}$, then the function Ξ_1 constructed above satisfies Ξ_1, $\Xi_1^{-1} \in \mathcal{R} \cap H^2$. ■

Proof of Theorem 5.2. Suppose that T_Φ is Fredholm. Let us first show that Φ admits a factorization of the form $\overline{\Psi_1}\Lambda\Psi_2$, where Ψ_1 and Ψ_2 are matrix functions such that $\Psi_j^{\pm 1} \in H^2$, $j = 1, 2$, and Λ is a diagonal matrix function of the form (5.2).

Put $L = \operatorname{Ker} T_\Phi$. We can apply Lemma 5.3 and find a subspace $L_\#$ and a diagonal polynomial matrix function P satisfying the conclusion of Lemma 5.3. Then $\operatorname{Ker} T_\Phi \big| L_\# = \operatorname{Ker} T_\Phi \big| PH^2(\mathbb{C}^n) = \{\mathbb{O}\}$. Hence, $\operatorname{Ker} T_{\Phi P} = \{\mathbb{O}\}$.

Since P is a diagonal matrix whose diagonal entries can have zeros only in $\mathbb{D}$, it follows that T_P is Fredholm, and so $T_{\Phi P} = T_\Phi T_P$ is a Fredholm operator with trivial kernel.

We can proceed in the same way with the operator $T_{\Phi P}^* = T_{P^*\Phi^*}$ and find a diagonal polynomial matrix function Q whose diagonal entries can have zeros only in $\mathbb{D}$ such that $T_{P^*\Phi^*Q}$ is a Fredholm operator with trivial kernel. Clearly, $T_{P^*\Phi^*Q}$ maps $H^2(\mathbb{C}^n)$ onto itself, and so it is invertible. Hence, $T_{Q^*\Phi P} = T_{P^*\Phi^*Q}^*$ is invertible.

By Theorem 5.1, there exist matrix functions Υ_1 and Υ_2 such that $\Upsilon_1^{\pm 1}$, $\Upsilon_2^{\pm 1} \in H^2$ and $Q^*\Phi P = \overline{\Upsilon_1}\Upsilon_2$. It follows that $\Phi = (Q^*)^{-1}\overline{\Upsilon_1}\Upsilon_2 P^{-1}$.

Consider the functions $\overline{(Q^*)^{-1}\overline{\Upsilon_1}} = (Q^t)^{-1}\Upsilon_1$ and $P\Upsilon_2^{-1} = (\Upsilon_2 P^{-1})^{-1}$. Clearly, they belong to $(\mathcal{R} + H^2)^{-1}$. Therefore by Theorem 5.4 there exist matrix functions Ξ_1, Ξ_2, Ω_1, Ω_2, Λ_1, Λ_2 such that $\Xi_1^{\pm 1}$, $\Xi_2^{\pm 1} \in H^2$, $\Omega_1^{\pm 1}$, $\Omega_2^{\pm 1} \in \mathcal{R} \cap H^2$, Λ_1 and Λ_2 are diagonal matrix functions of the form (5.2), and

$$(Q^t)^{-1}\Upsilon_1 = \Xi_1\Lambda_1\overline{\Omega_1}, \quad P\Upsilon_2^{-1} = \Xi_2\Lambda_2\overline{\Omega_2}.$$

Therefore

$$\Phi = (Q^*)^{-1}\overline{\Upsilon_1}\Upsilon_2 P^{-1} = \overline{\Xi_1}\,\overline{\Lambda_1}\,\overline{\Omega_1}\,\overline{\Omega_2^{-1}}\Lambda_2^{-1}\Xi_2^{-1}.$$

It is easy to see that $\overline{\Lambda_1}\,\overline{\Omega_1}\,\overline{\Omega_2^{-1}}\Lambda_2^{-1} \in \mathcal{R}^{-1}$. Applying Theorem 5.4 to $\left(\overline{\Lambda_1}\,\overline{\Omega_1}\,\overline{\Omega_2^{-1}}\Lambda_2^{-1}\right)^{-1}$, we can find matrix functions Δ_1, Λ, Δ_2 such that $\Delta_1^{\pm 1} \in \mathcal{R} \cap H^2$, $\Delta_2^{\pm 1} \in \mathcal{R} \cap H^2$, Λ is of the form (5.2), and

$$\left(\overline{\Lambda_1}\,\overline{\Omega_1}\,\overline{\Omega_2^{-1}}\Lambda_2^{-1}\right)^{-1} = \Delta_2^{-1}\Lambda^{-1}\overline{\Delta}_1^{-1}.$$

We now have

$$\Phi = \overline{\Psi_1}\Lambda\Psi_2, \tag{5.8}$$

where $\Psi_1 = \Xi_1\Delta_1$ and $\Psi_2 = \Delta_2\Xi_2^{-1}$. Clearly, Ψ_1 and Ψ_2 satisfy the requirements of Theorem 5.2.

It remains to prove that if $\Phi \in L^\infty(\mathbb{M}_{n,n})$ satisfies (5.8) with Ψ_1,Ψ_2, and Λ are as above, then T_Φ is Fredholm if and only if the operator B defined by (5.3) extends to a bounded operator on H^2.

We can represent Λ in the form $\Lambda = \Lambda_-\Lambda_+$, where

$$\Lambda_- = \begin{pmatrix} \bar{z}^{d_1} & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & \bar{z}^{d_2} & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & \bar{z}^{d_n} \end{pmatrix}, \quad \Lambda_+ = \begin{pmatrix} z^{m_1} & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & z^{m_2} & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & z^{m_n} \end{pmatrix}, \tag{5.9}$$

where $d_1, \cdots, d_n \in \mathbb{Z}_+$ and $m_1, \cdots, m_n \in \mathbb{Z}_+$, and

$$\min\{d_j, m_j\} = 0, \quad 1 \le j \le n.$$

Clearly, $k_j = m_j - d_j$ (see (5.2)). It is easy to see that $\Lambda_-\Lambda_+ = \Lambda_+\Lambda_-$.

Now we are in a position to complete the proof of Theorem 5.2.

Suppose that B is bounded. It is easy to see that $\Psi_2 Bf = \mathbb{P}_+\overline{\Lambda}\mathbb{P}_+\overline{\Psi}_1^{-1} f$ for any $f \in H^2(\mathbb{C}^n)$. It follows that $\mathbb{P}_+\overline{\Lambda}\mathbb{P}_+\overline{\Psi}_1^{-1} f \in H^1(\mathbb{C}^n)$ for any $f \in H^2(\mathbb{C}^n)$. Hence, $Bf = \Psi_2^{-1}\mathbb{P}_+\overline{\Lambda}\mathbb{P}_+\overline{\Psi}_1^{-1} f$ for any $f \in H^2(\mathbb{C}^n)$.

Let us show that the operators $BT_\Phi - I$ and $B^*T_{\Phi^*} - I$ have finite rank, which would imply that T_Φ is Fredholm.

Let p be a polynomial in $H^2(\mathbb{C}^n)$ and $f = \Psi_2^{-1}p \in H^2(\mathbb{C}^n)$. We have

$$\begin{aligned} BT_\Phi f &= \Psi_2^{-1}\mathbb{P}_+\overline{\Lambda}\mathbb{P}_+\overline{\Psi}_1^{-1}\mathbb{P}_+\overline{\Psi}_1\Lambda\Psi_2 f \\ &= \Psi_2^{-1}\mathbb{P}_+\overline{\Lambda}\mathbb{P}_+\Lambda\Psi_2 f = \Psi_2^{-1}\mathbb{P}_+\overline{\Lambda}\mathbb{P}_+\Lambda p. \end{aligned}$$

Consider the linear manifold

$$\mathcal{L} \stackrel{\text{def}}{=} \{p = \{p_j\}_{1\le j\le n} : \ g_j \in \boldsymbol{\mathcal{P}}_A, \ \bar{z}^{d_j} p_j \in H^2\}$$

and the subspace

$$\mathcal{M} \stackrel{\text{def}}{=} \{p = \{p_j\}_{1\le j\le n} : \ g_j \in \boldsymbol{\mathcal{P}}_A, \ \bar{z}^{d_j} p_j \in H^2_-\}. \tag{5.10}$$

Clearly, $\dim \mathcal{M} = d_1 + \cdots + d_n$. Obviously, $\Lambda p \in H^2(\mathbb{C}^n)$ for any $p \in \mathcal{L}$, and so $BT_\Phi\Psi_2^{-1}p = \Psi_2^{-1}p$ for $p \in \mathcal{L}$. It follows that $BT_\Phi f = f$ for any $f \in \operatorname{clos}\Psi_2^{-1}\mathcal{L}$.

We need the following simple fact.

Lemma 5.6. *Let Ψ be an $n\times n$ function on $\mathbb{T}$ such that $\Psi \in H^2(\mathbb{M}_{n,n})$ and $\Psi^{-1} \in H^2(\mathbb{M}_{n,n})$. Then Ψ is an outer function, i.e., the set*

$$\mathcal{K} \stackrel{\text{def}}{=} \{\Psi p : \ p = \{p_j\}_{1\le j\le n} : \ p_j \in \boldsymbol{\mathcal{P}}_A\}$$

is dense in $H^2(\mathbb{C}^n)$.

Proof. Suppose that $\varphi \in H^2(\mathbb{C}^n)$ and $\varphi \perp \mathcal{K}$. We have

$$0 = (\Psi p, \varphi) = \int_{\mathbb{T}} (\Psi^*(\zeta)\varphi(\zeta))^* \, p(\zeta) d\boldsymbol{m}(\zeta)$$

for any polynomial $p \in H^2(\mathbb{C}^n)$. It follows that $\Psi^*\varphi \in H^1_-(\mathbb{C}^n)$. Since $\Psi^{-1} \in H^2(\mathbb{M}_{n,n})$, it follows that $\varphi = (\Psi^*)^{-1}\Psi^*\varphi \in H^{2/3}_-(\mathbb{C}^n)$. Since $\varphi \in L^2(\mathbb{C}^n)$, we have $\varphi \in H^2_-(\mathbb{C}^n)$, which is possible only if $\varphi = \mathbb{O}$. ■

Now we are able to complete the proof of Theorem 5.2.

By Lemma 5.6, $\Psi_2^{-1}(\mathcal{L} + \mathcal{M})$ is dense in $H^2(\mathbb{C}^n)$. Since $BT_\Phi f = f$ on $\operatorname{clos}\Psi_2^{-1}\mathcal{L}$ and $\dim \Psi_2^{-1}\mathcal{M} = d_1 + \cdots + d_n$, it follows that

$$\operatorname{rank}(BT_\Phi - I) \le d_1 + \cdots + d_n < \infty.$$

Consider now the operator $B^*T_{\Phi^*} - I$. It is easy to see that

$$B^* f = (\Psi_1^{\mathrm{t}})^{-1}\mathbb{P}_+\Lambda\mathbb{P}_+(\Psi_2^*)^{-1} f$$

for any polynomial f. Now the proof of the fact that $\operatorname{rank}(B^*T_{\Phi^*} - I) < \infty$ is exactly the same as for the operator $BT_\Phi - I$.

Finally, suppose that T_Φ is Fredholm. Let us show that the operator B defined on the set of polynomials by (5.3) extends to a bounded operator on $H^2(\mathbb{C}^n)$.

Denote by X the space $\{f = \varphi + \mathbb{P}_+\psi : \ \varphi, \psi \in L^1\}$. We can endow X with the natural norm

$$\|f\|_X = \inf\{\|\varphi\|_{L^1} + \|\psi\|_{L^1} : \ f = \varphi + \mathbb{P}_+\psi\}.$$

Obviously, $\mathbb{P}_+$ is a bounded operator on X. Define the operator A on $H^2(\mathbb{C}^n)$ by

$$Af = \mathbb{P}_+\overline{\Lambda}\mathbb{P}_+\overline{\Psi}_1^{-1}f.$$

Clearly, $Af = \Psi_2 Bf$ if f is a polynomial in $H^2(\mathbb{C}^n)$. It is easy to see that A maps $H^2(\mathbb{C}^n)$ to $X(\mathbb{C}^n)$. We have

$$AT_\Phi f = \mathbb{P}_+\overline{\Lambda}\mathbb{P}_+\overline{\Psi}_1^{-1}\mathbb{P}_+\overline{\Psi}_1\Lambda\Psi_2 f = \mathbb{P}_+\overline{\Lambda}\mathbb{P}_+\Lambda\Psi_2 f.$$

It is easy to see that

$$AT_\Phi\Psi_2^{-1}p = p, \quad p \in \mathcal{L},$$

and so

$$AT_\Phi f = \Psi_2 f, \quad f \in \operatorname{clos}\Psi_2^{-1}\mathcal{L}. \tag{5.11}$$

We can define now Bf for each function $f \in H^2(\mathbb{C}^n)$ by $Bf = \Psi_2^{-1}Af$. It follows from (5.11) that $BT_\Phi f = f$ for any $f \in \operatorname{clos}\Psi_2^{-1}\mathcal{L}$. Since T_Φ is Fredholm, it follows that B maps the subspace $T_\Phi \operatorname{clos}\Psi_2^{-1}\mathcal{L}$ into $H^2(\mathbb{C}^n)$. By Lemma 5.6, $T_\Phi \operatorname{clos}\Psi_2^{-1}\mathcal{L}$ has finite codimension.

As we have already explained, $\mathbb{P}_+\overline{\Psi}_1^{-1}f$ is a polynomial for any polynomial f, and so B maps analytic vector polynomials into $H^2(\mathbb{C}^n)$. Let $\boldsymbol{\mathcal{P}}_m$ be the space of polynomials $\{p_j\}_{1\le j\le n}$ such that $\deg p_j \le m$. Since $\operatorname{codim} T_\Phi \operatorname{clos}\Psi_2^{-1}\mathcal{L} < \infty$, it follows that $\boldsymbol{\mathcal{P}}_m + T_\Phi \operatorname{clos}\Psi_2^{-1}\mathcal{L} = H^2(\mathbb{C}^n)$ for some $m \in \mathbb{Z}_+$. Hence, B is a bounded operator on $H^2(\mathbb{C}^n)$. ■

Let us now describe the kernel of T_Φ end evaluate $\operatorname{ind} T_\Phi$.

Theorem 5.7.

$$\operatorname{Ker} T_\Phi = \Psi_2^{-1}\mathcal{M},$$

where $\mathcal{M}$ *is defined by* (5.10).

Proof of Theorem 5.7. It is easy to see that $\Psi_2^{-1}\mathcal{M} \subset \operatorname{Ker} T_\Phi$. Suppose now that $f \in \operatorname{Ker} T_\Phi$. This means that $\overline{\Psi_1}\Lambda_-\Lambda_+\Psi_2 f \in H^2_-(\mathbb{C}^n)$. Since $\Psi_1^{-1} \in H^2(\mathbb{M}_{n,n})$, it follows that $\Lambda_-\Lambda_+\Psi_2 f \in H^1_-(\mathbb{C}^n)$. Hence, $\Lambda_+\Psi_2 f = \{p_j\}_{1\le j\le n}$, where p_j is a polynomial, $\deg p_j < d_j$ if $d_j > 0$ and $p_j = \mathbb{O}$ if $d_j = 0$. Since $\min\{d_j, m_j\} = 0$, $1 \le j \le n$, it follows that $\Psi_2 f = \{p_j\}_{1\le j\le n}$ which implies the result. ■

Corollary 5.8.

$$\dim \operatorname{Ker} T_\Phi = d_1 + \cdots + d_n, \quad \operatorname{ind} T_\Phi = -(k_1 + \cdots + k_n).$$

Proof. The first equality follows immediately from Theorem 5.7. If we apply it to $T_\Phi^* = T_{\Phi^*}$, we find that $\dim \operatorname{Ker} T_\Phi^* = m_1 + \cdots + m_n$ which implies the result. ■

Corollary 5.9. *Suppose that under the above assumptions T_Φ is Fredholm. Then T_Φ is invertible if and only if Λ is identically equal to I.*

We conclude this section with the following result.

Theorem 5.10. *Let $\Phi \in L^\infty(\mathbb{M}_{n,n})$ be the symbol of a Fredholm Toeplitz operator on $H^2(\mathbb{C}^n)$. Suppose that $k_1 \le k_2 \le \cdots \le k_n$ are the factorization indices of a Wiener–Hopf factorization of Φ. Then the indices k_j, $1 \le j \le n$, are uniquely determined by the function Φ itself.*

Proof. Let $m \in \mathbb{Z}$. Consider the function $z^m\Phi$. Clearly, $T_{z^m\Phi}$ is Fredholm and $z^m\Phi$ has a Wiener–Hopf factorization with indices

$$m + k_1, m + k_2, \cdots, m + k_n.$$

Consider the sequence $\{N_m\}_{m\in\mathbb{Z}}$ defined by

$$N_m = \dim \operatorname{Ker} T_{z^m\Phi}.$$

By Corollary 5.8,

$$N_m = 0, \quad m \ge -k_1, \quad \text{and} \quad N_{-k_1-1} > 0.$$

This uniquely determines k_1. Let $\nu_1 = N_{-k_1-1}$. It follows from Corollary 5.8 that ν_1 is the number of indices equal to k_1.

We have $k_1 = \cdots = k_{\nu_1}$ and if $\nu_1 < n$, then $k_{\nu_1+1} > k_1$, and so $k_1, \cdots, k_{\nu_1}$ are uniquely determined by the sequence $\{N_m\}_{m\in\mathbb{Z}}$.

It follows from Corollary 5.8 that

$$N_m = -\nu_1(k_1 + m), \quad -k_{\nu_1+1} \le m < -k_1,$$

and

$$N_{-k_{\nu_1+1}-1} > -\nu_1(k_1 - k_{\nu_1+1} - 1).$$

This uniquely determines k_{ν_1+1}. Put $\nu_2 = N_{-k_{\nu_1+1}-1} + \nu_1(k_1 - k_{\nu_1+1} - 1)$. It follows easily from Corollary 5.8 that ν_2 is the number of indices equal to k_{ν_1+1}.

Clearly, we can continue this process and determine all indices k_j, $1 \le j \le n$, from the sequence $\{N_m\}_{m\in\mathbb{Z}}$. ∎

6. Left Invertibility of Bounded Analytic Matrix Functions

The famous Carleson corona theorem (see Appendix 2.1) says that if Ω is a function in $H^\infty(\mathbb{C}^n)$ such that

$$\inf_{\zeta\in\mathbb{D}} \|\Omega(\zeta)\|_{\mathbb{C}^n} > 0, \tag{6.1}$$

then there exists a function Ξ in $H^\infty(\mathbb{C}^n)$ such that $\Xi^{\mathrm{t}}(\zeta)\Omega(\zeta) = 1$ for all $\zeta \in \mathbb{D}$. Later this result was generalized to matrix functions; however, it turns out that it does not generalize to the case of infinite matrix functions (see Concluding Remarks).

We consider here another condition on a bounded analytic matrix function which is necessary and sufficient for left invertibility. Instead of (6.1) we assume that the Toeplitz operator $T_{\overline{\Omega}}$ is left invertible. Unlike condition (6.1), it also works in the case of infinite matrix functions.

In this section we deal with functions taking values in the space $\mathbb{M}_{m,n}$ of $m \times n$ matrices. We identify $\mathbb{M}_{m,n}$ with the space of linear operators from $\mathbb{C}^n$ to $\mathbb{C}^m$ and equip $\mathbb{M}_{m,n}$ with the operator norm. We also admit the case of infinite matrix functions: if $n = \infty$, $\mathbb{C}^n$ is just the sequence space ℓ^2. As usual, I_n stands for the identity matrix in $\mathbb{M}_{n,n}$.

The following theorem gives a necessary and sufficient condition for the left invertibility of a bonded analytic matrix function. The main tool in the proof is the commutant lifting theorem.

Theorem 6.1. *Let m and n be positive integers or equal to ∞, and $K > 0$. Let Ω be a function in $H^\infty(\mathbb{M}_{m,n})$. The following are equivalent:*
(i) *there exists a function Ξ in $H^\infty(\mathbb{M}_{n,m})$ such that*

$$\Xi(\zeta)\Omega(\zeta) = I_n, \quad \zeta \in \mathbb{D}, \quad \textit{and} \quad \|\Xi\|_{H^\infty(\mathbb{M}_{n,m})} \le K; \tag{6.2}$$

(ii) *the Toeplitz operator $T_{\overline{\Omega}}$ satisfies*

$$\|T_{\overline{\Omega}} f\|_{H^2(\mathbb{C}^m)} \ge K^{-1} \|f\|_{H^2(\mathbb{C}^n)}, \quad f \in H^2(\mathbb{C}^n). \tag{6.3}$$

Proof. Suppose first that (6.2) holds for a function Ξ in $H^\infty(\mathbb{M}_{n,m})$. We have

$$T_{\overline{\Xi}} T_{\overline{\Omega}} = T_{\overline{\Xi\Omega}} = I.$$

Since $\|T_{\overline{\Xi}}\| = \|\Xi\|_{H^\infty(\mathbb{M}_{n,m})} \le K$, it follows that $T_{\overline{\Omega}}$ is left invertible and (6.3) holds.

Conversely, assume that (6.3) holds. Consider the Toeplitz operator $T_{\overline{\Omega}} : H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^m)$. Clearly,

$$S_m^* T_{\overline{\Omega}} = T_{\overline{\Omega}} S_n^*, \tag{6.4}$$

where S_n and S_m are multiplications by z on $H^2(\mathbb{C}^n)$ and $H^2(\mathbb{C}^m)$. Let $\mathcal{L}$ be the range of $T_{\overline{\Omega}}$. It follows from (6.3) that $\mathcal{L}$ is a closed subspace of $H^2(\mathbb{C}^m)$ and there exists a bounded linear operator $Q : \mathcal{L} \to H^2(\mathbb{C}^n)$ such that $QT_{\overline{\Omega}} = I$ and $\|Q\| \le K$. It follows easily from (6.4) that $\mathcal{L}$ is an invariant subspace of S_m^* and

$$QS_m^* \big| \mathcal{L} = S_n^* Q,$$

and so

$$(P_{\mathcal{L}} S_m \big| \mathcal{L}) Q^* = Q^* S_n, \tag{6.5}$$

where $P_{\mathcal{L}}$ is the orthogonal projection onto $\mathcal{L}$.

The operator S_n is isometric on $H^2(\mathbb{C}^n)$. It is easy to see that the operator S_m is an isometric dilation of $P_{\mathcal{L}} S_m \big| \mathcal{L}$. The operator Q^* intertwines S_n and $P_{\mathcal{L}} S_m \big| \mathcal{L}$. By the commutant lifting theorem (see Appendix 1.5), there exists a bounded linear operator R from $H^2(\mathbb{C}^n)$ to $H^2(\mathbb{C}^m)$ such that $\|R\| \le K$,

$$Q^* = P_{\mathcal{L}} R \big| \mathcal{L}, \tag{6.6}$$

and

$$S_m R = R S_n. \tag{6.7}$$

It follows easily from (6.7) that R has a block Toeplitz matrix. By Theorem 4.1, there exists a matrix function $\Psi \in L^\infty(\mathbb{M}_{m,n})$ such that $\|\Psi\|_{L^\infty} \le K$ and $R = T_\Psi$. It also follows from (6.7) that $Rz^k H^2(\mathbb{C}^n) \subset z^k H^2(\mathbb{C}^m)$, $k \in \mathbb{Z}_+$. Hence, $\Psi \in H^\infty(\mathbb{M}_{m,n})$. Since $\mathcal{L}$ is an invariant subspace of S_m^*, it follows that $\mathcal{L}$ is also an invariant subspace of $R^* = T_\Psi^*$, and it follows from (6.6) that

$$Q = P_{\mathcal{L}} R^* \big| \mathcal{L} = R^* \big| \mathcal{L}.$$

Therefore

$$I = QT_{\overline{\Omega}} = R^* T_{\overline{\Omega}} = T_{\Psi^*} T_{\overline{\Omega}}.$$

Hence, $\Psi^* \overline{\Omega} = I_n$. It remains to put $\Xi = \Psi^{\mathrm{t}}$. ■

Theorem 6.1 admits an "invariant" reformulation. We need the following notation. If $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces and $\Phi \in L^\infty(\mathcal{B}(\mathcal{H}, \mathcal{K}))$, we denote by $\Psi_\#$ the function in $L^\infty(\mathcal{B}(\mathcal{H}, \mathcal{K}))$ defined by

$$\Psi_\#(\zeta) = \Psi(\bar{\zeta}), \quad \zeta \in \mathbb{T}.$$

Theorem 6.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces, $K > 0$, and let $\Omega \in H^\infty(\mathcal{B}(\mathcal{H}, \mathcal{K}))$. The following are equivalent:*

(i) *there exists a function Ξ in $H^\infty(\mathcal{B}(\mathcal{K}, \mathcal{H}))$ such that*

$$\Xi(\zeta)\Omega(\zeta) = I, \quad \zeta \in \mathbb{D}, \quad \textit{and} \quad \|\Xi\|_{H^\infty(\mathcal{B}(\mathcal{K},\mathcal{H}))} \le K;$$

(ii) *the Toeplitz operator $T_{\Omega_\#}$ satisfies*

$$\|T_{\Omega_\#} f\|_{H^2(\mathcal{K})} \ge K^{-1} \|f\|_{H^2(\mathcal{H})}, \quad f \in H^2(\mathcal{H}).$$

It is easy to see that Theorem 6.2 is equivalent to Theorem 6.1.

Concluding Remarks

The systematic study of spectral properties of Toeplitz operators was started in the paper Brown and Halmos [1] in which Theorem 1.1 was obtained. Theorem 1.4 appeared in Coburn [1]. Theorem 1.6 is due to Wintner [1]. Theorem 1.7 can be found in Douglas [2]; see also Hartman and Wintner [1], where Corollary 1.8 was established. Theorem 1.9 is due to Brown and Halmos [1]. Theorems 1.11–1.13 were found by Widom [1] and Devinatz [1]. Theorem 1.14 was proved by N.K. Nikol'skii [2]. Finally, Theorem 1.15 is taken from Douglas and Sarason [1].

The fundamental invertibility criterion, Theorem 2.1, was found in Widom [1] and Devinatz [1]. The remaining results of §2 are well-known; see Peller and Khrushchëv [1].

Theorem 3.1 was found in Hartman and Wintner [1]. Theorem 3.3 was discovered independently by many authors; see Gohberg [1], Mikhlin [1]. Theorems 3.2 and 3.5 are due to Douglas, see Douglas [2]. Theorem 3.10 was obtained in Sarason [3]. Theorem 3.11 was proved in Widom [1] in the case of finitely many jumps and in Devinatz [1] in the general case. Let us also mention the paper Krein [2] in which the theory of Toeplitz operators whose symbols have absolutely convergent Fourier series was developed.

Theorems 4.1, 4.2, 4.4, and 4.6 are straightforward generalizations of the corresponding results in the scalar case. Theorem 4.8 is due to Douglas [3].

Theorems 5.1 and 5.2 were obtained in Simonenko [2]; see also Litvinchuk and Spitkovskii [1]. Theorems 5.7 and 5.10 are well-known.

Theorem 6.1 was proved in Arveson [3] (for $m = 1$), Schubert [1], and Sz.-Nagy and Foias [2]. Note that in Fuhrmann [1] the following matricial corona theorem was proved: if Ω is a matrix function in $H^\infty(\mathbb{M}_{m,n})$ such that $\|\Omega(\zeta)x\|_{\mathbb{C}^m} \ge \delta\|x\|_{\mathbb{C}^n}$ for any $\zeta \in \mathbb{D}$ and $x \in \mathbb{C}^n$, then Ω is left invertible in the space of bounded analytic matrix functions. Vasyunin (see Tolokonnikov [1]) generalized this result to the case $n = \infty$ (in the case $m = 1$ and $n = \infty$ this was done by Tolokonnikov [1] and Rosenblum [4]). However, Treil [5] showed that these results do not generalize to the case of an arbitrary bounded analytic operator function.

We also mention here the important and fruitful local theory of Toeplitz operator developed by Simonenko [1]. Another version of local theory was developed by Douglas [2].

A remarkable result by Widom [2] shows that the spectrum of Toeplitz operator is always connected; see also Douglas [2], where it was shown that the essential spectrum of a Toepliz operator is always connected.

The spectral structure of self-adjoint Toeplitz operator was completely described in Ismagilov [1] and Rosenblum [3]. Earlier Rosenblum [2] showed that a self-adjoint Toeplitz operator must have absolutely continuous spectral measure.

In Peller [7], [14], and [21] estimates of resolvents of Toeplitz operators for certain classes of symbols were obtained. Based on those estimates the existence of nontrivial invariant subspaces as well as some similarity results were found there. Note, however, that Treil [4] has shown that the resolvent of a Toeplitz operator with continuous symbol can grow arbitrarily fast. Interesting similarity theorems were obtained in Clark [4] and Yakubovich [1] for certain classes of Toeplitz operators.

Let us mention here the classical Riemann–Hilbert problem. Given functions g, $h \in L^2$, find functions $f_+ \in H^2$ and $f_- \in H^2_-$ such that

$$f_+ = gf_- + h.$$

It is easy to see that the study of the Riemann–Hilbert problem is equivalent to the study of equations involving Toeplitz operators. Note that a complete solution of the Riemann–Hilbert problem with data in Λ_α was obtained for the first time by Gakhov [1].

Finally, we mention the following books and survey articles on Toeplitz operators: Douglas [2], [3], Bötcher and Silbermann [1], Nikol'skii [2]-[4], Peller and Khrushchëv [1], and Litvinchuk and Spitkovskii [1].

4

Singular Values of Hankel Operators

In this chapter we study singular values of Hankel operators. The main result of the chapter is the fundamental theorem of Adamyan, Arov, and Krein. This theorem says that if Γ is a Hankel operator, then to evaluate the nth singular value $s_n(\Gamma)$ of Γ, there is no need to consider all operators of rank at most n, $s_n(\Gamma)$ is the distance from Γ to the set of *Hankel* operators of rank at most n. In §1 we prove the Adamyan–Arov–Krein theorem in the special case when $s_n(\Gamma)$ is greater than the essential norm of Γ. In §2 we reduce the general case to the case treated in §1. In §1 we also prove the uniqueness of the corresponding Hankel approximant of rank at most n under the same condition $s_n(\Gamma) > \|\Gamma\|_e$, and we obtain useful formulas for multiplicities of singular values of related Hankel operators. We prove a generalization of the Adamyan–Arov–Krein theorem to the case of vectorial Hankel operators in §3. We also obtain in §3 a formula for the essential norm of vectorial Hankel operators.

Finally, in §4 we consider relationships between the singular values of H_u and $H_{\bar{u}}$ for unimodular functions u. We obtain a very useful formula that relates these two Hankel operators which leads to interesting results about the singular values of H_u and $H_{\bar{u}}$. We also obtain in §4 similar results for vectorial Hankel operators.

1. The Adamyan–Arov–Krein Theorem

Recall that for a bounded linear operator T from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$ the singular values $s_m(T)$, $m \in \mathbb{Z}_+$, are defined by

$$s_m(T) = \inf\{\|T - R\| : \operatorname{rank} R \le m\}. \tag{1.1}$$

Clearly, $s_0(T) = \|T\|$ and $s_{m+1}(T) \le s_m(T)$. Put

$$s_\infty(T) = \lim_{m\to\infty} s_m(T).$$

It is easy to see that $s_\infty(T) = \|T\|_{\mathrm{e}}$, the *essential norm* of T. Clearly, T is compact if and only if $s_\infty(T) = 0$ (see Appendix 1.1).

The main aim of this and the next section is to prove that in the case of a Hankel operator T to find $s_m(T)$ we can consider the infimum in (1.1) over only the Hankel operators of rank at most m.

Theorem 1.1. *Let Γ be a Hankel operator from H^2 to H^2_-, $m \ge 0$. Then there exists a Hankel operator Γ_m of rank at most m such that*

$$\|\Gamma - \Gamma_m\| = s_m(\Gamma). \tag{1.2}$$

Since by Kronecker's theorem $\operatorname{rank}\Gamma_m \le m$ if and only if Γ_m has a rational symbol of degree at most m, Theorem 1.1 admits the following reformulation. Let $H^\infty_{(m)}$ be the set of functions f in L^∞ such that $\mathbb{P}_- f$ is a rational function of degree at most m. Clearly, $H^\infty_{(m)}$ can be identified with the set of meromorphic functions in $\mathbb{D}$ bounded near $\mathbb{T}$ and having at most m poles in $\mathbb{D}$ counted with multiplicities.

Theorem 1.2. *Let $\varphi \in L^\infty$, $m \in \mathbb{Z}_+$. Then there exists a function ψ in $H^\infty_{(m)}$ such that*

$$\|\varphi - \psi\|_\infty = s_m(H_\varphi). \tag{1.3}$$

Note that the problem of approximation of an L^∞ function on $\mathbb{T}$ by functions in $H^\infty_{(m)}$ is called the *Nehari–Takagi problem.*

We shall consider in this section the case when $s_m(\Gamma) > s_\infty(\Gamma)$. The proof in the case $s_m(\Gamma) = s_\infty(\Gamma)$ will be given in the next section. It turns out that in the case $s_m(\Gamma) > s_\infty(\Gamma)$ there exists a *unique* Hankel operator Γ_m of rank at most m that satisfies (1.2). In other words, the following theorem holds.

Theorem 1.3. *Let $\varphi \in L^\infty$, $m \in \mathbb{Z}_+$. If $s_m(H_\varphi) > s_\infty(H_\varphi)$, then there exists a unique ψ in $H^\infty_{(m)}$ that satisfies* (1.3).

Definition. Let T be a bounded linear operator from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$. If s is a singular value of T and $s > s_\infty(T)$, consider the subspaces

$$E_s^{(+)} = \{x \in \mathcal{H}_1 : T^*Tx = s^2 x\}, \quad E_s^{(-)} = \{y \in \mathcal{H}_2 : TT^*y = s^2 y\}.$$

Vectors in $E_s^{(+)}$ are called *Schmidt vectors of T* (or, more precisely, s-Schmidt vectors of T). Vectors in $E_s^{(-)}$ are called *Schmidt vectors of*

T^* (s-Schmidt vectors of T^*). Clearly, $x \in E_s^{(+)}$ if and only if $Tx \in E_s^{(-)}$. A pair $\{x, y\}$, $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$, is called a *Schmidt pair* of T (s-Schmidt pair) if $Tx = sy$ and $T^*y = sx$.

Proof of Theorem 1.1 in the case $s_m(\Gamma) > s_\infty(\Gamma)$. Put $s = s_m(\Gamma)$. If $s = \|\Gamma\|$, the result is trivial. Assume that $s < \|\Gamma\|$. Then there exist positive integers k, μ such that $k \le m \le k + \mu - 1$ and

$$s_{k-1}(\Gamma) > s_k(\Gamma) = \cdots = s_{k+\mu-1}(\Gamma) > s_{k+\mu}(\Gamma). \tag{1.4}$$

Clearly, it suffices to consider the case $m = k$.

Lemma 1.4. *Let $\{\xi_1, \eta_1\}$ and $\{\xi_2, \eta_2\}$ be s-Schmidt pairs of Γ. Then $\xi_1\bar{\xi}_2 = \eta_1\bar{\eta}_2$.*

To prove the lemma we need the following identity (see formula (1.1.13)):

$$\mathbb{P}_-(z^n\Gamma f) = \Gamma z^n f, \quad n \in \mathbb{Z}_+. \tag{1.5}$$

Proof of Lemma 1.4. Let $n \in \mathbb{Z}_+$. We have

$$\begin{aligned} \widehat{\xi_1\bar{\xi}_2}(-n) &= (z^n\xi_1, \xi_2) = \frac{1}{s}(z^n\xi_1, \Gamma^*\eta_2) = \frac{1}{s}(\Gamma z^n\xi_1, \eta_2) \\ &= \frac{1}{s}(\mathbb{P}_- z^n\Gamma\xi_1, \eta_2) = (z^n\eta_1, \eta_2) = \widehat{\eta_1\bar{\eta}_2}(-n) \end{aligned}$$

by (1.5). Similarly, $\widehat{\xi_1\bar{\xi}_2}(n) = \widehat{\eta_1\bar{\eta}_2}(n)$, $n \in \mathbb{Z}_+$, which implies $\xi_1\bar{\xi}_2 = \eta_1\bar{\eta}_2$. ■

Corollary 1.5. *Let $\{\xi, \eta\}$ be an s-Schmidt pair of Γ. Then the function*

$$\varphi_s = \frac{\eta}{\xi} \tag{1.6}$$

is unimodular and does not depend on the choice of $\{\xi, \eta\}$.

Proof. Let $\xi_1 = \xi_2 = \xi$, $\eta_1 = \eta_2 = \eta$ in Lemma 1.4. It follows that $|\xi|^2 = |\eta|^2$ and so η/ξ is unimodular for any Schmidt pair $\{\xi, \eta\}$.

Let $\{\xi_1, \eta_1\}$ and $\{\xi_2, \eta_2\}$ be s-Schmidt pairs of Γ. By Lemma 1.4, $\eta_1/\xi_1 = \bar{\xi}_2/\bar{\eta}_2$. Since η_2/ξ_2 is unimodular, it follows that $\eta_1/\xi_1 = \eta_2/\xi_2$. ■

We resume the proof of Theorem 1.1. Put

$$\Gamma_s = H_{s\varphi_s},$$

where φ_s is defined by (1.6). Clearly, $\|\Gamma_s\| \le s$. The result will be established if we show that $\operatorname{rank}(\Gamma - \Gamma_s) \le k$.

Let $\{\xi, \eta\}$ be an s-Schmidt pair of Γ. Let us show that it is also an s-Schmidt pair of Γ_s. Indeed

$$\Gamma_s\xi = s\mathbb{P}_-\frac{\eta}{\xi}\xi = s\eta, \quad \Gamma_s^*\eta = s\mathbb{P}_+\frac{\xi}{\eta}\eta = s\xi.$$

Let

$$E_+ = \{\xi \in H^2 : \ \Gamma^*\Gamma\xi = s^2\xi\}, \quad E_- = \{\eta \in H^2_- : \ \Gamma\Gamma^*\eta = s^2\eta\}$$

be the spaces of Schmidt vectors of Γ and Γ^*. It is easy to see that $\dim E_+ = \dim E_- = \mu$.

It follows easily from (1.5) that if $\Gamma\xi = \Gamma_s\xi$, then $\Gamma z^n\xi = \Gamma_s z^n\xi$ for any $n \in \mathbb{Z}_+$. Since $\Gamma_s \big| E_+ = \Gamma \big| E_+$, it follows that Γ and Γ_s coincide on the S-invariant subspace spanned by E_+, where S is multiplication by z on H^2. By Beurling's theorem this subspace has the form ϑH^2, where ϑ is an inner function (see Appendix 2.2). Denote by Θ multiplication by ϑ. We have $\Gamma\Theta = \Gamma_s\Theta$. The proof will be completed if we show that $\dim(H^2 \ominus \vartheta H^2) \le k$. Put $d = \dim(H^2 \ominus \vartheta H^2)$.

Lemma 1.6. *The singular value s of the operator $\Gamma\Theta$ has multiplicity at least $d + \mu$.*

Note that $s_j(\Gamma\Theta) \le s_j(\Gamma)\|\Theta\| = s_j(\Gamma)$ and so $s_\infty(\Gamma\Theta) < s$. It will follow from Lemma 1.6 that $d < \infty$.

Proof of Lemma 1.6. Let τ be an inner divisor of ϑ (i.e., $\vartheta\tau^{-1} \in H^\infty$). Let us show that for any $\xi \in E_+$

$$(\Gamma_s\Theta)^*(\Gamma_s\Theta)\bar{\tau}\xi = s^2\bar{\tau}\xi \in E_+. \tag{1.7}$$

Indeed it is easy to see that $\Gamma^*\bar{z}\bar{f} = \bar{z}\overline{\Gamma f}$ for any $f \in H^2$. Let J be the transformation on L^2 defined by $Jf = \bar{z}\bar{f}$. It follows that J maps E_+ onto E_-. Since $E_+ \subset \vartheta H^2$, we have $E_- \subset \bar{\vartheta} H^2_-$.

Let $\xi \in E_+$, $\eta = s^{-1}\Gamma\xi \in E_-$. We can represent η as $\eta = \bar{\vartheta}\eta_*$, where $\eta_* \in H^2_-$. We have

$$\begin{aligned}(\Gamma_s\Theta)^*(\Gamma_s\Theta)\bar{\tau}\xi &= (\Gamma_s\Theta)^* s\mathbb{P}_-\frac{\eta}{\xi}\vartheta\bar{\tau}\xi = s(\Gamma_s\Theta)^*\mathbb{P}_-\eta_*\bar{\tau} \\ &= s(\Gamma_s\Theta)^*\eta_*\bar{\tau} = s^2\mathbb{P}_+\frac{\xi}{\eta}\bar{\vartheta}\eta_*\bar{\tau} = s^2\bar{\tau}\xi,\end{aligned}$$

which proves (1.7).

Since $d = \dim(H^2 \ominus \vartheta H^2)$, it follows that for any $n < d$ we can find inner divisors $\{\vartheta_j\}_{1\le j\le n+1}$ of ϑ such that $\vartheta_{n+1} = \vartheta$, $\vartheta_{j+1}\vartheta_j^{-1} \in H^\infty$, $\vartheta_{j+1}\vartheta_j^{-1} \ne \text{const}$, and $\vartheta_1 \ne \text{const}$ (see Appendix 2.2). Then it follows from (1.7) that the subspace

$$E_j = \operatorname{span}\{E_+, \bar{\vartheta}_1 E_+, \cdots \bar{\vartheta}_j E_+\}, \quad 1 \le j \le n+1,$$

consists of eigenvectors of $(\Gamma\Theta)^*(\Gamma\Theta)$ corresponding to the eigenvalue s^2. Clearly, $E_1 \setminus E_+ \ne \varnothing$ and $E_{j+1} \setminus E_j \ne \varnothing$, $1 \le j \le n$. Therefore

$$\dim \operatorname{Ker}\big((\Gamma\Theta)^*\Gamma\Theta - s^2 I\big) \ge \dim E_{n+1} \ge \mu + n + 1.$$

Therefore the left-hand side is equal to ∞ if $d = \infty$ and is at least $\mu + d$ if $d < \infty$. ■

We can now complete the proof of Theorem 1.1. As we have already observed, $s_j(\Gamma\Theta) \le s_j(\Gamma)$. Thus if $d = \infty$, then it follows from Lemma 1.6 that s is a singular value of Γ of infinite multiplicity, and so $d < \infty$. Again, by Lemma 1.6 we have

$$s_{k+\mu}(\Gamma) < s_{k+\mu-1}(\Gamma) = \cdots = s_k(\Gamma) = s_{d+\mu-1}(\Gamma\Theta) \le s_{d+\mu-1}(\Gamma).$$

Therefore $d+\mu-1 < k+\mu$ and so $d \le k$, which completes the proof of Theorem 1.1 in the case $s_m(\Gamma) > s_\infty(\Gamma)$. ■

Remark. Note that $d = k$. For otherwise if $d < k$, then it would follow that $s_{k-1}(\Gamma) = s$, which contradicts (1.4).

Let $\Gamma = H_\varphi$, where $\varphi \in L^\infty$. Assuming (1.4) and $s = s_k(H_\varphi)$, we can consider the function

$$u = \varphi_s = \frac{1}{s} \cdot \frac{\Gamma\xi}{\xi}, \tag{1.8}$$

where $\xi \in E_+$, that is, $H_\varphi^* H_\varphi \xi = s^2 \xi$. Then u is a unimodular function and

$$s_k(H_\varphi) = \|\varphi - \psi\|_\infty,$$

where

$$\psi = \varphi - su. \tag{1.9}$$

It has been shown in the proof of Theorem 1.1 that $\operatorname{rank}\Gamma_\psi \le k$. Since $s_{k-1}(H_\varphi) > s_k(H_\varphi)$, it follows that $\operatorname{rank}\Gamma_\psi = k$.

Theorem 1.7. *Let $\Gamma = H_\varphi$ be a Hankel operator satisfying* (1.4) *and let u be the unimodular function defined by* (1.8). *Then*

$$\dim \operatorname{Ker} T_u = 2k + \mu.$$

Remark. Clearly, $\dim \operatorname{Ker} T_u$ is the multiplicity of the singular value 1 of H_u.

Proof. Let us first show that $\dim \operatorname{Ker} T_u \ge 2k+\mu$. Let ϑ be the greatest common divisor of functions in E_+. Let us prove that for any inner divisor τ of ϑ both τE_+ and $\bar\tau E_+$ are contained in $\operatorname{Ker} T_u$.

Let $\xi \in E_+$. Then

$$u = \frac{\Gamma\xi}{s\xi} = \frac{\eta}{\xi}.$$

Therefore

$$T_u \tau\xi = \mathbb{P}_+ \tau\eta = \mathbb{O},$$

since it has been shown in the proof of Theorem 1.1 that $\vartheta E_- \subset H^2_-$. We have

$$T_u \bar\tau\xi = \mathbb{P}_+ \bar\tau\eta = \mathbb{O}.$$

Let $\{\vartheta_j\}_{1\le j\le k}$ be inner divisors of ϑ such that $\vartheta_1 \ne \text{const}$, $\vartheta_j^{-1}\vartheta_{j+1} \in H^\infty$, $\vartheta_j^{-1}\vartheta_{j+1} \ne \text{const}$, $1 \le j \le k-1$, and $\vartheta_k = \vartheta$.

Consider the subspaces

$$\bar\vartheta_k E_+,\ \bar\vartheta_{k-1} E_+, \cdots, \bar\vartheta_1 E_+,\ E_+,\ \vartheta_1 E_+, \cdots, \vartheta_k E_+. \tag{1.10}$$

Clearly, $\dim \bar\vartheta_k E_+ = \dim E_+ = \mu$. Let F_j be the span of the j leftmost subspaces in (1.10), $1 \le j \le 2k+1$. Then $F_{j+1} \setminus F_j \ne \varnothing$, $1 \le j \le 2k$. It follows that $\dim F_{2n+1} \ge 2k+\mu$ and $F_{2k+\mu} \subset \operatorname{Ker} T_u$.

Suppose now that $\dim \operatorname{Ker} T_u > 2k + \mu$. It follows that 1 is a singular value of H_u of multiplicity greater than $2k + \mu$. Therefore s is a singular value of $H_{\varphi-\psi}$ of multiplicity greater than $2k + \mu$. We have

$$s = s_{2k+\mu}(H_{\varphi-\psi}) \le s_{k+\mu}(H_\varphi) + s_k(H_\psi)$$

(see Appendix 1.1). Since $\operatorname{rank} H_\psi = k$, it follows that $s_k(H_\psi) = 0$ and so $s_{k+\mu}(H_\varphi) \ge s$, which contradicts (1.4). ■

To prove uniqueness, we need the following lemma.

Lemma 1.8. *Let u be a unimodular function such that $N = \dim \operatorname{Ker} T_u > 0$. Let f be a function in $H^\infty_{(N-1)}$ such that $\mathbb{P}_- f \ne \mathbb{O}$. Then $\|H_{u+f}\| > 1$.*

Proof. Since $f \in H^\infty_{(N-1)}$, there exists a finite Blaschke product B of degree at most $N - 1$ such that $Bf \in H^\infty$.

Let $v = Bu$. Then

$$\dim \operatorname{Ker} T_v = \dim \left(\operatorname{Ker} T_u \bigcap BH^2 \right) > 0.$$

Clearly, $\|H_v g\|_2 = \|g\|_2$ for any $g \in \operatorname{Ker} T_v$, and by Theorem 1.4 of Chapter 1, v admits a representation

$$v = \bar{z}\bar{\vartheta}\frac{\bar{h}}{h},$$

where ϑ is an inner function and h is an outer function in H^2. Clearly, $\|H_u h\|_2 = \|h\|_2$.

Let us show that $H_u h \perp H_f h$. We have $H_u h = \bar{z}\bar{\vartheta}\bar{B}\bar{h}$. Therefore

$$(H_u h, H_f h) = (\bar{z}\bar{\vartheta}\bar{B}\bar{h}, fh) = (\bar{z}\bar{\vartheta}\bar{h}, Bfh) = 0,$$

since $Bf \in H^\infty$. Consequently,

$$\|H_{u+f} h\|^2 = \|h\|^2 + \|H_f h\|^2.$$

To complete the proof, it suffices to show that $H_f h \ne \mathbb{O}$. Suppose that $H_f h = \mathbb{O}$. Then $H_f z^n h = \mathbb{O}$ and since h is outer, it would follow that $H_f = \mathbb{O}$, which contradicts the assumption $\mathbb{P}_- f \ne \mathbb{O}$. ■

Proof of Theorem 1.3. Suppose that there exists a function q in $H^\infty_{(k+\mu-1)}$ such that $q \ne \psi$ and

$$\|\varphi - q\|_\infty = s_k(H_\varphi). \tag{1.11}$$

We have

$$\varphi - q = \varphi - \psi + \psi - q.$$

Let $u = (\varphi - \psi)/s_k(H_\varphi)$, $f = (\psi - q)/s_k(H_\varphi)$. By Theorem 1.7, $\dim \operatorname{Ker} T_u = 2k + \mu$. Clearly, $\psi \in H^\infty_{(k)}$, $q \in H^\infty_{(k+\mu-1)}$ and so $\psi - q \in H^\infty_{(2k+\mu-1)}$. If $\mathbb{P}_-(\psi - q) \ne \mathbb{O}$, it follows from Lemma 1.8 that $\|H_{u+f}\| > 1$ and so $\|H_{\varphi-q}\| > s_k(H_\varphi)$, which contradicts (1.11).

Thus $\psi - q \in H^\infty$. The Hankel operator $H_{\varphi-\psi}$ attains its norm on the unit ball of H^2, since for any $\xi \in E_+$

$$\|H_{\varphi-\psi}\xi\|_2 = s_k(H_\varphi) \cdot \|\xi\|_2 = \|H_{\varphi-\psi}\| \cdot \|\xi\|_2.$$

Therefore if (1.11) holds, we have

$$\|H_{\varphi-\psi}\| = \|\varphi - \psi\|_\infty = \|\varphi - \psi + (\psi - q)\|_\infty.$$

By Theorem 1.1.4 it follows that $\psi - q = \mathbb{O}$. ■

Remarks. 1. It follows from (1.9) that the function ψ in $H^\infty_{(m)}$ satisfying the equality $\|\varphi - \psi\|_\infty = s_m(H_\varphi)$ is given by

$$\psi = \varphi - \frac{H_\varphi \xi}{\xi}, \tag{1.12}$$

where ξ is an arbitrary nonzero function satisfying

$$H^*_\varphi H_\varphi = s^2_m \xi.$$

2. It will be shown in §7.1 that if $\varphi \in VMO$, then $\varphi - \psi \in QC$ and so it will follow from Theorem 1.7 that

$$\operatorname{wind}(\varphi - \psi) = -(2k + \mu).$$

2. The Case $s_m(\Gamma) = s_\infty(\Gamma)$

In this section we complete the proof of Theorem 1.1. To this end we construct an auxiliary Hankel operator $\tilde{\Gamma}$ for which $s_m(\tilde{\Gamma}) > s_\infty(\tilde{\Gamma})$ and apply Theorem 1.1 to $\tilde{\Gamma}$, which we can do because of the above inequality. To construct $\tilde{\Gamma}$ we make use of the one-step extension method, which has been used in Chapter 2 to establish the analog of the Nehari theorem for vectorial Hankel operators.

Proof of Theorem 1.1 in the case $s_m(\Gamma) = s_\infty(\Gamma)$. If $s_m(\Gamma) = s_0(\Gamma)$, the situation is trivial ($\Gamma_m = \mathbb{O}$). Assume that

$$s_{k-1}(\Gamma) > s_k(\Gamma) = s_m(\Gamma) = s_\infty(\Gamma).$$

It is sufficient to show that there exists a Hankel operator Γ_k of rank k such that $\|\Gamma - \Gamma_k\| = s_k(\Gamma)$.

Let ρ be a real number satisfying

$$s_{k-1}(\Gamma) > \rho > s_k(\Gamma). \tag{2.1}$$

Let us show that if ρ satisfies (2.1), then there exists a Hankel operator H_ψ such that $\operatorname{rank} H_\psi \le k$ and $\|\Gamma - H_\psi\| \le \rho$.

To do that, we reduce the situation to the case already treated in §1. Namely, we construct an auxiliary Hankel operator $\tilde{\Gamma}$ such that

$$\operatorname{Ker}(\tilde{\Gamma}^*\tilde{\Gamma} - \rho^2 I) \ne \{\mathbb{O}\} \tag{2.2}$$

and

$$s_{k-1}(\tilde{\Gamma}) \ge s_{k-1}(\Gamma) \ge s_k(\tilde{\Gamma}) \ge s_k(\Gamma) \ge s_{k+1}(\tilde{\Gamma}). \tag{2.3}$$

We are searching for such a $\tilde{\Gamma}$ in the form

$$\tilde{\Gamma}f = \bar{z}\Gamma f + (f,h)\bar{z}, \quad f \in H^2, \tag{2.4}$$

where $\mathcal{H} \in H^2$.

Let us find out under which conditions on h the operator $\tilde{\Gamma}$ defined by (2.4) is a Hankel operator. In other words, we have to characterize those h for which

$$\tilde{\Gamma}z^n = \mathbb{P}_- z^n \tilde{\Gamma}\mathbf{1}, \quad n \geq 0. \tag{2.5}$$

Let $\Gamma = H_\varphi$. It is easy to see that $\tilde{\Gamma}$ satisfies (2.5) if and only if

$$\overline{\hat{h}(n)} = \hat{\varphi}(-n), \quad n > 1.$$

If $\tilde{\Gamma} = H_{\varphi_1}$, then $\hat{\varphi}_1(-n) = \hat{\varphi}(-n+1)$, $n > 1$. Let $\alpha = \hat{\varphi}_1(-1)$. Then

$$h = \bar{\alpha} + z\mathbb{P}_+\bar{z}\bar{\varphi} = \bar{\alpha} + z\Gamma^*\bar{z}. \tag{2.6}$$

It follows from (2.4) that

$$\tilde{\Gamma}^*\tilde{\Gamma}f = \Gamma^*\Gamma f + (f,h)h, \quad f \in H^2.$$

Therefore $\tilde{\Gamma}^*\tilde{\Gamma}$ is a perturbation of $\Gamma^*\Gamma$ by a nonnegative rank one operator, which implies (2.3) (see Appendix 1.1).

Lemma 2.1. *For any $\rho \in (s_k(\Gamma), s_{k-1}(\Gamma))$ sufficiently close to $s_k(\Gamma)$ there exists $\alpha \in \mathbb{C}$ such that* (2.2) *holds.*

By Lemma 2.1, if ρ is sufficiently close to $s_k(\Gamma)$, then ρ is a singular value of $\tilde{\Gamma}$, and since (2.3) is satisfied, the only possibility is that $\rho = s_k(\tilde{\Gamma})$. Clearly, $s_\infty(\tilde{\Gamma}) = s_\infty(\Gamma) < \rho$. So it follows from the part of Theorem 1.1 already treated in §1 that there exists a Hankel operator H_{ψ_1} such that $\operatorname{rank} H_{\psi_1} = k$ and $\|\tilde{\Gamma} - H_{\psi_1}\| = s_k(\tilde{\Gamma}) = \rho$. As we have already observed, $\tilde{\Gamma} = H_{\varphi_1}$, where $\hat{\varphi}_1(n) = \hat{\varphi}(n+1)$, $n \leq -2$, and so

$$\Gamma = \mathbb{P}_-\mathcal{S}\tilde{\Gamma},$$

where $\mathcal{S}$ is multiplication by z on L^2.

Put $\psi = z\psi_1$. We have

$$\|\Gamma - H_\psi\| = \|\mathbb{P}_-\mathcal{S}\tilde{\Gamma} - \mathbb{P}_-\mathcal{S}H_{\psi_1}\| \leq \|\tilde{\Gamma} - H_{\psi_1}\| = \rho.$$

On the other hand, $\operatorname{rank} H_\psi = \operatorname{rank} \mathbb{P}_-\mathcal{S}H_{\psi_1} \leq \operatorname{rank} H_{\psi_1} = k$.

To complete the proof, we can take a sequence $\{\rho_n\}$ in $(s_k(\Gamma), s_{k-1}(\Gamma))$ that tends to $s_k(\Gamma)$ and construct a sequence of Hankel operators H_{φ_n} such that $\operatorname{rank} H_{\varphi_n} \leq k$ and $\|\Gamma - H_{\varphi_n}\| \leq \rho_n$.

Since the operators H_{φ_n} are uniformly bounded, there exists a subsequence weakly convergent to an operator, say Γ_m. It is easy to see that $\operatorname{rank} \Gamma_m \leq k$ and Γ_m is a Hankel operator. Clearly, $\|\Gamma - \Gamma_m\| \leq s_k(\Gamma)$, which completes the proof. ■

Proof of Lemma 2.1. Clearly, (2.2) is equivalent to the fact that there is a nonzero f in H^2 satisfying

$$(\rho^2 I - \Gamma^*\Gamma)f = (f, h)h. \tag{2.7}$$

Put

$$R_\rho = (\rho^2 I - \Gamma^*\Gamma)^{-1}$$

and

$$r(\rho) = (R_\rho \mathbf{1}, \mathbf{1}).$$

It is easy to see that there can be at most one ρ in $(s_k(\Gamma), s_{k-1}(\Gamma))$ such that $r(\rho) = 0$. Indeed, r is differentiable on $(s_k(\Gamma), s_{k-1}(\Gamma))$ and

$$\frac{dr(\rho)}{d\rho} = -2\rho(R_\rho^2 \mathbf{1}, \mathbf{1}) < 0, \quad \rho \in (s_k(\Gamma), s_{k-1}(\Gamma)),$$

and so r is strictly decreasing on $(s_k(\Gamma), s_{k-1}(\Gamma))$.

Lemma 2.2. *If* $r(\rho) \neq 0$, *then there exists* $\alpha \in \mathbb{C}$ *such that the function* h *defined by* (2.6) *satisfies*

$$(R_\rho h, h) = 1. \tag{2.8}$$

Let us first complete the proof of Lemma 2.1. Put $f = R_\rho h$, then f is nonzero and satisfies (2.7). ■

Proof of Lemma 2.2. Put $g = z\Gamma^*\bar{z}$. Then (2.6) means that $h = \bar{\alpha} + g$. It is easy to see that (2.8) is equivalent to the following equation:

$$|\alpha|^2 r(\rho) + 2\,\mathrm{Re}\,(\bar{\alpha}(R_\rho \mathbf{1}, g)) + (R_\rho g, g) = 1.$$

Multiplying both sides of this equation by $r(\rho)$, we see that it is equivalent to the following one:

$$|\alpha r(\rho) + (R_\rho \mathbf{1}, g)|^2 = r(\rho) + |(R_\rho \mathbf{1}, g)|^2 - r(\rho)(R_\rho g, g). \tag{2.9}$$

Let us prove that

$$r(\rho) + |(R_\rho \mathbf{1}, g)|^2 - r(\rho)(R_\rho g, g) = \frac{1}{\rho^2}. \tag{2.10}$$

If we do that, we can put for any complex τ of modulus 1

$$\alpha = \frac{\tau}{\rho r(\rho)} - \frac{(R_\rho \mathbf{1}, g)}{r(\rho)}.$$

Clearly, α satisfies (2.9).

Thus to complete the proof, we have to verify (2.10).

Let J be the operator on L^2 defined by $Jf = \bar{z}\bar{f}$, $f \in L^2$. Put $\Gamma' = SJ\Gamma$, where S is multiplication by z on H^2. Clearly, $\Gamma'\mathbf{1} = \overline{\Gamma\mathbf{1}} = g$. We are going to establish the identity

$$\Gamma' R_\rho \mathbf{1} = \rho^2 r(\rho) R_\rho g - \rho^2 (R_\rho g, \mathbf{1}) R_\rho \mathbf{1}. \tag{2.11}$$

Let us first show that (2.11) implies (2.10). Taking the inner product of both sides of (2.11) with $g = \Gamma'\mathbf{1}$, we obtain

$$\begin{aligned}\rho^2 r(\rho)(R_\rho g, g) - \rho^2|(R_\rho g, \mathbf{1})|^2 &= (\Gamma' R_\rho \mathbf{1}, \Gamma'\mathbf{1}) = (SJ\Gamma R_\rho \mathbf{1}, SJ\Gamma\mathbf{1})\\ &= (J\Gamma R_\rho \mathbf{1}, J\Gamma\mathbf{1}) = (\Gamma\mathbf{1}, \Gamma R_\rho \mathbf{1})\\ &= (\mathbf{1}, \Gamma^*\Gamma R_\rho \mathbf{1}) = \rho^2 r(\rho) - 1,\end{aligned}$$

which proves (2.10).

To establish (2.11), we apply to both sides of (2.11) the operator $R_\rho^{-1} = \rho^2 I - \Gamma^*\Gamma$ and find that it is equivalent to

$$R_\rho^{-1}\Gamma' R_\rho \mathbf{1} = \rho^2 r(\rho) g - \rho^2 (R_\rho g, \mathbf{1})\mathbf{1}. \tag{2.12}$$

Let P_0 be the orthogonal projection from H^2 onto the space of constant functions, i.e., $P_0 f = (f, \mathbf{1})\mathbf{1}$. Let us show that

$$(I - P_0) R_\rho^{-1}\Gamma' R_\rho \mathbf{1} = \rho^2 r(\rho) g \tag{2.13}$$

and

$$(R_\rho^{-1}\Gamma' R_\rho \mathbf{1}, \mathbf{1}) = -\rho^2 (R_\rho g, \mathbf{1}), \tag{2.14}$$

which will imply (2.12).

Let us first prove (2.13). It is easy to check that $J\Gamma = \Gamma^* J$ and $J\Gamma S = S^*\Gamma^* J$. Hence,

$$(\Gamma')^2 = SJ\Gamma SJ\Gamma = SS^*\Gamma^*\Gamma = (I - P_0)\Gamma^*\Gamma.$$

It follows that

$$(I - P_0) R_\rho^{-1} = (I - P_0)(\rho^2 I - (\Gamma')^2). \tag{2.15}$$

Hence,

$$\begin{aligned}(I - P_0) R_\rho^{-1}\Gamma' R_\rho \mathbf{1} &= (I - P_0)(\rho^2 I - (\Gamma')^2)\Gamma' R_\rho \mathbf{1} = \Gamma'(\rho^2 I - (\Gamma')^2) R_\rho \mathbf{1}\\ &= \Gamma'(I - P_0)(\rho^2 I - (\Gamma')^2) R_\rho \mathbf{1} + \Gamma' P_0 (\rho^2 - (\Gamma')^2) R_\rho \mathbf{1}.\end{aligned}$$

By (2.15)

$$\Gamma'(I - P_0)(\rho^2 I - (\Gamma')^2) R_\rho \mathbf{1} = \Gamma'(I - P_0)\mathbf{1} = \mathbb{O}.$$

Thus

$$\begin{aligned}(I - P_0) R_\rho^{-1}\Gamma' R_\rho \mathbf{1} &= \Gamma' P_0 (\rho^2 - (\Gamma')^2) R_\rho \mathbf{1}\\ &= \rho^2 \Gamma' P_0 R_\rho \mathbf{1} = \rho^2 r(\rho)\Gamma'\mathbf{1} = \rho^2 r(\rho) g,\end{aligned}$$

since $P_0\Gamma' = \mathbb{O}$.

It remains to prove (2.14). We have

$$\Gamma\Gamma' = J(J\Gamma S)J\Gamma = J(S^*\Gamma^* J)J\Gamma = JS^*(\rho^2 I - R_\rho^{-1}).$$

Consequently,

$$\begin{aligned}(R_\rho^{-1}\Gamma' R_\rho \mathbf{1}, \mathbf{1}) &= \rho^2(\Gamma' R_\rho \mathbf{1}, \mathbf{1}) - (\Gamma^*\Gamma\Gamma' R_\rho \mathbf{1}, \mathbf{1}) \\ &= -(JS^*(\rho^2 I - R_\rho^{-1})R_\rho \mathbf{1}, \Gamma\mathbf{1}) \\ &= -\rho^2(JS^* R_\rho \mathbf{1}, \Gamma\mathbf{1}) = -\rho^2(J\Gamma\mathbf{1}, S^* R_\rho \mathbf{1}) \\ &= -\rho^2(g, R_\rho \mathbf{1}),\end{aligned}$$

since $\Gamma' f \perp \mathbf{1}$ for any $f \in H^2$, which completes the proof. ■

3. Finite Rank Approximation of Vectorial Hankel Operators

In this section we present another approach to the problem of finite rank approximation of Hankel operators which works in both cases $s_m(\Gamma) > s_\infty(\Gamma)$ and $s_m(\Gamma) = s_\infty(\Gamma)$. Moreover, it also works in the case of vectorial Hankel operators. However, the proof given in §1 for the scalar case gives more information on the best approximant. As an application we find a formula for the essential norm of a vectorial Hankel operator.

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. The main result of this section is the following generalization of the Adamyan–Arov–Krein theorem.

Theorem 3.1. *Let $\Phi \in L^\infty(\mathcal{B}(\mathcal{H}, \mathcal{K}))$. Then*

$$s_n(H_\Phi) = \inf\{\|\Phi - \Psi\|_\infty : \operatorname{rank} H_\Psi \le n\}.$$

Remark. It is easy to see that "infimum" in the statement of Theorem 3.1 can be replaced with "minimum". Indeed, if $\{\Psi_j\}$ is a bounded sequence of functions in $L^\infty(\mathcal{B}(\mathcal{H}, \mathcal{K}))$ such that $\operatorname{rank} H_{\Psi_j} \le n$ and $\|\Phi - \Psi_j\|_\infty \to s_n(H_\Phi)$, there exists a subsequence $\{\Psi_{j_k}\}$ such that $\hat{\Psi}_{j_k}(m)$ converges weakly to $\hat{\Psi}(m)$, $m \in \mathbb{Z}$, for some $\Psi \in L^\infty(\mathcal{B}(\mathcal{H}, \mathcal{K}))$. It is easy to see that the sequence $\{H_{\Psi_{j_k}}\}$ converges to H_Ψ in the weak operator topology, and so $\operatorname{rank} H_\Psi \le n$. Clearly, $\|\Phi - \Psi\|_\infty = s_n(H_\Phi)$.

To prove Theorem 3.1, we make use of the following fact, which will be established after the proof of Theorem 3.1.

Theorem 3.2. *Let A be a bounded self-adjoint operator and S a bounded operator on a Hilbert space H. Let*

$$\mathcal{C} = \{x \in H : (Ax, x) \ge 0\}$$

and let $\mathcal{P}_- = \mathcal{E}((-\infty, 0))$, $\mathcal{P}_+ = \mathcal{E}([0, \infty))$, where $\mathcal{E}$ is the spectral measure of A. Suppose that the following conditions are satisfied:

(a) $A_- \stackrel{\text{def}}{=} A\,|\mathrm{Range}\,\mathcal{P}_-$ *is invertible;*

(b) $S\mathcal{C} \subset \mathcal{C}$*;*

(c) $\mathcal{P}_+ S \mathcal{P}_-$ *is compact.*

Then there exists a subspace $\mathcal{L}$ of H such that $S\mathcal{L} \subset \mathcal{L}$ and $\mathcal{L} \subset \mathcal{C}$, and $\mathcal{L}$ is a maximal (by inclusion) subspace of H contained in $\mathcal{C}$.

Put $H_- = \mathcal{P}_- H$, $H_+ = \mathcal{P}_+ H$, and $A_- = A\big|H_-$, $A_+ = A\big|H_+$. Denote by $\mathcal{M}$ the set of subspaces $\mathcal{L}$ of H such that $\mathcal{L} \subset \mathcal{C}$ and $\mathcal{P}_+\mathcal{L} = H_+$. Clearly, $\mathcal{M} \neq \varnothing$, since $H_+ \in \mathcal{M}$.

To deduce Theorem 3.1 from Theorem 3.2 we need the following lemma.

Lemma 3.3. *The following assertions hold:*

(i) *if $\mathcal{L} \in \mathcal{M}$, then there exists a unique operator $T_{\mathcal{L}} : H_+ \to H_-$ such that*

$$\mathcal{L} = \{x + T_{\mathcal{L}}x : \ x \in H_+\}; \tag{3.1}$$

(ii) *the set $\{T_{\mathcal{L}} : \mathcal{L} \in \mathcal{M}\}$ is a convex subset of $B(H_+, H_-)$, which is compact in the weak operator topology;*

(iii) *if $\mathcal{L}$ is a subspace of H, $\mathcal{L} \subset \mathcal{C}$, then $\mathcal{L}$ belongs to $\mathcal{M}$ if and only if it is a maximal subspace contained in $\mathcal{C}$;*

(iv) *if* $\operatorname{rank} A_- < \infty$, *$\mathcal{L}$ is a subspace of H, $\mathcal{L} \subset \mathcal{C}$, then it belongs to $\mathcal{M}$ if and only if* $\operatorname{codim} L = \operatorname{rank} A_-$.

First we deduce Theorem 3.1 from Theorem 3.2. Then we prove Lemma 3.3 and, finally, we prove Theorem 3.2.

Proof of Theorem 3.1. Let $H = H^2(\mathcal{H})$, $A = s_n^2(H_\Phi)I - H_\Phi^* H_\Phi$, and S is multiplication by z on $H^2(\mathcal{H})$. Clearly, in this case

$$\mathcal{C} = \left\{f \in H^2(\mathcal{H}) : \ \|H_\Phi f\| \le s_n(H_\Phi)\|f\|\right\}.$$

It is easy to see that if $f \in \mathcal{C}$, then

$$\|H_\Phi SF\| \le \|H_\Phi f\| \le s_n(H_\Phi)\|f\| = s_n(H_\Phi)\|Sf\|.$$

Hence $S\mathcal{C} \subset \mathcal{C}$.

Clearly, $\operatorname{rank} \mathcal{P}_- \le n$, and so conditions (a) and (c) in the statement of Theorem 3.2 are obviously satisfied.

By Theorem 3.2 and Lemma 3.3, (iii), (iv) there exists a subspace $\mathcal{L} \subset H^2(\mathcal{H})$ such that $S\mathcal{L} \subset \mathcal{L}$, $\operatorname{codim} \mathcal{L} \le n$, and $\mathcal{L} \subset \mathcal{C}$. Clearly, the last inclusion means that

$$\left\|H_\Phi\big|\mathcal{L}\right\| \le s_n(H_\Phi). \tag{3.2}$$

By Lemma 2.5.1, there exists a Blaschke–Potapov product B of finite degree such that $\mathcal{L} = BH^2(\mathcal{H})$. Therefore

$$\left\|H_\Phi\big|\mathcal{L}\right\| = \left\|H_\Phi\big|BH^2(\mathcal{H})\right\| = \|H_{\Phi B}\|,$$

and it follows from (3.2) that $\|H_{\Phi B}\| \le s_n(H_\Phi)$. By Theorem 2.2.2, there exists a function Ω in $H^\infty(\mathcal{H}, \mathcal{K})$ such that $\|H_{\Phi B}\| = \|\Phi B - \Omega\|_\infty$. Since B takes unitary values on T, we have $\|\Phi B - \Omega\|_\infty = \|\Phi - \Omega B^*\|_\infty$. Since $\mathcal{L} = BH^2(\mathcal{H}) \subset \operatorname{Ker} H_{\Omega B^*}$, it follows that

$$\operatorname{rank} H_{\Omega B^*} \le \operatorname{codim} \mathcal{L} \le n,$$

and so

$$\inf\{\|\Phi - \Psi\| : \ \operatorname{rank} H_\Psi \le n\} \le \|\Phi - \Omega B^*\|_\infty \le s_n(H_\Phi).$$

The opposite inequality

$$s_n(H_\Phi) \le \inf\{\|\Phi - \Psi\|_\infty : \ \operatorname{rank} H_\Psi \le n\}$$

is trivial. ■

Proof of Lemma 3.3. (i) Let $x \in \mathcal{C}$. Then $x = x_+ + x_-$, where $x_+ = \mathcal{P}_+ x$, $x_- = \mathcal{P}_- x$. We have

$$0 \le (Ax, x) = (A_+ x_+, x_+) + (A_- x_-, x_-) \le \|A_+\| \cdot \|x_+\|^2 - \|A_-^{-1}\|^{-1} \|x_-\|^2.$$

Therefore

$$\|x_-\| \le (\|A_+\| \cdot \|A_-^{-1}\|)^{1/2} \|x_+\|. \tag{3.3}$$

This implies that $\mathcal{P}_+ | \mathcal{L}$ is bounded from below, since it follows from (3.3) that for $x \in \mathcal{L}$

$$\|x\|^2 = \|x_+\|^2 + \|x_-\|^2 \le (1 + \|A_+\| \cdot \|A_-^{-1}\|) \|x_+\|^2.$$

Thus we can put

$$T_\mathcal{L} = \mathcal{P}_- (\mathcal{P}_+ \big| \mathcal{L})^{-1}, \tag{3.4}$$

which is well-defined on H_+. It is easy to check that (3.1) holds. The uniqueness of $T_\mathcal{L}$ is obvious.

(ii) The fact that the set $\{T_\mathcal{L} : \mathcal{L} \in \mathcal{M}\}$ is bounded follows from (3.3) and (3.4):

$$\|T_\mathcal{L}\| \le (\|A_+\| \cdot \|A_-^{-1}\|)^{1/2}$$

for any $\mathcal{L}$ in $\mathcal{M}$. Let us show that it is convex. Let $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathcal{M}$, $\lambda \in (0,1)$, and $T_\lambda \stackrel{\text{def}}{=} \lambda T_{\mathcal{L}_1} + (1-\lambda) T_{\mathcal{L}_2}$. If $x \in H_+$, we have

$$\begin{aligned} (A(x + T_\lambda x), x + T_\lambda x) &= (A_+ x, x) + (A_- T_\lambda x, T_\lambda x) \\ &\ge (A_+ x, x) + \min_{j=1,2} \left\{ (A_- T_{\mathcal{L}_j} x, T_{\mathcal{L}_j} x) \right\} \ge 0, \end{aligned}$$

since $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathcal{M}$. Consequently, $\{x + T_\lambda x : \ x \in H_+\} \in \mathcal{M}$ for $\lambda \in (0,1)$.

Since for any $x_+ \in H_+$, the set

$$\{x_- \in H_- : \ (A(x_+ + x_-), x_+ + x_-) \ge 0\}$$

is closed, it follows that the set $\{T_\mathcal{L} : \mathcal{L} \in \mathcal{M}\}$ is closed in the strong operator topology and since this set is convex, it is also closed in the weak operator topology (see Dunford and Schwartz [1], VI.1.5). Hence, it is weakly compact.

(iii) Let $\mathcal{L} \in \mathcal{M}$. We are going to show that $\mathcal{L}$ is a maximal subspace contained in $\mathcal{C}$.

Suppose that $\mathcal{L}'$ is a subspace of H and $\mathcal{L} \subset \mathcal{L}' \subset \mathcal{C}$. It follows that $\mathcal{P}_+ \mathcal{L}' = H_+$ and by (i)

$$\mathcal{L}' = \{x + T_{\mathcal{L}'} x : \ x \in H_+\}.$$

On the other hand,

$$\mathcal{L} = \{x + T_\mathcal{L} x : \ x \in H_+\}.$$

Since $\mathcal{L} \subset \mathcal{L}'$, it follows that $T_{\mathcal{L}'} = T_\mathcal{L}$ and so $\mathcal{L}' = \mathcal{L}$.

Suppose now that $\mathcal{L}$ is a maximal subspace of $\mathcal{C}$ but $\mathcal{P}_+\mathcal{L} \neq H_+$. Since it has been proved in (i) that $\mathcal{P}_+\big|\mathcal{L}$ is bounded from below, it follows that $\mathcal{P}_+\big|\mathcal{L}$ is a closed subspace of H_+.

Assume first that A_+ is invertible. Then $A_+\mathcal{P}_+L$ is a closed subspace of H_+ and $A_+\mathcal{P}_+L \neq H_+$. Put

$$\mathcal{L}' = \mathcal{L} + (H_+ \ominus A_+\mathcal{P}_+\mathcal{L}).$$

Clearly, $\mathcal{L}' \neq \mathcal{L}$. If $x \in \mathcal{L}$ and $f \in H_+ \ominus A_+\mathcal{P}_+\mathcal{L}$, we have

$$\begin{aligned}(A(x+f), x+f) &= (A_-x_-, x_-) + (A_+(x_+ + f), x_+ + f)\\ &= (A_-x_-, x_-) + (A_+x_+, x_+) + (A_+f, f) \geq (Ax, x) \geq 0.\end{aligned}$$

Hence, $\mathcal{L}' \subset \mathcal{C}$, which contradicts the assumption that $\mathcal{L}$ is maximal.

Let us now consider the general case. Put

$$A^{(n)} = A_-\mathcal{P}_- + \left(A_+ + \frac{1}{n}I_{H_+}\right)\mathcal{P}_+, \quad n \geq 1.$$

Then $A_+^{(n)}$ is invertible. Clearly,

$$\mathcal{L} \subset \mathcal{C}_n \stackrel{\text{def}}{=} \{x \in H : \ (A^{(n)}x, x) \geq 0\}.$$

By Zorn's lemma $\mathcal{L}$ is contained in a maximal subspace of $\mathcal{C}_n$ and since $A_+^{(n)}$ is invertible, we have already proved there exists $T_n : H_+ \to H_-$ such that

$$\mathcal{L} \subset \{x + T_nx : \ x \in H_+\} \subset \mathcal{C}_n.$$

Clearly, $\sup\limits_n \|T_n\| < \infty$. Let T be the weak limit of a subsequence of $\{T_n\}_{n\geq 1}$. Since $\mathcal{C}_{n+1} \subset \mathcal{C}_n$, we have

$$\mathcal{L} \subset \{x + Tx : \ x \in H_+\} \subset \bigcap_{n\geq 1} \mathcal{C}_n = \mathcal{C}.$$

Clearly, $\{x + Tx : \ x \in H_+\} \in \mathcal{M}$ and since $\mathcal{L}$ is maximal, it follows that

$$\mathcal{L} = \{x + Tx : \ x \in H_+\} \in \mathcal{M}.$$

(iv) Since $\operatorname{codim} H_+ = \operatorname{rank} A_-$, the result follows from the following equality, which is valid for any bounded operator $T : H_+ \to H_-$:

$$\operatorname{codim}\{x + Tx : \ x \in H_+\} = \operatorname{codim} H_+. \quad \blacksquare$$

To prove Theorem 3.2 we need the following fact.

Lemma 3.4. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces and let $\mathcal{X}$ be a convex subset of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ that is compact in the weak operator topology. Let α be a map from $\mathcal{X}$ to $\mathcal{B}(\mathcal{H}_1)$ and β a map from $\mathcal{X}$ to $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ that satisfy the following conditions:*

(1) *for any $X \in \mathcal{X}$ there exists $Y \in \mathcal{X}$ such that*

$$Y\alpha(X) = \beta(X);$$

(2) *the map $F : \mathcal{X} \times \mathcal{X} \to \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ defined by*

$$F(X, Y) = Y\alpha(X) - \beta(X) \tag{3.5}$$

is jointly continuous in the weak operator topology.

Then there exists $X_0 \in \mathcal{X}$ *such that*

$$X_0\alpha(X_0) = \beta(X_0).$$

Proof of Theorem 3.2. Put $\mathcal{H}_1 = H_+$, $\mathcal{H}_2 = H_-$,

$$\mathcal{X} = \{T_{\mathcal{L}} : \ L \in \mathcal{M}\},$$

$$\alpha(X) = \mathcal{P}_+ S\big|H_+ + \mathcal{P}_+ SX, \quad X \in \mathcal{X},$$

$$\beta(X) = \mathcal{P}_- S\big|H_+ + \mathcal{P}_- SX, \quad X \in \mathcal{X}.$$

By Lemma 3.3, $\mathcal{X}$ is convex and weakly compact. Since the operator $\mathcal{P}_+ S \mathcal{P}_-$ is compact and $\mathcal{P}_+ SX = \mathcal{P}_+ S\mathcal{P}_- X$, $X \in \mathcal{X}$, it follows that the function F defined by (3.5) satisfies condition (2) of Lemma 3.4.

Let us verify condition (1). Since $S\mathcal{C} \subset \mathcal{C}$ and any subspace contained in $\mathcal{C}$ is contained in a maximal subspace in $\mathcal{C}$ (i.e., a subspace from $\mathcal{M}$), it follows that for any $\mathcal{L} \in \mathcal{M}$ there exists $\mathcal{L}' \in \mathcal{M}$ such that

$$S\mathcal{L} \subset \mathcal{L}'. \tag{3.6}$$

If $x \in \mathcal{C}$, $\mathcal{L} \in \mathcal{M}$, then $x \in \mathcal{L}$ if and only if

$$T_{\mathcal{L}}\mathcal{P}_+ x = \mathcal{P}_- x. \tag{3.7}$$

Clearly,

$$S\mathcal{L} = \{Sx + ST_{\mathcal{L}}x : \ x \in H_+\}.$$

It follows from (3.7) that (3.6) is equivalent to the fact that

$$T_{\mathcal{L}'}(\mathcal{P}_+ Sx + \mathcal{P}_+ ST_{\mathcal{L}}x) = \mathcal{P}_- Sx + \mathcal{P}_- ST_{\mathcal{L}}x \tag{3.8}$$

for any $x \in H_+$, which means that condition (1) of Lemma 3.4 is satisfied.

It follows from Lemma 3.4 that there exists $\mathcal{L} \in \mathcal{M}$ such that (3.8) holds with $\mathcal{L}' = \mathcal{L}$, which means that $S\mathcal{L} \subset \mathcal{L}$. ■

Lemma 3.4 follows trivially from the following fact.

Theorem 3.5. *Let* W *be a locally convex Hausdorff space,* $\mathcal{X}$ *a compact convex subset of* W*, and* F *a continuous map from* $\mathcal{X} \times \mathcal{X}$ *to* W *such that*

$$F(x, \lambda y_1 + (1-\lambda)y_2) = \lambda F(x, y_1) + (1-\lambda)F(x, y_2), \ x, \ y_1, \ y_2 \in \mathcal{X}, \ 0 \le \lambda \le 1.$$

If for any $x \in \mathcal{X}$ *there exists* $y \in \mathcal{X}$ *such that* $F(x,y) = \mathbb{O}$*, then there exists* $x_0 \in \mathcal{X}$ *such that* $F(x_0, x_0) = \mathbb{O}$.

Note that if $F(x,y) = y - f(x)$, where f is a continuous map from $\mathcal{X}$ to $\mathcal{X}$, then Theorem 3.5 turns into Tikhonov's fixed point theorem (see Edwards [1], §3.6).

Lemma 3.6. *Let* $\mathcal{X}$ *be a compact convex subset in a locally convex space and let* Q *be a closed subset of* $\mathcal{X} \times \mathcal{X}$ *such that* $(x, x) \in Q$ *for any* $x \in \mathcal{X}$ *and the set* $\{y \in \mathcal{X} : \ (x, y) \notin Q\}$ *is convex for any* $x \in \mathcal{X}$*. Then there exists* $x_0 \in \mathcal{X}$ *such that* $(x_0, y) \in Q$ *for any* $y \in \mathcal{X}$*.*

Let us first deduce Theorem 3.5 from Lemma 3.6.

Proof of Theorem 3.5. Let $\{\rho_\nu\}$ be a set of seminorms that determines the topology in W. Put

$$G_\nu = \{x \in \mathcal{X} : \rho_\nu(F(x,x)) = 0\}.$$

Since $\mathcal{X}$ is compact, it is sufficient to show that for any finite set $\nu_1, \cdots, \nu_n$ the set

$$\bigcap_{k=1}^{n} G_{\nu_k} \neq \varnothing.$$

Fix $\nu_1, \cdots, \nu_n$ and put

$$Q = \left\{ (x,y) \in \mathcal{X} \times \mathcal{X} : \ \sum_{k=1}^{n} \rho_{\nu_k}(F(x,y)) \geq \sum_{k=1}^{n} \rho_{\nu_k}(F(x,x)) \right\}.$$

It is easy to see that Q satisfies the assumptions of Lemma 3.6. Therefore there exists $x_0 \in \mathcal{X}$ such that $(x_0, y) \in Q$ for any $y \in \mathcal{X}$.

By the assumptions of Theorem 3.5 there exists $y_0 \in \mathcal{X}$ such that $F(x_0, y_0) = \mathbb{O}$. By the definition of Q

$$\sum_{k=1}^{n} \rho_{\nu_k}(F(x_0,x_0)) \leq \sum_{k=1}^{n} \rho_{\nu_k}(F(x_0,y_0)) = 0,$$

so $\rho_{\nu_k}(F(x_0,x_0)) = 0$ for any k, $1 \leq k \leq n$, which means that

$$x_0 \in \bigcap_{k=1}^{n} G_{\nu_k}. \quad \blacksquare$$

To prove Lemma 3.6, we need one more lemma.

Lemma 3.7. *Let $t_1, t_2, \cdots, t_n$ be points in $\mathbb{R}^{n-1}$ such that their convex hull*

$$K = \operatorname{conv}\{t_1, t_2, \cdots, t_n\}$$

is an $(n-1)$-dimensional simplex (i.e., has nonempty interior in $\mathbb{R}^{n-1}$). Let $V_1, V_2, \cdots, V_n$ be subsets of K that are open in K. Suppose that for any subset $\{k_1, k_2, \cdots, k_l\}$ of $\{1, 2, \cdots, n\}$

$$\bigcap_{s=1}^{l} V_{j_s} \bigcap \operatorname{conv}\{t_{k_1}, t_{k_2}, \cdots, t_{k_l}\} = \varnothing. \tag{3.9}$$

Then $K \neq \bigcup_{j=1}^{n} V_j$.

Proof. Suppose that $K = \bigcup_{j=1}^{n} V_j$. Then there are closed sets $W_1, W_2, \cdots, W_n$ such that $W_j \subset V_j$, $1 \leq j \leq n$, and $K = \bigcup_{j=1}^{n} W_j$. Without loss of generality we can assume that

$$\sum_{j=1}^{n} t_j = 0.$$

It follows that any $n-1$ vectors from $\{t_1, t_2, \cdots, t_n\}$ are linearly independent.

Put

$$d_j(t) = \operatorname{dist}(t, W_j), \quad t \in K,$$

and

$$D_j(t) = \max\{s \geq 0 : \ t - s t_j \in K\}. \tag{3.10}$$

Let F be the continuous function on K defined by

$$F(t) = t - \frac{1}{n} \sum_{j=1}^{n} (\min\{d_j(t), D_j(t)\})\, t_j.$$

It follows from (3.10) that $F(t) \in K$ for any $t \in K$. By Brouwer's fixed point theorem (see Dunford and Schwartz [1], Ch. V, §12), there exists $t_0 \in K$ such that $F(t_0) = t_0$. Since $\bigcup_{j=1}^{n} W_j = K$, it follows that $d_j(t_0) = 0$ at least for one j. Since any $n-1$ vectors from the set $t_1, t_2, \cdots, t_n$ are linearly independent, it follows from the equality $F(t_0) = t_0$ that

$$\min\{d_j(t_0), D_j(t_0)\} = 0 \tag{3.11}$$

for $1 \leq j \leq n$.

Clearly, $d_j(t_0) = 0$ if and only if $t_0 \in W_j$, while $D_j(t_0) = 0$ if and only if $t \in \operatorname{conv}\{t_1, \cdots, t_{j-1}, t_{j+1}, \cdots, t_n\}$. Therefore it follows from (3.11) that there exists a subset $\{k_1, k_1, \cdots, k_l\}$ of $\{1, 2, \cdots, n\}$ such that

$$\bigcap_{s=1}^{l} W_{j_s} \bigcap \operatorname{conv}\{t_{k_1}, t_{k_2}, \cdots, t_{k_l}\} \neq \varnothing,$$

which contradicts (3.9). ■

Proof of Lemma 3.6. Let $y \in \mathcal{X}$. Put

$$Q_y \stackrel{\text{def}}{=} \{x \in \mathcal{X} : \ (x, y) \in Q\}.$$

We have to verify that $\bigcap_{y \in \mathcal{X}} Q_y \neq \varnothing$. Since $\mathcal{X}$ is compact, it is sufficient to show that $\bigcap_{j=1}^{n} Q_{y_j} \neq \varnothing$ for any finite set $y_1, y_2, \cdots, y_n$ in $\mathcal{X}$. The latter is equivalent to the fact that $\bigcup_{j=1}^{n} V_{y_j} \neq \mathcal{X}$, where $V_y \stackrel{\text{def}}{=} \mathcal{X} \setminus Q_y$. Clearly, V_y is open.

Let us show that for any $y_1, y_2, \cdots, y_n$ in $\mathcal{X}$

$$\bigcap_{j=1}^{n} V_{y_j} \bigcap \operatorname{conv}\{y_1, y_2, \cdots, y_n\} = \varnothing.$$

Indeed, if x belongs to the above intersection, then

$$y_j \in \{y \in \mathcal{X} : \ (x,y) \notin Q\}, \quad 1 \le j \le n,$$

and so by the hypotheses of Lemma 3.6, $(x,x) \notin Q$, which contradicts the assumption that Q contains the diagonal.

Let $t_1, t_2, \cdots, t_n \in \mathbb{R}^{n-1}$ be vectors satisfying the hypotheses of Lemma 3.7. Let

$$\tau : K = \operatorname{conv}\{t_1, t_2, \cdots, t_n\} \to \operatorname{conv}\{y_1, y_2, \cdots, y_n\} \subset \mathcal{X}$$

be the affine map defined by $\tau t_j = y_j$. Then the sets $V_j \stackrel{\text{def}}{=} \tau^{-1}(V_{y_j})$ satisfy the hypotheses of Lemma 3.7. It follows that $\bigcup\limits_{j=1}^{n} V_j \ne K$, so

$$\operatorname{conv}\{y_1, y_2, \cdots, y_n\} \not\subset \bigcup_{j=1}^{n} V_{y_j},$$

which completes the proof. ■

To conclude this section, we obtain a formula for the essential norm of a vectorial Hankel operator that is analogous to Theorem 1.5.3.

Theorem 3.8. *Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces and let $\Phi \in L^\infty((\mathcal{B}(\mathcal{H},\mathcal{K}))$. Then*

$$\|H_\Phi\|_{\mathrm{e}} = \operatorname{dist}_{L^\infty}\big(\Phi, H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K})) + C(\boldsymbol{\mathcal{C}}(\mathcal{H},\mathcal{K}))\big). \tag{3.12}$$

Proof. Clearly, by Theorems 2.2.2 and 2.4.1, (3.12) is equivalent to the fact that $\|H_\Phi\|_{\mathrm{e}}$ is equal to the distance from H_Φ to the set of compact vectorial Hankel operators from $H^2(\mathcal{H})$ to $H^2_-(\mathcal{K})$. Clearly, $\|H_\Phi\|_{\mathrm{e}}$ is equal to the distance from H_Φ to the set of finite rank operators. The result follows now from Theorem 3.1. ■

4. Relations between H_u and $H_{\bar{u}}$

The aim of this section is to obtain a useful formula that relates the Hankel operators H_u and $H_{\bar{u}}$ for a unimodular function u. This would allow us under certain circumstances to get information about properties of $H_{\bar{u}}$ from properties of H_u. In particular, we will be able to get information about the singular numbers of $H_{\bar{u}}$ from the singular values of H_u.

Let u be a unimodular function on $\mathbb{T}$. Then

$$H^*_{\bar{u}} H_{\bar{u}} T_u = T_u H^*_u H_u. \tag{4.1}$$

To prove this formula we make use of formula (3.1.5):

$$H^*_{\bar{u}} H_{\bar{u}} T_u = (I - T_u T_{\bar{u}}) T_u = T_u (I - T_{\bar{u}} T_u) = T_u H^*_u H_u.$$

To obtain consequences of formula (4.1) we need the notion of *polar decomposition*. Let T be an operator on a Hilbert space $\mathcal{H}$. Then T admits a representation in the form $T = \mathcal{U}(T^*T)^{1/2}$, where $\operatorname{Ker}\mathcal{U} = \operatorname{Ker} T$ and

$\mathcal{U}\,|\mathcal{H}\ominus\operatorname{Ker}\mathcal{U}$ is an isometry onto $\operatorname{clos}\operatorname{Range}T$ (see Halmos [2], Problem 134). $\mathcal{U}$ is called the *partially isometric factor of* T.

Lemma 4.1. *Let A and B be self-adjoint operators on Hilbert space and let T be an operator such that $AT=TB$. Then $A\mathcal{U}=\mathcal{U}B$, where $\mathcal{U}$ is the partially isometric factor of T.*

Proof. It follows from $AT=TB$ that $T^*A=BT^*$ and so

$$T^*TB=T^*AT=BT^*T.$$

Hence, B commutes with any function of the self-adjoint operator T^*T, in particular with $(T^*T)^{1/2}$. We have

$$A\mathcal{U}(T^*T)^{1/2}=\mathcal{U}(T^*T)^{1/2}B=\mathcal{U}B(T^*T)^{1/2}.$$

Multiplying on the right by $(T^*T)^{1/2}$, we obtain $A\mathcal{U}T^*T=\mathcal{U}BT^*T$. It follows that $A\mathcal{U}$ and $\mathcal{U}B$ coincide on $\operatorname{clos}\operatorname{Range}T^*T$. Let us verify that they also coincide on $\operatorname{Ker}T^*T=\operatorname{Ker}T$, which is the orthogonal complement to $\operatorname{clos}\operatorname{Range}T^*T$. Since $\operatorname{Ker}\mathcal{U}=\operatorname{Ker}T$ and since $\operatorname{Ker}T$ is B-invariant (the latter follows from $AT=TB$), we see that $A\mathcal{U}\big|\operatorname{Ker}T=\mathcal{U}B\big|\operatorname{Ker}T=\mathbb{O}$. ■

Applying Lemma 4.1 to (4.1), we obtain

$$H_{\bar{u}}^*H_{\bar{u}}\mathcal{U}_u=\mathcal{U}_uH_u^*H_u, \tag{4.2}$$

where $\mathcal{U}_u$ is the partially isometric factor of T_u.

Theorem 4.2. *Let u be a unimodular function on $\mathbb{T}$ such that $\operatorname{clos}T_uH^2=H^2$ and*

$$E=\left\{f\in H^2:\ H_u^*H_uf=f\right\}.$$

Then $H_{\bar{u}}^*H_{\bar{u}}$ is unitarily equivalent to $H_u^*H_u\,\big|H^2\ominus E$.

Proof. Note that $E=\operatorname{Ker}T_u=\operatorname{Ker}\mathcal{U}_u$ and $\mathcal{U}_u$ maps $H^2\ominus E$ isometrically onto H^2. It follows from (4.2) that

$$H_{\bar{u}}^*H_{\bar{u}}=\mathcal{U}_uH_u^*H_u\mathcal{U}_u^*=\mathcal{U}_u\left(H_u^*H_u\,\big|H^2\ominus E\right)U_u^*. \quad ■$$

Corollary 4.3. *Let u be a unimodular function such that*

$$\operatorname{Ker}T_u=\operatorname{Ker}T_u^*=\{\mathbb{O}\}.$$

Then $H_{\bar{u}}^*H_{\bar{u}}$ and $H_u^*H_u$ are unitarily equivalent.

The following assertion allows us to get information about the singular values of H_u from the singular values of $H_{\bar{u}}^*$.

Corollary 4.4. *Let u be a unimodular function in $H^\infty+C$ such that T_u has dense range. Then the Hankel operator $H_{\bar{u}}$ is compact and*

$$s_k(H_{\bar{u}})=s_{k+n}(H_u),\quad k\ge 0,$$

where $n=\dim\operatorname{Ker}T_u=\dim E$.

Proof. This follows immediately from Theorem 4.2 and the facts that

$$\left\|H_u^* H_u \,\middle|\, H^2 \ominus E\right\| < 1$$

and $H_u^* H_u f = f$ for $f \in E$. ■

In the next corollary $\boldsymbol{S}_p$, $0 < p < \infty$, is the Schatten–von Neumann class (see Appendix 1.1).

Corollary 4.5. *Let u be a unimodular function such that T_u has dense range in H^2. If H_u is compact, then so is $H_{\bar u}$. If $H_u \in \boldsymbol{S}_p$, $0 < p < \infty$, then $H_{\bar u} \in \boldsymbol{S}_p$ and $\|H_{\bar u}\|_{\boldsymbol{S}_p} < \|H_u\|_{\boldsymbol{S}_p}$.*

Corollary 4.6. *Let u be a unimodular function in $H^\infty + C$ such that $\operatorname{dist}_{L^\infty}(\bar u, H^\infty + C) < 1$. Then $u \in QC$.*

Proof. There is an integer n such that $\operatorname{dist}_{L^\infty}(z^n \bar u, H^\infty) < 1$. Without loss of generality we can assume that $n = 0$. By Theorem 1.11 of Chapter 1, $T_{\bar u}$ is left invertible and so $T_u H^2 = H^2$. Since H_u is compact by the Hartman theorem, $H_{\bar u}$ is also compact and again by the Hartman theorem $\bar u \in H^\infty + C$. ■

Corollary 4.7. *Let u be a unimodular function such that $\dim \operatorname{Ker} T_u < \infty$ and $\dim \operatorname{Ker} T_{\bar u} < \infty$. Then*

$$\|H_u\|_{\mathrm{e}} = \|H_{\bar u}\|_{\mathrm{e}}.$$

Proof. By Theorem 3.1.4, $\operatorname{Ker} T_u = \{\mathbb{O}\}$ or $\operatorname{Ker} T_{\bar u} = \{\mathbb{O}\}$. To be definite, suppose that $\operatorname{Ker} T_{\bar u} = \{\mathbb{O}\}$. Then T_u has dense range in H^2 and by Theorem 4.2, $H_{\bar u}^* H_{\bar u}$ is unitarily equivalent to the restriction of $H_u^* H_u$ to the orthogonal complement of $\operatorname{Ker} T_u$. By the hypotheses, $\dim \operatorname{Ker} T_u < \infty$, and so

$$\|H_{\bar u}\|_{\mathrm{e}} = \lim_{j \to \infty} s_j(H_{\bar u}) = \lim_{j \to \infty} s_j(H_u) = \|H_u\|_{\mathrm{e}}. \quad ■$$

Corollary 4.8. *Let u be a unimodular function such that the Toeplitz operator T_u is Fredholm. Then*

$$\|H_u\|_{\mathrm{e}} = \|H_{\bar u}\|_{\mathrm{e}}.$$

Let us now characterize those unimodular functions u for which T_u has dense range in H^2.

Theorem 4.9. *Let u be a unimodular function. Then $\operatorname{clos} T_u H^2 \neq H^2$ if and only if there exist an outer function h in H^2 and an inner function ϑ such that $u = z\vartheta h/\bar h$.*

Proof. Suppose that $\operatorname{clos} T_u H^2 \neq H^2$. Therefore $\operatorname{Ker} T_{\bar u} \neq \{\mathbb{O}\}$. Let f be a nonzero function in $\operatorname{Ker} T_{\bar u}$. Then $\bar u f \in H^2_-$. Let g be a function in H^2 such that $\bar z \bar g = \bar u f$. Since $|g| = |f|$ on $\mathbb{T}$, there exist an outer function h in H^2 and inner functions ϑ_1, ϑ_2 such that $f = \vartheta_1 h$, $g = \vartheta_2 h$. Then $u = z\vartheta_1\vartheta_2 h/\bar h$.

If $u = z\vartheta h/\bar h$, then obviously $h \in \operatorname{Ker} T_{\bar u}$. Hence, $\operatorname{clos} T_u H^2 \neq H^2$. ■

The following sufficient condition for T_u to have a dense range will be very useful.

Theorem 4.10. *Let $u = \bar{\vartheta}\bar{h}/h$, where ϑ is an inner function and h is an outer function in H^2. Then T_u has dense range in H^2.*

Proof. Suppose that $f \perp T_u H^2$ and $f \neq \mathbb{O}$. Then $(f, \bar{\vartheta}\bar{h}g/h) = 0$ for any g in H^2. Let $f = \tau k$, where τ is an inner function and k is an outer function. Take $g = \tau\vartheta h$. Then $(k, \bar{h}) = 0$, that is, $k(0)h(0) = 0$, which is impossible since both h and k are outer. ■

Consider now the case of vectorial Hankel operators with unitary-valued symbols. Let $\mathcal{H}$ be a separable Hilbert space and let U be a unitary-valued function with values in $\mathcal{B}(\mathcal{H})$. Then the following analog of formula (4.1) holds:

$$H_{U^*}^* H_{U^*} T_U = T_U H_U^* H_U.$$

This allows us to generalize Theorem 4.2 to the case of Hankel operators with unitary-valued symbols.

Theorem 4.11. *Let U be a unitary-valued function in $L^\infty(\mathcal{B}(\mathcal{H}))$ such that* $\operatorname{clos} T_U H^2(\mathcal{H}) = H^2(\mathcal{H})$ *and*

$$E = \left\{f \in H^2 : \ H_U^* H_U f = f\right\}.$$

Then $H_{U^*}^* H_{U^*}$ is unitarily equivalent to $H_U^* H_U \,\big|H^2 \ominus E$.

The proof of Theorem 4.11 is exactly the same as that of Theorem 4.2. It is also easy to see that Corollaries 4.3–4.5 also generalize very easily to the case of unitary-valued operator functions. Let us also state an analog of Corollary 4.8.

Corollary 4.12. *Let U be a unitary-valued matrix function such that the Toeplitz operator T_U is Fredholm. Then*

$$\|H_U\|_{\mathrm{e}} = \|H_{U^*}\|_{\mathrm{e}}.$$

Concluding Remarks

The results of §1 and §2 were obtained in Adamyan, Arov, and Krein [3]. Note that earlier Clark [1] proved a special case of the Adamyan–Arov–Krein theorem; namely, he considered the case of self-adjoint Hankel matrices. Later Butz [1] gave another proof of the Adamyan–Arov–Krein theorem.

Section 3 follows the paper Treil [3]. Note that the Adamyan–Arov–Krein theorem was generalized earlier to the case of finite matrix functions in Ball and Helton [1].

The results of §4 were obtained in Peller and Khrushchëv [1] for scalar functions; see Peller [17], where the case of operator functions was considered in connection with applications to vectorial stationary processes. Note that Corollary 4.5 for $p = 2$ was established in Solev [1].

In Le Merdy [1] the following generalization of the Adamyan–Arov–Krein theorem was found. It was shown that if H_φ is a Hankel operator from H^p

to H^q_-, $p \ge 2$, $1 < q \le 2$, then

$$\inf\{\|H_\varphi - K\|_{\mathcal{B}(H^p,H^q_-)} : \operatorname{rank} K \le n\} \le \operatorname{const} \inf\{\|H_\varphi - H_f\|_{\mathcal{B}(H^p,H^q_-)} : \operatorname{rank} H_f \le n\}.$$

In Baratchart and Seyfert [1] the authors proved a precise analog of the Adamyan–Arov–Krein theorem for Hankel operators from H^q, $q \in [2, \infty]$ to H^2_-. They proved that the distance to the set of operators of rank at most n equals the distance to the set of Hankel operators of rank at most n. In other words,

$$\sigma_n(H_\varphi) \stackrel{\text{def}}{=} \inf\{\|H_\varphi - K\|_{\mathcal{B}(H^q,H^2)} : \operatorname{rank} K \le n\}$$

is the distance in the norm of L^p from φ to the set of meromorphic functions with at most n poles in $\mathbb{D}$, where $1/p + 1/q = 1/2$. Moreover, in Baratchart and Seyfert [1] a precise analog of the Adamayan–Arov–Krein formula was obtained for the best meromorphic approximant in terms of "singular vectors" of the Hankel operator $H_\varphi : H^q \to H^2$.

It was also mentioned in Baratchart and Seyfert [1] that a precise analog of the Adamyan–Arov–Krein theorem can also be obtained for Hankel operators from H^q to the quotient space L^r/H^r, where $1/p+1/q+(r-1)/r = 1$. This allows one to express the numbers $\sigma_n(H_\varphi)$ in terms of the distance in L^p to the set of meromorphic functions with at most n poles in $\mathbb{D}$. Here p can be an arbitrary number in $[1, \infty)$. However, in this case no explicit formulas for the best meromorphic approximant in terms of "singular vectors" were found.

We also mention here the paper Prokhorov [3] in which the analog of the Adamyan–Arov–Krein theorem for Hankel operators from H^q to the quotient space L^r/H^r was obtained independently of the paper Baratchart and Seyfert [1].

5

Parametrization of Solutions of the Nehari Problem

For a Hankel operator Γ from H^2 to H^2_- and $\rho \geq \|\Gamma\|$, we consider in this section the problem of describing all symbols $\varphi \in L^\infty$ of Γ (i.e., $\Gamma = H_\varphi$) which satisfy the inequality $\|\varphi\|_\infty \leq \rho$. If φ_0 is a symbol of Γ, then as we have seen in §1.1, this problem is equivalent to the problem of finding all approximants $f \in H^\infty$ to φ_0 satisfying $\|\varphi_0 - f\|_\infty \leq \rho$. This problem is called the Nehari problem. If $\rho = \|\Gamma\|$, a solution φ of the Nehari problem (i.e., a symbol φ of Γ of norm at most ρ) is called *optimal*. If $\rho > \|\Gamma\|$, the solutions of the Nehari problem are called *suboptimal*). Clearly, the optimal solutions of the Nehari problem are the symbols of Γ of minimal norm.

In §1 we present the original approach by Adamyan, Arov, and Krein which is based on studying unitary extensions of isometric operators defined on a subspace of a Hilbert space. We obtain a uniqueness criterion and parametrize all solutions in the case of nonuniqueness. In particular, we show that in the case of nonuniqueness there are solutions of the Nehari problem of constant modulus equal to ρ.

In §2 we apply the results of §1 to the Nevanlinna–Pick interpolation problem. In case there are two distinct solutions of the Nevanlinna–Pick problem we show that there are solutions that are inner functions and we parametrize all solutions.

Another approach to the Nehari problem (also due to Adamyan, Arov, and Krein) is given in §3. This approach works only in the case $\rho > \|G\|$ and it also allows us to treat a more general problem, the so-called Nehari–Takagi problem. This problem can be described as follows. Suppose that Γ is a Hankel operator, $\rho > 0$, and $m \in \mathbb{Z}_+$ is such that $s_m(\Gamma) < \rho < s_{m-1}(\Gamma)$ if $m > 0$ and $s_0(\Gamma) < \rho$ if $m = 0$. The problem is to describe all functions

$\psi \in L^\infty$ such that $\|H_\psi\| \le \rho$ and $\operatorname{rank}(\Gamma - H_\psi) \le m$. Clearly, this problem is just the Nehari problem if $m = 0$. We parametrize in §3 all solutions of the Nehari problem.

In §4 we consider the Nehari problem for a block Hankel operator Γ and use the method of so-called one-step extensions developed by Adamyan, Arov, and Krein. We parametrize all solutions of the Nehari problem in the case of finite-dimensional spaces (i.e., in the case of Hankel operators with matricial symbols) and $\rho > \|\Gamma\|$.

Finally, in §5 we obtain a parametrization formula in the general case of vectorial Hankel operators. We use an approach that also depends on studying unitary extensions of isometric operators. However, it is different from the original approach by Adamyan, Arov, and Krein presented in §1, and it is based on the notion of the characteristic function of a unitary colligation.

Note that we return to the Nehari problem in Chapter 14.

1. Adamyan–Arov–Krein Parametrization in the Scalar Case

In this section we use the original approach by Adamyan, Arov, and Krein to the Nehari problem in the scalar case. We start with a Hankel operator Γ from H^2 to H^2_- and a number $\rho \ge \|\Gamma\|$, and consider the corresponding Nehari problem. We obtain a criterion for the uniqueness of a solution and find a parametrization formula for the solutions in the case there are at least two distinct solutions (this is always the case if $\rho > \|\Gamma\|$).

Consider the auxiliary operator A_ρ on L^2 defined by

$$A_\rho f = f + \frac{1}{\rho}(\Gamma f_+ + \Gamma^* f_-),$$

where

$$f_+ \stackrel{\text{def}}{=} \mathbb{P}_+ f, \quad f_- \stackrel{\text{def}}{=} \mathbb{P}_- f, \quad f \in H^2.$$

We introduce a new pseudo-inner product on L^2 by

$$(f, g)_\rho = (A_\rho f, g).$$

Let us show that $A_\rho \ge \mathbb{O}$. We have

$$\begin{aligned}(A_\rho f, f) &= \|f_+\|^2 + \|f_-\|^2 + \frac{2}{\rho}\operatorname{Re}(\Gamma f_+, f_-) \\ &\ge \|f_+\|^2 + \|f_-\|^2 - 2\|f_+\| \cdot \|f_-\| \ge 0.\end{aligned}$$

If $\rho > \|\Gamma\|$, then $(\cdot,\cdot)_\rho$ is equivalent to the standard inner product on L^2. The pseudo-inner product $(\cdot,\cdot)_\rho$ is nondegenerate on L^2 unless $\rho = \|\Gamma\|$ and the norm of Γ is attained on the unit ball of H^2. Indeed $(f, f)_\rho = 0$ if and only if

$$\|f_+\|^2 + \|f_-\|^2 + \frac{2}{\rho}\operatorname{Re}(\Gamma f_+, f_-) = 0,$$

which is equivalent to

$$\|\Gamma f_+\| = \rho\|f_+\|, \quad \Gamma f_+ = -\rho f_-.$$

Denote by $\mathcal{K}$ the Hilbert space obtained from L^2 by factorization and completion with respect to $(\cdot,\cdot)_\rho$. From now on we denote the inner product in $\mathcal{K}$ by $(\cdot,\cdot)_\mathcal{K}$, $(f,g)_\mathcal{K} = (f,g)_\rho$ for f, $g \in L^2$.

Let $\mathcal{S}$ be multiplication by z on L^2. Clearly, H^2 and H^2_- are isometrically imbedded in $\mathcal{K}$ and

$$\mathcal{K} = \operatorname{span}\{H^2, H^2_-\}$$

with respect to $(\cdot,\cdot)_\mathcal{K}$. It is also clear that $\mathcal{S}$ maps isometrically $\bar{z}H^2_- + H^2$ onto $H^2_- + zH^2$ in the metric of $\mathcal{K}$. Denote by V the closure of this isometric operator. Then the domain $\mathcal{D}_V$ and the range $\mathcal{R}_V$ of V are given by

$$\mathcal{D}_V = \operatorname{clos}_\mathcal{K}\{f \in L^2 : \ \hat{f}(-1) = 0\},$$
$$\mathcal{R}_V = \operatorname{clos}_\mathcal{K}\{f \in L^2 : \ \hat{f}(0) = 0\}.$$

We consider minimal unitary extensions of V. Let $\mathcal{K}$ be imbedded in a Hilbert space $\mathcal{H}$. A unitary operator U on $\mathcal{H}$ is called a *unitary extension* of V if

$$U|\mathcal{D}_V = V.$$

If in addition to that

$$\mathcal{H} = \operatorname{span}\{U^k\mathcal{H} : \ k \in \mathbb{Z}\},$$

the extension is called *minimal*.

Let U be a minimal unitary extension of V and $\mathcal{E}_U$ its spectral measure. Consider the operator-valued measure $\mathcal{E}_U$ on the space $\mathcal{K}$ defined by $E_U(\Delta) = \mathcal{P}\mathcal{E}_U(\Delta)|\mathcal{K}$, $\Delta \subset \mathbb{T}$, where $\mathcal{P}$ is the orthogonal projection onto $\mathcal{K}$. Define the scalar measure μ_U on $\mathbb{T}$ by

$$d\mu_U = \rho\bar{z}\, d(\mathcal{E}_U\mathbf{1}, \bar{z})_\mathcal{H} = \rho\bar{z}\, d(E_U\mathbf{1}, \bar{z})_\mathcal{K}. \tag{1.1}$$

Theorem 1.1. *For any minimal unitary extension U of V the measure μ_U is absolutely continuous with respect to Lebesgue measure and the density*

$$\psi = \frac{d\mu_U}{d\boldsymbol{m}} \tag{1.2}$$

satisfies

$$\|\psi\|_\infty \le \rho \quad \textit{and} \quad H_\psi = \Gamma. \tag{1.3}$$

Conversely if ψ satisfies (1.3), *then there exists a minimal unitary extension U of V such that ψ is given by* (1.2).

Theorem 1.1 describes the solutions of the Nehari problem with norm at most ρ in terms of minimal unitary extensions of V. It turns out that the map

$$U \mapsto \psi = \frac{d\mu_U}{d\boldsymbol{m}}$$

is a one-to-one correspondence between the minimal unitary extensions of V and the solutions of the Nehari problem with norm at most ρ if we identify equivalent extensions. By equivalent extensions we mean the following.

Let U_1 and U_2 be minimal extensions of V on spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. They are called *equivalent* if there is a unitary map W of $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $W\big|\mathcal{K}$ is the identity and

$$WU_1 = U_2W. \tag{1.4}$$

It is easy to see that if U_1 and U_2 are equivalent, then

$$E_{U_1} = E_{U_2}.$$

The following result shows that the map $U \mapsto \psi$, $\psi = \frac{d\mu_U}{d\boldsymbol{m}}$, is a one-to-one correspondence between the equivalence classes of minimal unitary extensions of V and the solutions of the Nehari problem with norm at most ρ.

Theorem 1.2. *Let U_1 and U_2 be minimal unitary extensions of V on $\mathcal{H}_1$ and $\mathcal{H}_2$. The following statements are equivalent:*

(i) *U_1 is equivalent to U_2;*

(ii) *$E_{U_1} = E_{U_2}$;*

(iii) *$\psi_1 = \psi_2$, where $\psi_j = \frac{d\mu_{U_j}}{d\boldsymbol{m}}$, $j = 1, 2$.*

Proof of Theorem 1.1. Let U be a minimal unitary extension of V and $\mathcal{E}_U$ its spectral measure. We have for all $k \ge 0$

$$U^k\mathbf{1} = z^k, \quad U^{-k}\bar{z} = \bar{z}^{k+1}.$$

Since U is unitary, the vectors $U^k\mathbf{1}$ are pairwise orthogonal for $k \in \mathbb{Z}$, and so are $U^k\bar{z}$. Therefore we have for any trigonometric polynomials f and g:

$$\begin{aligned}\left|\int_{\mathbb{T}} f(\zeta)\overline{g(\zeta)}(dE_U(\zeta)\mathbf{1}, \bar{z})_\rho\right| &= |\,(f(U)\mathbf{1}, g(U)\bar{z})_{\mathcal{H}}\,| \\ &\le \|f(U)\mathbf{1}\| \cdot \|g(U)\bar{z}\| = \|f\|_{L^2}\|g\|_{L^2}.\end{aligned}$$

Therefore μ_U is absolutely continuous with respect to Lebesgue measure and the density $\psi = \frac{d\mu_U}{d\boldsymbol{m}}$ satisfies

$$\psi \in L^\infty, \quad \|\psi\|_{L^\infty} \le \rho.$$

Let $k \ge 1$. We have

$$\begin{aligned}\hat{\psi}(-k) &= \int_{\mathbb{T}} \psi(\zeta)\zeta^k d\boldsymbol{m}(\zeta) = \rho\int_{\mathbb{T}} \zeta^{k-1}\frac{d}{d\boldsymbol{m}}(E_U(\zeta)\mathbf{1}, \bar{z})_{\mathcal{K}}\, d\boldsymbol{m}(\zeta) \\ &= \rho(U^{k-1}\mathbf{1}, \bar{z})_{\mathcal{H}} = \rho(z^{k-1}, \bar{z})_{\mathcal{K}} = \rho(A_\rho z^{k-1}, \bar{z}) = (\Gamma z^{k-1}, \bar{z}).\end{aligned}$$

So $\Gamma = H_\psi$.

Suppose that $\|\psi\|_\infty \le \rho$ and $\Gamma = H_\psi$. Put

$$\xi = \frac{1}{\rho}\psi, \quad \varkappa = 1 - |\xi|^2, \quad L^2_\varkappa = \operatorname{clos}_{L^2} \varkappa^{1/2} L^2, \quad \mathcal{H}_\psi = L^2 \oplus L^2_\varkappa.$$

Clearly,

$$L^2_\varkappa = \left\{ f \in L^2 : \; f \big| \{\zeta : |\xi(\zeta)| = 1\} = \mathbb{O} \right\}.$$

Define the imbedding $\mathcal{I} = \mathcal{I}_\psi$ of $\mathcal{K}$ in $\mathcal{H}_\psi$ on a dense subset of $\mathcal{K}$ by

$$\mathcal{I} f = (f_- + \xi f_+) \oplus \varkappa^{1/2} f_+, \quad f \in L^2.$$

Let us show that $\mathcal{I}$ extends to an isometric imbedding of $\mathcal{K}$ in $\mathcal{H}_\psi$. We have

$$\begin{aligned}
(\mathcal{I}f, \mathcal{I}g) &= (f_- + \xi f_+, g_- + \xi g_+) + (\varkappa^{1/2} f_+, \varkappa^{1/2} g_+) \\
&= (f_-, g_-) + \frac{1}{\rho}(f_-, \psi g_+) + \frac{1}{\rho}(\psi f_+, g_-) \\
&+ \frac{1}{\rho^2}(\psi f_+, \psi g_+) + \left(\left(1 - \frac{1}{\rho^2}|\psi|^2\right) f_+, g_+ \right) \\
&= (f_-, g_-) + (f_+, g_+) + \frac{1}{\rho}(f_-, \mathbb{P}_- \psi g_+) + \frac{1}{\rho}(\mathbb{P}_- \psi f_+, g_-) \\
&= (f, g) + \frac{1}{\rho}(f_-, \Gamma g_+) + \frac{1}{\rho}(\Gamma f_+, g_-) = (f, g)_{\mathcal{K}}.
\end{aligned}$$

Let U be the unitary operator defined on $\mathcal{H}_\psi$ by

$$U(f \oplus g) = zf \oplus zg.$$

It is evident that $U \big| \mathcal{I}\mathcal{D}_V = \mathcal{I}V_\rho \big| \mathcal{D}_V$. So U is a unitary extension of V under the identification of $\mathcal{K}$ with $\mathcal{I}\mathcal{K}$. Let us show that U is minimal. Indeed,

$$\begin{aligned}
U^n \mathcal{I} \bar{z}^k &= z^{n-k} \oplus \mathbb{O}, \quad k \ge 2, \; n \in \mathbb{Z}; \\
U^n \mathcal{I} z^k &= \xi z^{n+k} \oplus \varkappa^{1/2} z^{n+k}, \quad k \ge 0, \; n \in \mathbb{Z}.
\end{aligned}$$

Clearly,

$$\operatorname{span}\{U^n \mathcal{I} \bar{z}^k : \; k \ge 2, \; n \in \mathbb{Z}\} \cup \{U^n z^k : \; k \ge 0, \; n \in \mathbb{Z}\} = \mathcal{H}_\psi.$$

Let $\mathcal{E}_U$ be the spectral measure of U. Then $\mathcal{E}_U(\Delta)$ is multiplication by χ_Δ on $\mathcal{H}_\psi$. Therefore

$$(\mathcal{E}_U(\Delta)\mathcal{I}\mathbf{1}, \mathcal{I}\bar{z})_{\mathcal{H}_\psi} = \int_\Delta \zeta\, \xi(\zeta) d\boldsymbol{m}(\zeta).$$

It follows that

$$\psi = \rho \bar{z} \frac{d}{d\boldsymbol{m}} (\mathcal{E}_U \mathcal{I}\mathbf{1}, \mathcal{I}\bar{z})_{\mathcal{H}_\psi} = \rho \bar{z} \frac{d}{d\boldsymbol{m}} (E_U \mathbf{1}, \bar{z})_{\mathcal{K}},$$

which completes the proof. ■

Proof of Theorem 1.2. If U_1 and U_2 are equivalent minimal unitary extensions of V and $W : \mathcal{H}_1 \to \mathcal{H}_2$ is a unitary map such that $W \big| \mathcal{K} = I_{\mathcal{K}}$ and (1.4) holds, then for any f, g in $\mathcal{K}$

$$(U_1^k f, g)_{\mathcal{H}_1} = (W^* U_2^k W f, g)_{\mathcal{H}_1} = (U_2^k f, g)_{\mathcal{H}_2}.$$

This implies

$$(\mathcal{E}_{U_1}(\Delta)f, g)_{\mathcal{H}_1} = (\mathcal{E}_{U_2}(\Delta)f, g)_{\mathcal{H}_2}, \quad f,\, g \in \mathcal{K},$$

which proves (i)⇒(ii).

The implication (ii)⇒(iii) is trivial. Let us establish (iii)⇒(i).

We have from (iii):

$$(\mathcal{E}_{U_1}\mathbf{1}, \bar{z})_{\mathcal{H}_1} = (\mathcal{E}_{U_2}\mathbf{1}, \bar{z})_{\mathcal{H}_2},$$

which implies

$$(U_1^k\mathbf{1}, \bar{z})_{\mathcal{H}_1} = (U_2^k\mathbf{1}, \bar{z})_{\mathcal{H}_2}, \quad k \in \mathbb{Z}. \tag{1.5}$$

Since U_1 and U_2 are extensions of V, it follows that

$$U_1^k\mathbf{1} = U_2^k\mathbf{1} = z^k, \quad k \ge 0, \tag{1.6}$$

$$U_1^{-k}\bar{z} = U_2^{-k}\bar{z} = \bar{z}^{1+k}, \quad k \ge 0. \tag{1.7}$$

By minimality

$$\mathcal{H}_1 = \operatorname{span}\{U_1^k\mathbf{1},\, U_1^k\bar{z},\ k \in \mathbb{Z}\}, \tag{1.8}$$

$$\mathcal{H}_2 = \operatorname{span}\{U_2^k\mathbf{1},\, U_2^k\bar{z},\ k \in \mathbb{Z}\}. \tag{1.9}$$

Define the operator $W : \mathcal{H}_1 \to \mathcal{H}_2$ by

$$WU_1^k\mathbf{1} = U_2^k\mathbf{1}, \quad WU_1^k\bar{z} = U_2^k\bar{z}, \quad k \in \mathbb{Z}.$$

It follows from (1.8) and (1.9) that to show that W is a unitary operator from $\mathcal{H}_1$ to $\mathcal{H}_2$, it is sufficient to prove that

$$(U_1^k\mathbf{1}, U_1^m\mathbf{1})_{\mathcal{H}_1} = (U_2^k\mathbf{1}, U_2^m\mathbf{1})_{\mathcal{H}_2}, \tag{1.10}$$

$$(U_1^k\bar{z}, U_1^m\bar{z})_{\mathcal{H}_1} = (U_2^k\bar{z}, U_2^m\bar{z})_{\mathcal{H}_2}, \tag{1.11}$$

$$(U_1^k\bar{z}, U_1^m\mathbf{1})_{\mathcal{H}_1} = (U_2^k\bar{z}, U_2^m\mathbf{1})_{\mathcal{H}_2}. \tag{1.12}$$

Clearly, (1.12) follows directly from (1.5). To establish (1.10) and (1.11), we assume (to be definite) that $k \ge m$. Then

$$\begin{aligned}(U_1^k\mathbf{1}, U_1^m\mathbf{1})_{\mathcal{H}_1} &= (U_1^{k-m}\mathbf{1}, \mathbf{1})_{\mathcal{H}_1} = (z^{k-m}, \mathbf{1})_{\mathcal{K}} \\ &= (U_2^{k-m}\mathbf{1}, \mathbf{1})_{\mathcal{H}_2} = (U_2^k\mathbf{1}, U_2^m\mathbf{1})_{\mathcal{H}_2},\end{aligned}$$

$$\begin{aligned}(U_1^k, \bar{z}, U_1^m\bar{z})_{\mathcal{H}_1} &= (\bar{z}, U^{m-k}\bar{z})_{\mathcal{H}_1} = (\bar{z}, \bar{z}^{1+k+m})_{\mathcal{K}} \\ &= (\bar{z}, U_2^{m-k}\bar{z})_{\mathcal{H}_2} = (U_2^k\bar{z}, U_2^m\bar{z})_{\mathcal{H}_2}.\end{aligned}$$

Thus W is unitary. It follows from (1.6) and (1.7) that $W\big|\mathcal{D}_V = I_{\mathcal{D}_V}$. The identity $WU_1 = U_2W$ follows directly from (1.8), (1.9), and the definition of W. ■

Uniqueness

We are now going to consider the question of uniqueness of solutions ψ of the Nehari problem with $\|\psi\|_\infty \le \rho$. Define the deficiency indices of the operator V by

$$d_1 = \dim \mathcal{K} \ominus \mathcal{D}_V, \quad d_2 = \dim \mathcal{K} \ominus \mathcal{R}_V.$$

Clearly, $0 \le d_1,\, d_2 \le 1$.

Lemma 1.3.

$$d_1 = d_2.$$

Proof. Let J be the (real-linear) operator on L^2 defined by

$$Jf = \bar{z}\bar{f}.$$

Clearly, $JH^2 = H^2_-$, $J^2 = I$, and $\|Jf\|_2 = \|f\|_2$ for any f in L^2. Let us show that

$$\Gamma J h = J\Gamma^* h, \quad h \in H^2_-.$$

Indeed, let $\Gamma = H_\varphi$, $\varphi \in L^\infty$. We have

$$\Gamma J h = H_\varphi J h = \mathbb{P}_- \varphi \bar{z}\bar{h} = J\mathbb{P}_+ J(\varphi \bar{z}\bar{h}) = J\mathbb{P}_+ \bar{\varphi} h = J\Gamma^* h.$$

In a similar way one can prove that

$$\Gamma^* J h = J\Gamma h, \quad h \in H^2.$$

It follows that

$$(Jf, Jg)_\mathcal{K} = (f, g)_\mathcal{K}, \quad f,\, g \in H^2.$$

Indeed, let $f_+ = \mathbf{P}_+ f$, $f_- = \mathbb{P}_- f$. We have

$$\begin{aligned}(Jf, Jg)_\mathcal{K} &= (A_\rho Jf, Jg) = (Jf, Jg) + \frac{1}{\rho}(\Gamma J f_-, Jg) + \frac{1}{\rho}(\Gamma^* J f_+, Jg) \\ &= (f, g) + \frac{1}{\rho}(J\Gamma^* f_-, Jg) + \frac{1}{\rho}(J\Gamma f_+, Jg) = (f, g)_\mathcal{K}.\end{aligned}$$

Since obviously $J\mathbf{1} = \bar{z}$ and

$$J\{f \in L^2 : \hat{f}(-1) = 0\} = \{f \in L^2 : \hat{f}(0) = 0\},$$

it follows that

$$J\mathcal{D}_V = \mathcal{R}_V,$$

and so $\mathcal{D}_V$ coincides with $\mathcal{K}$ if and only if so does $\mathcal{R}_V$. This implies that $d_1 = d_2$. ■

Lemma 1.4. *The following statements are equivalent:*

(i) *there is a unique solution ψ of the Nehari problem with $\|\psi\|_\infty \le \rho$;*

(ii) $d_1 = d_2 = 0$.

Proof. Suppose that $d_1 = d_2 = 0$. Then $\mathcal{D}_V$ and $\mathcal{R}_V$ coincide with $\mathcal{K}$ and so V extends by continuity to a unitary operator on $\mathcal{K}$. Clearly, such an extension is unique.

If $d_1 = d_2 = 1$, then $\dim \mathcal{K} \ominus \mathcal{D}_V = \dim \mathcal{K} \ominus \mathcal{R}_V = 1$. Let $g_1 \in \mathcal{K} \ominus \mathcal{D}_V$, $g_2 \in \mathcal{K} \ominus \mathcal{R}_V$, $\|g_1\|_\rho = \|g_2\|_\rho = 1$. Given $\gamma \in \mathbb{T}$ we can define a unitary extension $U^{(\gamma)}$ of V by

$$U^{(\gamma)} f = f, \quad f \in \mathcal{D}_V, \quad U^{(\gamma)} g_1 = \gamma g_2. \tag{1.13}$$

Clearly, $U^{(\gamma)}$ extends by linearity to a unitary extension of V on $\mathcal{K}$ and $U^{(\gamma_1)} \neq U^{(\gamma_2)}$ for distinct γ_1 and γ_2. ■

If $\rho > \|\Gamma\|$, then $(\cdot,\cdot)_\rho$ is equivalent to the initial inner product in L^2. So $\bar{z} \notin \mathcal{D}_V$ and by Lemma 1.4, there are infinitely many solutions of the Nehari problem. If $\rho = \|\Gamma\|$ and the norm of Γ is attained on the unit ball of H^2 (the latter is fulfilled if Γ is compact), then as we have seen in §1.1, there exists a unique solution φ of the Nehari problem and $|\varphi(\zeta)| = \rho$ almost everywhere on $\mathbb{T}$.

However, we will see in this section that if $\|\Gamma\| = \rho$ and Γ does not attain its norm on the unit ball of H^2, the Nehari problem may have a unique solution or infinitely many solutions.

The following theorem establishes a criterion of the uniqueness of a solution of the Nehari problem.

Theorem 1.5. *The following statements are equivalent:*

(i) *there is a unique solution ψ of the Nehari problem with $\|\psi\|_\infty \le \rho$;*

(ii) $\lim\limits_{r\to\rho+} ((r^2 I - \Gamma^*\Gamma)^{-1}\mathbf{1}, \mathbf{1}) = \infty$*;*

(iii) $\mathbf{1} \notin \operatorname{Range}(\rho^2 I - \Gamma^*\Gamma)^{1/2}$.

We need the following elementary lemma.

Lemma 1.6. *Let A be a nonnegative operator on a Hilbert space $\mathcal{H}$,*

$$(f,g)_A \stackrel{\text{def}}{=} (Af, g), \quad \|f\|_A = (f,f)_A^{1/2}, \ f \in \mathcal{H}.$$

Given $h \in \mathcal{H}$, the linear functional $f \mapsto (f,h)$ is continuous in $\|\cdot\|_A$ if and only if

$$\lim_{s\to 0+} \big((A + sI)^{-1}h, h\big) < \infty,$$

which in turn is equivalent to the fact that $h \in A^{1/2}\mathcal{H}$.

Proof of Lemma 1.6. Let $h = A^{1/2}x,\ x \in \mathcal{H}$. We have

$$\begin{aligned} |(f,h)| &= |(f, A^{1/2}x)| = |(A^{1/2}f, x)| \le \|x\| \cdot \|A^{1/2}f\| \\ &= \|x\| \cdot |(A^{1/2}f, A^{1/2}f)|^{1/2} = \|x\| \cdot |(Af,f)|^{1/2} = \|x\| \cdot \|f_A\|. \end{aligned}$$

Thus the functional $f \mapsto (f,h)$ is continuous in $\|\cdot\|_A$.

On the other hand, if $f \mapsto (f,h)$ is a continuous linear functional with norm c, then

$$\begin{aligned} ((A+sI)^{-1}h, h) &\le c\,\|(A+sI)^{-1}h\|_A = c\,(A(A+sI)^{-1}h, h)^{1/2} \\ &\le c\,\|h\|^{1/2}\|A(A+sI)^{-1}h\|^{1/2} \le c\,\|h\|, \end{aligned}$$

since $t(t+s)^{-1} \le 1$ for $t \ge 0$, and so $\|A(A+sI)^{-1}\| \le c$.

Suppose now that $\lim_{s\to 0+}((A+sI)^{-1}h,h)<\infty$. Then

$$\lim_{s\to 0+}\|(A+sI)^{-1/2}h\|<\infty.$$

There exists a sequence $\{s_n\}_{n\ge 0}$ of positive numbers such that $\lim_{n\to\infty} s_n=0$ and the sequence $\{(A+s_nI)^{-1/2}h\}_{n\ge 0}$ weakly converges to an element x of $\mathcal{H}$. Then obviously,

$$\text{w-}\lim_{n\to\infty} A^{1/2}(A+s_nI)^{-1/2}h=A^{1/2}x.$$

On the other hand, it follows easily from spectral theorem that

$$A^{1/2}(A+s_nI)^{-1/2}h\to h$$

in the strong operator topology, which implies that $h=A^{1/2}x$. ■

Proof of Theorem 1.5. By Lemmas 1.3 and 1.4, (i) is equivalent to the fact that $\mathcal{R}_\rho=\mathcal{K}_\rho$. The latter means that the linear functional $f\mapsto\hat f(0)$ is unbounded in $\|\cdot\|_\rho$. By Lemma 1.6 this is true if and only if

$$\lim_{s\to 0}\big((A_\rho+sI)^{-1}\mathbf{1},\mathbf{1}\big)=\infty. \tag{1.14}$$

Let $f_s=(A_\rho+sI)^{-1}\mathbf{1}$. We have by the definition of A_ρ

$$\mathbf{1}=(1+s)f_s+\frac{1}{\rho}\Gamma\mathbb{P}_+f_s+\frac{1}{\rho}\Gamma^*\mathbb{P}_-f_s.$$

Since $\mathbf{1}\in H^2$, it follows that

$$(1+s)\mathbb{P}_-f_s+\frac{1}{\rho}\Gamma\mathbb{P}_+f_s=\mathbb{O} \tag{1.15}$$

and

$$\mathbf{1}=(1+s)\mathbb{P}_+f_s+\frac{1}{\rho}\Gamma^*\mathbb{P}_-f_s. \tag{1.16}$$

Substituting $\mathbb{P}_-f_s$ from (1.15) in (1.16), we obtain

$$\mathbf{1}=(1+s)\mathbb{P}_+f_s-\frac{1}{(1+s)\rho^2}\Gamma^*\Gamma\mathbb{P}_+f_s,$$

and so

$$\mathbb{P}_+f_s=(1+s)\rho^2\left((1+s)^2\rho^2I-\Gamma^*\Gamma)^{-1}\right)\mathbf{1}.$$

We have

$$\begin{aligned}\big((A_\rho+sI)^{-1}\mathbf{1},\mathbf{1}\big) &= (f_s,\mathbf{1})=(\mathbb{P}_+f_s,\mathbf{1})\\ &= (1+s)\rho^2\left((1+s)^2\rho^2I-\Gamma^*\Gamma)^{-1}\mathbf{1},\mathbf{1}\right).\end{aligned}$$

Thus (1.14) is equivalent to the fact that

$$\lim_{s\to 0+}\left((1+s)^2\rho^2I-\Gamma^*\Gamma)^{-1}\mathbf{1},\mathbf{1}\right)=\infty,$$

which is obviously equivalent to (ii). The fact that (ii) and (iii) are equivalent follows from Lemma 1.6. ■

Let $d_0 = d_1 = 1$. Then by Theorem 1.1, the solutions of the Nehari problem are in a one-to-one correspondence with the minimal unitary extensions of V. There can be two different types of extensions. Extensions of the first type are defined on the space $\mathcal{K}$ while extensions of the second type are defined on a space larger than $\mathcal{K}$. It is easy to see that all extensions of the first type are described by (1.13); they can be parametrized by the complex numbers of modulus one.

First we are going to study solutions of the Nehari problem that correspond to minimal unitary extensions on the space $\mathcal{K}$.

Canonical Functions

A solution ψ of the Nehari problem with norm at most ρ is called a *ρ-canonical function* if the operator V has deficiency indices equal to 1 and the corresponding minimal unitary extension of V is an operator on $\mathcal{K}$. We shall call ρ-canonical functions *canonical* if this does not lead to confusion.

Theorem 1.7. *If ψ is a canonical function, then $|\psi| = \rho$ almost everywhere.*

Proof. As in the proof of Theorem 1.1 put

$$\xi = \frac{1}{\rho}\psi, \quad \varkappa = 1 - |\xi|^2, \quad L^2_\varkappa = \operatorname{clos}_{L^2} \varkappa^{1/2} L^2, \quad \mathcal{H}_\psi = L^2 \oplus L^2_\varkappa,$$
$$\mathcal{I} f = (f_- + \xi f_+) \oplus \varkappa^{1/2} f_+, \quad f \in L^2.$$

Then $\mathcal{I}$ extends to an isometric imbedding of $\mathcal{K}$ in $\mathcal{H}_\psi$. Put

$$\tilde{\mathcal{K}} = \mathcal{I}\mathcal{K}, \quad \mathcal{H}_+ = \mathcal{I}H^2, \quad \mathcal{H}_- = \mathcal{I}H^2_-, \quad \tilde{\mathcal{R}}_V = \mathcal{I}\mathcal{R}_V.$$

Then

$$\mathcal{H}_- = H^2_- \oplus \{\mathbb{O}\}, \quad \mathcal{H}_+ = \{\xi f \oplus \varkappa^{1/2} f : f \in H^2\}.$$

It follows that

$$\tilde{\mathcal{K}} = \operatorname{clos}(\mathcal{H}_- + \mathcal{H}_+), \quad \tilde{\mathcal{R}}_V = \operatorname{clos}(\mathcal{H}_- + z\mathcal{H}_+).$$

Let us describe $\mathcal{H}_\psi \ominus \tilde{\mathcal{R}}_V$. Let $f \oplus g \in \mathcal{H}_\psi \ominus \tilde{\mathcal{R}}_V$. Since $f \oplus g \perp \mathcal{H}_-$, it follows that $f \in H^2$. The fact that $f \oplus g \perp z\mathcal{H}_+$ means that

$$(f \oplus g, z\xi\omega \oplus z\varkappa^{1/2}\omega)_{\mathcal{H}_\psi} = (f, z\xi\omega) + (g, z\varkappa^{1/2}\omega) = (\bar{\xi} f + \varkappa^{1/2} g, z\omega) = 0$$

for any ω in H^2. That is, $\bar{\xi} f + \varkappa^{1/2} g \in zH^2_-$. Thus

$$\mathcal{H}_\psi \ominus \tilde{\mathcal{R}}_V = \{f \oplus g : \ f \in H^2, \ \bar{\xi} f + \varkappa^{1/2} g \in zH^2_-\}. \tag{1.17}$$

By the assumption $\dim(\mathcal{H}_\psi \ominus \tilde{\mathcal{R}}_V) = 1$. Let us show that this is possible only if $\varkappa = 0$. This would imply that $|\psi| = \rho$ almost everywhere.

Let $f \oplus g \in \mathcal{H}_\psi \ominus \tilde{\mathcal{R}}_V$. Then $f \in H^2$, $\eta \stackrel{\text{def}}{=} \bar{\xi} f + \varkappa^{1/2} g \in zH^2_-$. Suppose that $\varkappa \neq \mathbb{O}$. Then there exist a set Δ of positive measure and a positive number δ such that $\varkappa \geq \delta$ on Δ.

It is easy to construct a nonconstant function α in H^∞ such that $\operatorname{Im}\alpha(\zeta)=0$ for $\zeta\notin\Delta$. Indeed, let $\chi=\chi_\Delta$. Put

$$\alpha=\exp\left(-(-\tilde{\chi}+\mathrm{i}\chi)^2\right).$$

Clearly,

$$|\alpha|=\exp\left(-\operatorname{Re}(-\tilde{\chi}+\mathrm{i}\chi)^2\right)=\exp(\chi^2-\tilde{\chi}^2)\in L^\infty.$$

On the other hand, for $\zeta\notin\Delta$

$$\alpha(\zeta)=\exp\left(\tilde{\chi}^2(\zeta)\right)\in\mathbb{R}.$$

So $\alpha\in H^\infty$ and $\operatorname{Im}\alpha(\zeta)=0$ for $\zeta\notin\Delta$.

We can define now the functions $f_*,\ \eta_*,\ g_*$ by

$$f_*=\alpha f,\quad \eta_*=\bar{\alpha}\eta,$$

$$g_*(\zeta)=\begin{cases}\varkappa^{-1/2}(\zeta)\left(\eta_*(\zeta)-\overline{\xi(\zeta)}f_*(\zeta)\right), & \zeta\in\Delta,\\ \alpha(\zeta)g(\zeta), & \zeta\notin\Delta.\end{cases}$$

Let us show that $f_*\oplus g_*\in\mathcal{H}_\psi\ominus\tilde{\mathcal{R}}_\rho$. Clearly, $f_*\in H^2$, $\eta_*\in zH^2_-$, and $g_*\in L^2_\varkappa$. We have to prove that $\bar{\xi}f_*+\varkappa^{1/2}g_*=\eta_*$. Let $\zeta\notin\Delta$. Then

$$\begin{aligned}\overline{\xi(\zeta)}f_*(\zeta)+\varkappa^{1/2}(\zeta)g_*(\zeta) &= \overline{\xi(\zeta)}\alpha(\zeta)f(\zeta)+\varkappa^{1/2}(\zeta)\alpha(\zeta)g(\zeta)\\ &= \alpha(\zeta)\eta(\zeta)=\overline{\alpha(\zeta)}\eta(\zeta)=\eta_*(\zeta),\end{aligned}$$

since $\alpha(\zeta)$ is real for $\zeta\notin\Delta$. Let $\zeta\in\Delta$. We have

$$\begin{aligned}\overline{\xi(\zeta)}f_*(\zeta)+\varkappa^{1/2}(\zeta)g_*(\zeta) &= \overline{\xi(\zeta)}\alpha(\zeta)f(\zeta)+\overline{\alpha(\zeta)}\eta(\zeta)-\overline{\xi(\zeta)}\alpha(\zeta)f(\zeta)\\ &= \overline{\alpha(\zeta)}\eta(\zeta)=\eta_*(\zeta),\end{aligned}$$

which proves that $f_*\oplus g_*\in\mathcal{H}_\psi\ominus\tilde{\mathcal{R}}_V$ (see (1.17)).

Clearly, $f\oplus g$ and $f_*\oplus g_*$ are linear independent, which contradicts the fact that $\dim\mathcal{H}_\psi\oplus\tilde{\mathcal{R}}_V=1$. ■

Theorem 1.8. *Let ψ be a canonical function. Then ψ admits a representation*

$$\psi=\rho\, h/\bar{h},\tag{1.18}$$

where h is an outer function in H^2.

Proof. Let $\xi=\psi/\rho$. By Theorem 1.7, $|\xi(\zeta)|=1$ almost everywhere on $\mathbb{T}$ and $\mathcal{H}_\psi=L^2$. If $f\in\mathcal{H}_\psi\ominus\tilde{\mathcal{R}}_V$, then

$$\bar{\xi}f=\eta,\quad f\in H^2,\ \eta\in zH^2_-.$$

It follows that $\xi=f/\eta$. The functions f and $\bar{\eta}$ belong to H^2 and have the same moduli. So f and η admit factorizations

$$f=k\vartheta_1,\quad \eta=\bar{k}\bar{\vartheta}_2,$$

where k is an outer function in H^2, and ϑ_1 and ϑ_2 are inner functions. Then $\xi = \vartheta_1\vartheta_2 k/\bar{k}$. If $\vartheta_1\vartheta_2 \neq \text{const}$, then k and $\vartheta_1\vartheta_2 k$ belong to $\mathcal{H}_\psi \ominus \tilde{\mathcal{R}}_V$. Indeed,

$$\bar{\xi}k = \frac{\eta k}{f} = \bar{k}\bar{\vartheta}_2\bar{\vartheta}_1 \in zH^2_-,$$

$$\xi\vartheta_1\vartheta_2 k = \frac{\eta\vartheta_1\vartheta_2 k}{f} = \bar{k} \in zH^2_-.$$

Clearly, k and $\vartheta_1\vartheta_2 k$ are linear independent, which contradicts the fact that $\mathcal{H}_\psi \ominus \tilde{\mathcal{R}}$ is one-dimensional.

Thus $\vartheta_1\vartheta_2 = c$, $|c| = 1$, which implies

$$\xi = c\,k/\bar{k}.$$

Let $\zeta \in \mathbb{T}$, $\zeta^2 = \bar{c}$. Put $h = \zeta k$. Clearly, h is an outer function and $\xi = h/\bar{h}$. ■

Remark. A representation of ψ in the form (1.18) is unique up to a multiplicative constant. Moreover, if

$$\psi = \rho\varphi_1/\bar{\varphi}_2,$$

where φ_1 and φ_2 are in H^2, then φ_1 and φ_2 are scalar multiples of h.

Indeed, in this case $\bar{\xi}\varphi_1 = \bar{\varphi}_2$, so $\varphi_1 \in \mathcal{H}_\psi \ominus \tilde{\mathcal{R}}_V$, which implies that $\varphi_1/h = \text{const}$. Since φ_2 has the same modulus as φ_1, it follows that $\varphi_2 = \vartheta\varphi_1$, where ϑ is an inner function. If $\vartheta \neq \text{const}$, then the same reasoning as in the proof of Theorem 1.8 shows that $\dim\, \mathcal{H}_\psi \ominus \tilde{\mathcal{R}}_V > 1$. ■

It follows from Theorem 1.7 that if the Nehari problem has at least two distinct solutions, then there exists a solution with constant modulus. As we have seen in §1.1, in the case $\rho = \|\Gamma\|$ and Γ attains its norm on the unit ball of H^2 there is a unique solution of the Nehari problem and it has constant modulus. The following example shows that in the case of uniqueness the solution does not necessarily have constant modulus.

Example. Let

$$\psi = \chi\bar{\vartheta},$$

where χ is the characteristic function of

$$I = \{e^{\mathrm{i}t} : -\alpha < t < \alpha\}, \quad 0 < \alpha < \pi,$$

$$\vartheta(\zeta) = \exp\frac{\zeta + 1}{\zeta - 1}.$$

Let us show that ψ is the only solution of the Nehari problem with $\rho = 1$ and $\Gamma = H_\psi$.

Suppose that f is a nonzero function in H^∞ such that $\|\psi - f\|_\infty \le 1$. Then $\|\chi - \vartheta f\|_\infty \le 1$ and so $|(\vartheta f)(\zeta) - 1| \le 1$ for $\zeta \in I$. It follows that $\operatorname{Re}(\vartheta f)(\zeta) \ge 0$ for $\zeta \in I$. Since $\vartheta f \in H^\infty$, the inner factor of ϑf extends analytically across I (see Garnett [1], Ch. II, Prob. 14). However, ϑ does not extend analytically across I. ■

Theorem 1.9. *Let ψ be a solution of the Nehari problem for Γ such that $|\psi(\zeta)| = \|\Gamma\| = \rho$ almost everywhere on $\mathbb{T}$. Then the Nehari problem for $\Gamma = H_\psi$ has only one solution with norm at most ρ if and only if ψ does not admit a representation*

$$\psi = f/\bar{g}, \quad f,\ g \in H^2. \tag{1.19}$$

Proof. Let $\xi = \psi/\rho$. Then $\varkappa = 0$ a.e. on $\mathbb{T}$. So $\mathcal{H}_\psi = L^2$ and

$$\mathcal{H}_\psi \ominus \tilde{\mathcal{R}}_V = \{f \in H^2 :\ \bar{\xi} f \in zH^2_-\}. \tag{1.20}$$

If $\mathcal{H}_\psi \neq \tilde{\mathcal{R}}_V$ and f is a nonzero function in $\mathcal{H}_\psi \ominus \tilde{\mathcal{R}}_V$, then ψ satisfies (1.19) with $g = \frac{1}{\rho}\xi\bar{f} \in H^2_-$.

Conversely, if ψ is represented by (1.19), then $f \in \mathcal{H}_\psi \ominus \tilde{\mathcal{R}}_V$, and so the Nehari problem has infinitely many solutions. ■

The following theorem contains simple facts about canonical functions.

Theorem 1.10. *Let ψ be a solution of the Nehari problem for Γ (i.e., $H_\psi = \Gamma$). The following assertions hold:*

(i) *If ψ admits a representation*

$$\psi = \rho\, h/\bar{h},$$

where $h \in H^2$, $1/h \in H^2$, then ψ is a canonical function.

(ii) *If ψ takes values in an arc of the circle $\{\zeta : |\zeta| = \rho\}$ of angle less than π, then ψ is a canonical function.*

(iii) *Let $\psi = \rho h/\bar{h}$, where $h \in H^2$. Then ψ is a ρ-canonical function with $\|H_\psi\| < \rho$ if and only if the Toeplitz operator $T_{\bar{h}/h}$ i s invertible on H^2.*

Proof. (i). It follows from (1.20) that $f \in \mathcal{H}_\psi \ominus \tilde{\mathcal{R}}_V$ if and only if $f \in \operatorname{Ker} T_{\bar{z}\bar{h}/h}$. Let us show that $\dim \operatorname{Ker} T_{\bar{z}\bar{h}/h} = 1$. Obviously, $h \in \operatorname{Ker} T_{\bar{z}\bar{h}/h}$.

We have $T_{\bar{z}\bar{h}/h} = T_{\bar{z}} T_{\bar{h}/h}$. Since $\dim \operatorname{Ker} T_{\bar{z}} = 1$, it remains to prove that $\operatorname{Ker} T_{\bar{h}/h} = \{\mathbb{O}\}$ or, which is equivalent, that $T^*_{\bar{h}/h} = T_{h/\bar{h}}$ has dense range. Let $g = 1/h \in H^2$. Clearly, g is outer and $T_{h/\bar{h}} = T_{\bar{g}/g}$. Now $T_{\bar{g}/g}$ has dense range by Theorem 4.4.10, which proves that

$$\dim \mathcal{H}_\psi \ominus \tilde{\mathcal{R}}_V = \dim \mathcal{K} \ominus \mathcal{R}_V = 1.$$

Hence, $\mathcal{H}_\psi = \mathcal{I}_\psi \mathcal{K}$ and so ψ is canonical.

(ii). In this case ψ can be represented in the form

$$\psi = \rho \exp \mathrm{i}(c + \alpha),$$

where $c \in \mathbb{R}$ and α is a real-valued function, $\|\alpha\|_\infty < \pi/2$. Let

$$h = \exp \frac{1}{2}(-\tilde{\alpha} + \mathrm{i}(\alpha + c)).$$

Then by Zygmund's theorem $h \in H^2$, $1/h \in H^2$ (see Appendix 2.1). Obviously, $\psi = \rho h/\bar{h}$. The result follows from (i).

(iii). If $T_{\bar{h}/h}$ is invertible, then by Theorem 3.1.11, $\|H_{h/\bar{h}}\| < 1$, and so $\|H_\psi\| < \rho$. In this case $\dim \operatorname{Ker} T_{\bar{z}\bar{h}/h} = 1$ and as in (ii) we can show that $\dim \mathcal{H}_\psi \ominus \tilde{\mathcal{R}}_V = 1$.

Suppose that $\dim \mathcal{H}_\psi \ominus \tilde{\mathcal{R}}_V = 1$ and $\|H_\psi\| < \rho$. Then $\dim \operatorname{Ker} T_{\bar{z}\bar{h}/h} = 1$. Therefore $\operatorname{Ker} T_{\bar{h}/h} = \{\mathbb{O}\}$, since if $T_{\bar{h}/h} f = \mathbb{O}$, then $h + zf \in \operatorname{Ker} T_{\bar{z}\bar{h}/h}$, which contradicts the fact that $\dim \operatorname{Ker} T_{\bar{z}\bar{h}/h} = 1$. Hence, $T_{h/\bar{h}}$ has dense range.

Let $u = \bar{h}/h$. Then T_u has dense range in H^2. Indeed, let $h = g\vartheta$, where g is outer and ϑ is inner. Then $u = \bar{\vartheta}^2 \bar{g}/g$ and by Theorem 4.4.10, T_u has dense range in H^2.

It follows now from Corollary 4.4.3 that $\|H_u\| = \|H_{\bar{u}}\|$. Since $\|H_\psi\| < \rho$, it follows that $\|H_{\bar{u}}\| < 1$. Hence, $\|H_u\| < 1$, which implies that T_u is invertible (see Theorem 3.1.11). ■

Now we can give an example of a ρ-canonical function ψ for which $\|\Gamma_\psi\| = \rho$. In this case the Nehari problem has infinitely many solutions and so ψ has different best approximations by bounded analytic functions. In §1.1 we have already constructed an example of a function φ in L^∞, which has different best approximations. We have shown here that for that function φ, there exists a function f in H^∞ such that $\|\varphi - f\|_\infty = \|\Gamma_\varphi\|$ and $\varphi - f$ is a canonical function. The following example produces an explicit canonical function ψ satisfying the above requirements.

Example. Let h be an outer function such that $h \in H^2$, $1/h \in H^2$, but $|h|^2$ does not satisfy the Helson–Szegö condition (see §3.2). To produce such a function h, it suffices to pick any positive function w such that $w \in L^1$, $1/w \in L^1$, but $w \notin L^{1+\varepsilon}$ with any positive ε and to take h to be an outer function with modulus $w^{1/2}$ (see the remark after the proof of Theorem 2.5). Then by Theorem 2.2.5, the Toeplitz operator $T_{\bar{h}/h}$ is noninvertible. By Theorem 4.4.10 and Corollary 4.4.3, $\|H_{\bar{h}/h}\| = \|H_{h/\bar{h}}\| = 1$. Therefore $\|H_{h/\bar{h}}\| = 1$; otherwise $T_{\bar{h}/h}$ would be invertible by Theorem 3.1.11.

It follows now from Theorem 1.10, (i) and (iii) that $\psi = \rho h/\bar{h}$ is a ρ-canonical function and $\|H_\psi\| = \rho$. ■

Parametrization of Canonical Solutions

We are going to describe all canonical solutions of the Nehari problem in the case of nonuniqueness. In this case there exists a canonical solution. Let us fix a canonical solution ψ:

$$H_\psi = \Gamma, \quad \|\psi\|_\infty = \rho.$$

It follows from Theorem 1.8 that ψ admits a representation

$$\psi = \rho h/\bar{h},$$

where h is an outer function in H^2.

Consider the function G defined by

$$G = |h|^2 + \mathrm{i}\widetilde{|h|^2}.$$

G is an outer function and $\operatorname{Re} G(\zeta) \geq \mathbb{O}$, $\zeta \in \mathbb{D}$. Let k be the function analytic in $\mathbb{D}$ defined by

$$k = \frac{G-1}{G+1}. \tag{1.21}$$

Clearly,

$$\frac{1+k}{1-k} = G. \tag{1.22}$$

Since $\operatorname{Re} G \geq \mathbb{O}$, it follows that $k \in H^\infty$ and $\|k\|_\infty \leq 1$.

The following theorem gives a parametrization of all canonical solutions of the Nehari problem.

Theorem 1.11. *Let $\psi = \rho h/\bar{h}$ be a canonical function of the Nehari problem, where h is an outer function in H^2 of norm one and let k be the function defined by* (1.21). *Then for any $\mathfrak{w} \in \mathbb{T}$ the function φ defined by*

$$\varphi(\zeta) = \rho \frac{h(\zeta)}{\overline{h(\zeta)}} \cdot \frac{1-k(\zeta)}{1-\overline{k(\zeta)}} \cdot \frac{\mathfrak{w} - \overline{k(\zeta)}}{1-\mathfrak{w}k(\zeta)} \tag{1.23}$$

is ρ-canonical.

Conversely, if φ is a ρ-canonical function, then there exists a complex number $\mathfrak{w}$ of modulus one such that (1.23) *holds.*

Proof. Let φ be a canonical function. Let U_φ and U_ψ be the minimal unitary extensions of V that correspond to φ and ψ. Let

$$E_\varphi = \mathcal{E}_{U_\varphi} = E_{U_\varphi}, \quad E_\psi = \mathcal{E}_{U_\psi} = E_{U_\psi}$$

be the spectral measures of U_φ and U_ψ. It follows from (1.1) and (1.2) that

$$\varphi - \psi = \rho \bar{z} \frac{d((E_\varphi - E_\psi)\mathbf{1}, \bar{z})_{\mathcal{K}}}{d\boldsymbol{m}}. \tag{1.24}$$

We have

$$(\mathbb{P}_+(\varphi - \psi))(\lambda) = \int_{\mathbb{T}} \frac{\varphi(\tau) - \psi(\tau)}{1 - \lambda\bar{\tau}} d\boldsymbol{m}(\tau), \quad \lambda \in \mathbb{D}.$$

Substituting $\varphi - \psi$ from (1.24), we obtain

$$(\mathbb{P}_+(\varphi - \psi))(\lambda) = \rho \int_{\mathbb{T}} \frac{d((E_\varphi - E_\psi)\mathbf{1}, \bar{z})_{\mathcal{K}}(\lambda)}{\tau - \lambda} = \rho \left(R_\lambda^{(\varphi)} - R_\lambda^{(\psi)}\mathbf{1}, \bar{z} \right)_{\mathcal{K}},$$

where $R_\lambda^{(\varphi)} = (U_\varphi - \lambda I)^{-1}$, $R_\lambda^{(\psi)} = (U_\psi - \lambda I)^{-1}$ are the resolvents of U_φ and U_ψ.

Since $H_\varphi = H_\psi$, it follows that $\mathbb{P}_-\varphi = \mathbb{P}_-\psi$ and so for $\zeta \in \mathbb{T}$

$$\begin{aligned} (\varphi - \psi)(\zeta) &= \lim_{r \to 1-} \left(\mathbb{P}_+(\varphi - \psi)\right)(r\zeta) \\ &= \rho \lim_{r \to 1-} \left(\left(R_{r\zeta}^{(\varphi)} - R_{r\zeta}^{(\psi)} \right) \mathbf{1}, \bar{z} \right)_{\mathcal{K}}. \end{aligned} \tag{1.25}$$

We identify $\mathcal{K}$ and $\mathcal{H}_\psi = L^2$ with the help of the imbedding $\mathcal{I}_\psi$ (since ψ is canonical, $\mathcal{I}_\psi$ is onto):

$$\mathcal{I}_\psi f = f_- + \xi f_+, \quad f \in L^2 \cap \mathcal{K}, \tag{1.26}$$

where $\xi = \rho^{-1}\psi$, $f_- = \mathbb{P}_- f$, $f_+ = \mathbb{P}_+ f$.

To avoid confusion, we introduce the notation

$$\tilde{U}_\psi = \mathcal{I}_\psi U_\psi \mathcal{I}_\psi^{-1}, \quad \tilde{U}_\varphi = \mathcal{I}_\psi U_\varphi \mathcal{I}_\psi^{-1}$$

($\tilde{U}_\varphi$ is defined in terms of $\mathcal{I}_\psi$ rather than $\mathcal{I}_\varphi$!).

We have $\xi = h/\bar{h}$. It is easily seen from (1.20) that

$$L^2 \ominus \tilde{\mathcal{R}}_V = \{\lambda h : \lambda \in \mathbb{C}\}.$$

It is evident that $\bar{z}h = \tilde{U}_\psi^* h \in \mathcal{H}_\psi \ominus \tilde{\mathcal{D}}_V$.

The operator $\tilde{U}_\varphi$ coincides with $\tilde{U}_\psi$ on $L^2 \ominus \tilde{\mathcal{D}}_V$. So $\tilde{U}_\varphi f = zf$, $f \perp \tilde{\mathcal{D}}_V$. Clearly,

$$\tilde{U}_\varphi(L^2 \ominus \tilde{\mathcal{D}}_V) = L^2 \ominus \tilde{\mathcal{R}}_V.$$

Since $\tilde{U}_\varphi$ is unitary, there exists a complex number $\mathfrak{w}$ of modulus one such that

$$\tilde{U}_\varphi \bar{z} h = \bar{\mathfrak{w}} h.$$

Therefore $\tilde{U}_\varphi$ acts on L^2 as follows:

$$\tilde{U}_\varphi f = zf + (\bar{\mathfrak{w}} - 1)(f, \bar{z}h)h, \quad f \in L^2. \tag{1.27}$$

Formula (1.25) can be rewritten in the following way. Let

$$\tilde{R}_\lambda^{(\varphi)} = \mathcal{I}_\psi R_\lambda^{(\varphi)} \mathcal{I}_\psi^{-1}, \quad \tilde{R}_\lambda^{(\psi)} = \mathcal{I}_\psi R_\lambda^{(\psi)} \mathcal{I}_\psi^{-1}$$

be the resolvents of $\tilde{U}_\varphi$ and $\tilde{U}_\psi$. Clearly, $\mathcal{I}_\psi \mathbf{1} = \xi$, $\mathcal{I}_\psi \bar{z} = \bar{z}$.

Then

$$(\varphi - \psi)(\zeta) = \rho \lim_{r \to 1-} \left(\left(\tilde{R}_{r\zeta}^{(\varphi)} - \tilde{R}_{r\zeta}^{(\psi)} \right) \xi, \bar{z} \right), \quad \zeta \in \mathbb{T}. \tag{1.28}$$

Let $\lambda = r\zeta$. We have

$$\begin{aligned} \tilde{R}_\lambda^{(\varphi)} - \tilde{R}_\lambda^{(\psi)} &= (\tilde{U}_\varphi - \lambda I)^{-1} - (\tilde{U}_\psi - \lambda I)^{-1} \\ &= (\tilde{U}_\varphi - \lambda I)^{-1} \left(I - (\tilde{U}_\varphi - \lambda I)(\tilde{U}_\psi - \lambda I)^{-1} \right) \\ &= \tilde{R}_\lambda^{(\varphi)} (\tilde{U}_\psi - \tilde{U}_\varphi) \tilde{R}_\lambda^{(\psi)}. \end{aligned} \tag{1.29}$$

Therefore by (1.27)

$$\left(\tilde{R}_\lambda^{(\varphi)} - \tilde{R}_\lambda^{(\psi)} \right) \xi = (1 - \bar{\mathfrak{w}}) \left(\tilde{R}_\lambda^{(\psi)} \xi, \bar{z}h \right) \tilde{R}_\lambda^{(\varphi)} h. \tag{1.30}$$

Let us show that

$$\tilde{R}_\lambda^{(\varphi)} h = \frac{\tilde{R}_\lambda^{(\psi)} h}{1 - (1 - \bar{\mathfrak{w}}) \left(\tilde{R}_\lambda^{(\psi)} h, \bar{z}h \right)}. \tag{1.31}$$

We have

$$\left(1-(1-\bar{\mathfrak{w}})\left(\tilde{R}_\lambda^{(\psi)}h,\bar{z}h\right)\right)\tilde{R}_\lambda^{(\varphi)}h=\tilde{R}_\lambda^{(\varphi)}h-\tilde{R}_\lambda^{(\varphi)}\left(\tilde{U}_\psi-\tilde{U}_\varphi\right)\tilde{R}_\lambda^{(\psi)}h$$

by (1.27). Formula (1.31) follows now from (1.29). Therefore we have from (1.30)

$$\left(\tilde{R}_\lambda^{(\varphi)}-\tilde{R}_\lambda^{(\psi)}\right)\xi=\frac{(1-\bar{\mathfrak{w}})\left(\tilde{R}_\lambda^{(\psi)}\xi,\bar{z}h\right)\tilde{R}_\lambda^{(\psi)}h}{1-(1-\bar{\mathfrak{w}})\left(\tilde{R}_\lambda^{(\psi)}h,\bar{z}h\right)}.$$

It follows that

$$\left(\left(\tilde{R}_\lambda^{(\varphi)}-\tilde{R}_\lambda^{(\psi)}\right)\xi,\bar{z}\right)=\frac{(1-\bar{\mathfrak{w}})\left(\tilde{R}_\lambda^{(\psi)}\xi,\bar{z}h\right)\left(\tilde{R}_\lambda^{(\psi)}h,\bar{z}\right)}{1-(1-\bar{\mathfrak{w}})\left(\tilde{R}_\lambda^{(\psi)}h,\bar{z}h\right)}.$$

We have

$$\begin{aligned}\left(\tilde{R}_\lambda^{(\psi)}\xi,\bar{z}h\right)&=\rho\int_{\mathbb{T}}\frac{h(\tau)}{\overline{h(\tau)}}\frac{\tau\overline{h(\tau)}}{\tau-\lambda}d\boldsymbol{m}(\tau)=\rho\int_{\mathbb{T}}\frac{\tau h(\tau)}{\tau-\lambda}d\boldsymbol{m}(\tau)=\rho h(\lambda),\\ \left(\tilde{R}_\lambda^{(\psi)}h,\bar{z}\right)&=\int_{\mathbb{T}}\frac{\tau h(\tau)}{\tau-\lambda}d\boldsymbol{m}(\tau)=h(\lambda),\\ \left(\tilde{R}_\lambda^{(\psi)}h,\bar{z}h\right)&=\int_{\mathbb{T}}\frac{h(\tau)}{\tau-\lambda}\tau\overline{h(\tau)}d\boldsymbol{m}(\tau)=\int_{\mathbb{T}}\frac{|h(\tau)|^2}{1-\lambda\bar{\tau}}d\boldsymbol{m}(\tau).\end{aligned}$$

Let k be the function defined by (1.21). Then

$$\frac{1+k(\lambda)}{1-k(\lambda)}=G(\lambda)=\int_{\mathbb{T}}\frac{\tau+\lambda}{\tau-\lambda}|h(\tau)|^2d\boldsymbol{m}(\tau).$$

Since $\|h\|_2=1$, we have

$$\begin{aligned}\frac{1}{1-k(\lambda)}&=\frac{1}{2}\left(1+\frac{1+k(\lambda)}{1-k(\lambda)}\right)\\&=\frac{1}{2}\int_{\mathbb{T}}\left(\frac{\tau+\lambda}{\tau-\lambda}|h(\tau)|^2+|h(\tau)|^2\right)d\boldsymbol{m}(\tau)\\&=\int_{\mathbb{T}}\frac{|h(\tau)|^2}{1-\lambda\bar{\tau}}d\boldsymbol{m}(\tau).\end{aligned}$$

Thus

$$\begin{aligned}\left((\tilde{R}_\lambda^{(\varphi)}-\tilde{R}_\lambda^{(\psi)})\xi,\bar{z}\right)&=\rho\frac{(1-\bar{\mathfrak{w}})(h(\lambda))^2}{1-(1-\bar{\mathfrak{w}})/(1-k(\lambda))}\\&=\rho\frac{(\mathfrak{w}-1)(h(\lambda))^2(1-k(\lambda))}{1-\mathfrak{w}k(\lambda)}.\end{aligned}$$

Let $\zeta\in\mathbb{T}$. It follows from (1.28) that

$$\varphi(\zeta)=\rho\frac{h(\zeta)}{\overline{h(\zeta)}}+\rho\frac{(\mathfrak{w}-1)(1-k(\zeta))(h(\zeta))^2}{1-\mathfrak{w}k(\zeta)}.$$

Note that

$$|h(\zeta)|^2 = \operatorname{Re}\frac{1+k(\zeta)}{1-k(\zeta)} = \frac{1-|k(\zeta)|^2}{|1-k(\zeta)|^2}. \tag{1.32}$$

Therefore

$$\varphi(\zeta) = \rho\frac{h(\zeta)}{\overline{h(\zeta)}}\left(1 + \frac{(\mathfrak{w}-1)(1-k(\zeta))(1-|k(\zeta)|^2)}{(1-\mathfrak{w}k(\zeta))|1-k(\zeta)|^2}\right),$$

and after elementary transformations we obtain

$$\varphi(\zeta) = \rho\frac{h(\zeta)}{\overline{h(\zeta)}}\frac{1-k(\zeta)}{1-\overline{k(\zeta)}}\frac{\mathfrak{w}-\overline{k(\zeta)}}{1-\mathfrak{w}k(\zeta)}.$$

Suppose now that $\mathfrak{w} \in \mathbb{T}$ and φ is given by (1.23). Then we can define an extension of V as follows. Let $\mathcal{H} = L^2$ and let $\mathcal{I}_\psi$ be the imbedding of $\mathcal{K}$ in L^2 given by (1.26). Define the unitary operator U on L^2 by

$$Uf = zf + (\bar{\mathfrak{w}}-1)(f, zh)h, \quad f \in L^2.$$

Then U is a minimal unitary extension of $\mathcal{I}_\psi V \mathcal{I}_\psi^{-1}$. Let φ_1 be the solution of the Nehari problem that corresponds to U. Then φ_1 is a canonical function and $U = \widetilde{U}_{\varphi_1}$.

It follows from the part of the theorem already proved that

$$\varphi_1 = \rho\frac{h}{\bar{h}}\frac{1-k}{1-\bar{k}}\frac{\mathfrak{w}-\bar{k}}{1-\mathfrak{w}k}.$$

Thus $\varphi_1 = \varphi$ and φ is a canonical solution of the Nehari problem. ■

Let $\omega = (1-k)h$. Then ω is an outer function and formula (1.23) can be rewritten as follows:

$$\varphi = \rho\frac{\omega}{\bar{\omega}}\frac{\mathfrak{w}-\bar{k}}{1-\mathfrak{w}k}, \quad \mathfrak{w} \in \mathbb{T}. \tag{1.33}$$

In this parametrization the function k depends on the choice of a canonical function ψ. However, the following result shows that k is determined uniquely by the Hankel operator Γ up to a unimodular constant.

Theorem 1.12. *Let ψ_1 and ψ_2 be canonical solutions of the Nehari problem,*

$$\psi_1 = \rho\frac{h_1}{\bar{h}_1}, \quad \psi_2 = \rho\frac{h_2}{\bar{h}_2},$$

where h_1 and h_2 are outer functions in H^2 of norm one, and let k_1 and k_2 be the functions defined by

$$\frac{1+k_j}{1-k_j} = |h_j|^2 + \mathrm{i}\widetilde{|h_j|^2}, \quad j = 1, 2.$$

Then k_1/k_2 is a unimodular constant.

If

$$\omega_j = (1-k_j)h_j, \quad j = 1, 2,$$

then ω_1/ω_2 is a unimodular constant.

Proof. Let U_1 and U_2 be the minimal unitary extensions of V on the space $\mathcal{K}$ that correspond to ψ_1 and ψ_2. Then

$$U_1|\mathcal{D}_V = U_2|\mathcal{D}_V. \tag{1.34}$$

Let $x \in \mathcal{K} \ominus \mathcal{D}_V$, $\|x\| = 1$, $y = U_1 x$. Then $y \in \mathcal{K} \ominus \mathcal{R}_V$ and

$$U_2 x = \gamma y \tag{1.35}$$

for some unimodular constant γ.

Consider the unitary map $\mathcal{I}_{\psi_1}$ of $\mathcal{K}$ onto L^2 given by

$$\mathcal{I}_{\psi_1} f = f_- + \frac{1}{\rho}\psi_1 f_+, \quad f \in \mathcal{K} \cap L^2.$$

Then the operator $\tilde{U}_{\psi_1} = \mathcal{I}_{\psi_1} U_1 \mathcal{I}_{\psi_1}^{-1}$ is multiplication by z on L^2. As we have seen in the proof of Theorem 1.11, $\mathcal{I}_{\psi_1} y = ch_1$, $\mathcal{I}_{\psi_1} x = c\bar{z}h_1$, where c is a unimodular constant.

By the definition of k_1 we have

$$\frac{1 + k_1(\zeta)}{1 - k_1(\zeta)} = \int_{\mathbb{T}} \frac{\tau + \zeta}{\tau - \zeta} |h_1(\tau)|^2 d\boldsymbol{m}(\tau).$$

Since $\|h_1\|_2 = 1$, it follows that

$$\begin{aligned}\frac{k_1(\zeta)}{1 - k_1(\zeta)} &= \zeta \int_{\mathbb{T}} \frac{1}{\tau - \zeta} |h_1(\tau)|^2 d\boldsymbol{m}(\tau) = \zeta \left(\left(\tilde{U}_{\psi_1} - \zeta I \right)^{-1} h_1, h_1 \right) \\ &= \zeta \left((\tilde{U}_{\psi_1} - \zeta I)^{-1} ch_1, ch_1 \right) = \zeta \left((U_1 - \zeta I)^{-1} y, y \right)_{\mathcal{K}}.\end{aligned}$$

It is also clear that

$$\frac{1}{1 - k_1(\zeta)} = \int_{\mathbb{T}} \frac{\tau}{\tau - \zeta} |h_1(\tau)|^2 d\boldsymbol{m}(\tau) = \left(U_1 (U_1 - \zeta I)^{-1} y, y \right)_{\mathcal{K}}. \tag{1.36}$$

If we do the same with ψ_2, we obtain

$$\begin{aligned}\frac{k_2(\zeta)}{1 - k_2(\zeta)} &= \zeta \left((U_2 - \zeta I)^{-1} U_2 x, U_2 x \right)_{\mathcal{K}} \\ &= \zeta \left((U_2 - \zeta I)^{-1} \gamma y, \gamma y \right)_{\mathcal{K}} \\ &= \zeta \left((U_2 - \zeta I)^{-1} y, y \right)_{\mathcal{K}}.\end{aligned} \tag{1.37}$$

It is easy to see from (1.34) and (1.35) that

$$(U_2 - \zeta I)^{-1} f - (U_1 - \zeta I)^{-1} f = -(U_2 - \zeta I)^{-1} (U_2 - U_1)(U_1 - \zeta I)^{-1} f$$

$$= (1 - \gamma) \left((U_1 - \zeta I)^{-1} f, x \right)_{\mathcal{K}} (U_2 - \zeta I)^{-1} y, \quad f \in \mathcal{K}.$$

Therefore

$$\begin{aligned}
\frac{k_2(\zeta)}{1-k_2(\zeta)} - \frac{k_1(\zeta)}{1-k_1(\zeta)} &= \zeta\left((U_2-\zeta I)^{-1}y - (U_1-\zeta I)^{-1}y, y\right)_{\mathcal{K}} \\
&= (1-\gamma)\zeta\left((U_1-\zeta I)^{-1}y, x\right)_{\mathcal{K}}\left((U_1-\zeta I)^{-1}y, x\right)_{\mathcal{K}} \\
&= (1-\gamma)\frac{k_2(\zeta)}{1-k_2(\zeta)}\left((U_2-\zeta I)^{-1}y, y\right)_{\mathcal{K}} \\
&= (1-\gamma)\frac{k_2(\zeta)}{1-k_2(\zeta)}\left(U_1(U_1-\zeta I)^{-1}y, y\right)_{\mathcal{K}} \\
&= (1-\gamma)\frac{k_2(\zeta)}{1-k_2(\zeta)}\frac{1}{1-k_1(\zeta)}
\end{aligned}$$

by (1.37) and (1.36). This easily implies that $k_1(\zeta) = \gamma k_2(\zeta)$.

Let us now show that ω_1 and ω_2 differ by a unimodular constant. Since ω_1 and ω_2 are outer, it is sufficient to show that $|\omega_1| = |\omega_2|$. It follows from (1.22) that

$$|h_j|^2 = \operatorname{Re}\frac{1+k_j}{1-k_j} = \frac{1-|k_j|^2}{|1-k_j|^2}, \quad j = 1, 2,$$

and so

$$|\omega_1|^2 = |1-k_1|^2 \cdot |h_1|^2 = 1-|k_1|^2 = 1-|k_2|^2 = |1-k_2|^2|h_2|^2 = |\omega_2|^2. \quad \blacksquare$$

Parametrization of All Solutions

We have proved that each unimodular constant $\mathfrak{w}$ determines by formula (1.33) a canonical solution of the Nehari problem. The following result claims that if $\mathfrak{w}$ ranges over the unit ball of H^∞, formula (1.33) describes all solutions.

Theorem 1.13. *Let Γ be a Hankel operator, $\|\Gamma\| \le \rho$. Suppose that the set of solutions*

$$\{\varphi \in L^\infty : \ H_\varphi = \Gamma, \ \|\varphi\| \le \rho\} \tag{1.38}$$

of the Nehari problem contains two different functions. Let $\psi = \rho h/\bar{h}$ be a canonical solution, where h is an outer function in H^2 of norm one, let k be the function defined by (1.21), *and let $\omega = (1-k)h$.*

Then the map

$$\mathfrak{w} \mapsto \varphi_{\mathfrak{w}} = \rho\frac{\omega}{\bar{\omega}}\frac{\mathfrak{w}-\bar{k}}{1-\mathfrak{w}k} \tag{1.39}$$

is an isomorphism from the unit ball of H^∞ onto the set (1.38) *of solutions of the Nehari problem.*

Remark. It is easy to see that the map (1.39) is an isomorphism between the unit ball of L^∞ and the ball of radius ρ of L^∞.

Proof. Let us first prove that $\varphi_{\mathfrak{w}}$ is a solution of the Nehari problem for any $\mathfrak{w}$ in the unit ball of H^∞. Clearly,

$$\varphi_{\mathbf{1}} = \rho\frac{h}{\bar{h}}\frac{1-k}{1-\bar{k}}\frac{1-\bar{k}}{1-k} = \psi.$$

Let $\mathfrak{w} \in H^\infty$ and $\|\mathfrak{w}\|_{H^\infty} \le 1$. We have

$$\begin{aligned}\varphi_{\mathbf{1}} - \varphi_{\mathfrak{w}} &= \rho\frac{h}{\bar{h}}\left(1 - \frac{1-k}{1-\bar{k}}\frac{\mathfrak{w}-\bar{k}}{1-\mathfrak{w}k}\right)\\ &= \rho\frac{h^2}{|h|^2}\left(1 - \frac{1-k}{1-\bar{k}}\frac{\mathfrak{w}-\bar{k}}{1-\mathfrak{w}k}\right)\\ &= \rho\frac{h^2}{|h|^2}\frac{(1-\mathfrak{w})(1-|k|^2)}{(1-\bar{k})(1-\mathfrak{w}k)}.\end{aligned}$$

Bearing in mind that

$$|h|^2 = \frac{1-|k|^2}{|1-k|^2}$$

(see (1.32)), we obtain

$$\varphi_{\mathbf{1}} - \varphi_{\mathfrak{w}} = \rho h^2\frac{(1-\mathfrak{w})|1-k|^2}{(1-\bar{k})(1-\mathfrak{w}k)} = \rho h^2\frac{(1-\mathfrak{w})(1-k)}{1-\mathfrak{w}k}. \tag{1.40}$$

Since $\|\mathfrak{w}\|_\infty \le 1$ and $\|k\|_\infty \le 1$, it follows that $\operatorname{Re}(1-\mathfrak{w}k) \ge \mathbb{O}$ and so $1-\mathfrak{w}k$ is an outer function (see Appendix 1.2). Obviously, $\varphi_{\mathbf{1}} - \varphi_{\mathfrak{w}} \in L^\infty$, which implies that $\varphi_{\mathbf{1}} - \varphi_{\mathfrak{w}} \in H^\infty$. This proves that $H_{\varphi_{\mathfrak{w}}} = H_{\varphi_{\mathbf{1}}}$, so $\varphi_{\mathfrak{w}}$ is a solution of the Nehari problem.

Suppose now that φ is a solution of the Nehari problem, that is, $H_\varphi = \Gamma$ and $\|\varphi\|_\infty \le \rho$. Then as we have noticed above, there is a function $\mathfrak{w}$ in the unit ball of L^∞ such that $\varphi = \varphi_{\mathfrak{w}}$, which means that $f \stackrel{\text{def}}{=} \varphi_{\mathbf{1}} - \varphi_{\mathfrak{w}} \in H^\infty$. For our convenience we can assume without loss of generality that $\rho = 1$.

We have

$$|1 - f(\zeta)\overline{\varphi_{\mathbf{1}}(\zeta)}| = |\varphi_{\mathbf{1}}(\zeta) - f(\zeta)| = |\varphi_{\mathfrak{w}}(\zeta)| \le 1, \quad \zeta \in \mathbb{T}.$$

Therefore $f\varphi_{\mathbf{1}}$ takes values in $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \ge 0\}$. Hence, there exists a real-valued function α such that

$$-\frac{\pi}{2} \le \alpha(\zeta) \le \frac{\pi}{2}, \quad \zeta \in \mathbb{T},$$

and

$$f(\zeta)\overline{\varphi_{\mathbf{1}}(\zeta)} = |f(\zeta)|\exp \mathrm{i}\alpha(\zeta), \quad \zeta \in \mathbb{T}. \tag{1.41}$$

Since $|1 - f(\zeta)\overline{\varphi_{\mathbf{1}}(\zeta)}| \le 1$, it follows elementarily that

$$|f(\zeta)| = |f(\zeta)\overline{\varphi_{\mathbf{1}}(\zeta)}| \le 2\cos\alpha(\zeta), \quad \zeta \in \mathbb{T}. \tag{1.42}$$

Put

$$g = \exp(\tilde{\alpha} - \mathrm{i}\alpha). \tag{1.43}$$

Then g is an outer function. Consider the harmonic extensions of α and $\tilde{\alpha}$ to the unit disk; we keep the same notation α and $\tilde{\alpha}$ for them. Then

$$g(\zeta) = \exp(\tilde{\alpha}(\zeta) - \mathrm{i}\alpha(\zeta)), \quad \zeta \in \mathbb{D}.$$

Clearly,

$$-\frac{\pi}{2} \le \alpha(\zeta) \le \frac{\pi}{2}, \quad \zeta \in \mathbb{D}.$$

Therefore $\operatorname{Re} g(\zeta) \ge 0$ for $\zeta \in \mathbb{D}$.

Let us show that $gf \in H^1$. We have by (1.42)

$$|g(\zeta)f(\zeta)| \le 2|g(\zeta)| \cos \alpha(\zeta) = 2 \operatorname{Re}\, g(\zeta), \quad \zeta \in \mathbb{T}.$$

The function $\operatorname{Re} g$ is a nonnegative harmonic function in $\mathbb{D}$ and so

$$\begin{aligned}\int_{\mathbb{T}} |g(\zeta)f(\zeta)| d\boldsymbol{m}(\zeta) &\le 2\int_{\mathbb{T}} \operatorname{Re} g(\zeta) d\boldsymbol{m}(\zeta) \\ &\le 2 \liminf_{r\to 1-} \int_{\mathbb{T}} \operatorname{Re} g(r\zeta) d\boldsymbol{m}(\zeta) = 2 \operatorname{Re} g(0)\end{aligned}$$

(see Appendix 2.1). Being the product of an outer function and a function in H^∞, the function gf belongs to H^1.

We are going to establish that gf is a positive scalar multiple of h^2. Being in H^1, the function gf admits a factorization $gf = \vartheta h_1^2$, where ϑ is an inner function and h_1 is an outer function in H^2. Then

$$\frac{\vartheta h_1}{\bar{h}_1} = \frac{\vartheta h_1^2}{|h_1|^2} = \frac{gf}{|h_1|^2} = \frac{gf}{|gf|}.$$

It follows from (1.41) and (1.43) that

$$gf\bar{\varphi}_{\mathbf{1}} = |gf\bar{\varphi}_{\mathbf{1}}| = |gf|,$$

which implies

$$\frac{\vartheta h_1}{\bar{h}_1} = \varphi_{\mathbf{1}} = \frac{h}{\bar{h}}.$$

However, being a canonical function, $\varphi_{\mathbf{1}}$ admits a unique (up to a constant factor) representation of the form $\varphi_{\mathbf{1}} = \eta_1/\bar{\eta}_2$, η_1, $\eta_2 \in H^2$ (see the Remark after Theorem 1.8). Therefore $\vartheta h_1 = ch$, $\bar{h}_1 = c\bar{h}$. Thus

$$gf = \vartheta h_1^2 = |c|^2 h^2.$$

It follows that

$$\operatorname{Re} \frac{h^2(\zeta)}{f(\zeta)} \ge 0, \quad \zeta \in \mathbb{D}, \tag{1.44}$$

since $\operatorname{Re} g(\zeta) \ge 0$ in $\mathbb{D}$.

Recall that by (1.40)

$$f = \varphi_{\mathbf{1}} - \varphi_{\mathfrak{w}} = h^2 \frac{(1-\mathfrak{w})(1-k)}{1-\mathfrak{w}k},$$

which implies

$$\frac{h^2(\zeta)}{f(\zeta)} = \frac{1}{2}\left(\frac{1+k(\zeta)}{1-k(\zeta)} + \frac{1+\mathfrak{w}(\zeta)}{1-\mathfrak{w}(\zeta)}\right), \quad \zeta \in \mathbb{T}. \tag{1.45}$$

Since

$$\operatorname{Re}\frac{1+k(\zeta)}{1-k(\zeta)} = |h(\zeta)|^2, \quad \zeta \in \mathbb{T},$$

we obtain

$$\operatorname{Re}\frac{h^2(\zeta)}{f(\zeta)} - \frac{1}{2}|h(\zeta)|^2 = \frac{1+\mathfrak{w}(\zeta)}{1-\mathfrak{w}(\zeta)} \geq 0, \quad \zeta \in \mathbb{T}. \tag{1.46}$$

Let

$$v(\zeta) \stackrel{\text{def}}{=} \operatorname{Re}\frac{h^2(\zeta)}{f(\zeta)}, \quad w(\zeta) \stackrel{\text{def}}{=} \frac{1}{2}|f(\zeta)|^2, \quad \zeta \in \mathbb{T}.$$

Obviously, $w \in L^1$. However, it follows from (1.44) that the function v on $\mathbb{T}$, being the boundary-value function of the positive harmonic function $\operatorname{Re}(h^2/f)$, also belongs to L^1 (see Appendix 2.1). Consider the harmonic extensions of v and w to the unit disk, keeping for them the same notation v and w. We have

$$w(\zeta) = \frac{1}{2}\operatorname{Re}\frac{1+k(\zeta)}{1-k(\zeta)}, \quad \zeta \in \mathbb{D}. \tag{1.47}$$

It follows from the positivity of $\operatorname{Re}(h^2/f)$ that

$$\operatorname{Re}\frac{h^2(\zeta)}{f(\zeta)} \geq v(\zeta), \quad \zeta \in \mathbb{D}, \tag{1.48}$$

(see Appendix 2.1).

Inequality (1.46) means that $v(\zeta) - w(\zeta) \geq 0$, $\zeta \in \mathbb{T}$, which implies

$$v(\zeta) - w(\zeta) \geq 0, \quad \zeta \in \mathbb{D},$$

and together with (1.47) and (1.48) this yields

$$\operatorname{Re}\frac{h^2(\zeta)}{f(\zeta)} - \frac{1}{2}\operatorname{Re}\frac{1+k(\zeta)}{1-k(\zeta)} \geq 0, \quad \zeta \in \mathbb{D}. \tag{1.49}$$

Let

$$\eta(\zeta) \stackrel{\text{def}}{=} \frac{h^2(\zeta)}{f(\zeta)} - \frac{1}{2}\frac{1+k(\zeta)}{1-k(\zeta)}.$$

It follows from (1.49) that $\operatorname{Re}\eta(\zeta) \geq 0$, $\zeta \in \mathbb{D}$. Therefore η is an outer function and we obtain from (1.45)

$$\eta(\zeta) = \frac{1+\mathfrak{w}(\zeta)}{1-\mathfrak{w}(\zeta)}, \quad \zeta \in \mathbb{T}.$$

Hence, the function $\mathfrak{w}$ analytically extends to $\mathbb{D}$ by

$$\mathfrak{w}(\zeta) = \frac{\eta(\zeta)-1}{\eta(\zeta)+1}, \quad \zeta \in \mathbb{D}.$$

Since $\operatorname{Re}\eta(\zeta) \geq 0$, $\zeta \in \mathbb{D}$, it follows that $|\mathfrak{w}(\zeta)| \leq 1$, $\zeta \in \mathbb{D}$. ■

Remark. We have just proved that formula (1.39) parametrizes all solutions of the Nehari problem when $\mathfrak{w}$ ranges over the unit ball of H^∞. Theorem 1.11 claims that to describe the canonical functions, we have to substitute in (1.39) unimodular constants $\mathfrak{w}$. It is easy to see that a solution

φ in (1.39) has constant modulus ρ is and only if $\mathfrak{w}$ is an inner function. This allows us to construct solutions of the Nehari problem of constant modulus ρ that are not canonical.

2. Parametrization of Solutions of the Nevanlinna–Pick Problem

We apply in this section the results of §1 to the Nevanlinna–Pick interpolation problem. We consider here the case of infinitely many distinct points ζ_j, $j \ge 0$, in $\mathbb{D}$. The case of finitely many points can be treated in the same way. In a similar way one can consider the case of multiple interpolating, i.e., interpolating not only values of a function but also values of its derivatives up to a certain order.

Let ζ_j, $j \ge 0$, be a sequence of distinct points in $\mathbb{D}$ and let w_j, $j \ge 0$, be complex numbers satisfying $|w_j| < 1$. We are looking for functions $f \in H^\infty$ satisfying:

$$f(\zeta_j) = w_j, \quad j \in \mathbb{Z}_+, \qquad \text{and} \qquad \|f\|_{H^\infty} \le 1. \tag{2.1}$$

We consider the case when this Nevanlinna–Pick interpolation problem has at least two distinct solutions. Clearly, in this case the sequence $\{\zeta_j\}_{j\ge 0}$ must satisfy the Blaschke condition

$$\sum_{j\ge 0}(1 - |\zeta_j|) < \infty.$$

We show that if the interpolation problem (2.1) has at least two distinct solutions, then there exist inner solutions of the problem (2.1) and we parametrize all solutions of (2.1).

We denote by B a Blaschke product with simple zeros at the ζ_j, $j \ge 0$. Suppose that f_0 is a function in H^∞ interpolating the values w_j at ζ_j, i.e., $f(\zeta_j) = w_j$, $j \ge 0$. Then all functions g in H^∞ satisfying $g(\zeta_j) = w_j$, $j \ge 0$, have the form $g = f_0 - Bq$, where $q \in H^\infty$. We have

$$\|g\|_\infty = \|f_0 - Bq\|_\infty = \|\overline{B} f_0 - q\|_\infty,$$

and so by the Nehari theorem,

$$\min_{h\in H^\infty} \|f_0 - Bq\|_\infty = \|H_{\overline{B} f_0}\|.$$

It is easy to see that the Nevanlinna–Pick interpolation problem reduces to the Nehari problem. Indeed, let $\varphi = \overline{B} f_0$. and $\|\varphi - q\|_\infty \le 1$, then $f_0 - Bq$ is a solution of (2.1). Clearly, all solutions of (2.1) can be obtained in this way.

Thus a necessary and sufficient condition for the problem (2.1) to have a solution is $\|H_{\overline{B} f_0}\| \le 1$. It is easy to see that if $\|H_{\overline{B} f_0}\| < 1$, then (2.1) has infinitely many distinct solutions.

However, the problem (2.1) can have distinct solutions even if $\|H_{\overline{B}f_0}\| = 1$. Indeed, in §1.1 we have constructed an example of a Hankel operator with distinct symbols of minimal norm. We can use the same example now. Recall that in that example ω is the conformal map of $\mathbb{D}$ onto the domain

$$\left\{\zeta \in \mathbb{C}: \ |\zeta| + \frac{|1-\zeta|}{2} < 1\right\}.$$

We have shown there that if B is an infinite Blaschke product whose zeros have an accumulation point at 1, then $\operatorname{dist}_{L^\infty}(\overline{B}\omega, H^\infty) = \|\omega\|_\infty = 1$ and $\|\overline{B}\omega - (1-\omega)/2\|_\infty = 1$. Suppose now that B has simple zeros ζ_j, $j \ge 0$, and consider the interpolation problem (2.1) with $w_j = \omega(\zeta_j)$. Then ω and $\omega - B(1-\omega)/2$ are distinct solutions of (2.1).

It is easy to see from the above example that if B is an infinite Blaschke product with simple zeros ζ_j, $j \ge 0$, then there exists $f \in H^\infty$ such that $\|H_{\overline{B}f}\| = 1$ and the problem (2.1) with $w_j = f(\zeta_j)$ has distinct solutions of norm 1.

Theorem 2.1. *If the interpolation problem* (2.1) *has at least two distinct solutions, there exists an inner function that solves* (2.1).

Proof. Consider the Nehari problem for the function $\varphi = \overline{B}f_0$. Since it has distinct solutions, the corresponding partial isometry V defined at the beginning of §1 has deficiency indices equal to 1, and so there exists a canonical solution ψ of the Nehari problem. By Theorem 1.7, a canonical solution ψ has modulus one almost everywhere. Let $\psi = \overline{B}f_0 - q$. Then $f = f_0 - Bq$ is a solution of (2.1) and has modulus one almost everywhere. Thus f is an inner solution of (2.1). ■

The following example shows that if (2.1) has one solution, it does not have to be inner.

Example. Let $\{\zeta_j\}_{j\ge 0}$ be a sequence satisfying the Blaschke condition and such that $\lim_{j\to\infty} \zeta_j = 1$. Let J be an open arc of $\mathbb{T}$ that contains 1 and is not equal to $\mathbb{T}$. Let f_0 be a function invertible in H^∞ and such that $|f_0(\zeta)| = 1$ for $\zeta \in J$, $\|f_0\|_\infty = 1$, but $|f_0|$ is not identically equal to 1 on $\mathbb{T}$. Put $w_j = f(\zeta_j)$, $j \ge 0$. Clearly, f_0 is a solution of (2.1). Suppose that $f = f_0 - Bq$, $q \in H^\infty$, is another solution. Then $\|f_0 - Bq\|_\infty \le 1$, and so $|1 - (Bq/f_0)(\zeta)| \le 1$ for $\zeta \in J$. Hence, $\operatorname{Re}(Bq/f_0)(\zeta) \ge 0$ for $\zeta \in J$. Then the inner factor of Bq/f_0 extends analytically across J (see Garnett [1], Ch. II, Ex. 14). However, B does not extend analytically across J, and so $q = \mathbb{O}$. Thus f_0 is the only solution of (2.1), and f_0 is not inner.

Let us now apply Theorem 1.13 to obtain a parametrization formula for the solutions of (2.1) under the assumption that it has distinct solutions. We consider the Nehari problem for $\varphi = \overline{B}f_0$. Let $\psi = h/\bar{h}$ be a canonical solution of the Nehari problem, where h is an outer function in H^2. As in

§1 we consider the function k defined by

$$\frac{1+k}{1-k}=|h|^2+\mathrm{i}\widetilde{|h|^2}.$$

Then $k\in H^\infty$ and $\|k\|_\infty\le 1$ (see §1). Consider the inner solution $\vartheta=\psi B$ of (2.1). The following result parametrizes all solutions of (2.1) in the case of nonuniqueness.

Theorem 2.2. *Suppose that the Nevanlinna–Pick interpolation problem* (2.1) *has distinct solutions. Then there exist functions* α_1, α_2, *and* α_3 *in* H^∞ *such that the set of solutions of* (2.1) *is given by*

$$\left\{\frac{\alpha_1+\alpha_2\beta}{1-\alpha_3\beta}:\ \beta\in H^\infty,\ \|\beta\|_\infty\le 1\right\}. \tag{2.2}$$

Moreover, for almost all $\zeta\in\mathbb{T}$ *the map*

$$\frac{\alpha_1(\zeta)+\alpha_2(\zeta)z}{1-\alpha_3(\zeta)z} \tag{2.3}$$

is a conformal map of the unit disc onto itself.

Proof. Let ϑ, k, and h be as above. By Theorem 1.13, all solutions of (2.1) are given by

$$\left\{B\cdot\frac{h}{\bar h}\cdot\frac{1-k}{1-\bar k}\cdot\frac{\beta-\bar k}{1-\beta k}:\ \beta\in H^\infty,\ \|\beta\|_\infty\le 1\right\}.$$

It follows from (1.40) that f is a solution of (2.1) if and only if

$$f=\vartheta-Bh^2\frac{(1-\beta)(1-k)}{1-\beta k}$$

for some β in the unit ball of H^∞. It follows from the definition of k that

$$h^2(1-k)=\frac{2h^2}{|h|^2+1+\mathrm{i}\widetilde{|h|^2}},$$

and so $h^2(1-k)\in H^\infty$. Thus the set of solutions of the Nevanlinna–Pick problem (2.1) is given by

$$\left\{\frac{\vartheta-Bh^2(1-k)+\beta(Bh^2(1-k)-\vartheta k)}{1-\beta k}:\ \beta\in H^\infty,\ \|\beta\|_\infty\le 1\right\}.$$

We can now put

$$\alpha_1=\vartheta-Bh^2(1-k),\quad \alpha_2=Bh^2(1-k)-\vartheta k,\quad \alpha_3=-k.$$

The fact that for almost all $\zeta\in\mathbb{T}$ the map (2.3) is a conformal map of $\mathbb{D}$ onto itself is an immediate consequence of Theorem 1.11. However, this can be verified straightforwardly if we use the fact that $\vartheta=Bh/\bar h$. Indeed,

$$\begin{aligned}&\frac{\vartheta-Bh^2(1-k)+\beta(Bh^2(1-k)-\vartheta k)}{1-\beta k}\\&=(Bh^2(1-k)-\vartheta k)\frac{\frac{-\vartheta+Bh^2(1-k)}{(Bh^2(1-k)-\vartheta k)}-\beta}{1-\beta k}.\end{aligned}$$

It follows from (1.21) and (1.22) that

$$|h|^2 = \frac{1-|k|^2}{|1-k|^2} \tag{2.4}$$

whence it is easy to see that on $\mathbb{T}$

$$\big|(Bh^2(1-k)-\vartheta k\big| = \big|(Bh^2(1-k)-Bh/\bar{h}k\big| = \big||h|^2(1-k)-k\big| = 1.$$

On the other hand, on $\mathbb{T}$

$$\frac{-\vartheta + Bh^2(1-k)}{Bh^2(1-k)-\vartheta k} = \frac{-Bh/\bar{h} + Bh^2(1-k)}{Bh^2(1-k)-Bkh/\bar{h}} = \frac{-1+|h|^2(1-k)}{|h|^2(1-k)-k} = \bar{k},$$

which is an immediate consequence of (2.4). Thus we have on $\mathbb{T}$

$$\frac{\vartheta - Bh^2(1-k)+\beta(Bh^2(1-k)-\vartheta k)}{1-\beta k} = (Bh^2(1-k)-\vartheta k)\frac{\bar{k}-\beta}{1-\beta k},$$

which completes the proof. ■

It follows from Theorem 1.11 that all canonical solutions of the Nehari problem for $\bar{B}f_0$ can be obtained by substituting in (2.2) all complex numbers β of modulus one. Clearly, this produces inner solutions of the Nevanlinna–Pick interpolation problem. However, there are other inner solutions. The following theorem parametrizes all inner solutions.

Theorem 2.3. *The inner solutions of of the Nevanlinna–Pick interpolation problem* (2.2) *correspond to the inner functions β in* (2.2).

Proof. The result is an immediate consequence of the fact that for almost all $\zeta \in \mathbb{T}$ the map (2.3) is a conformal map of $\mathbb{D}$ onto itself. ■

3. Parametrization of Solutions of the Nehari–Takagi Problem

For a function φ in L^∞, $\rho > 0$, and $m \in \mathbb{Z}_+$, the Nehari–Takagi approximation problem is to approximate φ by functions f in $H^\infty_{(m)}$ so that $\|\varphi - f\|_\infty \le \rho$. Recall (see §4.1) that $H^\infty_{(m)}$ is the set of bounded functions whose antianalytic part is a rational function of degree at most m. By the Adamyan–Arov–Krein theorem (see §4.1 and §4.2), this problem is solvable if and only if $s_m(H_\varphi) \le \rho$. We are going to parametrize all solutions of the Nehari–Takagi problem in the *suboptimal case* (i.e., $s_m(H_\varphi) < \rho$). Moreover, we consider the case when $s_m(H_\varphi) < \rho < s_{m-1}(H_\varphi)$ ($s_m(H_\varphi) < \rho$ if $m = 0$). If $m = 0$, we deal with analytic approximation and thus we arrive at the Nehari problem. All solutions of the Nehari problem in the case of nonuniqueness have been parametrized in §1. We offer here a different approach that also works for $m = 0$ and $\rho > \|H_\varphi\|$ in which case it leads to a parametrization of all solutions of the Nehari problem.

Using the Adamyan–Arov–Krein theorem mentioned above, one can reformulate the Nehari–Takagi problem in the following way. Let

$\Gamma : H^2 \to H^2_-$ be a bounded Hankel operator, $\rho > 0$, and $k \in \mathbb{Z}_+$. The problem is to parametrize all functions $f \in H^\infty_{(m)}$ such that $\|\Gamma - H_f\| \le \rho$.

We assume in this section that

$$s_m(\Gamma) < \rho < s_{m-1}(\Gamma) \tag{3.1}$$

(in the case $m = 0$ we simply assume that $s_m(\Gamma) < \rho$).

We are going to use the notation introduced in §4.2. Recall that

$$R_\rho \stackrel{\text{def}}{=} (\rho^2 I - \Gamma^*\Gamma)^{-1}, \quad r(\rho) \stackrel{\text{def}}{=} (R_\rho \mathbf{1}, \mathbf{1}),$$

and J is the real-linear operator on L^2 defined by $Jg = \bar{z}\bar{g}$.

We define the functions p_ρ and q_ρ in H^2 by

$$p_\rho = \rho R_\rho \mathbf{1}, \quad q_\rho = z R_\rho \Gamma^* \bar{z}. \tag{3.2}$$

We need the following properties of p_ρ and q_ρ.

Lemma 3.1. *Let Γ, ρ, p_ρ, and q_ρ be as above. The following holds:*

$$\Gamma p_\rho = \rho J S^* q_\rho = \rho \bar{q}_\rho, \quad \Gamma q_\rho = \rho J S^* p_\rho, \tag{3.3}$$

and

$$|p_\rho|^2 - |q_\rho|^2 = r(\rho) \quad \text{on} \quad \mathbb{T}. \tag{3.4}$$

Proof. Let us show that

$$J\Gamma R_\rho = R_\rho \Gamma^* J. \tag{3.5}$$

Indeed, it is easy to see that $J\Gamma = \Gamma^* J$, and so

$$J\Gamma R_\rho = \Gamma^* J R_\rho.$$

Multiplying this equality by $\rho^2 I - \Gamma^*\Gamma$ on the left, we obtain

$$(\rho^2 I - \Gamma^*\Gamma) J\Gamma R_\rho = \rho^2 \Gamma^* J R_\rho - \Gamma^*\Gamma\Gamma^* J R_\rho = \Gamma^*(\rho^2 I - \Gamma\Gamma^*) J R_\rho.$$

Using the obvious fact that $\Gamma\Gamma^* J = J\Gamma^*\Gamma$, we have

$$(\rho^2 I - \Gamma^*\Gamma) J\Gamma R_\rho = \Gamma^* J(\rho^2 I - \Gamma^*\Gamma) R_\rho = \Gamma^* J,$$

which implies (3.5).

Let us now establish (3.3). It follows from (3.5) that

$$\Gamma p_\rho = \rho \Gamma R_\rho \mathbf{1} = \rho J R_\rho \Gamma^* \bar{z} = \rho J \bar{z} q_\rho = \rho J S^* q_\rho,$$

since $q_\rho \in zH^2$. Obviously, $J\bar{z}q_\rho = \bar{q}_\rho$.

To prove the second equality in (3.3), we observe the following equalities:

$$(\rho^2 I - \Gamma^*\Gamma)^{-1}\Gamma^* = \Gamma^*(\rho^2 I - \Gamma\Gamma^*)^{-1} \tag{3.6}$$

and

$$J(\rho^2 I - \Gamma\Gamma^*)^{-1} = (\rho^2 I - \Gamma^*\Gamma)^{-1} J. \tag{3.7}$$

To verify (3.6), it suffices to multiply it by $\rho^2 I - \Gamma^*\Gamma$ on the left and by $\rho^2 I - \Gamma\Gamma^*$ on the right. To verify (3.7), it suffices to multiply it on the right by $\rho^2 I - \Gamma\Gamma^*$.

We have

$$\begin{aligned}\Gamma q_\rho &= \Gamma z R_\rho \Gamma^* \bar{z} = \Gamma z \Gamma^* (\rho^2 I - \Gamma\Gamma^*)^{-1} \bar{z} = \mathbb{P}_- z \Gamma\Gamma^* (\rho^2 I - \Gamma\Gamma^*)^{-1} \bar{z} \\ &= -\mathbb{P}_- z (\rho^2 I - \Gamma\Gamma^*)(\rho^2 I - \Gamma\Gamma^*)^{-1} \bar{z} + \rho^2 \mathbb{P}_- z (\rho^2 I - \Gamma\Gamma^*)^{-1} \bar{z} \\ &= \rho^2 J^2 \mathbb{P}_- z (\rho^2 I - \Gamma\Gamma^*)^{-1} \bar{z} = \rho^2 J S^* R_\rho J \bar{z} = \rho^2 J S^* R_\rho \mathbf{1} = \rho J S^* p_\rho.\end{aligned}$$

Finally, to prove (3.4), it suffices to show that

$$\left(p_\rho, (S^*)^j p_\rho\right) = \left(q_\rho, (S^*)^j q_\rho\right), \quad j \ge 1, \tag{3.8}$$

and

$$\|p_\rho\|_2^2 - \|q_\rho\|_2^2 = r(\rho). \tag{3.9}$$

Using (3.3), we obtain

$$\begin{aligned}\rho p_\rho - \mathbf{1} &= \left(\rho^2 R_\rho - I\right) \mathbf{1} = \left(\rho^2 R_\rho - \left(\rho^2 I - \Gamma^*\Gamma\right) R_\rho\right) \mathbf{1} \\ &= \Gamma^*\Gamma R_\rho \mathbf{1} = \Gamma^* J \bar{z} q_\rho = \Gamma^* \bar{q}_\rho.\end{aligned}$$

Hence, for $j \ge 1$,

$$\rho (S^*)^j p_\rho = (S^*)^j \Gamma^* \bar{q}_\rho = \Gamma^* \bar{z}^j \bar{q}_\rho.$$

Thus for $j \ge 1$,

$$\begin{aligned}\rho \left(p_\rho, (S^*)^j p_\rho\right) &= (p_\rho, \Gamma^* \bar{z}^j \bar{q}_\rho) = (\Gamma p_\rho, \bar{z}^j \bar{q}_\rho) \\ &= \rho (J \bar{z} q_\rho, \bar{z}^j \bar{q}_\rho) = \rho (\bar{q}_\rho, \bar{z}^j \bar{q}_\rho) = \rho \left(q_\rho, (S^*)^j q_\rho\right).\end{aligned}$$

It remains to verify (3.9). We have by (3.3),

$$\begin{aligned}(p_\rho, p_\rho) - (q_\rho, q_\rho) &= (p_\rho, p_\rho) - \frac{1}{\rho^2} (\Gamma p_\rho, \Gamma p_\rho) \\ &= \frac{1}{\rho^2} \left((\rho^2 I - \Gamma^*\Gamma) p_\rho, p_\rho\right) = (R_\rho \mathbf{1}, \mathbf{1}) = r(\rho). \quad \blacksquare\end{aligned}$$

Now let $\alpha \in \mathbb{C}$ and consider the Hankel operator $\tilde{\Gamma} = \tilde{\Gamma}^{(\alpha)}$ defined as in §4.2:

$$\tilde{\Gamma}^{(\alpha)} = H_{\varphi_\alpha},$$

where φ_α is a function such that $\hat{\varphi}_\alpha(j) = \hat{\varphi}(j+1)$ for $j \le -2$ and $\hat{\varphi}_\alpha(-1) = \alpha$.

Let us make the assumption that

$$r(\rho) = (R_\rho \mathbf{1}, \mathbf{1}) \ne 0.$$

It has been shown in the proof of Lemma 4.2.2 that ρ^2 is a simple eigenvalue of the operator $\tilde{\Gamma}^*\tilde{\Gamma}$ if $\alpha = \alpha_\tau$ is given by

$$\alpha_\tau = \frac{\tau}{\rho r(\rho)} - \frac{(R_\rho \mathbf{1}, z\Gamma^* \bar{z})}{r(\rho)}, \tag{3.10}$$

where $|\tau| = 1$. Moreover, the function g_τ defined by

$$g_\tau = \bar{\alpha}_\tau R_\rho \mathbf{1} + R_\rho z \Gamma^* \bar{z} = R_\rho (\bar{\alpha}_\tau + z \Gamma^* \bar{z}) \tag{3.11}$$

is an eigenfunction of $\tilde{\Gamma}^*\tilde{\Gamma}$ that corresponds to the eigenvalue ρ^2.

Lemma 3.2. *The eigenfunction g_τ defined by* (3.11) *satisfies*

$$\tau\rho^2 r(\rho) g_\tau = p_\rho + \tau q_\rho. \tag{3.12}$$

Proof. By (3.11),

$$\tau\rho^2 r(\rho) g_\tau = \tau\bar{\alpha}_\tau \rho^2 r(\rho) R_\rho \mathbf{1} + \tau\rho^2 r(\rho) R_\rho z \Gamma^* \bar{z}.$$

We have from (3.10)

$$\tau\bar{\alpha}_\tau \rho^2 r(\rho) = \rho - \rho^2\tau(R_\rho z\Gamma^*\bar{z}, \mathbf{1}),$$

and so

$$\tau\rho^2 r(\rho) g_\tau = \rho R_\rho \mathbf{1} - \rho^2\tau(R_\rho z\Gamma^*\bar{z}, \mathbf{1}) R_\rho \mathbf{1} + \tau\rho^2 r(\rho) R_\rho z\Gamma^*\bar{z}. \tag{3.13}$$

By formula (2.11) of Chapter 4, we have

$$\rho^2 r(\rho) R_\rho z\Gamma^*\bar{z} = \rho^2 (R_\rho z\Gamma^*\bar{z}, \mathbf{1}) R_\rho \mathbf{1} + SJ\Gamma R_\rho \mathbf{1},$$

and so by (3.5),

$$\rho^2 r(\rho) R_\rho z\Gamma^*\bar{z} = \rho^2 (R_\rho z\Gamma^*\bar{z}, \mathbf{1}) R_\rho \mathbf{1} + z R_\rho \Gamma^*\bar{z}.$$

It follows now from (3.13) that

$$\begin{aligned} \tau\rho^2 r(\rho) g_\tau &= \rho R_\rho \mathbf{1} - \rho^2\tau(R_\rho z\Gamma^*\bar{z}, \mathbf{1}) R_\rho \mathbf{1} + \rho^2\tau(R_\rho z\Gamma^*\bar{z}, \mathbf{1}) R_\rho \mathbf{1} \\ &+ \tau z R_\rho \Gamma^*\bar{z} = \rho R_\rho \mathbf{1} + \tau z R_\rho \Gamma^*\bar{z}, \end{aligned}$$

which proves (3.12). ∎

We start with the case $m = 0$, i.e., we first consider the Nehari problem. We have already obtained in §1 a parametrization formula in this case. Nevertheless, here we give another approach that works if $\rho > \|\Gamma\|$, and we parametrize all solutions of the Nehari problem in terms of the functions p_ρ and q_ρ defined by (3.2). Clearly, in this case $r(\rho) = ((\rho^2 I - \Gamma^*\Gamma)^{-1}\mathbf{1}, \mathbf{1}) > 0$.

We need the following lemma.

Lemma 3.3. *Suppose that $\rho > \|\Gamma\|$. Then for any υ in the unit ball of H^∞, the function $p_\rho + \upsilon q_\rho$ is outer.*

To prove Lemma 3.3 we need one more lemma.

Lemma 3.4. *Let p and q be functions in H^2 such that $p + q$ is outer and $|p(\zeta)|^2 - |q(\zeta)|^2 \ge \delta > 0$ for almost all $\zeta \in \mathbb{T}$. Then p is outer.*

Proof. We have

$$\frac{|p|}{|p+q|} \le \frac{|p| \cdot |p+q|}{|p|^2 - |q|^2} \le \frac{2}{\delta}|p|^2 \in L^1,$$

and since $p + q$ is outer, it follows that $p(p+q)^{-1} \in H^1$. To prove that p is outer, it suffices to show that $\operatorname{Re} p(p+q)^{-1} \ge \mathbb{O}$ on $\mathbb{T}$ (see Appendix 2.1). This is obvious, since

$$\operatorname{Re}\frac{p+q}{p} = 1 + \operatorname{Re}\frac{q}{p} \ge 1 - \frac{|q|}{|p|} \ge \mathbb{O}. \quad ∎$$

Proof of Lemma 3.3. Let $\tau \in \mathbb{C}$ and $|\tau| = 1$. Let $\alpha = \alpha_\tau$ be the number defined by (3.10) and let $\tilde{\Gamma}^{(\alpha)}$ be the corresponding Hankel operator. Then $\left\|\tilde{\Gamma}^{(\alpha)}\right\| = \rho$, ρ is a singular value of $\tilde{\Gamma}^{(\alpha)}$ of multiplicity 1 (see formula (2.3) of Chapter 4). It follows from Lemma 3.2 that $p_\rho + \tau q_\rho$ is a maximizing vector of $\tilde{\Gamma}^{(\alpha)}$. It is easy to see that $p_\rho + \tau q_\rho$ is an outer function, since otherwise we could divide it by its inner factor and obtain another maximizing vector of $\tilde{\Gamma}^{(\alpha)}$, which is impossible since the multiplicity of the singular value ρ is one. It follows now from Lemma 3.4 and (3.4) that p_ρ is outer.

Finally, suppose that $\|\upsilon\|_{H^\infty} \le 1$. Since p_ρ is outer, it follows from (3.4) that $\upsilon q_\rho / p_\rho \in H^\infty$ and $\|\upsilon q_\rho/p_\rho\|_\infty \le 1$. Hence,

$$\operatorname{Re}\left(\mathbf{1} + \frac{\upsilon q_\rho}{p_\rho}\right) \ge \mathbb{O} \quad \text{on} \quad \mathbb{T},$$

and so

$$\mathbf{1} + \frac{\upsilon q_\rho}{p_\rho} = \frac{p_\rho + \upsilon q_\rho}{p_\rho}$$

is outer which implies that $p_\rho + \upsilon q_\rho$ is outer. ■

Theorem 3.5. *Suppose that $\rho > \|\Gamma\|$. A function ψ is a solution of the Nehari problem*

$$H_\psi = \Gamma, \quad \|\psi\|_\infty \le \rho$$

if and only if it has the form

$$\psi = \rho \frac{\upsilon \bar{p}_\rho + \bar{q}_\rho}{p_\rho + \upsilon q_\rho} \tag{3.14}$$

for a function υ in the unit ball of H^∞.

Proof. Let τ be a complex number of modulus 1. Suppose first that ψ is defined by (3.14), where υ is the constant function identically equal to τ. Clearly, $\|\psi\|_\infty = \rho$. Let us show that $H_\psi = \Gamma$. Let $\alpha = \alpha_\tau$ be the number defined by (3.10) and let $\tilde{\Gamma} = \tilde{\Gamma}^{(\alpha)}$ be the corresponding Hankel operator. Since by Lemma 3.2, $p_\rho + \tau q_\rho$ is a maximizing vector of $\tilde{\Gamma}$, the function

$$\frac{\tilde{\Gamma}(p_\rho + \tau q_\rho)}{p_\rho + \tau q_\rho}$$

is a symbol of the Hankel operator $\tilde{\Gamma}$ (see Theorem 1.1.4). Let us show that

$$\tilde{\Gamma}(p_\rho + \tau q_\rho) = \rho \bar{z}(\bar{q}_\rho + \tau \bar{p}_\rho). \tag{3.15}$$

Indeed, since ρ is a singular value of $\tilde{\Gamma}$ of multiplicity 1, it follows from Theorem 1.1.4 that

$$\tilde{\Gamma}(p_\rho + \tau q_\rho) = c\rho \bar{z}(\bar{p}_\rho + \bar{\tau}\bar{q}_\rho) = c\bar{\tau}\bar{z}(\bar{q}_\rho + \tau\bar{p}_\rho) \tag{3.16}$$

for some complex number c of modulus 1. We have to prove that $c\bar{\tau} = 1$. Using formulas (2.4) and (2.6) of Chapter 4, we obtain

$$\tilde{\Gamma}(p_\rho + \tau q_\rho) = \bar{z}\Gamma(p_\rho + \tau q_\rho) + (p_\rho + \tau q_\rho, \bar{\alpha} + z\Gamma^* \bar{z})\bar{z}.$$

It follows easily from (3.3) that

$$\tilde{\Gamma}(p_\rho + \tau q_\rho) = \rho\bar{z}(\bar{q}_\rho + \tau\bar{p}_\rho) + d\bar{z}$$

for some $d \in \mathbb{C}$. It is easy to see from (3.2) that $\bar{q}_\rho + \tau\bar{p}_\rho$ is not a constant function. On the other hand, if we compare the jth Fourier coefficients of $c\bar{\tau}\bar{z}(\bar{q}_\rho + \tau\bar{p}_\rho)$ and $\tilde{\Gamma}(p_\rho + \tau q_\rho)$ for $j \le -2$, we see from (3.16), that $c\bar{\tau} = 1$. This proves (3.15).

We can conclude now that the function

$$\rho\bar{z}\frac{\bar{q}_\rho + \tau\bar{p}_\rho}{p_\rho + \tau q_\rho}$$

is a symbol of $\tilde{\Gamma}$, i.e.,

$$\tilde{\Gamma} = H_{\rho\bar{z}\frac{\bar{q}_\rho + \tau\bar{p}_\rho}{p_\rho + \tau q_\rho}}.$$

Since $\Gamma = \tilde{\Gamma}S$, it follows that

$$\Gamma = H_{z\rho\bar{z}\frac{\bar{q}_\rho + \tau\bar{p}_\rho}{p_\rho + \tau q_\rho}} = H_{\rho\frac{\bar{q}_\rho + \tau\bar{p}_\rho}{p_\rho + \tau q_\rho}}.$$

In other words, the function $\rho(\bar{q}_\rho + \tau\bar{p}_\rho)(p_\rho + \tau q_\rho)^{-1}$ is a symbol of Γ.

Suppose now that υ is in the unit ball of H^∞. Obviously, for almost all $\zeta \in \mathbb{T}$ the function

$$\tau \mapsto \frac{\bar{q}_\rho(\zeta) + \tau\bar{p}_\rho(\zeta)}{p_\rho(\zeta) + \tau q_\rho(\zeta)} \tag{3.17}$$

maps $\mathbb{T}$ onto itself. By (3.4), $|q_\rho(\zeta)/p_\rho(\zeta)| \le 1$ for almost all $\zeta \in \mathbb{T}$, and so the map (3.17) is a conformal map of $\mathbb{D}$ onto itself. Thus the function

$$\varphi_\upsilon \stackrel{\text{def}}{=} \rho\frac{\upsilon\bar{p}_\rho + \bar{q}_\rho}{p_\rho + \upsilon q_\rho}$$

satisfies $\|\varphi_\upsilon\|_\infty \le \rho$. To show that φ_υ is a symbol of Γ, it suffices to prove that

$$\varphi_\upsilon - \varphi_\tau \stackrel{\text{def}}{=} \varphi_\upsilon - \rho\frac{\tau\bar{p}_\rho + \bar{q}_\rho}{p_\rho + \tau q_\rho} \in H^\infty.$$

We obtain from (3.4)

$$\varphi_\upsilon - \varphi_\tau = \rho r(\rho)\frac{\upsilon - \tau}{(p_\rho + \upsilon q_\rho)(p_\rho + \tau q_\rho)} \in H^\infty, \tag{3.18}$$

since $\varphi_\upsilon - \varphi_\tau \in L^\infty$ and the function $(p_\rho + \upsilon q_\rho)(p_\rho + \tau q_\rho)$ is outer by Lemma 3.3.

Suppose now that $\psi \in L^\infty$, $\|\psi\|_\infty \le \rho$, and $\Gamma = H_\psi$. Since (3.17) is a conformal map of $\mathbb{D}$ onto itself, it follows that there exists a function υ in the unit ball of L^∞ such that

$$\psi = \rho\frac{\upsilon\bar{p}_\rho + \bar{q}_\rho}{p_\rho + \upsilon q_\rho}.$$

We have to show that $\upsilon \in H^\infty$. Let τ be a complex number of modulus 1. Since $\Gamma = H_{\varphi_\tau}$, it follows that

$$\rho\frac{\upsilon\bar{p}_\rho + \bar{q}_\rho}{p_\rho + \upsilon q_\rho} - \varphi_\tau = \rho r(\rho)\frac{\upsilon - \tau}{(p_\rho + \upsilon q_\rho)(p_\rho + \tau q_\rho)} \stackrel{\text{def}}{=} \rho r(\rho)\varkappa_\tau \in H^\infty. \quad (3.19)$$

Let $\mathcal{L} \stackrel{\text{def}}{=} \{\omega \in H^1 : \upsilon\omega \in H^1\}$. Clearly, $\mathcal{L}$ is a closed subspace of H^1 that is invariant under multiplication by z. Next, $\mathcal{L} \neq \{\mathbb{O}\}$, which is a consequence of the following easily verifiable equality:

$$\upsilon(\mathbf{1} - \varkappa_\tau q_\rho(p_\rho + \tau q_\rho)) = \varkappa_\tau p_\rho(p_\rho + \tau q_\rho) + \tau.$$

(Note that $\mathbf{1} - \varkappa_\tau q_\rho(p_\rho + \tau q_\rho)$ cannot be the zero function for all $\tau \in \mathbb{T}$.) Then $\mathcal{L}$ has the form $\mathcal{L} = \vartheta H^1$ for an inner function ϑ (see Appendix 2.2). Let $h \stackrel{\text{def}}{=} \vartheta\upsilon \in H^\infty$. It is easy to see that h and ϑ are coprime; otherwise if they had a nonconstant common inner factor $\vartheta_\#$, we would have the inclusion $\bar{\vartheta}_\#\vartheta H^1 \subset \mathcal{L}$, which is impossible.

It follows from the definition of $\varkappa_\tau$ (3.19) that

$$\varkappa_\tau(p_\rho + \tau q_\rho)(\vartheta p_\rho + hq_\rho) = \vartheta(\upsilon - \tau) = h - \tau\vartheta.$$

It follows that the inner factor $(\vartheta p_\rho + hq_\rho)_{(\mathrm{i})}$ of $\vartheta p_\rho + hq_\rho$ is a divisor of $h - \tau\vartheta$ for any $\tau \in \mathbb{T}$. Hence, $(\vartheta p_\rho + hq_\rho)_{(\mathrm{i})}$ is a common divisor of h and ϑ. However, h and ϑ are coprime, and so the function $\vartheta p_\rho + hq_\rho$ is outer. By (3.4),

$$|\vartheta p_\rho|^2 - |hq_\rho|^2 \geq |p_\rho|^2 - |q_\rho|^2 = r(\rho) \quad \text{on} \quad \mathbb{T}.$$

It follows now from Lemma 3.4 that ϑp_ρ is outer, and so ϑ is constant, which means that $\upsilon = \bar{\vartheta}h \in H^\infty$. ■

We proceed now to the case when $m > 0$ and $s_m(\Gamma) < \rho < s_{m-1}(\Gamma)$. Recall that for a function $\psi \in H^1$ we denote by $\psi_{(\mathrm{o})}$ its outer factor and by $\psi_{(\mathrm{i})}$ the inner factor of ψ (see Appendix 2.1). Recall also that for an inner function ϑ its degree $\deg\vartheta$ is the dimension of $H^2 \ominus \vartheta H^2$ and $\deg\vartheta < \infty$ if and only if ϑ is a finite Blaschke product (see Appendix 2.1).

We need the following version of Rouché's theorem.

Lemma 3.6. *Let ξ and η be functions in H^1 such that $|\eta(\zeta)| \leq |\xi(\zeta)|$, $\zeta \in \mathbb{T}$. Then $\deg(\xi + \eta)_{(\mathrm{i})} \leq \deg\xi_{(\mathrm{i})}$. If, in addition, $\frac{\xi}{\xi+\eta} \in L^1$, then $\deg\xi_{(\mathrm{i})} \leq \deg(\xi + \eta)_{(\mathrm{i})}$.*

Proof. It is easy to see that

$$\frac{\eta}{\xi_{(\mathrm{o})}} \in H^\infty \quad \text{and} \quad \frac{\xi + \eta}{\xi_{(\mathrm{o})}} \in H^\infty,$$

and we can replace the functions ξ and η with $\xi/\xi_{(\mathrm{o})}$ and η with $\eta/\xi_{(\mathrm{o})}$. Thus without loss of generality we may assume that ξ is inner and $\|\eta\|_\infty \leq 1$. Put $\vartheta \stackrel{\text{def}}{=} (\xi + \eta)_{(\mathrm{i})}$.

Assume that $\deg\xi < \deg\vartheta$. Then $\dim(H^2 \ominus \xi H^2) < \dim(H^2 \ominus \vartheta H^2)$. Hence, there exists a nonzero function ω in $\xi H^2 \cap (H^2 \ominus \vartheta H^2)$. Then $\psi \stackrel{\text{def}}{=} \bar{\xi}\omega \in H^2$. On the other hand, $\omega\bar{\vartheta} \in H^2_-$.

Consider the Hankel operator $H_{\bar\vartheta\xi}$. Clearly, $\|H_{\bar\vartheta\xi}\| \le 1$. On the other hand,

$$\|H_{\bar\vartheta\xi}\psi\|_2 = \|\mathbb{P}_-\bar\vartheta\xi\bar\xi\omega\|_2 = \|\mathbb{P}_-\bar\vartheta\omega\|_2 = \|\bar\vartheta\omega\|_2 = \|\psi\|_2.$$

Thus $\|H_{\bar\vartheta\xi}\| = 1$ and $H_{\bar\vartheta\xi}$ attains its norm on the unit ball of H^2. By Theorem 1.1.4, $H_{\bar\vartheta\xi}$ has only one symbol $\bar\vartheta\xi$ of norm 1. However,

$$\|\bar\vartheta\xi - (\xi+\eta)_{(\mathrm{o})}\|_\infty = \|\xi - \vartheta(\xi+\eta)_{(\mathrm{o})}\|_\infty = \|\xi - (\xi+\eta)\|_\infty = \|\xi\|_\infty = 1,$$

and so both $\bar\vartheta\xi$ and $\bar\vartheta\xi - (\xi+\eta)_{(\mathrm{o})}$ are symbols of $H_{\bar\vartheta\xi}$ of minimal norm, and we have got a contradiction.

Suppose now that $\frac{\xi}{\xi+\eta} \in L^1$. Assume that $\deg\xi > \deg\vartheta$. Then ϑ is a finite Blaschke product. Let $\Gamma \stackrel{\text{def}}{=} H_{\bar\vartheta\xi}$. Let us show that $\|\Gamma\| < 1$.

Indeed, since ϑ is a finite Blaschke product, $\operatorname{rank}\Gamma < \infty$ (see Corollary 1.3.3). If $\|\Gamma\| = 1$, then there is a nonzero function $w \in H^2$ such that $\bar\vartheta\xi w \in H^2_-$, and so $\operatorname{Ker} T_{\bar\vartheta\xi} \ne \{\mathbb{O}\}$. Consider now the Toeplitz operator $T^*_{\bar\vartheta\xi} = T_{\bar\xi\vartheta} = T_{\bar\xi}T_\vartheta$. We have $\operatorname{Range} T_\vartheta = \vartheta H^2$ and

$$\operatorname{Ker} T_{\bar\xi} = K_\xi = H^2 \ominus \xi H^2.$$

By our assumption, $\dim K_\xi = \deg\xi > \operatorname{codim}\vartheta H^2 = \deg\vartheta$, and so $\operatorname{Ker} T_{\bar\xi\vartheta} \ne \{\mathbb{O}\}$, which contradicts Theorem 3.1.4.

Obviously, $\bar\vartheta\xi + \bar\vartheta\eta \in H^\infty$, and so

$$\Gamma = H_{\bar\vartheta\xi} = H_{-\bar\vartheta\eta}.$$

Hence, Γ has two distinct symbols $\bar\vartheta\xi$ and $-\bar\vartheta\eta$ of norm at most 1. Put $\rho = 1$ and apply Theorem 3.5. Thus there exist functions υ and $\upsilon_\#$ in the unit ball of H^1 such that

$$\bar\vartheta\xi = \frac{\upsilon\bar p_1 + \bar q_1}{p_1 + \upsilon q_1} \quad \text{and} \quad -\bar\vartheta\eta = \frac{\upsilon_\#\bar p_1 + \bar q_1}{p_1 + \upsilon_\# q_1}.$$

Recall that for almost all $\zeta \in \mathbb{T}$

$$\tau \mapsto \frac{\tau\bar p_1(\zeta) + \bar q_1(\zeta)}{p_1(\zeta) + \tau q_1(\zeta)}$$

is a conformal map of $\mathbb{D}$ onto itself, and since $\bar\vartheta\xi$ is a unimodular function, it follows that υ is also unimodular, i.e., υ is an inner function.

We claim that p_1 and q_1 are rational functions with poles outside the closed unit disk. Indeed, since $\operatorname{rank}\Gamma < \infty$, the range of Γ consists of rational functions (see the Remark at the end of §1.3). Then by (3.3), q_1 is rational and S^*p_1 is rational, and so p_1 is also rational.

Now it follows easily from (3.4) that the function $(p_1 + \upsilon q_1)(p_1 + \upsilon_\# q_1)$ is invertible in H^∞. We have by (3.4)

$$\begin{aligned}(\xi+\eta)_{(\mathrm{o})} &= \bar\vartheta\xi - (-\bar\vartheta\eta) = \frac{(\upsilon - \upsilon_\#)(|p_1|^2 - |q_1|^2)}{(p_1 + \upsilon q_1)(p_1 + \upsilon_\# q_1)} \\ &= r(1)\frac{\upsilon - \upsilon_\#}{(p_1 + \upsilon q_1)(p_1 + \upsilon_\# q_1)}.\end{aligned}$$

By Lemma 3.3, the functions $p_1 + \upsilon q_1$ and $p_1 + \upsilon_\# q_1$ are outer, which implies that $\upsilon - \upsilon_\#$ is outer. Moreover,

$$\left|\frac{\upsilon}{\upsilon - \upsilon_\#}\right| = \frac{1}{|\upsilon - \upsilon_\#|} = \frac{1}{r(1)}|(p_1 + \upsilon q_1)(p_1 + \upsilon_\# q_1)|^{-1}\left|\frac{\xi}{\xi + \eta}\right| \in L^1.$$

On the other hand, since $|\upsilon_\#| \le |\upsilon|$ on $\mathbb{T}$, it is easy to see that

$$\operatorname{Re}\frac{\upsilon}{\upsilon - \upsilon_\#} \ge 0 \quad \text{on} \quad \mathbb{T},$$

and so the function $\upsilon(\upsilon - \upsilon_\#)^{-1}$ is outer, and since $\upsilon - \upsilon_\#$ is outer, we find that υ is outer. However, υ is inner, and so $\upsilon(\zeta) = \tau$ almost everywhere on $\mathbb{T}$, where $|\tau| = 1$. Then

$$\bar{\vartheta}\xi = \frac{\tau\bar{p}_1 + \bar{q}_1}{p_1 + \tau q_1} = \tau\frac{\overline{p_1 + \tau q_1}}{p_1 + \tau q_1}.$$

Again, it follows from (3.4) that $p_1 + \tau q_1$ has no zeros on $\mathbb{T}$, and so $\bar{\vartheta}\xi$ is a rational function, which implies that ξ is a rational function, i.e., ξ is a finite Blaschke product. Since $p_1 + \tau q_1$ is invertible in H^∞, it follows that the Toeplitz operators $T_{(p_1+\tau q_1)^{-1}}$ and $T_{\overline{p_1+\tau q_1}}$ are invertible on H^2, and so

$$T_{\bar{\vartheta}\xi} = \tau T_{\overline{p_1+\tau q_1}} T_{(p_1+\tau q_1)^{-1}}$$

is invertible. We now have

$$0 = \operatorname{ind} T_{\bar{\vartheta}\xi} = \operatorname{ind} T_{\bar{\vartheta}} + \operatorname{ind} T_\xi = \deg\vartheta - \deg\xi,$$

which completes the proof. ■

Corollary 3.7. *Suppose that* $s_m(\Gamma) < \rho < s_{m-1}(\Gamma)$ *and* $r(\rho) > 0$. *Then* $\deg(p_\rho)_{(\mathrm{i})} = m$. *Moreover, for any inner function* ϑ *and any* υ *in the unit ball of* H^∞,

$$\deg(\vartheta p_\rho + \upsilon q_\rho)_{(\mathrm{i})} = \deg\vartheta + m. \tag{3.20}$$

Proof. By (3.4),

$$|\vartheta p_\rho|^2 - |\upsilon q_\rho|^2 \ge |p_\rho|^2 - |q_\rho|^2 = r(\rho) > 0 \quad \text{on} \quad \mathbb{T}.$$

Next,

$$\left|\frac{\vartheta p_\rho}{\vartheta p_\rho + \upsilon q_\rho}\right| \le \frac{|p_\rho|(|\vartheta p_\rho| + |\upsilon q_\rho|)}{|\vartheta p_\rho|^2 - |\upsilon q_\rho|^2} \le \frac{1}{r(\rho)}|p_\rho|(|\vartheta p_\rho| + |\upsilon q_\rho|),$$

and so $\vartheta p_\rho(\vartheta p_\rho + \upsilon q_\rho)^{-1} \in L^1$. It follows now from Lemma 3.6 that

$$\deg(\vartheta p_\rho + \upsilon q_\rho)_{(\mathrm{i})} = \deg(\vartheta p_\rho)_{(\mathrm{i})} = \deg\vartheta + \deg(p_\rho)_{(\mathrm{i})}.$$

In particular, for any complex number τ of modulus 1

$$\deg(\vartheta p_\rho + \tau q_\rho)_{(\mathrm{i})} = \deg(p_\rho)_{(\mathrm{i})}.$$

Now let $\alpha = \alpha_\tau$ be the number defined by (3.10) and $\tilde{\Gamma} = \tilde{\Gamma}^{(\alpha)}$ be the corresponding Hankel operator. Then $\rho = s_m(\tilde{\Gamma})$ (see §4.2) and by Lemma 3.2, $p_\rho + \tau q_\rho$ is a Schmidt function of $\tilde{\Gamma}$, which corresponds to the singular value ρ. By the Remark after the proof of Lemma 4.1.6, $\deg(p_\rho)_{(\mathrm{i})} = m$, which completes the proof. ■

Now we are ready to parametrize the solutions of the Nehari–Takagi problem. We start with the case $r(\rho) > 0$.

Theorem 3.8. *Let m be a positive integer. Suppose that $s_m(\Gamma) < \rho < s_{m-1}(\Gamma)$ and $r(\rho) > 0$. A function $\psi \in L^\infty$ satisfies the conditions*

$$\operatorname{rank}(\Gamma - H_\psi) \le m \quad \textit{and} \quad \|\psi\|_\infty \le \rho \tag{3.21}$$

if and only if it can be represented in the form

$$\psi = \varphi_\upsilon \stackrel{\text{def}}{=} \rho \frac{\upsilon \bar{p}_\rho + \bar{q}_\rho}{p_\rho + \upsilon q_\rho}$$

for a function υ in the unit ball of H^∞. Moreover, $\upsilon \mapsto \varphi_\upsilon$ is a one-to-one map of the unit ball of L^∞ onto the ball of radius ρ of L^∞.

Proof. Obviously, for almost all $\zeta \in \mathbb{T}$ the map

$$\tau \mapsto \frac{\upsilon \bar{p}_\rho(\zeta) + \bar{q}_\rho(\zeta)}{p_\rho(\zeta) + \upsilon q_\rho(\zeta)} \tag{3.22}$$

maps $\mathbb{T}$ onto itself. By (3.4), $|q_\rho(\zeta)/p_\rho(\zeta)| < 1$ almost everywhere on $\mathbb{T}$, and so the map (3.21) is a conformal map of $\mathbb{D}$ onto itself, which implies that $\upsilon \mapsto \varphi_\upsilon$ is a one-to-one map of the unit ball of L^∞ onto the ball of radius ρ of L^∞.

Let us show that for any complex number τ of modulus 1 the function

$$\varphi_\tau \stackrel{\text{def}}{=} \rho \frac{\tau \bar{p}_\rho + \bar{q}_\rho}{p_\rho + \tau q_\rho}$$

satisfies $\operatorname{rank}(\Gamma - H_{\varphi_\tau}) \le m$. The proof of this fact is similar to the corresponding fact for $m = 0$ that has been established in the proof of Theorem 3.5. Indeed, let

$$\varkappa \stackrel{\text{def}}{=} \frac{\tilde{\Gamma}(p_\rho + \tau q_\rho)}{p_\rho + \tau q_\rho},$$

where $\tilde{\Gamma} = \tilde{\Gamma}^{(\alpha)}$ and $\alpha = \alpha_\tau$ is defined by (3.10). Recall that $p_\rho + \tau q_\rho$ is a Schmidt function of $\tilde{\Gamma}$ that corresponds to the singular value ρ. It has been proved in §4.1 that $\operatorname{rank}(\tilde{\Gamma} - H_\varkappa) = m$. It has been proved in §4.1 that J maps the space of singular vectors of $\tilde{\Gamma}$ corresponding to the singular value ρ onto the space of singular vectors of $\tilde{\Gamma}^*$ corresponding to the same singular value. Since the multiplicity of the singular value ρ is 1, it follows that $\tilde{\Gamma}(p_\rho + \tau q_\rho) = \operatorname{const} J(p_\rho + \tau q_\rho)$. In the same way as in the proof of Theorem 3.5 we can now show that

$$\tilde{\Gamma}(p_\rho + \tau q_\rho) = \rho \bar{z}(\bar{q}_\rho + \tau \bar{p}_\rho). \tag{3.23}$$

We have

$$\operatorname{rank}(\Gamma - H_{\varphi_\tau}) = \operatorname{rank}(\Gamma - H_{z\varkappa}) = \operatorname{rank}\big((\tilde{\Gamma} - H_\varkappa)S\big) \le \operatorname{rank}\big(\tilde{\Gamma} - H_\varkappa\big) = m. \tag{3.24}$$

Suppose that ψ satisfies (3.21). Then $\psi = \varphi_\upsilon$, where υ is in the unit ball of L^∞. Let us show that $\upsilon \in H^\infty$. Let $\Gamma = H_\varphi$, where $\varphi \in L^\infty$. Then there exists an inner function θ such that

$$\deg\theta \le m \quad \text{and} \quad \theta(\varphi - \varphi_\upsilon) \in H^\infty. \tag{3.25}$$

By Corollary 3.7, for any complex number τ of modulus 1

$$\deg(p_\rho + \tau q_\rho)_{(\mathrm{i})} = m. \tag{3.26}$$

Let us show that

$$p_\rho + \tau q_\rho \in \operatorname{Ker}(\Gamma - H_{\varphi_\tau}).$$

Indeed, by (3.23), $\Gamma(p_\rho + \tau q_\rho) = \rho\mathbb{P}_-(\tau\bar{p}_\rho + \bar{q}_\rho)$. On the other hand, it follows from the definition of φ_τ that $H_{\varphi_\tau}(p_\rho + \tau q_\rho) = \rho\mathbb{P}_-(\tau\bar{p}_\rho + \bar{q}_\rho)$.

Since the kernel of a Hankel operator is an invariant subspace under multiplication by z, we have $(p_\rho + \tau q_\rho)_{(\mathrm{i})}H^2 \subset \operatorname{Ker}(\Gamma - H_{\varphi_\tau})$. It follows now from (3.24) and (3.26) that

$$\operatorname{Ker}(\Gamma - H_{\varphi_\tau}) = (p_\rho + \tau q_\rho)_{(\mathrm{i})}H^2. \tag{3.27}$$

We now have from (3.25) and (3.27)

$$h_\tau \stackrel{\text{def}}{=} \theta(p_\rho + \tau q_\rho)_{(\mathrm{i})}(\varphi_\upsilon - \varphi_\tau) \in H^\infty.$$

As in (3.18),

$$\varphi_\upsilon - \varphi_\tau = \rho r(\rho)\frac{\upsilon - \tau}{(p_\rho + \upsilon q_\rho)(p_\rho + \tau q_\rho)}. \tag{3.28}$$

Now it is an elementary exercise to verify the following equality:

$$\upsilon\big(\rho r(\rho)\theta - h_\tau q_\rho(p_\rho + \tau q_\rho)_{(\mathrm{o})}\big) = \tau\rho r(\rho)\theta + h_\tau p_\rho(p_\rho + \tau q_\rho)_{(\mathrm{o})}. \tag{3.29}$$

Clearly, $\rho r(\rho)\theta - h_\tau q_\rho(p_\rho + \tau q_\rho)_{(\mathrm{o})} \ne \mathbb{O}$, since otherwise we would have $p_\rho = -\tau q_\rho$, which contradicts (3.4).

As in the proof of Theorem 3.5, consider the closed subspace $\mathcal{L}$ of H^1 defined by $\mathcal{L} \stackrel{\text{def}}{=} \{\omega \in H^1 : \upsilon\omega \in H^1\}$. It follows from (3.29) that $\mathcal{L} \ne \{\mathbb{O}\}$. Then $\mathcal{L}$ has the form $\mathcal{L} = \vartheta H^1$ for an inner function ϑ (see Appendix 2.2). Thus the function $g \stackrel{\text{def}}{=} \vartheta\upsilon$ belongs to H^∞, $\|g\|_\infty \le 1$, and g and ϑ are coprime.

It follows from (3.28) that

$$h_\tau(p_\rho + \tau q_\rho)_{(\mathrm{o})}(\vartheta p_\rho + g q_\rho) = \rho r(\rho)\theta(g - \tau\vartheta),$$

and so $(\vartheta p_\rho + g q_\rho)_{(\mathrm{i})}$ is a divisor of $\theta(g - \tau\vartheta)$. Since τ is an arbitrary number of modulus 1 and ϑ is coprime with g, it follows that $(\vartheta p_\rho + g q_\rho)_{(\mathrm{i})}$ is a divisor of θ. Hence, by (3.25),

$$\deg(\vartheta p_\rho + g q_\rho)_{(\mathrm{i})} \le \deg\theta \le m.$$

On the other hand, by Corollary 3.7,

$$\deg(\vartheta p_\rho + g q_\rho)_{(\mathrm{i})} = \deg \vartheta + m.$$

Thus $\deg \vartheta = 0$, and so $\upsilon = g/\vartheta \in H^\infty$.

Suppose now that $\upsilon \in H^\infty$ and $\|\upsilon\|_\infty \le 1$. Let us show that if τ_1 and τ_2 are distinct numbers of modulus 1, then $p_\rho + \tau_1 q_\rho$ and $p_\rho + \tau_2 q_\rho$ have no common zeros. Indeed, we have from (3.28)

$$\begin{aligned} \varphi_{\tau_1} - \varphi_{\tau_2} &= \frac{\rho r(\rho)(\tau_1 - \tau_2)}{(p_\rho + \tau_1 q_\rho)(p_\rho + \tau_2 q_\rho)} \\ &= \rho r(\rho) \frac{(\tau_1 - \tau_2)\overline{(p_\rho + \tau_1 q_\rho)}_{(\mathrm{i})}\overline{(p_\rho + \tau_2 q_\rho)}_{(\mathrm{i})}}{(p_\rho + \tau_1 q_\rho)_{(\mathrm{o})}(p_\rho + \tau_2 q_\rho)_{(\mathrm{o})}}. \end{aligned}$$

Clearly, $(p_\rho + \tau_1 q_\rho)^{-1}_{(\mathrm{o})}(p_\rho + \tau_2 q_\rho)^{-1}_{(\mathrm{o})}$ is a bounded outer function, and so

$$\operatorname{Ker}\left(H_{\varphi_{\tau_1}} - H_{\varphi_{\tau_2}}\right) = (p_\rho + \tau_1 q_\rho)_{(\mathrm{i})}(p_\rho + \tau_2 q_\rho)_{(\mathrm{i})} H^2. \tag{3.30}$$

On the other hand, by (3.27),

$$\operatorname{Ker}(\Gamma - H_{\varphi_{\tau_j}}) = (p_\rho + \tau_j q_\rho)_{(\mathrm{i})} H^2, \quad j = 1,\ 2. \tag{3.31}$$

Clearly,

$$\operatorname{Ker}(\Gamma - H_{\varphi_{\tau_1}}) \cap \operatorname{Ker}(\Gamma - H_{\varphi_{\tau_2}}) \subset \operatorname{Ker}\left(H_{\varphi_{\tau_1}} - H_{\varphi_{\tau_2}}\right),$$

and (3.30) and (3.31) imply that $(p_\rho + \tau_1 q_\rho)_{(\mathrm{i})}$ and $(p_\rho + \tau_2 q_\rho)_{(\mathrm{i})}$ have no common zeros.

Let $\varphi \in L^\infty$ and $\Gamma = H_\varphi$. It follows from (3.27) and (3.28) that $\mathbb{P}_-(\varphi - \varphi_\upsilon)$ is a rational function with poles in the zeros of $(p_\rho + \tau q_\rho)(p_\rho + \upsilon q_\rho)$ counted with multiplicities. However, $\mathbb{P}_-(\varphi - \varphi_\upsilon)$ does not depend on the choice of τ and we have just proved that for distinct τ_1 and τ_2 the functions $(p_\rho + \tau_1 q_\rho)$ and $(p_\rho + \tau_2 q_\rho)$ have no common zeros. Thus $\mathbb{P}_-(\varphi - \varphi_\upsilon)$ is a rational function with poles in the zeros of $p_\rho + \upsilon q_\rho$. It follows now from Corollary (3.7) that $\operatorname{rank}(\Gamma - H_{\varphi_\upsilon}) \le m$. ■

Consider now the case $r(\rho) < 0$.

Theorem 3.9. *Let m be a positive integer. Suppose that $s_m(\Gamma) < \rho < s_{m-1}(\Gamma)$ and $r(\rho) < 0$. A function $\psi \in L^\infty$ satisfies the conditions*

$$\operatorname{rank}(\Gamma - H_\psi) \le m \quad \textit{and} \quad \|\psi\|_\infty \le \rho$$

if and only if it can be represented in the form

$$\psi = \varphi_\upsilon \stackrel{\text{def}}{=} \rho \frac{\bar{p}_\rho + \upsilon \bar{q}_\rho}{\upsilon p_\rho + q_\rho}$$

for a function υ in the unit ball of H^∞. Moreover, $\upsilon \mapsto \varphi_\upsilon$ is a one-to-one map of the unit ball of L^∞ onto the ball of radius ρ of L^∞.

The proof of Theorem 3.9 is the same as the proof of Theorem 3.8. We simply have to interchange the roles of p_ρ and q_ρ. In particular, in this case the analog of Corollary 3.7 is the following:

$$\deg(\vartheta q_\rho + \upsilon p_\rho)_{(\mathrm{i})} = \deg \vartheta + m$$

for any υ in the unit ball of H^∞ and any inner ϑ.

We proceed now to the remaining case $r(\rho) = 0$. By (3.4), $|p_\rho| = |q_\rho|$. However, much more can be said in this case. It follows from formula (2.11) of Chapter 4 and from (3.5) that

$$-\rho^2(R_\rho z\Gamma^*\bar{z}, \mathbf{1})R_\rho\mathbf{1} + \rho^2 r(\rho)R_\rho z\Gamma^*\bar{z} = zR_\rho\Gamma^*\bar{z},$$

and since $r(\rho) = 0$, we have from (3.2)

$$-\rho(R_\rho z\Gamma^*\bar{z}, \mathbf{1})p_\rho = q_\rho,$$

i.e., p_ρ and q_ρ are linear dependent. Thus

$$p_\rho + \tau_\rho q_\rho = \mathbb{O} \tag{3.32}$$

for some complex number τ_ρ of modulus 1.

Consider now on the interval $(s_m(\Gamma), s_{m-1}(\Gamma))$ the operator function

$$t \mapsto R_t = (t^2 I - \Gamma^*\Gamma)^{-1}.$$

Clearly, it is differentiable. It follows that the vector functions

$$t \mapsto p_t \quad \text{and} \quad t \mapsto q_t$$

are also differentiable on $(s_m(\Gamma), s_{m-1}(\Gamma))$. Here by the derivatives we understand the derivatives of the above operator and vector functions as functions of t. It is easy to see that

$$R_t' = -2tR_t^2. \tag{3.33}$$

By (3.3), we have for $\tau \in \mathbb{T}$

$$\Gamma(pt + \tau q_t) = tJS^*q_t + t\tau JS^*p_t = t\tau JS^*(p_t + \tau q_t).$$

Hence,

$$\Gamma(p_t' + \tau q_t') = \tau JS^*(p_t + \tau q_t) + \tau tJS^*(p_t' + \tau q_t').$$

Substituting $t = \rho$ and $\tau = \tau_\rho$, we obtain

$$\Gamma(p_\rho' + \tau_\rho q_\rho') = \rho\tau_\rho JS^*(p_\rho' + \tau q_\rho').$$

Now put

$$w_\rho = p_\rho' + \tau q_\rho'. \tag{3.34}$$

Clearly, $w_\rho \in H^2$ and

$$\Gamma w_\rho = \rho\tau_\rho JS^*w_\rho. \tag{3.35}$$

Let us show that p_ρ and w_ρ are linearly independent. Indeed,

$$p_\rho(0) = \rho(R_\rho\mathbf{1}, \mathbf{1}) = r(\rho) = 0.$$

On the other hand,

$$w_\rho(0) = (w_\rho, \mathbf{1}) = (p_\rho' + \tau_\rho q_\rho', \mathbf{1}).$$

By (3.2), $q_t(0) = 0$ for any $t \in (s_m(\Gamma), s_{m-1}(\Gamma))$, and so $q'_\rho(0) = 0$. On the other hand,

$$(p'_\rho, \mathbf{1}) = \rho(R'_\rho \mathbf{1}, \mathbf{1}) = -2\rho^2(R^2_\rho \mathbf{1}, \mathbf{1})$$

by (3.33). Thus $w_\rho(0) = -2\rho^2(R_\rho \mathbf{1}, R_\rho \mathbf{1}) < 0$.

By (3.3) and (3.32), we have

$$\Gamma p_\rho = \rho J S^* q_\rho = -\rho \tau_\rho J S^* p_\rho. \tag{3.36}$$

Now we obtain from (3.35) and (3.36)

$$\Gamma(w_\rho + \mathrm{i}sp_\rho) = \rho\tau_\rho J S^*(w_\rho + \mathrm{i}sp_\rho), \quad s \in \mathbb{R}. \tag{3.37}$$

For $s \in \mathbb{R}$ we find the number $\alpha = \alpha_{(s)} \in \mathbb{C}$ from the equation

$$-\rho\tau_\rho(J(w_\rho + \mathrm{i}sp_\rho), \mathbf{1}) + (w_\rho + \mathrm{i}sp_\rho, \bar{\alpha} + z\Gamma^*\bar{z}) = 0. \tag{3.38}$$

The equation is solvable, since $(w_\rho, \mathbf{1}) \neq 0$ and $(p_\rho, \mathbf{1}) = \mathbb{O}$. As before, consider the Hankel operator $\tilde{\Gamma} = \tilde{\Gamma}^{(\alpha)}$ defined by

$$\tilde{\Gamma} g = \bar{z}\Gamma g + (g, \bar{\alpha} + z\Gamma^*\bar{z})\bar{z}, \quad g \in H^2.$$

It follows easily from (3.37) and (3.38) that

$$\begin{aligned}
\tilde{\Gamma}(w_\rho + \mathrm{i}sp_\rho) &= \rho\tau_\rho \bar{z} J S^*(w_\rho + \mathrm{i}sp_\rho) + (w_\rho + \mathrm{i}sp_\rho, \bar{\alpha} + z\Gamma^*\bar{z}) \\
&= \rho\tau_\rho J(w_\rho + \mathrm{i}sp_\rho) - \rho\tau_\rho(J(w_\rho + \mathrm{i}sp_\rho), \mathbf{1}) \\
&+ (w_\rho + \mathrm{i}sp_\rho, \bar{\alpha} + z\Gamma^*\bar{z}) \\
&= \rho\tau_\rho J(w_\rho + \mathrm{i}sp_\rho).
\end{aligned}$$

Hence,

$$\tilde{\Gamma}^*\tilde{\Gamma}(w_\rho + \mathrm{i}sp_\rho) = \rho^2(w_\rho + \mathrm{i}sp_\rho).$$

Consequently, $\rho = s_m(\tilde{\Gamma})$ and since $s_m(\Gamma) < \rho$, we can prove in the same way as in the proof of Theorem 3.8 (see (3.27)) that

$$\operatorname{Ker}(\Gamma - H_{\varphi_s}) = (w_\rho + \mathrm{i}sp_\rho)_{(\mathrm{i})} H^2,$$

where

$$\varphi_s = \rho\tau_\rho \frac{\bar{w}_\rho - \mathrm{i}s\bar{p}_\rho}{w_\rho + \mathrm{i}sp_\rho} = \frac{\frac{1-\mathrm{i}s}{1+\mathrm{i}s}(\overline{w_\rho - p_\rho}) + \overline{w_\rho + p_\rho}}{w_\rho - p_\rho + \frac{1-\mathrm{i}s}{1+\mathrm{i}s}(w_\rho + p_\rho)}.$$

Note that

$$\left| \frac{1 - \mathrm{i}s}{1 + \mathrm{i}s} \right| = 1.$$

Lemma 3.10. *Suppose that $r(\rho) = 0$. Then the function w_ρ defined by* (3.34) *satisfies*

$$\operatorname{Re} w_\rho(\zeta)\overline{p_\rho(\zeta)} = -\rho\|R_\rho \mathbf{1}\|^2 \quad \textit{for almost all} \quad \zeta \in \mathbb{T}.$$

Proof. In the proof of Lemma 3.1 we have shown (see (3.8) and(3.9)) that for any $t \in (s_m(\Gamma), s_{m-1}(\Gamma))$ the following equalities hold:

$$(p_t, (S^*)^j p_t) - (q_t, (S^*)^j q_t) = 0, \quad j \geq 1,$$

and

$$(p_t, p_t) - (q_t, q_t) = (R_t \mathbf{1}, \mathbf{1}).$$

Let us differentiate these equalities and put $t = \rho$. Then using (3.32), we obtain

$$\left(p'_\rho + \tau_\rho q'_\rho, (S^*)^j p_\rho\right) + \left(p_\rho, (S^*)^j (p'_\rho + \tau q'_\rho)\right) = 0, \quad j \geq 1,$$

and

$$(p'_\rho + \tau_\rho q'_\rho, p_\rho) + (p_\rho, p'_\rho + \tau q'_\rho) = -2\rho \|R_\rho \mathbf{1}\|^2$$

(see (3.33)). Hence, the jth Fourier coefficient of $\operatorname{Re} w_\rho(\zeta)\overline{p_\rho(\zeta)}$ is zero for $j \neq 0$ and $-\rho\|R_\rho \mathbf{1}\|^2$ for $j = 0$. ■

Theorem 3.11. *Let m be a positive integer. Suppose that $s_m(\Gamma) < \rho < s_{m-1}(\Gamma)$ and $r(\rho) = 0$. A function $\psi \in L^\infty$ satisfies the conditions*

$$\operatorname{rank}(\Gamma - H_\psi) \leq m \quad \textit{and} \quad \|\psi\|_\infty \leq \rho$$

if and only if it can be represented in the form

$$\psi = \varphi_v \stackrel{\text{def}}{=} \rho\tau_\rho \frac{v(\overline{w_\rho - p_\rho}) + \overline{w_\rho + p_\rho}}{w_\rho - p_\rho + v(w_\rho + p_\rho)}$$

for a function v in the unit ball of H^∞. Moreover, $v \mapsto \varphi_v$ is a one-to-one map of the unit ball of L^∞ onto the ball of radius ρ of L^∞.

Proof. The fact that $v \mapsto \varphi_v$ is a one-to-one map of the unit ball of L^∞ onto the ball of radius ρ of L^∞ is a simple consequence of the equality

$$|w_\rho(\zeta) - p_\rho(\zeta)|^2 - |w_\rho(\zeta) + p_\rho(\zeta)|^2 = 4\rho\|R_\rho \mathbf{1}\|^2 > 0,$$

which, in turn, follows immediately from Lemma 3.10.

Next, as in Corollary 3.7 one can prove that for any inner function ϑ and any function v in the unit ball of H^∞

$$\deg(\vartheta(w_\rho - p_\rho) + v(w_\rho + p_\rho)_{(\mathrm{i})} = \deg \vartheta + m.$$

The rest of the proof is the same as the proof of Theorem 3.8. ■

4. Parametrization via One-Step Extension

We consider here the Nehari problem for vectorial Hankel operators. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $\Gamma = \{\Omega_{j+k}\}_{j,k \geq 0}$ be a block Hankel matrix, where the Ω_j, $j \in \mathbb{Z}_+$, are bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ such that Γ determines a bounded linear operator from $\ell^2(\mathcal{H})$ to $\ell^2(\mathcal{K})$. Suppose that $\|\Gamma\| \leq \rho$. The Nehari problem in this context is to describe all functions $\Xi \in L^\infty(\mathcal{B}(\mathcal{H}, \mathcal{K}))$ such that $\hat{\Xi}(j) = \Omega_j$, $j \in \mathbb{Z}_+$. As we have already discussed in §2.2, this problem is equivalent to the problem of describing all extensions $\{\Omega_j\}_{j\in\mathbb{Z}}$ of the sequence $\{\Omega_j\}_{j\ge 0}$ such that the operator matrix $\Gamma = \{\Omega_{j+k}\}_{j,k\in\mathbb{Z}}$ determines a bounded linear operator from $\ell^2_{\mathbb{Z}}(\mathcal{H})$ to $\ell^2_{\mathbb{Z}}(\mathcal{K})$ of norm at most ρ.

Note that the adjoint operator Γ^* is also a Hankel operator from $\ell^2(\mathcal{K})$ to $\ell^2(\mathcal{H})$ and it has Hankel matrix $\{\Omega^*_{j+k}\}_{j,k\ge 0}$.

One-Step Extension

We begin with the *problem of one-step extension*, i.e., we want to describe the operators $\Omega : \mathcal{H} \to \mathcal{K}$ such that the sequence $\Omega, \Omega_0, \Omega_1, \Omega_2, \cdots$ determines a bounded block Hankel matrix

$$\tilde{\Gamma} = \begin{pmatrix} \Omega & \Omega_0 & \Omega_1 & \cdots \\ \Omega_0 & \Omega_1 & \Omega_2 & \cdots \\ \Omega_1 & \Omega_2 & \Omega_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.1}$$

of norm at most ρ. In this case we say that $\tilde{\Gamma}$ is a *one-step ρ-extension of* Γ. We parametrize the set of all such operators Ω. Then we obtain a parametrization of all solutions of the Nehari problem in the case when the spaces $\mathcal{H}$ and $\mathcal{K}$ are finite-dimensional and $\|\Gamma\| < \rho$.

We consider the natural imbedding $\mathbf{J}_{\mathcal{H}}$ into $\ell^2(\mathcal{H})$ defined by

$$\mathbf{J}_{\mathcal{H}} x \stackrel{\text{def}}{=} (x, \mathbb{O}, \mathbb{O}, \cdots), \quad x \in \mathcal{H}.$$

Clearly,

$$\mathbf{J}^*_{\mathcal{H}}(x_0, x_1, x_2, \cdots) = x_0$$

and

$$\mathbf{J}_{\mathcal{H}}\mathbf{J}^*_{\mathcal{H}}(x_0, x_1, x_2, \cdots) = (x_0, \mathbb{O}, \mathbb{O}, \cdots).$$

Similarly, we define the imbedding $\mathbf{J}_{\mathcal{K}}$ of $\mathcal{K}$ into $\ell^2(\mathcal{K})$.

For $\rho > \|\Gamma\|$ we consider the following operators:

$$R_\rho \stackrel{\text{def}}{=} (\rho^2 I - \Gamma^*\Gamma)^{-1}, \quad R_{*\rho} \stackrel{\text{def}}{=} (\rho^2 I - \Gamma\Gamma^*)^{-1},$$

$$G_\rho \stackrel{\text{def}}{=} (\mathbf{J}^*_{\mathcal{H}} R_\rho \mathbf{J}_{\mathcal{H}})^{-1/2}, \quad G_{*\rho} \stackrel{\text{def}}{=} (\mathbf{J}^*_{\mathcal{K}} R_{*\rho} \mathbf{J}_{\mathcal{K}})^{-1/2}.$$

Here we use the same notation I for the identity operator on different Hilbert spaces. Sometimes to avoid confusion we write I_X for the identity operator on a space X.

Recall that $S_{\mathcal{H}}$ and $S_{\mathcal{K}}$ are shifts on $\ell^2(\mathcal{H})$ and $\ell^2(\mathcal{K})$. The following identity characterizes the vectorial Hankel operators:

$$S^*_{\mathcal{K}}\Gamma = \Gamma S_{\mathcal{H}}. \tag{4.2}$$

It is easy to see that the one-step extension $\tilde{\Gamma}$ defined by (4.1) satisfies the following equality:

$$\tilde{\Gamma} = \mathbf{J}_{\mathcal{K}}\Omega\mathbf{J}^*_{\mathcal{H}} + S_{\mathcal{K}}\Gamma + \mathbf{J}_{\mathcal{K}}\mathbf{J}^*_{\mathcal{K}}\Gamma S^*_{\mathcal{H}}. \tag{4.3}$$

The following result parametrizes all one-step ρ-extensions in the case $\rho > \|\Gamma\|$.

Theorem 4.1. *Let $\rho > \|\Gamma\|$. An operator $\Omega : \mathcal{H} \to \mathcal{K}$ determines a one step ρ-extension $\tilde{\Gamma}$ defined by* (4.1) *if and only if Ω admits a representation*

$$\Omega = -\mathbf{J}_{\mathcal{K}}^* \Gamma S_{\mathcal{H}}^* R_\rho \mathbf{J}_{\mathcal{H}} G_\rho^2 + \frac{1}{\rho} G_{*\rho} \mathcal{E} G_\rho, \tag{4.4}$$

where $\mathcal{E} : \mathcal{H} \to \mathcal{K}$ and $\|\mathcal{E}\| \le 1$.

Proof. Replacing Γ with $\frac{1}{\rho}\Gamma$, we can reduce the general case to the case $\rho = 1$. Let $R = R_1$, $R_* = R_{*1}$, $G = G_1$, and $G_* = G_{*1}$. Then $R = D_\Gamma^{-2}$ and $R_* = D_{\Gamma^*}^{-2}$ (see the notation in §2.1). Now we can deduce Theorem 4.1 from Theorem 2.1.5. We have

$$\tilde{\Gamma} = \begin{pmatrix} \Omega & C \\ B & A \end{pmatrix},$$

where

$$A = \Gamma S_{\mathcal{H}} = S_{\mathcal{K}}^* \Gamma = \begin{pmatrix} \Omega_1 & \Omega_2 & \cdots \\ \Omega_2 & \Omega_3 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

$$B = \Gamma \mathbf{J}_{\mathcal{H}} = \begin{pmatrix} \Omega_0 \\ \Omega_1 \\ \vdots \end{pmatrix}, \quad \text{and} \quad C = \mathbf{J}_{\mathcal{K}}^* \Gamma = \begin{pmatrix} \Omega_0 & \Omega_1 & \cdots \end{pmatrix}.$$

Then $B = D_{A^*} K$ and $C = L D_A$, where the operators K and L are the same as in Theorem 2.1.5. By Theorem 2.1.5, $\|\tilde{\Gamma}\| \le 1$ if and only if Ω admits a representation

$$\Omega = -L A^* K + D_{L^*} \mathcal{E} D_K,$$

with $\|\mathcal{E}\| \le 1$.

To compute D_K, we first observe that

$$\begin{aligned} D_{A^*}^2 &= I - \Gamma S_{\mathcal{H}} S_{\mathcal{H}}^* \Gamma^* = D_{\Gamma^*}^2 + \Gamma \mathbf{J}_{\mathcal{H}} \mathbf{J}_{\mathcal{H}}^* \Gamma^* \\ &= D_{\Gamma^*} \left(I + D_{\Gamma^*}^{-1} \Gamma \mathbf{J}_{\mathcal{H}} \mathbf{J}_{\mathcal{H}}^* \Gamma^* D_{\Gamma^*}^{-1} \right) D_{\Gamma^*}. \end{aligned}$$

Hence,

$$D_{A^*}^{-2} = D_{\Gamma^*}^{-1} \left(I + D_{\Gamma^*}^{-1} \Gamma \mathbf{J}_{\mathcal{H}} \mathbf{J}_{\mathcal{H}}^* \Gamma^* D_{\Gamma^*}^{-1} \right)^{-1} D_{\Gamma^*}^{-1}.$$

Since $K = D_{A^*}^{-1} \Gamma \mathbf{J}_{\mathcal{H}}$, it follows that

$$\begin{aligned} D_K^2 &= I - \mathbf{J}_{\mathcal{H}}^* \Gamma^* D_{A^*}^{-2} \Gamma \mathbf{J}_{\mathcal{H}} \\ &= I - \mathbf{J}_{\mathcal{H}}^* \Gamma^* D_{\Gamma^*}^{-1} \left(I + D_{\Gamma^*}^{-1} \Gamma \mathbf{J}_{\mathcal{H}} \mathbf{J}_{\mathcal{H}}^* \Gamma^* D_{\Gamma^*}^{-1} \right)^{-1} D_{\Gamma^*}^{-1} \Gamma \mathbf{J}_{\mathcal{H}} \\ &= (I + \mathbf{J}_{\mathcal{H}} \Gamma^* D_{\Gamma^*}^{-2} \Gamma \mathbf{J}_{\mathcal{H}})^{-1} = (\mathbf{J}_{\mathcal{H}} D_\Gamma^{-2} \mathbf{J}_{\mathcal{H}})^{-1} = G^2. \end{aligned}$$

Here we have used the following elementary and easily verifiable facts:

$$T^*(I + TT^*)^{-1} T = I - (I + T^*T)^{-1}$$

and

$$T^*(I - TT^*)^{-1}T = (I - T^*T)^{-1} - I \tag{4.5}$$

(the operator on the left-hand side is invertible if and only if the operator on the right-hand side is).

Similarly, one can show that $D_{L^*} = G_*$.

Finally, let us compute LA^*K. Since $L = \mathbf{J}_\mathcal{K}^*\Gamma D_A^{-1}$, we have

$$\begin{aligned} LA^*K &= \mathbf{J}_\mathcal{K}^*\Gamma D_A^{-1}A^*D_{A^*}^{-1}\Gamma\mathbf{J}_\mathcal{H} = \mathbf{J}_\mathcal{K}^*\Gamma A^*D_{A^*}^{-2}\Gamma\mathbf{J}_\mathcal{H} \\ &= \mathbf{J}_\mathcal{K}^*\Gamma S_\mathcal{H}^*\Gamma^*D_{\Gamma^*}^{-1}\left(I + D_{\Gamma^*}^{-1}\Gamma\mathbf{J}_\mathcal{H}\mathbf{J}_\mathcal{H}^*\Gamma^*D_{\Gamma^*}^{-1}\right)^{-1}D_{\Gamma^*}^{-1}\Gamma\mathbf{J}_\mathcal{H} \\ &= \mathbf{J}_\mathcal{K}^*\Gamma S_\mathcal{H}^*\Gamma^*D_{\Gamma^*}^{-2}\Gamma\mathbf{J}_\mathcal{H}(I + \mathbf{J}_\mathcal{H}^*\Gamma^*D_{\Gamma^*}^{-2}\Gamma\mathbf{J}_\mathcal{H})^{-1} \\ &= \mathbf{J}_\mathcal{K}^*\Gamma S_\mathcal{H}^*(R - I)\mathbf{J}_\mathcal{H}(\mathbf{J}_\mathcal{H}^*R\mathbf{J}_\mathcal{H})^{-1} = \mathbf{J}_\mathcal{K}^*\Gamma S_\mathcal{H}^*R\mathbf{J}_\mathcal{H}G^2, \end{aligned}$$

since, clearly, $S_\mathcal{H}^*\mathbf{J}_\mathcal{H} = \mathbb{O}$. Here we have used (4.5) and another elementary identity:

$$(I + TT^*)^{-1}T = T(I + T^*T)^{-1}.$$

This completes the proof. ■

The set of operators Ω that admit a representation of the form (4.4) is an operator quasiball. Operator quasiballs can be defined as follows. If Λ is a nonnegative operator on $\mathcal{K}$, Π is a nonnegative operator on $\mathcal{H}$, and O is a bounded linear operator from $\mathcal{H}$ to $\mathcal{K}$, the set of operators of the form

$$\boldsymbol{B} = \{O + \Lambda\mathcal{E}\Pi : \ \mathcal{E} \in \mathcal{B}(\mathcal{H},\mathcal{K}), \ \|\mathcal{E}\| \le 1\}$$

is called an *operator quasiball*. The operator O is called the *center* of $\boldsymbol{B}$, Λ is called a *left semiradius* and Π is called a *right semiradius* of $\boldsymbol{B}$. The quasiball $\boldsymbol{B}$ is nontrivial, i.e., it consists of more than one element if and only if both Λ and Π are nonzero operators. Clearly, the semiradii are not uniquely determined by the quasiball. However, if the quasiball is nontrivial, then the left and right semiradii are determined modulo positive scalar constant multiples. It is also easy to see that the center is uniquely determined by the quasiball.

The center of the quasiball in (4.4) is $-\mathbf{J}_\mathcal{K}^*\Gamma S_\mathcal{H}^*R_\rho\mathbf{J}_\mathcal{H}G_\rho^2$. If we apply formula (4.4) to the operator Γ^* and take the adjoint, we obtain another expression for the center: $-G_{*\rho}^2\mathbf{J}_\mathcal{K}^*R_{*\rho}S_\mathcal{K}\Gamma\mathbf{J}_\mathcal{H}$. Hence,

$$-\mathbf{J}_\mathcal{K}^*\Gamma S_\mathcal{H}^*R_\rho\mathbf{J}_\mathcal{H}G_\rho^2 = -G_{*\rho}^2\mathbf{J}_\mathcal{K}^*R_{*\rho}S_\mathcal{K}\Gamma\mathbf{J}_\mathcal{H} \tag{4.6}$$

is the center of the quasiball (4.4).

If $\boldsymbol{B}_1 \supset \boldsymbol{B}_2 \supset \boldsymbol{B}_3 \supset \cdots$ is a sequence of operator quasiballs, then their intersection $\bigcap\limits_{j\ge0}\boldsymbol{B}_j$ is also an operator quasiball. Indeed, let O_j be the center of $\boldsymbol{B}_j$. It can easily be seen that one can choose the semiradii Λ_j and Π_j of $\boldsymbol{B}_j$ so that $\Lambda_j \ge \Lambda_{j+1}$ and $\Pi_j \ge \Pi_{j+1}$. Let O be the limit in the weak operator topology of a subsequence of $\{O_j\}_{j\ge1}$. It can be shown that the sequences $\{\Lambda_j\}_{j\ge1}$ and $\{\Pi_j\}_{j\ge1}$ converge in the weak operator topology to operators Λ and Π and

$$\bigcap_{j\ge0}\boldsymbol{B}_j=\{O+\Lambda\mathcal{E}\Pi:\ \mathcal{E}\in\mathcal{B}(\mathcal{H},\mathcal{K}),\ \|\mathcal{E}\|\le1\}.$$

Moreover, it is not hard to see that the sequence $\{O_j\}_{j\ge1}$ converges in the weak operator topology to O. We refer the reader to Shmul'yan [1] for the facts on operator quasiballs mentioned above.

Consider now the case $\rho=\|\Gamma\|$, i.e., we consider one-step extensions of minimal norm. Let $\{\rho_j\}_{j\ge1}$ be a sequence of positive numbers such that $\rho_j>\|\Gamma\|$ and $\rho_j\to\|\Gamma\|$ as $j\to\infty$. An operator $\Omega:\mathcal{H}\to\mathcal{K}$ determines a one-step extension of minimal norm if and only if it belongs to the intersection of the quasiballs

$$\boldsymbol{B}_j\stackrel{\text{def}}{=}\left\{-\mathbf{J}_\mathcal{K}^*\Gamma S_\mathcal{H}^*R_{\rho_j}\mathbf{J}_\mathcal{H}G_{\rho_j}^2+\frac{1}{\rho_j}G_{*\rho_j}\mathcal{E}G_{\rho_j}:\ \|\mathcal{E}\|_{\mathcal{B}(\mathcal{H},\mathcal{K})}\le1\right\}.$$

It is easy to see that the sequences $\{\rho_j^{-1/2}G_{*\rho_j}\}_{j\ge1}$ and $\{\rho_j^{-1/2}G_{\rho_j}\}_{j\ge1}$ are nonincreasing and we obtain the following result.

Theorem 4.2. *An operator $\Omega:\mathcal{H}\to\mathcal{K}$ determines a one-step extension of minimal norm if and only if it belongs to the operator quasiball with center*

$$-\lim_{\rho\to\|\Gamma\|+}\mathbf{J}_\mathcal{K}^*\Gamma S_\mathcal{H}^*R_\rho\mathbf{J}_\mathcal{H}G_\rho^2$$

and semiradii

$$\frac{1}{\|\Gamma\|}\lim_{\rho\to\|\Gamma\|+}\mathbf{J}_\mathcal{K}^*R_{*\rho}\mathbf{J}_\mathcal{K}$$

and

$$\frac{1}{\|\Gamma\|}\lim_{\rho\to\|\Gamma\|+}\mathbf{J}_\mathcal{H}^*R_\rho\mathbf{J}_\mathcal{H},$$

the limits being taken in the strong operator topology.

Now we are able to describe the operators Γ with a unique one-step extension of minimal norm.

Theorem 4.3. *Let Γ be a vectorial Hankel operator from $\ell^2(\mathcal{H})$ to $\ell^2(\mathcal{K})$. Then it has a unique one-step extension of minimal norm if and only if at least one of the following conditions* (i) *and* (ii) *is satisfied:*

(i) $s\text{-}\lim_{\rho\to\|\Gamma\|+}G_\rho=\mathbb{O}$;

(ii) $s\text{-}\lim_{\rho\to\|\Gamma\|+}G_{*\rho}=\mathbb{O}$.

Moreover, (i) *is equivalent to the condition*

(i′) $\operatorname{Range}(\|\Gamma\|^2I-\Gamma^*\Gamma)^{1/2}\bigcap\mathbf{J}_\mathcal{H}\mathcal{H}=\{\mathbb{O}\}$,

and (ii) *is equivalent to the condition*

(ii′) $\operatorname{Range}(\|\Gamma\|^2I-\Gamma\Gamma^*)^{1/2}\bigcap\mathbf{J}_\mathcal{K}\mathcal{K}=\{\mathbb{O}\}$.

Proof. In view of the above discussion it remains to show that (i)⇔(i′) and (ii)⇔(ii′). Let us establish the equivalence of (i) and (i′). To verify the equivalence of (ii) and (ii′), it suffices to interchange the roles of Γ and Γ^*.

Suppose that (i) is satisfied and x is a vector in $\mathcal{H}$ such that

$$\mathbf{J}_{\mathcal{H}}x \in \operatorname{Range}(\|\Gamma\|^2 I - \Gamma^*\Gamma)^{1/2},$$

i.e.,

$$\mathbf{J}_{\mathcal{H}}x = (\|\Gamma\|^2 I - \Gamma^*\Gamma)^{1/2} g, \quad g \in \ell^2(\mathcal{H}).$$

Then for $\rho > \|\Gamma\|$ we have

$$\begin{aligned}\|G_\rho^{-1}x\|^2 &= \big((\rho^2 I - \Gamma^*\Gamma)^{-1}\mathbf{J}_{\mathcal{H}}x, \mathbf{J}_{\mathcal{H}}x\big)\\ &= \Big\|(\rho^2 I - \Gamma^*\Gamma)^{-1/2}(\|\Gamma\|^2 I - \Gamma^*\Gamma)^{1/2} g\Big\|^2_{\ell^2(\mathcal{H})} \to \|g\|^2\end{aligned}$$

as $\rho \to \|\Gamma\|+$, by the spectral theorem. Thus the family $y_\rho \stackrel{\text{def}}{=} G_\rho^{-1}x$, $\rho > \|\Gamma\|$, is bounded. Hence, it follows from (i) that $G_\rho y_\rho \to \mathbb{0}$ as $\rho \to \|\Gamma\|+$ in the weak operator topology. However, $G_\rho y_\rho = x$, which proves that (i)$\Rightarrow$(i$'$).

Conversely, suppose that (i) does not hold. Since $G_{\rho_1} \ge G_{\rho_2}$ for $\rho_1 \ge \rho_2$ and the operators G_ρ are nonnegative, it is well known (see Halmos [2], Problem 120) that there exists a strong limit G of G_ρ as $\rho \to \|\Gamma\|+$. Let $w \in \mathcal{H}$ such that

$$x \stackrel{\text{def}}{=} Gw = \lim_{\rho \to \|\Gamma\|+} G_\rho w \ne \mathbb{O}.$$

Let us show that

$$\|G_\rho^{-1}x\| = \|G_\rho^{-1}Gw\| \le \|w\|. \tag{4.7}$$

Clearly, $\|G_\rho^{-1}G\| = \|GG_\rho^{-1}\|$. It is easy to see that $G_{\rho_1}^2 \ge G_{\rho_2}^2$ for $\rho_1 \ge \rho_2$, and so $G_\rho^2 \ge G^2$ for $\rho \ge \|\Gamma\|$. Thus

$$\|GG_\rho^{-1}v\|^2 = (G^2 G_\rho^{-1}v, G_\rho^{-1}v) \le (G_\rho^2 G_\rho^{-1}v, G_\rho^{-1}v) = \|v\|^2,$$

which proves (4.7).

We have

$$\big((\rho^2 I - \Gamma^*\Gamma)^{-1}\mathbf{J}_{\mathcal{H}}x, \mathbf{J}_{\mathcal{H}}x\big) = \|G_\rho^{-1}x\|^2 \le \text{const},$$

and it follows from Lemma 5.1.6 that $\mathbf{J}_{\mathcal{H}}x \in \operatorname{Range}(\|\Gamma\|^2 I - \Gamma^*\Gamma)^{1/2}$. ■

Corollary 4.4. *Suppose that* $\|\Gamma\|^2$ *is an eigenvalue of* $\Gamma^*\Gamma$ *and*

$$\operatorname{clos} \mathbf{J}_{\mathcal{H}}^* \operatorname{Ker}(\|\Gamma\|^2 I - \Gamma^*\Gamma) = \mathcal{H}; \tag{4.8}$$

then Γ *has a unique one-step extension of minimal norm.*

Proof. Suppose that $x \in \mathcal{H}$ and $\mathbf{J}_{\mathcal{H}}x = (\|\Gamma\|^2 I - \Gamma^*\Gamma)^{1/2} f$ for some $f \in \ell^2(\mathcal{H})$. Then $\mathbf{J}_{\mathcal{H}}x \perp \operatorname{Ker}(\|\Gamma\|^2 I - \Gamma^*\Gamma)$. By (4.8), $x = \mathbb{O}$. ■

Remark. Another sufficient condition for uniqueness can be obtained if we interchange the roles of Γ and Γ^*.

We obtain now some useful identities that will be used in this section. In addition to (4.2) we need the following easily verifiable commutation relation:

$$\Gamma S_{\mathcal{H}}^* = S_{\mathcal{K}}\Gamma + \mathbf{J}_{\mathcal{K}}\mathbf{J}_{\mathcal{K}}^*\Gamma S_{\mathcal{H}}^* - S_{\mathcal{K}}\Gamma\mathbf{J}_{\mathcal{H}}\mathbf{J}_{\mathcal{H}}^*. \tag{4.9}$$

It follows easily from (4.2) and (4.9) that

$$(\rho^2 I - \Gamma\Gamma^*)S_{\mathcal{K}} = S_{\mathcal{K}}(\rho^2 I - \Gamma\Gamma^*) + S_{\mathcal{K}}\Gamma\mathbf{J}_{\mathcal{H}}\mathbf{J}_{\mathcal{H}}^*\Gamma^* - \mathbf{J}_{\mathcal{K}}\mathbf{J}_{\mathcal{K}}^*\Gamma S_{\mathcal{H}}^*\Gamma^*.$$

This implies the following identity:

$$S_{\mathcal{K}}R_{*\rho} = R_{*\rho}S_{\mathcal{K}} + R_{*\rho}S_{\mathcal{K}}\Gamma\mathbf{J}_{\mathcal{H}}\mathbf{J}_{\mathcal{H}}^*\Gamma^*R_{*\rho} - R_{*\rho}\mathbf{J}_{\mathcal{K}}\mathbf{J}_{\mathcal{K}}^*\Gamma S_{\mathcal{H}}^*\Gamma^*R_{*\rho}. \quad (4.10)$$

If we apply (4.10) to the operator Γ^*, we obtain

$$S_{\mathcal{H}}R_{\rho} = R_{\rho}S_{\mathcal{H}} + R_{\rho}S_{\mathcal{H}}\Gamma^*\mathbf{J}_{\mathcal{K}}\mathbf{J}_{\mathcal{K}}^*\Gamma R_{\rho} - R_{\rho}\mathbf{J}_{\mathcal{H}}\mathbf{J}_{\mathcal{H}}^*\Gamma^*S_{\mathcal{K}}^*\Gamma R_{\rho}. \quad (4.11)$$

We also need the following simple formula:

$$\Gamma^*R_{*\rho}\Gamma = \rho^2 R_{\rho} - I. \quad (4.12)$$

To prove (4.12), we observe that

$$\Gamma^*R_{*\rho}\Gamma(\rho^2 I - \Gamma^*\Gamma) = \Gamma^*R_{*\rho}(\rho^2 I - \Gamma\Gamma^*)\Gamma = \Gamma^*\Gamma = \rho^2 I - (\rho^2 I - \Gamma^*\Gamma).$$

Multiplying this on the right by R_{ρ}, we obtain (4.12). If we apply (4.12) to Γ^*, we find that

$$\Gamma R_{\rho}\Gamma^* = \rho^2 R_{*\rho} - I. \quad (4.13)$$

Finally, we need the formula

$$\tilde{\Gamma}\tilde{\Gamma}^* = (\mathbf{J}_{\mathcal{K}}\Omega + S_{\mathcal{K}}\Gamma\mathbf{J}_{\mathcal{H}})(\Omega^*\mathbf{J}_{\mathcal{K}}^* + \mathbf{J}_{\mathcal{H}}^*\Gamma^*S_{\mathcal{K}}^*) + \Gamma\Gamma^*. \quad (4.14)$$

Indeed,

$$\begin{aligned}
\tilde{\Gamma}\tilde{\Gamma}^* &= \begin{pmatrix} \Omega & \Omega_0 & \Omega_1 & \cdots \\ \Omega_0 & \Omega_1 & \Omega_2 & \cdots \\ \Omega_1 & \Omega_2 & \Omega_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \Omega^* & \Omega_0^* & \Omega_1^* & \cdots \\ \Omega_0^* & \Omega_1^* & \Omega_2^* & \cdots \\ \Omega_1^* & \Omega_2^* & \Omega_3^* & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= \begin{pmatrix} \Omega \\ \Omega_0 \\ \vdots \end{pmatrix} \begin{pmatrix} \Omega^* & \Omega_0^* & \cdots \end{pmatrix} \\
&\quad + \begin{pmatrix} \Omega_0 & \Omega_1 & \cdots \\ \Omega_1 & \Omega_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \Omega_0^* & \Omega_1^* & \cdots \\ \Omega_1^* & \Omega_2^* & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \\
&= (\mathbf{J}_{\mathcal{K}}\Omega + S_{\mathcal{K}}\Gamma\mathbf{J}_{\mathcal{H}})(\Omega^*\mathbf{J}_{\mathcal{K}}^* + \mathbf{J}_{\mathcal{H}}^*\Gamma^*S_{\mathcal{K}}^*) + \Gamma\Gamma^*.
\end{aligned}$$

Let us introduce now the following important operators.

$$\begin{array}{lll}
P_{\rho}: & \mathcal{H} \to \ell^2(\mathcal{H}), & P_{\rho} = \rho R_{\rho}\mathbf{J}_{\mathcal{H}}G_{\rho}; \\
Q_{\rho}: & \mathcal{H} \to \ell^2(\mathcal{K}), & Q_{\rho} = S_{\mathcal{K}}\Gamma R_{\rho}\mathbf{J}_{\mathcal{H}}G_{\rho}; \\
P_{*\rho}: & \mathcal{K} \to \ell^2(\mathcal{K}), & P_{*\rho} = \rho R_{*\rho}\mathbf{J}_{\mathcal{K}}G_{*\rho}; \\
Q_{*\rho}: & \mathcal{K} \to \ell^2(\mathcal{H}), & Q_{*\rho} = S_{\mathcal{H}}\Gamma^*R_{*\rho}\mathbf{J}_{\mathcal{K}}G_{*\rho}.
\end{array}$$

One-Step Canonical Extensions

Without loss of generality we may assume that $\dim \mathcal{H} \le \dim \mathcal{K}$. Otherwise we could study Γ^* rather than Γ. If $\rho > \|\Gamma\|$, we say that a one-step ρ-extension $\tilde{\Gamma}$ is *canonical* if the operator $\mathcal{E} : \mathcal{H} \to \mathcal{K}$ in (4.4) is an isometric operator, i.e., $\mathcal{E}^*\mathcal{E} = I_{\mathcal{H}}$.

Theorem 4.5. *Let $\rho > \|\Gamma\|$ and let $\tilde{\Gamma}$ be a canonical one-step ρ-extension of Γ determined by an isometry $\mathcal{E}$ in* (4.4). *Then the following equalities hold:*

$$\tilde{\Gamma}(P_\rho + Q_{*\rho}\mathcal{E}) = \rho(Q_\rho + P_{*\rho}\mathcal{E}), \quad \tilde{\Gamma}^*(Q_\rho + P_{*\rho}\mathcal{E}) = \rho(P_\rho + Q_{*\rho}\mathcal{E}). \tag{4.15}$$

Moreover,

$$\operatorname{Ker}(\rho^2 I - \tilde{\Gamma}^*\tilde{\Gamma}) = (P_\rho + Q_{*\rho}\mathcal{E})\mathcal{H}, \quad \operatorname{Ker}(\rho^2 I - \tilde{\Gamma}\tilde{\Gamma}^*) \supset (Q_\rho + P_{*\rho}\mathcal{E})\mathcal{H}, \tag{4.16}$$

and

$$\left(\|\Gamma\|^2, \rho^2\right) \bigcap \sigma\left(\tilde{\Gamma}^*\tilde{\Gamma}\right) = \left(\|\Gamma\|^2, \rho^2\right) \bigcap \sigma\left(\tilde{\Gamma}\tilde{\Gamma}^*\right) = \varnothing.$$

Remark. It is easy to see that

$$\mathbf{J}^*_{\mathcal{H}}(P_\rho + Q_{*\rho}\mathcal{E})\mathcal{H} = \mathcal{H}.$$

Indeed, it follows from the definition of P_ρ, Q_ρ, $P_{*\rho}$, $Q_{*\rho}$ that

$$\mathbf{J}^*_{\mathcal{H}}(P_\rho + Q_{*\rho}\mathcal{E})\mathcal{H} = \mathbf{J}^*_{\mathcal{H}}P_\rho\mathcal{H} = \rho\mathbf{J}^*_{\mathcal{H}}R_\rho\mathbf{J}_{\mathcal{H}}G_\rho\mathcal{H} = \rho G_\rho^{-1}\mathcal{H} = \mathcal{H}.$$

Proof of Theorem 4.5. Let us show that for any operator $\mathcal{E}$ from $\mathcal{H}$ to $\mathcal{K}$ the following equalities hold:

$$\Gamma(P_\rho + Q_{*\rho}\mathcal{E}) = \rho S^*_{\mathcal{K}}(Q_\rho + P_{*\rho}\mathcal{E}); \quad \Gamma^*(Q_\rho + P_{*\rho}\mathcal{E}) = \rho S^*_{\mathcal{H}}(P_\rho + Q_{*\rho}\mathcal{E}). \tag{4.17}$$

Indeed, by the definition of P_ρ, Q_ρ, $P_{*\rho}$, $Q_{*\rho}$, we have

$$\begin{aligned}
\Gamma(P_\rho + Q_{*\rho}\mathcal{E}) &= \rho\Gamma R_\rho\mathbf{J}_{\mathcal{H}}G_\rho + \Gamma S_{\mathcal{H}}\Gamma^* R_{*\rho}\mathbf{J}_{\mathcal{K}}G_{*\rho}\mathcal{E} \\
&= \rho S^*_{\mathcal{K}}Q_\rho + S^*_{\mathcal{K}}\Gamma\Gamma^* R_{*\rho}\mathbf{J}_{\mathcal{K}}G_{*\rho}\mathcal{E} \\
&= \rho S^*_{\mathcal{K}}Q_\rho + S^*_{\mathcal{K}}((\Gamma\Gamma^* - \rho^2 I) + \rho^2 I)R_{*\rho}\mathbf{J}_{\mathcal{K}}G_{*\rho}\mathcal{E} \\
&= \rho S^*_{\mathcal{K}}Q_\rho + S^*_{\mathcal{K}}(-I + \rho^2 R_{*\rho})\mathbf{J}_{\mathcal{K}}G_{*\rho}\mathcal{E} = \rho S^*_{\mathcal{K}}(Q_\rho + P_{*\rho}\mathcal{E}),
\end{aligned}$$

since obviously, $S^*_{\mathcal{K}}\mathbf{J}_{\mathcal{K}}G_{*\rho}\mathcal{E} = \mathbb{O}$. The second equality in (4.17) can be proved in exactly the same way.

Using (4.3), we obtain

$$\begin{aligned}
\tilde{\Gamma}(P_\rho + Q_{*\rho}\mathcal{E}) &= \mathbf{J}_{\mathcal{K}}\Omega\mathbf{J}^*_{\mathcal{H}}(P_\rho + Q_{*\rho}\mathcal{E}) \\
&+ S_{\mathcal{K}}\Gamma(P_\rho + Q_{*\rho}\mathcal{E}) + \mathbf{J}_{\mathcal{K}}\mathbf{J}^*_{\mathcal{K}}\Gamma S^*_{\mathcal{H}}(P_\rho + Q_{*\rho}\mathcal{E}).
\end{aligned}$$

By (4.17),

$$\begin{aligned} S_{\mathcal{K}}\Gamma(P_\rho + Q_{*\rho}\mathcal{E}) &= \rho S_{\mathcal{K}} S_{\mathcal{K}}^*(Q_\rho + P_{*\rho}\mathcal{E}) \\ &= \rho(Q_\rho + P_{*\rho}\mathcal{E}) - \rho \mathbf{J}_{\mathcal{K}}\mathbf{J}_{\mathcal{K}}^*(Q_\rho + P_{*\rho}\mathcal{E}) \\ &= \rho(Q_\rho + P_{*\rho}\mathcal{E}) - \rho \mathbf{J}_{\mathcal{K}}\mathbf{J}_{\mathcal{K}}^* P_{*\rho}\mathcal{E} \\ &= \rho(Q_\rho + P_{*\rho}\mathcal{E}) - \rho^2 \mathbf{J}_{\mathcal{K}} G_{*\rho}^{-1}\mathcal{E}. \end{aligned}$$

Next,

$$\mathbf{J}_{\mathcal{K}}\mathbf{J}_{\mathcal{K}}^*\Gamma S_{\mathcal{H}}^*(P_\rho + Q_{*\rho}\mathcal{E}) = \rho \mathbf{J}_{\mathcal{K}}\mathbf{J}_{\mathcal{K}}^*\Gamma S_{\mathcal{H}}^* R_\rho \mathbf{J}_{\mathcal{H}} G_\rho + \mathbf{J}_{\mathcal{K}}\mathbf{J}_{\mathcal{K}}^*\Gamma\Gamma^* R_{*\rho}\mathbf{J}_{\mathcal{K}} G_{*\rho}\mathcal{E},$$

and

$$\begin{aligned} \mathbf{J}_{\mathcal{K}}\mathbf{J}_{\mathcal{K}}^*\Gamma\Gamma^* R_{*\rho}\mathbf{J}_{\mathcal{K}} G_{*\rho}\mathcal{E} &= \mathbf{J}_{\mathcal{K}}\mathbf{J}_{\mathcal{K}}^*((\Gamma\Gamma^* - \rho^2 I) + \rho^2 I) R_{*\rho}\mathbf{J}_{\mathcal{K}} G_{*\rho}\mathcal{E} \\ &= -\mathbf{J}_{\mathcal{K}} G_{*\rho}\mathcal{E} + \rho^2 \mathbf{J}_{\mathcal{K}}\mathbf{J}_{\mathcal{K}}^* R_{*\rho}\mathbf{J}_{\mathcal{K}} G_{*\rho}\mathcal{E} \\ &= -\mathbf{J}_{\mathcal{K}} G_{*\rho}\mathcal{E} + \rho^2 \mathbf{J}_{\mathcal{K}} G_{*\rho}^{-1}\mathcal{E}. \end{aligned}$$

It follows from (4.4) that

$$\begin{aligned} \mathbf{J}_{\mathcal{K}}\Omega\mathbf{J}_{\mathcal{H}}^*(P_\rho + Q_{*\rho}\mathcal{E}) &= \mathbf{J}_{\mathcal{K}}\Omega\mathbf{J}_{\mathcal{H}}^* P_\rho \\ &= -\mathbf{J}_{\mathcal{K}}\mathbf{J}_{\mathcal{K}}^*\Gamma S_{\mathcal{H}}^* R_\rho \mathbf{J}_{\mathcal{H}} G_\rho^2 \mathbf{J}_{\mathcal{H}}^* P_\rho + \frac{1}{\rho}\mathbf{J}_{\mathcal{K}} G_{*\rho}\mathcal{E} G_\rho \mathbf{J}_{\mathcal{H}}^* P_\rho \\ &= -\rho \mathbf{J}_{\mathcal{K}}\mathbf{J}_{\mathcal{K}}^*\Gamma S_{\mathcal{H}}^* R_\rho \mathbf{J}_{\mathcal{H}} G_\rho + \mathbf{J}_{\mathcal{K}} G_{*\rho}\mathcal{E}. \end{aligned}$$

This proves the first equality in (4.15). To establish the second equality we use the following analog of (4.3) for $\tilde{\Gamma}^*$:

$$\tilde{\Gamma}^* = \mathbf{J}_{\mathcal{H}}\Omega^*\mathbf{J}_{\mathcal{K}}^* + S_{\mathcal{H}}\Gamma^* + \mathbf{J}_{\mathcal{H}}\mathbf{J}_{\mathcal{H}}^*\Gamma^* S_{\mathcal{K}}^*.$$

We have

$$\begin{aligned} \tilde{\Gamma}^*(Q_\rho + P_{*\rho}\mathcal{E}) &= \mathbf{J}_{\mathcal{H}}\Omega^*\mathbf{J}_{\mathcal{K}}^*(Q_\rho + P_{*\rho}\mathcal{E}) \\ &+ S_{\mathcal{H}}\Gamma^*(Q_\rho + P_{*\rho}\mathcal{E}) + \mathbf{J}_{\mathcal{H}}\mathbf{J}_{\mathcal{H}}^*\Gamma^* S_{\mathcal{K}}^*(Q_\rho + P_{*\rho}\mathcal{E}). \end{aligned}$$

By (4.17),

$$\begin{aligned} S_{\mathcal{H}}\Gamma^*(Q_\rho + P_{*\rho}\mathcal{E}) &= \rho S_{\mathcal{H}} S_{\mathcal{H}}^*(P_\rho + Q_{*\rho}\mathcal{E}) \\ &= \rho(P_\rho + Q_{*\rho}\mathcal{E}) - \rho \mathbf{J}_{\mathcal{H}}\mathbf{J}_{\mathcal{H}}^*(P_\rho + Q_{*\rho}\mathcal{E}) \\ &= \rho(P_\rho + Q_{*\rho}\mathcal{E}) - \rho^2 \mathbf{J}_{\mathcal{H}}\mathbf{J}_{\mathcal{H}}^* R_\rho \mathbf{J}_{\mathcal{H}} G_\rho \\ &= \rho(P_\rho + Q_{*\rho}\mathcal{E}) - \rho^2 \mathbf{J}_{\mathcal{H}} G_\rho^{-1}. \end{aligned}$$

Next,

$$\mathbf{J}_{\mathcal{H}}\mathbf{J}_{\mathcal{H}}^*\Gamma^* S_{\mathcal{K}}^*(Q_\rho + P_{*\rho}\mathcal{E}) = \mathbf{J}_{\mathcal{H}}\mathbf{J}_{\mathcal{H}}^*\Gamma^*\Gamma R_\rho \mathbf{J}_{\mathcal{H}} G_\rho + \rho \mathbf{J}_{\mathcal{H}}\mathbf{J}_{\mathcal{H}}^*\Gamma^* S_{\mathcal{K}}^* R_{*\rho}\mathbf{J}_{\mathcal{K}} G_{*\rho}\mathcal{E}$$

and

$$\begin{aligned}
\mathbf{J}_{\mathcal{H}}\mathbf{J}_{\mathcal{H}}^*\Gamma^*\Gamma R_\rho \mathbf{J}_{\mathcal{H}}G_\rho &= \mathbf{J}_{\mathcal{H}}\mathbf{J}_{\mathcal{H}}^*((\Gamma^*\Gamma-\rho^2 I)+\rho^2 I)R_\rho\mathbf{J}_{\mathcal{H}}G_\rho \\
&= -\mathbf{J}_{\mathcal{H}}G_\rho+\rho^2\mathbf{J}_{\mathcal{H}}\mathbf{J}_{\mathcal{H}}^*R_\rho\mathbf{J}_{\mathcal{H}}G_\rho \\
&= -\mathbf{J}_{\mathcal{H}}G_\rho+\rho^2\mathbf{J}_{\mathcal{H}}G_\rho^{-1}.
\end{aligned}$$

Finally, it follows from (4.4) and (4.6) that

$$\begin{aligned}
\mathbf{J}_{\mathcal{H}}\Omega^*\mathbf{J}_{\mathcal{K}}^*(Q_\rho+P_{*\rho}\mathcal{E}) &= \mathbf{J}_{\mathcal{H}}\Omega^*\mathbf{J}_{\mathcal{K}}^*P_{*\rho}\mathcal{E} \\
&= -\rho\mathbf{J}_{\mathcal{H}}\mathbf{J}_{\mathcal{H}}^*\Gamma^*S_{\mathcal{K}}^*R_{*\rho}\mathbf{J}_{\mathcal{K}}G_{*\rho}^2\mathbf{J}_{\mathcal{K}}^*R_{*\rho}\mathbf{J}_{\mathcal{K}}G_{*\rho}\mathcal{E} \\
&+ \mathbf{J}_{\mathcal{H}}G_\rho\mathcal{E}^*G_{*\rho}\mathbf{J}_{\mathcal{K}}^*R_{*\rho}\mathbf{J}_{\mathcal{K}}G_{*\rho}\mathcal{E} \\
&= -\rho\mathbf{J}_{\mathcal{H}}\mathbf{J}_{\mathcal{H}}^*\Gamma^*S_{\mathcal{K}}^*R_{*\rho}\mathbf{J}_{\mathcal{K}}G_{*\rho}\mathcal{E}+\mathbf{J}_{\mathcal{H}}G_\rho\mathcal{E}^*\mathcal{E} \\
&= -\rho\mathbf{J}_{\mathcal{H}}\mathbf{J}_{\mathcal{H}}^*\Gamma^*S_{\mathcal{K}}^*R_{*\rho}\mathbf{J}_{\mathcal{K}}G_{*\rho}\mathcal{E}+\mathbf{J}_{\mathcal{H}}G_\rho.
\end{aligned}$$

This proves the second equality in (4.15).

Obviously, (4.15) implies that

$$(P_\rho+Q_{*\rho}\mathcal{E})\mathcal{H}\subset\operatorname{Ker}(\rho^2 I-\tilde{\Gamma}^*\tilde{\Gamma})$$

and

$$(Q_\rho+P_{*\rho}\mathcal{E})\mathcal{H}\subset\operatorname{Ker}(\rho^2 I-\tilde{\Gamma}\tilde{\Gamma}^*).$$

It follows from the Remark preceding the proof of this theorem that if

$$(P_\rho+Q_{*\rho}\mathcal{E})\mathcal{H}\neq\operatorname{Ker}(\rho^2 I-\tilde{\Gamma}^*\tilde{\Gamma}),$$

then there exists an eigenvector f of $\tilde{\Gamma}^*\tilde{\Gamma}$ with eigenvalue ρ^2 such that $f\notin(P_\rho+Q_{*\rho}\mathcal{E})\mathcal{H}$ and $\mathbf{J}_{\mathcal{H}}^*f=\mathbb{O}$. Let $g=S_{\mathcal{H}}^*f\in\ell^2(\mathcal{H})$. Then

$$\|\tilde{\Gamma}f\|=\|\tilde{\Gamma}\|\cdot\|f\|=\rho\|g\|.$$

On the other hand, it is easy to see that

$$\|\tilde{\Gamma}f\|=\|\tilde{\Gamma}S_{\mathcal{H}}g\|=\|\Gamma g\|,$$

and so $\|\Gamma g\|\geq\rho\|g\|$, which contradicts the assumption $\|\Gamma\|<\rho$. The second part of (4.16) is obvious.

To complete the proof, we need the following elementary result.

Lemma 4.6. *Let Y and Z be Banach spaces, and let $T_1:Y\to Z$ and $T_2:Z\to Y$ be bounded linear operators. If λ is a nonzero complex number, then λ belongs to the spectrum of T_1T_2 if and only if λ belongs to the spectrum of T_2T_1.*

Proof. Let λ be a nonzero complex number. It has to be proved that if $T_1T_2-\lambda I$ is invertible, then $T_2T_1-\lambda I$ is. Clearly, without loss of generality we may assume that $\lambda=1$. We claim that $T_2(T_1T_2-I)^{-1}T_1-I$ is the inverse of T_2T_1-I. Indeed, we have

$$(T_2(T_1T_2-I)^{-1}T_1-I)(T_2T_1-I)=T_2(T_1T_2-I)^{-1}(T_1T_2-I)T_1-T_2T_1+I=I$$

and

$$(T_2T_1-I)(T_2(T_1T_2-I)^{-1}T_1-I) = T_2(T_1T_2-I)(T_1T_2-I)^{-1}T_1-T_2T_1+I = I.$$

■

By Lemma 4.6, it is sufficient to show that $(\|\Gamma\|^2, \rho^2)$ is contained in the resolvent set of $\tilde{\Gamma}\tilde{\Gamma}^*$.

Suppose that $\lambda > \|\Gamma\|$. We have by (4.14)

$$\lambda^2 I - \tilde{\Gamma}\tilde{\Gamma}^* = \lambda^2 I - \Gamma\Gamma^* - (\mathbf{J}_{\mathcal{K}}\Omega + S_{\mathcal{K}}\Gamma\mathbf{J}_{\mathcal{H}})(\Omega^*\mathbf{J}_{\mathcal{K}}^* + \mathbf{J}_{\mathcal{H}}^*\Gamma^* S_{\mathcal{K}}^*).$$

Multiplying this equality on the left and on the right by $(\lambda^2 I - \Gamma\Gamma^*)^{-1/2}$, we find that $\lambda^2 I - \tilde{\Gamma}\tilde{\Gamma}^*$ is invertible if and and only if the operator

$$I - (\lambda^2 I - \Gamma\Gamma^*)^{-1/2}(\mathbf{J}_{\mathcal{K}}\Omega + S_{\mathcal{K}}\Gamma\mathbf{J}_{\mathcal{H}})(\Omega^*\mathbf{J}_{\mathcal{K}}^* + \mathbf{J}_{\mathcal{H}}^*\Gamma^* S_{\mathcal{K}}^*)(\lambda^2 I - \Gamma\Gamma^*)^{-1/2}$$

is invertible. It follows from Lemma 4.6 that this is equivalent to the invertibility of the operator

$$\mathcal{T}(\lambda) \stackrel{\text{def}}{=} I_{\mathcal{H}} - (\Omega^*\mathbf{J}_{\mathcal{K}}^* + \mathbf{J}_{\mathcal{H}}^*\Gamma^* S_{\mathcal{K}}^*)R_{*\lambda}(\mathbf{J}_{\mathcal{K}}\Omega + S_{\mathcal{K}}\Gamma\mathbf{J}_{\mathcal{H}})$$

on $\mathcal{H}$. Let us compute $\mathcal{T}(\rho)$. Using (4.4) and (4.6), we obtain

$$\begin{aligned}
(\Omega^*\mathbf{J}_{\mathcal{K}}^* + \mathbf{J}_{\mathcal{H}}^*\Gamma^* S_{\mathcal{K}}^*)R_{*\rho}\mathbf{J}_{\mathcal{K}}\Omega &= -\mathbf{J}_{\mathcal{H}}^*\Gamma^* S_{\mathcal{K}}^* R_{*\rho}\mathbf{J}_{\mathcal{K}} G_{*\rho}^2 \mathbf{J}_{\mathcal{K}}^* R_{*\rho}\mathbf{J}_{\mathcal{K}}\Omega \\
&+ \frac{1}{\rho} G_\rho \mathcal{E}^* G_{*\rho}\mathbf{J}_{\mathcal{K}}^* R_{*\rho}\mathbf{J}_{\mathcal{K}}\Omega + \mathbf{J}_{\mathcal{H}}^*\Gamma^* S_{\mathcal{K}}^* R_{*\rho}\mathbf{J}_{\mathcal{K}}\Omega \\
&= -\mathbf{J}_{\mathcal{H}}^*\Gamma^* S_{\mathcal{K}}^* R_{*\rho}\mathbf{J}_{\mathcal{K}}\Omega + \frac{1}{\rho} G_\rho \mathcal{E}^* G_{*\rho}^{-1}\mathbf{J}_{\mathcal{K}}\Omega \\
&+ \mathbf{J}_{\mathcal{H}}^*\Gamma^* S_{\mathcal{K}}^* R_{*\rho}\mathbf{J}_{\mathcal{K}}\Omega = \frac{1}{\rho} G_\rho \mathcal{E}^* G_{*\rho}^{-1}\mathbf{J}_{\mathcal{K}}\Omega \\
&= -\frac{1}{\rho} G_\rho \mathcal{E}^* G_{*\rho}^{-1}\mathbf{J}_{\mathcal{K}} G_{*\rho}^2 \mathbf{J}_{\mathcal{K}}^* R_{*\rho} S_{\mathcal{K}}\Gamma\mathbf{J}_{\mathcal{H}} \\
&+ \frac{1}{\rho^2} G_\rho \mathcal{E}^* G_{*\rho}^{-1}\mathbf{J}_{\mathcal{K}} G_{*\rho}\mathcal{E} G_\rho \\
&= -\frac{1}{\rho} G_\rho \mathcal{E}^* G_{*\rho}\mathbf{J}_{\mathcal{K}}^* R_{*\rho} S_{\mathcal{K}}\Gamma\mathbf{J}_{\mathcal{H}} + \frac{1}{\rho^2} G_\rho \mathcal{E}^*\mathcal{E} G_\rho
\end{aligned}$$

and

$$\begin{aligned}
\Omega^*\mathbf{J}_{\mathcal{K}}^* R_{*\rho} S_{\mathcal{K}}\Gamma\mathbf{J}_{\mathcal{H}} &= -\mathbf{J}_{\mathcal{H}}^*\Gamma^* S_{\mathcal{K}}^* R_{*\rho}\mathbf{J}_{\mathcal{K}} G_{*\rho}^2 \mathbf{J}_{\mathcal{K}}^* R_{*\rho} S_{\mathcal{K}}\Gamma\mathbf{J}_{\mathcal{H}} \\
&+ \frac{1}{\rho} G_\rho \mathcal{E}^* G_{*\rho}\mathbf{J}_{\mathcal{K}}^* R_{*\rho} S_{\mathcal{K}}\Gamma\mathbf{J}_{\mathcal{H}}.
\end{aligned}$$

Since $\mathcal{E}^*\mathcal{E} = I_{\mathcal{H}}$, we have

$$\begin{aligned}
\mathcal{T}(\rho) &= I_{\mathcal{H}} - \frac{1}{\rho^2} G_\rho^2 + \mathbf{J}_{\mathcal{H}}^*\Gamma^* S_{\mathcal{K}}^* R_{*\rho}\mathbf{J}_{\mathcal{K}} G_{*\rho}^2 \mathbf{J}_{\mathcal{K}}^* R_{*\rho} S_{\mathcal{K}}\Gamma\mathbf{J}_{\mathcal{H}} \\
&- \mathbf{J}_{\mathcal{H}}^*\Gamma^* S_{\mathcal{K}}^* R_{*\rho} S_{\mathcal{K}}\Gamma\mathbf{J}_{\mathcal{H}}.
\end{aligned}$$

Let us show that $\mathcal{T}(\rho) = \mathbb{O}$.

By (4.10), we have

$$R_{*\rho} = S_{\mathcal{K}}^* R_{*\rho} S_{\mathcal{K}} + S_{\mathcal{K}}^* R_{*\rho} S_{\mathcal{K}} \Gamma \mathbf{J}_{\mathcal{H}} \mathbf{J}_{\mathcal{H}}^* \Gamma^* R_{*\rho} - S_{\mathcal{K}}^* R_{*\rho} \mathbf{J}_{\mathcal{K}} \mathbf{J}_{\mathcal{K}}^* \Gamma S_{\mathcal{H}}^* \Gamma^* R_{*\rho}.$$

Hence,

$$\begin{aligned} \mathbf{J}_{\mathcal{H}}^* \Gamma^* S_{\mathcal{K}}^* R_{*\rho} S_{\mathcal{K}} \Gamma \mathbf{J}_{\mathcal{H}} &= \mathbf{J}_{\mathcal{H}}^* \Gamma^* R_{*\rho} \Gamma \mathbf{J}_{\mathcal{H}} \\ &- \mathbf{J}_{\mathcal{H}}^* \Gamma^* S_{\mathcal{K}}^* R_{*\rho} S_{\mathcal{K}} \Gamma \mathbf{J}_{\mathcal{H}} \mathbf{J}_{\mathcal{H}}^* \Gamma^* R_{*\rho} \Gamma \mathbf{J}_{\mathcal{H}} \\ &+ \mathbf{J}_{\mathcal{H}}^* \Gamma^* S_{\mathcal{K}}^* R_{*\rho} \mathbf{J}_{\mathcal{K}} \mathbf{J}_{\mathcal{K}}^* \Gamma S_{\mathcal{H}}^* \Gamma^* R_{*\rho} \Gamma \mathbf{J}_{\mathcal{H}}. \end{aligned} \tag{4.18}$$

Using (4.12), we obtain from (4.18)

$$\begin{aligned} \mathbf{J}_{\mathcal{H}}^* \Gamma^* S_{\mathcal{K}}^* R_{*\rho} S_{\mathcal{K}} \Gamma \mathbf{J}_{\mathcal{H}} &= -I + \rho^2 G_\rho^{-2} + \mathbf{J}_{\mathcal{H}}^* \Gamma^* S_{\mathcal{K}}^* R_{*\rho} S_{\mathcal{K}} \Gamma \mathbf{J}_{\mathcal{H}} \\ &- \rho^2 \mathbf{J}_{\mathcal{H}}^* \Gamma^* S_{\mathcal{K}}^* R_{*\rho} S_{\mathcal{K}} \Gamma \mathbf{J}_{\mathcal{H}} G_\rho^{-2} \\ &- \mathbf{J}_{\mathcal{H}}^* \Gamma^* S_{\mathcal{K}}^* R_{*\rho} \mathbf{J}_{\mathcal{K}} \mathbf{J}_{\mathcal{K}}^* \Gamma S_{\mathcal{H}}^* \mathbf{J}_{\mathcal{H}} \\ &+ \rho^2 \mathbf{J}_{\mathcal{H}}^* \Gamma^* S_{\mathcal{K}}^* R_{*\rho} \mathbf{J}_{\mathcal{K}} \mathbf{J}_{\mathcal{K}}^* \Gamma S_{\mathcal{H}}^* R_\rho \mathbf{J}_{\mathcal{H}}. \end{aligned}$$

Since obviously, $S_{\mathcal{H}}^* \mathbf{J}_{\mathcal{H}} = \mathbb{O}$, it follows that

$$\rho^2 \mathbf{J}_{\mathcal{H}}^* \Gamma^* S_{\mathcal{K}}^* R_{*\rho} S_{\mathcal{K}} \Gamma \mathbf{J}_{\mathcal{H}} = -G_\rho^2 + \rho^2 I + \rho^2 \mathbf{J}_{\mathcal{H}}^* \Gamma^* S_{\mathcal{K}}^* R_{*\rho} \mathbf{J}_{\mathcal{K}} \mathbf{J}_{\mathcal{K}}^* \Gamma S_{\mathcal{H}}^* R_\rho \mathbf{J}_{\mathcal{H}} G_\rho^2.$$

Dividing this equality by ρ^2 and applying (4.6), we see that $\mathcal{T}(\rho) = \mathbb{O}$.

Let us show that for $\lambda \in (\|\Gamma\|, \rho)$,

$$(-\mathcal{T}(\lambda)x, x) \geq \delta \|x\|^2, \quad x \in \mathcal{H}, \tag{4.19}$$

for some positive number δ. Indeed, it follows from the spectral theorem that

$$\big((R_{*\lambda} - R_{*\rho})f, f\big) \geq \frac{\rho^2 - \lambda^2}{\lambda^2 \rho^2} \|f\|^2, \quad f \in \ell^2(\mathcal{K}), \quad \lambda \in (\|\Gamma\|, \rho).$$

Now let $f = (\mathbf{J}_{\mathcal{K}} \Omega + S_{\mathcal{K}} \Gamma \mathbf{J}_{\mathcal{H}})x$. Then

$$\big((R_{*\lambda} - R_{*\rho})f, f\big) = \big(\mathcal{T}(\rho) - \mathcal{T}(\lambda))x, x\big) = (-\mathcal{T}(\lambda)x, x)$$

and

$$\begin{aligned} \|f\|_{\ell^2(\mathcal{K})} &= \|(\mathbf{J}_{\mathcal{K}} \Omega + S_{\mathcal{K}} \Gamma \mathbf{J}_{\mathcal{H}})x\|_{\ell^2(\mathcal{K})} \\ &= \|R_{*\rho}^{-1/2} R_{*\rho}^{1/2} (\mathbf{J}_{\mathcal{K}} \Omega + S_{\mathcal{K}} \Gamma \mathbf{J}_{\mathcal{H}})x\|_{\ell^2(\mathcal{K})} \\ &\geq \|R_{*\rho}^{1/2}\|^{-1} \|R_{*\rho}^{1/2} (\mathbf{J}_{\mathcal{K}} \Omega + S_{\mathcal{K}} \Gamma \mathbf{J}_{\mathcal{H}})x\|_{\ell^2(\mathcal{K})} \\ &= \|R_{*\rho}^{1/2}\|^{-1} \Big| \big((\mathbf{J}_{\mathcal{K}} \Omega + S_{\mathcal{K}} \Gamma \mathbf{J}_{\mathcal{H}})^* R_{*\rho} (\mathbf{J}_{\mathcal{K}} \Omega + S_{\mathcal{K}} \Gamma \mathbf{J}_{\mathcal{H}})x, x\big) \Big|^{1/2} \\ &= \|R_{*\rho}^{1/2}\|^{-1} \Big| \big((I - \mathcal{T}(\rho))x, x\big) \Big|^{1/2} = \|R_{*\rho}^{1/2}\|^{-1} \|x\|_{\mathcal{H}}, \end{aligned}$$

which implies (4.19). Thus $\mathcal{T}(\lambda)$ is invertible for $\lambda \in (\|\Gamma\|, \rho)$. ■

It follows from Theorem 4.6 that if $\tilde{\Gamma}$ is a canonical one-step ρ-extension of Γ, then $\|\tilde{\Gamma}\| = \rho$.

Now we can show that a canonical one-step ρ-extension has a unique one-step extension of minimal norm.

Theorem 4.7. *Let $\rho > \|\Gamma\|$ and let $\tilde{\Gamma}$ be a canonical one step ρ-extension of Γ. Then $\tilde{\Gamma}$ has a unique one-step ρ-extension.*

Proof. This is an immediate consequence of Corollary 4.4 and the Remark followed by the proof of Theorem 4.5. ■

Complete ρ-Extensions

If $\Gamma = \{\Omega_{j+k}\}_{j,k\geq 0} : \ell^2(\mathcal{H}) \to \ell^2(\mathcal{K})$ is a vectorial Hankel operator such that $\|\Gamma\| \leq \rho$ and $\{\Omega_j\}_{j\in\mathbb{Z}}$ is an extension of the sequence $\{\Omega_j\}_{j\geq 0}$ such that $\Omega_j \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $j \in \mathbb{Z}$, and the operator

$$\Gamma_{\#} = \{\Omega_{j+k}\}_{j,k\geq\in\mathbb{Z}} : \ell^2_{\mathbb{Z}}(\mathcal{H}) \to \ell^2_{\mathbb{Z}}(\mathcal{K})$$

has norm at most ρ, we say that $\Gamma_{\#}$ is a *complete ρ-extension* (or simply *ρ-extension*) of Γ. It is evident that $\|\Gamma_{\#}\| \geq \|\Gamma\|$. We have proved in §2.2 that each bounded vectorial Hankel operator Γ has a complete extension of norm $\|\Gamma\|$. Moreover, in §2.2 a complete extension has been constructed by constructing first a one-step extension, then a one step extension of the one-step extension, etc. It turns out that a complete extension of norm $\|\Gamma\|$ is unique if and only if a one-step extension of norm $\|\Gamma\|$ is unique.

Theorem 4.8. *Let $\Gamma = \{\Omega_{j+k}\}_{j,k\geq 0}$ be a bounded block Hankel matrix. The following are equivalent:*

(i) Γ *has a unique one-step extension of minimal norm;*

(ii) Γ *has a unique complete extension of minimal norm.*

Proof. The implication (ii)$\Rightarrow$(i) is obvious. Let us prove that (i)$\Rightarrow$(ii). It is sufficient to show that if Γ has a unique one-step extension $\tilde{\Gamma}$ of norm $\|\Gamma\|$, then $\tilde{\Gamma}$ also has a unique one-step extension of minimal norm. Let $\tilde{\tilde{\Gamma}}$ be a one-step extension of $\tilde{\Gamma}$ such that $\left\|\tilde{\tilde{\Gamma}}\right\| = \|\Gamma\|$. Since $\Gamma = S^*_{\mathcal{K}}\tilde{\Gamma}$, we have

$$\Gamma^*\Gamma = \tilde{\Gamma}^* S_{\mathcal{K}} S^*_{\mathcal{K}} \tilde{\Gamma} = \tilde{\Gamma}^*\tilde{\Gamma} - \tilde{\Gamma}^* \mathbf{J}_{\mathcal{K}} \mathbf{J}^*_{\mathcal{K}} \tilde{\Gamma} \geq \tilde{\Gamma}^*\tilde{\Gamma}.$$

Hence, $(\rho^2 I - \Gamma^*\Gamma)^{-1} \leq (\rho^2 I - \tilde{\Gamma}^*\tilde{\Gamma})^{-1}$, $\rho > \|\Gamma\|$. Consequently,

$$G_\rho^{-2} = \mathbf{J}^*_{\mathcal{H}}(\rho^2 I - \Gamma^*\Gamma)^{-1}\mathbf{J}_{\mathcal{H}} \leq \mathbf{J}^*_{\mathcal{H}}(\rho^2 I - \tilde{\Gamma}^*\tilde{\Gamma})^{-1}\mathbf{J}_{\mathcal{H}} \stackrel{\text{def}}{=} \tilde{G}_\rho^{-2}.$$

It follows that $\tilde{G}^2_\rho \leq G^2_\rho$. By the Heinz inequality (see Appendix 1.7), $\tilde{G}_\rho \leq G_\rho$. If we interchange the roles of Γ and Γ^*, we obtain the inequality $\tilde{G}_{*\rho} \leq G_{*\rho}$, where $\tilde{G}_{*\rho} \stackrel{\text{def}}{=} \left(\mathbf{J}^*_{\mathcal{K}}(\rho^2 I - \tilde{\Gamma}^*\tilde{\Gamma})^{-1}\mathbf{J}_{\mathcal{K}}\right)^{-1/2}$. The result follows now from Theorem 4.3 (see conditions (i) and (ii)). ■

Theorems 4.3 and 4.8 give a necessary and sufficient condition for the uniqueness of a complete extension of minimal norm. Note that in the scalar case the same result has been proved in §5.1 (see Theorem 1.5).

We can obtain now an analog of the uniqueness result in Theorem 1.1.4 for vectorial Hankel operators.

Corollary 4.9. *Suppose that ρ^2 is an eigenvalue of $\Gamma^*\Gamma$ and* (4.8) *holds. Then Γ has a unique extension of minimal norm.*

Proof. This is an immediate consequence of Corollary 4.4 and Theorem 4.8. ■

Note that another sufficient condition for uniqueness can be obtained if we interchange the roles of Γ and Γ^*.

We now consider a condition stronger than (4.8). Namely, we assume that

$$\mathbf{J}_{\mathcal{H}}^* \big| \operatorname{Ker}(\|\Gamma\|^2 I - \Gamma^*\Gamma)$$

is a one-to-one map of the eigenspace $\operatorname{Ker}(\|\Gamma\|^2 I - \Gamma^*\Gamma)$ of $\Gamma^*\Gamma$ onto $\mathcal{H}$. Let $V : \mathcal{H} \to \operatorname{Ker}(\|\Gamma\|^2 I - \Gamma^*\Gamma)$ be the inverse of this operator. We consider V as an operator from $\mathcal{H}$ to $\ell^2(\mathcal{H})$. Let $W : \mathcal{H} \to \ell^2(\mathcal{K})$ be the operator defined by

$$W = \frac{1}{\rho}\Gamma V.$$

Clearly,

$$\Gamma V = \rho W \quad \text{and} \quad \Gamma^* W = \frac{1}{\rho}\Gamma^*\Gamma V = \rho V. \tag{4.20}$$

We denote by $\Gamma_\#$ the complete extension $\Gamma_\#$ of norm $\|\Gamma\|$, which is unique by Corollary 4.9. Then $\Gamma_\# = \{\Omega_{j+k}\}_{j,k\in\mathbb{Z}}$, where $\Omega_j = \hat{\Xi}(j)$ for an operator function $\Xi \in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ such that $\|\Xi\|_{L^\infty} = \|\Gamma\|$ (see §2.2).

As usual we identify in a natural way the spaces $\ell^2(\mathcal{H})$ and $\ell^2(\mathcal{K})$ with the Hardy classes $H^2(\mathcal{H})$ and $H^2(\mathcal{K})$. We also identify the spaces $\ell^2_{\mathbb{Z}}(\mathcal{H})$ and $\ell^2_{\mathbb{Z}}(\mathcal{K})$ with $L^2(\mathcal{H})$ and $L^2(\mathcal{K})$. Under this identification we can write

$$(\Gamma_\# f)(\zeta) = \Xi(\zeta) f\big(\bar{\zeta}\big), \quad \zeta \in \mathbb{T}, \quad f \in L^2(\mathcal{H}).$$

We have

$$\|\mathbb{P}_+\Gamma_\# Vx\| = \|\Gamma Vx\| = \rho\|Vx\| = \|\tilde{\Gamma}\| \cdot \|Vx\|, \quad x \in \mathcal{H}.$$

Hence, $\Gamma V = \mathbb{P}_+\Gamma_\# V = \Gamma_\# V$. Similarly, $\Gamma^* W = \mathbb{P}_+\Gamma_\#^* W = \Gamma_\#^* W$. Thus

$$\Gamma_\# V = \rho W \quad \text{and} \quad \Gamma_\#^* W = \rho V. \tag{4.21}$$

Let $Vx = \{V_j x\}_{j\ge 0}$ and $Wx = \{W_j x\}_{j\ge 0}$, where $V_j : \mathcal{H} \to \mathcal{H}$ and $W_j : \mathcal{H} \to \mathcal{K}$ are the operators defined by

$$V_j = \mathbf{J}_{\mathcal{H}}^*(S_{\mathcal{H}}^*)^j V \quad \text{and} \quad W_j = \mathbf{J}_{\mathcal{K}}^*(S_{\mathcal{K}}^*)^j W.$$

The following identity holds:

$$V^* S_{\mathcal{H}}^j V = W^*(S_{\mathcal{K}}^*)^j W. \tag{4.22}$$

Indeed,

$$W^*(S_{\mathcal{K}}^*)^j W = \frac{1}{\rho} W^*(S_{\mathcal{K}}^*)^j \Gamma V = \frac{1}{\rho} W^*\Gamma S_{\mathcal{H}}^j V = V^* S_{\mathcal{H}}^j V.$$

It follows from the definition of V that $V_0 = I_{\mathcal{H}}$.

Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be the operator functions defined by

$$\boldsymbol{V}x = \sum_{j\ge 0} z^j V_j x \quad \text{and} \quad \boldsymbol{W}x = \sum_{j\ge 0} z^j W_j x, \quad x \in \mathcal{H}.$$

It is easy to see that (4.20) and (4.21) can be rewritten in the following way:

$$\mathbb{P}_+ \Xi \boldsymbol{V}(\bar{z}) = \rho \boldsymbol{W}, \quad \mathbb{P}_+ \Xi^* \boldsymbol{W}(\bar{z}) = \rho \boldsymbol{V}$$

and

$$\Xi \boldsymbol{V}(\bar{z}) = \rho \boldsymbol{W}, \quad \Xi^* \boldsymbol{W}(\bar{z}) = \rho \boldsymbol{V}. \tag{4.23}$$

For x and y in $\mathcal{H}$ we consider the functions $v_{x,y} = \big(\boldsymbol{V}^*(z)\boldsymbol{V}(z)x, y\big)$ and $w_{x,y} = \big(\boldsymbol{W}^*(z)\boldsymbol{W}(z)x, y\big)$ on the unit circle. We have

$$\hat{v}_{x,y}(j) = \int_{\mathbb{T}} \big(\boldsymbol{V}(\zeta)x, \zeta^j \boldsymbol{V}(\zeta)y\big) d\boldsymbol{m}(\zeta) = (Vx, S_{\mathcal{H}}^j Vy) = (x, V^* S_{\mathcal{H}}^j Vy),$$

for $j \ge 0$ and

$$\hat{v}_{x,y}(j) = \int_{\mathbb{T}} \big(\zeta^{-j} \boldsymbol{V}(\zeta)x, \boldsymbol{V}(\zeta)y\big) d\boldsymbol{m}(\zeta) = (S_{\mathcal{H}}^{-j} Vx, Vy) = (V^* S_{\mathcal{H}}^{-j} Vx, y),$$

for $j \le 0$. Similarly,

$$\hat{w}_{x,y}(j) = \int_{\mathbb{T}} \big(\boldsymbol{W}(\zeta)x, \zeta^j \boldsymbol{W}(\zeta)y\big) d\boldsymbol{m}(\zeta) = \big(W^*(S_{\mathcal{H}}^*)^j Wx, y\big), \quad j \ge 0,$$

and

$$\hat{w}_{x,y}(j) = \int_{\mathbb{T}} \big(\zeta^{-j} \boldsymbol{W}(\zeta)x, \boldsymbol{W}(\zeta)y\big) d\boldsymbol{m}(\zeta) = \big(x, W^*(S_{\mathcal{H}}^*)^{-j} Wy\big), \quad j \le 0.$$

By (4.22), we obtain

$$\hat{w}_{x,y}(j) = \hat{v}_{x,y}(-j), \quad j \in \mathbb{Z}.$$

This means that $w_{x,y}(z) = v_{x,y}(\bar{z})$, and so

$$\boldsymbol{W}^*(z)\boldsymbol{W}(z) = \boldsymbol{V}^*(\bar{z})\boldsymbol{V}(\bar{z}). \tag{4.24}$$

If the space $\mathcal{H}$ is finite-dimensional, $\det \boldsymbol{V}$ belongs to the Hardy class $H^{2/\dim \mathcal{H}}$, and since $\det \boldsymbol{V}(0) = \det X_0 = 1$, it is a nonzero function. Therefore for almost all $\zeta \in \mathbb{T}$ the operator $\boldsymbol{V}(\zeta)$ is invertible, and by (4.23), we have

$$\Xi(\zeta) = \rho \boldsymbol{W}(\zeta)\boldsymbol{V}^{-1}\big(\bar{\zeta}\big), \quad \zeta \in \mathbb{T}.$$

It now follows from (4.23) and (4.24) that the function $\frac{1}{\rho}\Xi$ takes isometric values almost everywhere on $\mathbb{T}$.

Suppose now that Γ is a Hankel operator from $\ell^2(\mathcal{H})$ to $\ell^2(\mathcal{K})$ and $\rho > \|\Gamma\|$ (we do not assume here that $\dim \mathcal{H} < \infty$). Let $\tilde{\Gamma}$ be a canonical one-step ρ-extension that corresponds to an isometry $\mathcal{E} : \mathcal{H} \to \mathcal{K}$ in (4.4). By Theorem 4.5 and the Remark following its statement, $\mathbf{J}_{\mathcal{H}}^*$ is a one-to-one map of $\operatorname{Ker}(\rho^2 I - \tilde{\Gamma}^*\tilde{\Gamma})$ onto $\mathcal{H}$, and formulas (4.15) hold. Now the operators $P_\rho + Q_{*\rho}$ and $Q_\rho + P_{*\rho}$ can play the roles of the operators V and W above. Note, however, that in this case

$$(P_\rho + Q_{*\rho})_0 \stackrel{\text{def}}{=} \mathbf{J}_{\mathcal{H}}^*(P_\rho + Q_{*\rho})$$

does not have to be equal to $I_{\mathcal{H}}$, but we do not need it! We have used only the fact that V_0 is invertible on $\mathcal{H}$, and the fact that $(P_\rho + Q_{*\rho})_0$ is invertible on $\mathcal{H}$ is an immediate consequence of the definition of P_ρ and $Q_{*\rho}$.

As we have already observed, $\tilde{\Gamma}$ has a unique complete extension of norm ρ, which we denote by $\tilde{\Gamma}_\#$. It is easy to see that $\Gamma_\# \stackrel{\text{def}}{=} \tilde{\Gamma}_\# \mathcal{S}_{\mathcal{H}} = \mathcal{S}_{\mathcal{K}}^* \tilde{\Gamma}_\#$ is a complete ρ-extension of Γ. (Recall that $\mathcal{S}_{\mathcal{H}}$ and $\mathcal{S}_{\mathcal{K}}$ are bilateral shifts on $\ell^2_{\mathbb{Z}}(\mathcal{H})$ and $\ell^2_{\mathbb{Z}}(\mathcal{K})$.) Thus by (4.15), we have

$$\Gamma_\#(P_\rho + Q_{*\rho}\mathcal{E}) = \rho \mathcal{S}_{\mathcal{K}}^*(Q_\rho + P_{*\rho}\mathcal{E}), \quad \Gamma_\#^* \mathcal{S}_{\mathcal{K}}^*(Q_\rho + P_{*\rho}\mathcal{E}) = \rho(P_\rho + Q_{*\rho}\mathcal{E}). \tag{4.25}$$

We define the operator functions $\boldsymbol{P}_\rho$, $\boldsymbol{Q}_\rho$, $\boldsymbol{P}_{*\rho}$, and $\boldsymbol{Q}_{*\rho}$ by

$$\boldsymbol{P}_\rho x = \sum_{j \ge 0} z^j (P_\rho)_j x, \quad \boldsymbol{Q}_\rho x = \sum_{j \ge 0} z^j (Q_\rho)_j x, \quad x \in \mathcal{H},$$

and

$$\boldsymbol{P}_{*\rho} x = \sum_{j \ge 0} z^j (P_{*\rho})_j x, \quad \boldsymbol{Q}_{*\rho} x = \sum_{j \ge 0} z^j (Q_{*\rho})_j x, \quad x \in \mathcal{H},$$

where as in the case of operators V and W above,

$$P_\rho x = \{(P_\rho)_j x\}_{j \ge 0}, \quad Q_\rho x = \{(Q_\rho)_j x\}_{j \ge 0},$$

$$P_{*\rho} x = \{(P_{*\rho})_j x\}_{j \ge 0}, \quad \text{and} \quad Q_{*\rho} x = \{(Q_{*\rho})_j x\}_{j \ge 0}.$$

Let Ξ be an operator function in $L^\infty(\mathcal{B}(\mathcal{H}, \mathcal{K}))$ such that $\hat{\Xi}(j) = \Omega_j$, $j \ge 0$. As we have already mentioned, $\|\Xi\|_{L^\infty} = \rho$ and $(\Gamma_\# f)(z) = \Xi(z) f(\bar{z})$, $f \in L^2(\mathcal{H})$. We obtain from (4.25)

$$\Xi(z)\big(\boldsymbol{P}_\rho(\bar{z}) + \boldsymbol{Q}_{*\rho}(\bar{z})\mathcal{E}\big) = \rho \bar{z}\big(\boldsymbol{Q}_\rho(z) + \boldsymbol{P}_{*\rho}(z)\mathcal{E}\big). \tag{4.26}$$

and

$$\Xi^*(z) = \bar{z}\big(\boldsymbol{Q}_\rho(z) + \boldsymbol{P}_{*\rho}(z)\mathcal{E}\big) = \rho\big(\boldsymbol{P}_\rho(\bar{z}) + \boldsymbol{Q}_{*\rho}(\bar{z})\mathcal{E}\big).$$

Again, if $\dim \mathcal{H} < \infty$, we obtain from (4.26) the following formula for Ξ:

$$\Xi(z) = \rho \bar{z}\big(\boldsymbol{Q}_\rho(z) + \boldsymbol{P}_{*\rho}(z)\mathcal{E}\big)\big(\boldsymbol{P}_\rho(\bar{z}) + \boldsymbol{Q}_{*\rho}(\bar{z})\mathcal{E}\big)^{-1}. \tag{4.27}$$

Such complete ρ-extensions are called *canonical*.

Properties of the Functions $\boldsymbol{P}_\rho$, $\boldsymbol{P}_{*\rho}$, $\boldsymbol{Q}_\rho$, and $\boldsymbol{Q}_{*\rho}$

We establish here some important properties of $\boldsymbol{P}_\rho$, $\boldsymbol{P}_{*\rho}$, $\boldsymbol{Q}_\rho$, and $\boldsymbol{Q}_{*\rho}$ which we are going to use later in this section.

Theorem 4.10. *Suppose that $\rho > \|\Gamma\|$. Then the following identities hold for almost all $\zeta \in \mathbb{T}$:*

$$\boldsymbol{P}_\rho^*(\zeta)\boldsymbol{P}_\rho(\zeta) - \boldsymbol{Q}_\rho^*(\bar{\zeta})\boldsymbol{Q}_\rho(\bar{\zeta}) = I_{\mathcal{H}}; \tag{4.28}$$

$$\boldsymbol{P}_{*\rho}^*(\bar{\zeta})\boldsymbol{P}_{*\rho}(\bar{\zeta}) - \boldsymbol{Q}_{*\rho}^*(\zeta)\boldsymbol{Q}_{*\rho}(\zeta) = I_{\mathcal{K}}; \tag{4.29}$$

$$\boldsymbol{P}_{*\rho}^*(\bar{\zeta})\boldsymbol{Q}_\rho(\bar{\zeta}) - \boldsymbol{Q}_{*\rho}^*(\zeta)\boldsymbol{P}_\rho(\zeta) = \mathbb{O}. \tag{4.30}$$

Proof. Let us establish the following formulas:

$$P_\rho^* S_{\mathcal{H}}^j P_\rho - Q_\rho^*(S_{\mathcal{K}}^*)^j Q_\rho = \begin{cases} I_{\mathcal{H}}, & j = 0, \\ \mathbb{O}, & j > 0, \end{cases} \tag{4.31}$$

$$P_{*\rho}^* S_{\mathcal{K}}^j P_{*\rho} - Q_{*\rho}^*(S_{\mathcal{H}}^*)^j Q_{*\rho} = \begin{cases} I_{\mathcal{K}}, & j = 0, \\ \mathbb{O}, & j > 0, \end{cases} \tag{4.32}$$

and

$$P_{*\rho}^* S_{\mathcal{K}}^j Q_\rho - Q_{*\rho}^*(S_{\mathcal{H}}^*)^j P_\rho = P_{*\rho}^*(S_{\mathcal{K}}^*)^j Q_\rho - Q_{*\rho}^* S_{\mathcal{H}}^j P_\rho = \mathbb{O}, \quad j \ge 0. \tag{4.33}$$

It follows easily from definition that

$$S_{\mathcal{K}}\Gamma P_\rho = \rho Q_\rho \quad \text{and} \quad \Gamma^* S_{\mathcal{K}}^* Q_\rho = \rho P_\rho - \mathbf{J}_{\mathcal{H}} G_\rho.$$

Hence, $P_\rho^* = \frac{1}{\rho}(G_\rho \mathbf{J}_{\mathcal{H}}^* + Q_\rho^* S_{\mathcal{K}}\Gamma)$, and so for $j > 0$ we have

$$\begin{aligned} P_\rho^* S_{\mathcal{H}}^j P_\rho &= \frac{1}{\rho}(G_\rho \mathbf{J}_{\mathcal{H}}^* + Q_\rho^* S_{\mathcal{K}}\Gamma) S_{\mathcal{H}}^j P_\rho = \frac{1}{\rho} Q_\rho^* S_{\mathcal{K}}\Gamma S_{\mathcal{H}}^j P_\rho \\ &= \frac{1}{\rho} Q_\rho^* S_{\mathcal{K}}(S_{\mathcal{K}}^*)^j \Gamma P_\rho = \frac{1}{\rho} Q_\rho^* S_{\mathcal{K}} S_{\mathcal{K}}^* (S_{\mathcal{K}}^*)^j S_{\mathcal{K}}\Gamma P_\rho = Q_\rho^*(S_{\mathcal{K}}^*)^j Q_\rho. \end{aligned}$$

On the other hand,

$$\begin{aligned} P_\rho^* P_\rho - Q_\rho^* Q_\rho &= P_\rho^* P_\rho - \frac{1}{\rho^2} P_\rho^* \Gamma^* \Gamma P_\rho \\ &= \frac{1}{\rho^2} P_\rho^* (\rho^2 I - \Gamma^*\Gamma) P_\rho = G_\rho \mathbf{J}_{\mathcal{H}}^* R_\rho \mathbf{J}_{\mathcal{H}} G_\rho = I_{\mathcal{H}}. \end{aligned}$$

This proves (4.31).

Using the identities

$$S_{\mathcal{H}}\Gamma^* P_{*\rho} = \rho Q_{*\rho} \quad \text{and} \quad \Gamma S_{\mathcal{H}}^* Q_{*\rho} = \rho P_{*\rho} - \mathbf{J}_{\mathcal{K}} G_{*\rho},$$

we can prove in the same way (4.32).

Let us establish (4.33). We have for $j \geq 0$

$$\begin{aligned}
P_{*\rho}^* S_{\mathcal{K}}^j Q_\rho &= \frac{1}{\rho}(G_{*\rho}\mathbf{J}_{\mathcal{K}}^* + Q_{*\rho}^* S_{\mathcal{H}}\Gamma^*)S_{\mathcal{K}}^j Q_\rho = \frac{1}{\rho} Q_{*\rho}^* S_{\mathcal{H}}\Gamma^* S_{\mathcal{K}}^{j+1}\Gamma R_\rho \mathbf{J}_{\mathcal{H}} G_\rho \\
&= \frac{1}{\rho} Q_{*\rho}^* S_{\mathcal{H}}(S_{\mathcal{H}}^*)^{j+1}\Gamma^*\Gamma R_\rho \mathbf{J}_{\mathcal{H}} G_\rho = \frac{1}{\rho} Q_{*\rho}^*(S_{\mathcal{H}}^*)^j \Gamma^*\Gamma R_\rho \mathbf{J}_{\mathcal{H}} G_\rho \\
&= \frac{1}{\rho} Q_{*\rho}^*(S_{\mathcal{H}}^*)^j(\Gamma^*\Gamma - \rho^2 I + \rho^2 I)R_\rho \mathbf{J}_{\mathcal{H}} G_\rho = Q_{*\rho}^*(S_{\mathcal{H}}^*)^j P_\rho,
\end{aligned}$$

since obviously,

$$\frac{1}{\rho} Q_{*\rho}^*(S_{\mathcal{H}}^*)^j \mathbf{J}_{\mathcal{H}} G_\rho = \mathbb{O}.$$

Finally, for $j > 0$ we obtain

$$\begin{aligned}
Q_{*\rho}^* S_{\mathcal{H}}^j P_\rho &= \frac{1}{\rho} P_{*\rho}^*\Gamma S_{\mathcal{H}}^{j-1} P_\rho = \frac{1}{\rho} P_{*\rho}^*(S_{\mathcal{K}}^*)^{j-1}\Gamma P_\rho \\
&= \frac{1}{\rho} P_{*\rho}^*(S_{\mathcal{K}}^*)^j S_{\mathcal{K}}\Gamma P_\rho = P_{*\rho}^*(S_{\mathcal{K}}^*)^j Q_\rho,
\end{aligned}$$

which completes the proof of (4.33).

Now it is easy to deduce (4.28), (4.29), and (4.30) from (4.31), (4.32), and (4.33). Let us show how to obtain, for example, (4.28). We compare the Fourier coefficients of the functions $\boldsymbol{P}_\rho^*(z)\boldsymbol{P}_\rho(z)$ and $\boldsymbol{Q}_\rho^*(\bar{z})\boldsymbol{Q}_\rho(\bar{z})$. Since both functions take self-adjoint values, it is sufficient to compare their jth Fourier cofficients with $j \geq 0$. Let $x,\ y \in \mathcal{H}$ and $j > 0$. Then

$$\begin{aligned}
\int_{\mathbb{T}} \big(\boldsymbol{P}_\rho^*(\zeta)\boldsymbol{P}_\rho(\zeta)x, y\big)\bar{\zeta}^j d\boldsymbol{m}(\zeta) &= \int_{\mathbb{T}} \big(\boldsymbol{P}_\rho^*(\zeta)\boldsymbol{P}_\rho(\zeta)x, \zeta^j \boldsymbol{P}_\rho(\zeta)y\big) d\boldsymbol{m}(\zeta) \\
&= (x, P_\rho^* S_{\mathcal{H}}^j P_\rho y) = (x, Q_\rho^*(S_{\mathcal{K}}^*)^j Q_\rho y) \\
&= \int_{\mathbb{T}} \big(\boldsymbol{Q}_\rho^*(\zeta)\boldsymbol{Q}_\rho(\zeta)x, y\big)\zeta^j d\boldsymbol{m}(\zeta) \\
&= \int_{\mathbb{T}} \big(\boldsymbol{Q}_\rho^*(\bar{\zeta})\boldsymbol{Q}_\rho(\bar{\zeta})x, y\big)\bar{\zeta}^j d\boldsymbol{m}(\zeta).
\end{aligned}$$

For $j = 0$ we have

$$\begin{aligned}
\int_{\mathbb{T}} \big(\boldsymbol{P}_\rho^*(\zeta)\boldsymbol{P}_\rho(\zeta)x, y\big) d\boldsymbol{m}(\zeta) &- \int_{\mathbb{T}} \big(\boldsymbol{Q}_\rho^*(\bar{\zeta})\boldsymbol{Q}_\rho(\bar{\zeta})x, y\big) d\boldsymbol{m}(\zeta) \\
&= (P_\rho^* P_\rho x, y) - (Q_\rho^* Q_\rho x, y) = (x, y).
\end{aligned}$$

Formulas (4.29) and (4.30) can be deduced from (4.32) and (4.33) in a similar way. ■

Suppose now that the spaces $\mathcal{H}$ and $\mathcal{K}$ are finite-dimensional and $\dim \mathcal{H} \le \dim \mathcal{K}$. Consider the block matrix function U_ρ on $\mathbb{T}$ defined by

$$U_\rho(z) = \begin{pmatrix} \boldsymbol{P}_\rho(z) & \boldsymbol{Q}_{*\rho}(z) \\ \boldsymbol{Q}_\rho(\bar{z}) & \boldsymbol{P}_{*\rho}(\bar{z}) \end{pmatrix} \tag{4.34}$$

and the matrix

$$J \stackrel{\text{def}}{=} \begin{pmatrix} I_{\mathcal{H}} & \mathbb{O} \\ \mathbb{O} & -I_{\mathcal{K}} \end{pmatrix}. \tag{4.35}$$

It is easy to see that (4.28)–(4.30) mean that

$$U_\rho^*(\zeta) J U_\rho(\zeta) = J \quad \text{for almost all } \zeta \in \mathbb{T}. \tag{4.36}$$

In other words, the matrix function U_ρ is *J-unitary*. It is easy to see that the matrix function U_ρ^* is also J-unitary. Indeed, by (4.36), $U_\rho(\zeta) = J(U_\rho^*(\zeta))^{-1}J$, and so

$$U_\rho(\zeta) J U_\rho^*(\zeta) = J(U_\rho^*(\zeta))^{-1} J J U_\rho^*(\zeta) = J(U_\rho^*(\zeta))^{-1} U_\rho^*(\zeta) = J. \tag{4.37}$$

Considering all entries in (4.37), we obtain the following identities

$$\boldsymbol{P}_\rho(\zeta)\boldsymbol{P}_\rho^*(\zeta) - \boldsymbol{Q}_{*\rho}(\zeta)\boldsymbol{Q}_{*\rho}^*(\zeta) = I_{\mathcal{H}}; \tag{4.38}$$

$$\boldsymbol{P}_{*\rho}(\bar{\zeta})\boldsymbol{P}_{*\rho}^*(\bar{\zeta}) - \boldsymbol{Q}_\rho(\bar{\zeta})\boldsymbol{Q}_\rho^*(\bar{\zeta}) = I_{\mathcal{K}}; \tag{4.39}$$

$$\boldsymbol{P}_\rho(\zeta)\boldsymbol{Q}_\rho^*(\bar{\zeta}) - \boldsymbol{Q}_{*\rho}(\zeta)\boldsymbol{P}_{*\rho}^*(\bar{\zeta}) = \mathbb{O}. \tag{4.40}$$

It follows from (4.28) and (4.29) (as well as from (4.38) and (4.39)) that the operator functions $\boldsymbol{P}_\rho(z)$ and $\boldsymbol{P}_{*\rho}(\bar{z})$ are invertible. Moreover, $\|\boldsymbol{P}_\rho^{-1}(\zeta)\| \le 1$ and $\|\boldsymbol{P}_{*\rho}^{-1}(\bar{\zeta})\| \le 1$ almost everywhere on $\mathbb{T}$.

Consider now the operator function $\mathcal{X}_\rho$ on $\mathbb{T}$ defined by

$$\mathcal{X}_\rho(z) = \boldsymbol{P}_\rho^{-1}(z)\boldsymbol{Q}_{*\rho}(z) = \boldsymbol{Q}_\rho^*(\bar{z})(\boldsymbol{P}_{*\rho}^*)^{-1}(\bar{z}) \tag{4.41}$$

(see (4.40)).

Theorem 4.11. *The function $\mathcal{X}_\rho$ has the following properties:*

$$\mathcal{X}_\rho(0) = \mathbb{O}; \quad \|\mathcal{X}_\rho(\zeta)\| < 1, \quad \zeta \in \mathbb{D}; \quad \textit{and} \quad (1 - \|\mathcal{X}_\rho(z)\|)^{-1} \in L^1. \tag{4.42}$$

The operator functions $\boldsymbol{P}_\rho$, $\boldsymbol{P}_{*\rho}$, $\boldsymbol{Q}_\rho$, and $\boldsymbol{Q}_{*\rho}$ are uniquely determined by $\mathcal{X}_\rho$, and the functions $\boldsymbol{P}_\rho$ and $\boldsymbol{P}_{*\rho}$ satisfy

$$\boldsymbol{P}_\rho^{-1} \in H^\infty; \quad \boldsymbol{P}_{*\rho}^{-1}(\bar{z}) \in H^\infty.$$

Proof. The fact that $\mathcal{X}_\rho(0) = \mathbb{O}$ is an immediate consequence of the definition of $\boldsymbol{Q}_{*\rho}$. It is easy to see from (4.38) that for almost all $\zeta \in \mathbb{T}$

$$\|\mathcal{X}_\rho(\zeta)\|^2 = \|\mathcal{X}_\rho(\zeta)\mathcal{X}_\rho^*(\zeta)\| = \|I_{\mathcal{H}} - \boldsymbol{P}_\rho^{-1}(\zeta)(\boldsymbol{P}_\rho^*)^{-1}(\zeta)\| = 1 - \|\boldsymbol{P}_\rho(\zeta)\|^{-2} < 1.$$

Hence,

$$(1 - \|\mathcal{X}_\rho(\zeta)\|)^{-1} = (1 + \|\mathcal{X}_\rho(\zeta)\|)\|\boldsymbol{P}_\rho(\zeta)\|^2 \le 2\|\boldsymbol{P}_\rho(\zeta)\|^2, \quad \zeta \in \mathbb{T},$$

and so

$$\int_{\mathbb{T}} (1 - \|\mathcal{X}_\rho(\zeta)\|)^{-1} d\boldsymbol{m}(\zeta) \le 2 \int_{\mathbb{T}} \|\boldsymbol{P}_\rho(\zeta)\|^2 d\boldsymbol{m}(\zeta) < \infty.$$

This completes the proof of (4.42).

Now let $\mathcal{E}$ be an isometry from $\mathcal{H}$ to $\mathcal{K}$ and let

$$\Xi_{\mathcal{E}}(z) \stackrel{\text{def}}{=} \rho\bar{z}\big(\boldsymbol{Q}_\rho(z) + \boldsymbol{P}_{*\rho}(z)\mathcal{E}\big)\big(\boldsymbol{P}_\rho(\bar{z}) + \boldsymbol{Q}_{*\rho}(\bar{z})\mathcal{E}\big)^{-1}. \tag{4.43}$$

We have already shown that $\hat{\Xi}_{\mathcal{E}}(j) = \Omega_j$, $j \ge 0$. It follows from (4.28)–(4.30) that $\Xi_{\mathcal{E}}$ also admits the following representation:

$$\Xi_{\mathcal{E}}(z) = \rho\bar{z}\big(\boldsymbol{P}^*_{*\rho}(z) + \mathcal{E}\boldsymbol{Q}^*_\rho(z)\big)^{-1}\big(\boldsymbol{Q}^*_{*\rho}(\bar{z}) + \mathcal{E}\boldsymbol{P}^*_\rho(\bar{z})\big). \tag{4.44}$$

Moreover, if $\mathcal{E}$ is an arbitrary contractive operator from $\mathcal{H}$ to $\mathcal{K}$ (i.e., $\|\mathcal{E}\| \le 1$), then the operator functions defined by (4.43) and (4.44) coincide.

It is easy to see that if $\mathcal{E}_1$ and $\mathcal{E}_2$ are contractive operators from $\mathcal{H}$ to $\mathcal{K}$, and we use (4.43) for $\Xi_{\mathcal{E}_1}$ and (4.44) for $\Xi_{\mathcal{E}_2}$, then the following identity follows from (4.28)–(4.30):

$$\Xi_{\mathcal{E}_1}(z) - \Xi_{\mathcal{E}_2}(z) = \rho\bar{z}\big(\boldsymbol{P}^*_{*\rho}(z) + \mathcal{E}_2\boldsymbol{Q}^*_\rho(z)\big)^{-1}(\mathcal{E}_1 - \mathcal{E}_2)\big(\boldsymbol{P}_\rho(\bar{z}) + \boldsymbol{Q}_{*\rho}(\bar{z})\mathcal{E}_1\big)^{-1}. \tag{4.45}$$

If $\mathcal{E}_1$ and $\mathcal{E}_2$ are isometries, then it follows from (4.45) that

$$\bar{z}\big(\boldsymbol{P}^*_{*\rho}(z) + \mathcal{E}_2\boldsymbol{Q}^*_\rho(z)\big)^{-1}(\mathcal{E}_1 - \mathcal{E}_2) \in H^2_-(\mathcal{B}(\mathcal{H}, \mathcal{K}))$$

and

$$\bar{z}(\mathcal{E}_1 - \mathcal{E}_2)\big(\boldsymbol{P}_\rho(\bar{z}) + \boldsymbol{Q}_{*\rho}(\bar{z})\mathcal{E}_1\big)^{-1} \in H^2_-(\mathcal{B}(\mathcal{H}, \mathcal{K})).$$

If $\mathcal{E}_1$ is an arbitrary isometry, we can put $\mathcal{E}_2 = -\mathcal{E}_1$ and we find that

$$\bar{z}\big(\boldsymbol{P}_\rho(\bar{z}) + \boldsymbol{Q}_{*\rho}(\bar{z})\mathcal{E}_1\big)^{-1} \in H^2_-(\mathcal{B}(\mathcal{H})). \tag{4.46}$$

On the other hand, given an isometry $\mathcal{E}_2$ and an arbitrary vector b in $\mathcal{K}$, we can find an isometry $\mathcal{E}_1$ such that $b \in \operatorname{Range}(\mathcal{E}_1 - \mathcal{E}_1)$. Thus it is easy to see that

$$\bar{z}\big(\boldsymbol{P}^*_{*\rho}(z) + \mathcal{E}_2\boldsymbol{Q}^*_\rho(z)\big)^{-1} \in H^2_-(\mathcal{B}(\mathcal{K})).$$

Note that for finite-dimensional spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and a space X of functions on $\mathbb{T}$, we say that an operator function F belongs to $X(\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2))$ (or simply $F \in X$) if $(Fa, b) \in X$ for any $a \in \mathcal{H}_1$ and $b \in \mathcal{H}_2$.

For an isometry $\mathcal{E}$ from $\mathcal{H}$ to $\mathcal{K}$ we define the function $\mathcal{G}_{\mathcal{E}}$ by

$$\mathcal{G}_{\mathcal{E}}(z) = \big(\boldsymbol{P}_\rho(z) + \boldsymbol{Q}_{*\rho}(z)\mathcal{E}\big)^{-1}\big(\boldsymbol{P}_\rho(z) - \boldsymbol{Q}_{*\rho}(z)\mathcal{E}\big).$$

It is easy to see from (4.46) that $\mathcal{G}_{\mathcal{E}} \in H^1$. By the definition of $\mathcal{X}_\rho$, we have for almost all $\zeta \in \mathbb{T}$:

$$\mathcal{G}_{\mathcal{E}}(\zeta) = (I_{\mathcal{H}} + \mathcal{X}_\rho(\zeta)\mathcal{E})^{-1}(I_{\mathcal{H}} - \mathcal{X}_\rho(\zeta)\mathcal{E}).$$

Since $\|\mathcal{X}_\rho(\zeta)\| < 1$ for almost all $\zeta \in \mathbb{T}$, it is easy to see that

$$\mathcal{G}^*_{\mathcal{E}}(\zeta) + \mathcal{G}_{\mathcal{E}}(\zeta) > \mathbb{O}, \quad \text{a.e. on } \mathbb{T}.$$

Taking the Poisson integral, we find that

$$\mathcal{G}_{\mathcal{E}}^{*}(\zeta) + \mathcal{G}_{\mathcal{E}}(\zeta) > \mathbb{O}, \quad \zeta \in \mathbb{D}. \tag{4.47}$$

Define the operator function $\mathcal{Y}$ by

$$\mathcal{Y}(\zeta) = (I_{\mathcal{H}} + \mathcal{G}_{\mathcal{E}}(\zeta))^{-1}(I_{\mathcal{H}} - \mathcal{G}_{\mathcal{E}}(\zeta)), \quad \zeta \in \mathbb{D}.$$

It is well known (see e.g., Sz.-Nagy and Foias [1], Ch. IV, §4) and easy to verify that that (4.47) implies that $\|\mathcal{Y}(\zeta)\| < 1$, $\zeta \in \mathbb{D}$. It is also clear from the definition of $\mathcal{Y}$ that

$$\mathcal{G}_{\mathcal{E}}(\zeta) = (I_{\mathcal{H}} + \mathcal{Y}(\zeta))^{-1}(I_{\mathcal{H}} - \mathcal{Y}(\zeta)), \quad \zeta \in \mathbb{D}.$$

It follows that the boundary values of $\mathcal{Y}$ coincide with the function $\mathcal{X}_{\rho}\mathcal{E}$ almost everywhere on $\mathbb{T}$. Thus $\mathcal{X}_{\rho}\mathcal{E} \in H^{\infty}$. Since $\mathcal{E}$ is an arbitrary isometry, it follows that $\mathcal{X}_{\rho} \in H^{\infty}$. Next,

$$\begin{aligned}&(I_{\mathcal{H}} + \mathcal{X}_{\rho}(\zeta)\mathcal{E})\big(\boldsymbol{P}_{\rho}(\zeta) + \boldsymbol{Q}_{*\rho}(\zeta)\mathcal{E}\big)^{-1} \\ &= (I_{\mathcal{H}} + \boldsymbol{P}_{\rho}^{-1}(\zeta)\boldsymbol{Q}_{*\rho}(\zeta)\mathcal{E})\big(\boldsymbol{P}_{\rho}(\zeta) + \boldsymbol{Q}_{*\rho}(\zeta)\mathcal{E}\big)^{-1} = \boldsymbol{P}_{\rho}^{-1}(\zeta)\end{aligned}$$

for almost all $\zeta \in \mathbb{T}$. It follows that $\boldsymbol{P}_{\rho}^{-1} \in H^2$ and since it is bounded on $\mathbb{T}$, it follows that $\boldsymbol{P}_{\rho}^{-1} \in H^{\infty}$. Similarly, $\boldsymbol{P}_{*\rho}^{-1}(\bar{z}) \in H^{\infty}$.

Finally, by (4.41) and (4.38), we have for almost all $\zeta \in \mathbb{T}$

$$\mathcal{X}_{\rho}(\zeta)\mathcal{X}_{\rho}^{*}(\zeta) = \boldsymbol{P}_{\rho}^{-1}(\zeta)\boldsymbol{Q}_{*\rho}(\zeta)\boldsymbol{Q}_{*\rho}^{*}(\zeta)(\boldsymbol{P}^{*})_{\rho}^{-1}(\zeta) = I_{\mathcal{H}} - \boldsymbol{P}_{\rho}^{-1}(\zeta)(\boldsymbol{P}^{*})_{\rho}^{-1}(\zeta).$$

Hence,

$$I_{\mathcal{H}} - \mathcal{X}_{\rho}(\zeta)\mathcal{X}_{\rho}^{*}(\zeta) = \boldsymbol{P}_{\rho}^{-1}(\zeta)(\boldsymbol{P}^{*})_{\rho}^{-1}(\zeta), \quad \zeta \in \mathbb{T}.$$

Thus $\boldsymbol{P}_{\rho}^{-1}$ is a solution of the factorization problem

$$I_{\mathcal{H}} - \mathcal{X}_{\rho}\mathcal{X}_{\rho}^{*} = FF^{*}.$$

It is easy to see that this factorization problem has a unique solution F such that $F \in H^2$, $F^{-1} \in H^2$, and $F(0) > \mathbb{O}$. Indeed, if F_1 and F_2 are both solutions of the factorization problems satisfying the above conditions, we have

$$F_2^{-1}F_1 = F_2^{*}(F_1^{*})^{-1}.$$

The left-hand side of this equality belongs to H^1 while the right-hand side belongs to $\overline{H^1}$. Thus $F_2^{-1}F_1 = C$, where C is a constant operator. It follows that $F_2CC^{*}F_2^{*} = F_2F_2^{*}$, and so $CC^{*} = I$. Since, $C = F_2^{-1}(0)F_1(0)$, and both F_1 and F_2 are positive, we have

$$CC^{*} = F_2^{-1}(0)F_1(0)F_1(0)F_2^{-1}(0) = I.$$

Hence, $F_2^2(0) = F_1^2(0)$, which implies $C = I$.

Thus $\boldsymbol{P}_{\rho}^{-1}$ is uniquely determined by $\mathcal{X}_{\rho}$. Since $\mathcal{X}_{\rho}(z) = \boldsymbol{P}_{\rho}^{-1}(z)\boldsymbol{Q}_{*\rho}(z)$, $\boldsymbol{Q}_{*\rho}(z)$ is also uniquely determined by $\mathcal{X}_{\rho}$.

Similarly, it can be shown that

$$I_{\mathcal{K}} - \mathcal{X}_{\rho}^{*}(z)\mathcal{X}_{\rho}(z) = \boldsymbol{P}_{*\rho}^{-1}(\bar{z})(\boldsymbol{P}_{*\rho}^{*})^{-1}(\bar{z}),$$

and the functions $\boldsymbol{P}_{*\rho}$ and $\boldsymbol{Q}_{\rho}$ are uniquely determined by $\mathcal{X}_{\rho}$. ■

Lemma 4.12. *The following equality holds:*

$$\det \boldsymbol{P}_\rho(\zeta) = \det \boldsymbol{P}^*_{*\rho}(\bar{\zeta}), \quad \zeta \in \mathbb{D}. \tag{4.48}$$

Proof. Since, $\boldsymbol{P}_\rho \in H^2$, it follows that $\det \boldsymbol{P}_\rho \in H^{2/n}$. Next, $\boldsymbol{P}_\rho^{-1} \in H^\infty$, and so $(\det \boldsymbol{P}_\rho)^{-1} \in H^\infty$. Hence, $\det \boldsymbol{P}_\rho$ is an outer function. It follows from the definition of $\boldsymbol{P}_\rho$ that $\det \boldsymbol{P}_\rho(0) > 0$. Similarly, $\det \boldsymbol{P}^*_{*\rho}(\bar{z})$ is an outer function, and it takes a positive value at 0. By (4.38) and (4.29), we have for almost all $\zeta \in \mathbb{T}$

$$\begin{aligned} |\det \boldsymbol{P}_\rho(\zeta)|^2 &= \det(I_{\mathcal{H}} + \boldsymbol{Q}_{*\rho}(\zeta)\boldsymbol{Q}^*_{*\rho}(\zeta)) \\ &= \det(I_{\mathcal{K}} + \boldsymbol{Q}^*_{*\rho}(\zeta)\boldsymbol{Q}_{*\rho}(\zeta)) = |\det \boldsymbol{P}^*_{*\rho}(\bar{\zeta})|^2. \end{aligned}$$

Since both $\det \boldsymbol{P}_\rho(z)$ and $\det \boldsymbol{P}^*_{*\rho}(\bar{z})$ are outer functions with positive values at 0, we obtain (4.48). ■

Consider now the operator quasiball (4.5). Obviously, the left and right semiradii Λ_ρ and Π_ρ of this quasi-ball admit the following representations:

$$\Lambda_\rho = \rho^{1/2}\boldsymbol{P}_\rho(0), \quad \Pi_\rho = \rho^{1/2}\boldsymbol{P}^*_{*\rho}(0).$$

By (4.48), we have

$$\det \Lambda_\rho = \det \Pi_\rho.$$

It follows that $\lim\limits_{\rho \to \|\Gamma\|+} \Lambda_\rho$ is invertible if and only if $\lim\limits_{\rho \to \|\Gamma\|+} \Pi_\rho$ is invertible.

Parametrization of Complete Extensions

Now we are ready to obtain a parametrization formula for complete extensions in the case $\dim \mathcal{H} < \infty$, $\dim \mathcal{K} < \infty$, and $\rho > \|\Gamma\|$. As we have already mentioned, the problem of describing all complete ρ-extensions of Γ is equivalent to the problem of describing the function $\Xi \in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ satisfying

$$\hat{\Xi}(j) = \Omega_j, \quad j \geq 0, \quad \text{and} \quad \|\Xi\|_{L^\infty} \leq \rho. \tag{4.49}$$

Theorem 4.13. *Suppose that* $\dim \mathcal{H} < \infty$, $\dim \mathcal{K} < \infty$, *and* $\rho > \|\Gamma\|$. *An operator function* Ξ *satisfies* (4.49) *if and only if* Ξ *admits a representation*

$$\Xi(z) = \Xi_{\mathcal{E}}(z) \stackrel{\text{def}}{=} \rho\bar{z}\big(\boldsymbol{Q}_\rho(z) + \boldsymbol{P}_{*\rho}(z)\mathcal{E}(\bar{z})\big)\big(\boldsymbol{P}_\rho(\bar{z}) + \boldsymbol{Q}_{*\rho}(\bar{z})\mathcal{E}(\bar{z})\big)^{-1} \tag{4.50}$$

for an operator function $\mathcal{E}$ *in the unit ball of* $H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$.

Recall that if $\mathcal{E}$ is an isometry from $\mathcal{H}$ to $\mathcal{K}$, then $\Xi_{\mathcal{E}}$ determines a canonical complete ρ-extension. We identify here a contractive operator $\mathcal{E}$ from $\mathcal{H}$ to $\mathcal{K}$ with the constant function identically equal to $\mathcal{E}$.

We need the following well-known lemma.

Lemma 4.14. *Let* J *be the block matrix defined by* (4.35). *Suppose that the block matrix* $A = \{A_{jk}\}_{j,k=1,2}$ *is* J*-unitary, i.e.,* $A^*JA = J$. *Then* *the operator $A_{11} + A_{12}\mathcal{E}$ is invertible in $\mathcal{B}(\mathcal{H})$ for any $\mathcal{E}$ in the unit ball of $\mathcal{B}(\mathcal{H},\mathcal{K})$ and the transformation*

$$\mathcal{E} \mapsto (A_{21} + A_{22}\mathcal{E})(A_{11} + A_{12}\mathcal{E})^{-1} \tag{4.51}$$

maps the unit ball of $\mathcal{B}(\mathcal{H},\mathcal{K})$ onto itself.

Proof. We have

$$\begin{aligned} A^*JA &= \begin{pmatrix} A_{11}^* & -A_{21}^* \\ A_{12}^* & -A_{22}^* \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\ &= \begin{pmatrix} A_{11}^*A_{11} - A_{21}^*A_{21} & A_{11}^*A_{12} - A_{21}^*A_{22} \\ A_{12}^*A_{11} - A_{22}^*A_{21} & A_{12}^*A_{12} - A_{22}^*A_{22} \end{pmatrix} = J. \end{aligned}$$

Hence, we have the following identities:

$$\begin{array}{ll} A_{11}^*A_{11} = A_{21}^*A_{21} + I_{\mathcal{H}}, & A_{11}^*A_{12} = A_{21}^*A_{22}, \\ A_{12}^*A_{11} = A_{22}^*A_{21}, & A_{12}^*A_{12} = A_{22}^*A_{22} - I_{\mathcal{K}}. \end{array} \tag{4.52}$$

As we have already observed, the matrix A^* is also J-unitary, and this means that

$$\begin{array}{ll} A_{11}A_{11}^* = A_{12}A_{12}^* + I_{\mathcal{H}}, & A_{11}A_{21}^* = A_{12}A_{22}^*, \\ A_{21}A_{11}^* = A_{22}A_{12}^*, & A_{21}A_{21}^* = A_{22}A_{22}^* - I_{\mathcal{K}}. \end{array} \tag{4.53}$$

Let $\mathcal{E} : \mathcal{H} \to \mathcal{K}$ and $\|\mathcal{E}\| \le 1$. By (4.53),

$$\|A_{11}^*x\|^2 = \|x\|^2 + \|A_{12}^*x\|^2, \quad a \in \mathcal{H}.$$

In particular, A_{11} is invertible. Hence, if $x \ne \mathbb{O}$, we have

$$\begin{aligned} \|(A_{11} + A_{12}\mathcal{E})^*x\| &\ge \|A_{11}^*x\| - \|\mathcal{E}^*A_{12}^*x\| \ge \|A_{11}^*x\| - \|A_{12}^*x\| \\ &= \frac{\|A_{11}^*x\|^2 - \|A_{12}^*x\|^2}{\|A_{11}^*x\| + \|A_{12}^*x\|} = \frac{\|x\|^2}{\|A_{11}^*x\| + \|A_{12}^*x\|} \ge \|x\|, \end{aligned}$$

and so $A_{11} + A_{12}\mathcal{E}$ is invertible.

Now let $\mathcal{E}_1$ be a contractive operator from $\mathcal{H}$ to $\mathcal{K}$ and

$$\mathcal{E}_2 = (A_{21} + A_{22}\mathcal{E}_1)(A_{11} + A_{12}\mathcal{E}_1)^{-1}.$$

Let us show that $\|\mathcal{E}_2\| \le 1$. This is equivalent to the inequality

$$\big((A_{21} + A_{22}\mathcal{E}_1)^*(A_{21} + A_{22}\mathcal{E}_1)(A_{11} + A_{12}\mathcal{E}_1)^{-1}x, (A_{11} + A_{12}\mathcal{E}_1)^{-1}x\big) \le \|x\|^2 \tag{4.54}$$

for any $x \in \mathcal{H}$. Put $(A_{11} + A_{12}\mathcal{E}_1)^{-1}x = y$. Clearly, (4.54) is equivalent to the inequality

$$\big((A_{21} + A_{22}\mathcal{E}_1)^*(A_{21} + A_{22}\mathcal{E}_1)y, y\big) \le \big((A_{11} + A_{12}\mathcal{E}_1)^*(A_{11} + A_{12}\mathcal{E}_1)y, y\big)$$

for any $y \in \mathcal{H}$. Using (4.52), we obtain

$$\begin{aligned}
&\big((A_{11}+A_{12}\mathcal{E}_1)^*(A_{11}+A_{12}\mathcal{E}_1)y,y\big)-\big((A_{21}+A_{22}\mathcal{E}_1)^*(A_{21}+A_{22}\mathcal{E}_1)y,y\big)\\
&=(A_{11}^*A_{11}y,y)+(\mathcal{E}_1^*A_{12}^*A_{12}\mathcal{E}_1y,y)+(A_{11}^*A_{12}\mathcal{E}_1y,y)+(\mathcal{E}_1^*A_{12}^*A_{11}y,y)\\
&-(A_{21}^*A_{21}y,y)-(\mathcal{E}_1^*A_{22}^*A_{22}\mathcal{E}_1y,y)-(A_{21}^*A_{22}\mathcal{E}_1y,y)-(\mathcal{E}_1^*A_{22}^*A_{21}y,y)\\
&=\|y\|^2-\|\mathcal{E}_1y\|^2\ge 0,
\end{aligned}$$

which proves (4.22). To show that our transformation maps the unit ball of $\mathcal{B}(\mathcal{H},\mathcal{K})$ onto itself, we observe that

$$\mathcal{E}_1^*=(-A_{21}^*+A_{11}^*\mathcal{E}_2^*)(A_{22}^*-A_{12}^*\mathcal{E}_2^*)^{-1},$$

and we have to show that $\|\mathcal{E}_1^*\|\le 1$ whenever $\|\mathcal{E}_2^*\|\le 1$. Now consider the matrix

$$J_{\#}=\begin{pmatrix} I_{\mathcal{K}} & \mathbb{O}\\ \mathbb{O} & -I_{\mathcal{H}}\end{pmatrix}.$$

It follows easily from (4.52) that the matrix

$$A_{\#}\stackrel{\text{def}}{=}\begin{pmatrix} A_{22}^* & -A_{12}^*\\ -A_{21}^* & A_{11}^*\end{pmatrix}$$

is $J_{\#}$-unitary, i.e., $A_{\#}^*J_{\#}A_{\#}=J_{\#}$. It remains to apply what we have already proved to the matrix $A_{\#}$. ■

Proof of Theorem 4.13. Since the matrix $U_\rho(\zeta)$ defined by (4.34) is J-unitary for almost all $\zeta\in\mathbb{T}$, it follows from Lemma 4.14 that the transformation (4.51) maps the unit ball of $L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ onto the ball of radius ρ of the space $L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$. It remains to show that $\Xi=\Xi_{\mathcal{E}}$ satisfies (4.49) if and only if $\mathcal{E}\in H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$.

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be functions in the unit ball of $L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$. In the same way as in (4.45) for constant matrix functions we can show that the following formula still holds for our contractive matrix functions $\mathcal{E}_1$ and $\mathcal{E}_2$:

$$\begin{aligned}
&\Xi_{\mathcal{E}_1}(z)-\Xi_{\mathcal{E}_2}(z)\\
=\;&\rho\bar z\big(\boldsymbol{P}_{*\rho}^*(z)+\mathcal{E}_2(\bar z)\boldsymbol{Q}_\rho^*(z)\big)^{-1}(\mathcal{E}_1(\bar z)-\mathcal{E}_2(\bar z))\big(\boldsymbol{P}_\rho(\bar z)+\boldsymbol{Q}_{*\rho}(\bar z)\mathcal{E}_1(\bar z)\big)^{-1}.
\end{aligned}$$

Suppose first that $\|\mathcal{E}_j\|_{L^\infty}<1$. Then by Theorem 4.11,

$$\big(\boldsymbol{P}_\rho+\boldsymbol{Q}_{*\rho}\mathcal{E}_1\big)^{-1}=(I_{\mathcal{H}}+\mathcal{X}_\rho\mathcal{E}_1)^{-1}\boldsymbol{P}_\rho^{-1}\in H^\infty.$$

Similarly,

$$\big(\boldsymbol{P}_{*\rho}^*(\bar z)+\mathcal{E}_2(z)\boldsymbol{Q}_\rho^*(\bar z)\big)^{-1}\in H^\infty,$$

and so

$$\Xi_{\mathcal{E}_1}(z)-\Xi_{\mathcal{E}_2}(z)\in\bar z\overline{H^\infty}. \tag{4.55}$$

Suppose now that $\mathcal{E}_1$ and $\mathcal{E}_2$ are arbitrary functions in the unit ball of $H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$. Since

$$\sup_{r\in(0,1)}\|\Xi_{r\mathcal{E}_1}-\Xi_{r\mathcal{E}_2}\|_{L^\infty}\le 2\rho$$

and for almost all $\zeta \in \mathbb{T}$,

$$\lim_{r\to 1-} \big(\Xi_{r\mathcal{E}_1}(\zeta) - \Xi_{r\mathcal{E}_2}(\zeta)\big) = \Xi_{\mathcal{E}_1}(\zeta) - \Xi_{\mathcal{E}_2}(\zeta),$$

it follows that (4.55) holds for arbitrary $\mathcal{E}_1$ and $\mathcal{E}_2$ in the unit ball of $H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$.

Since we know that $\hat{\Xi}_{\mathcal{E}}(j) = \Omega_j$, $j \in \mathbb{Z}_+$, if $\mathcal{E}$ is a constant isometry, it follows from (4.55) that the same is true for an arbitrary $\mathcal{E}$ in the unit ball of $H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$.

Suppose now that $\hat{\Xi}_{\mathcal{E}}(j) = \Omega_j$, $j \in \mathbb{Z}_+$, for some $\mathcal{E}$ in the unit ball of $L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$. Let us show that $\mathcal{E} \in H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$. Put $\mathcal{E}_1 = \mathcal{E}$ and $\mathcal{E}_2 = \mathbb{O}$. Then

$$\Xi_{\mathcal{E}_1}(z) - \Xi_{\mathcal{E}_2}(z) = \rho\bar{z}\big(\boldsymbol{P}^*_{*\rho}(z)\big)^{-1}\mathcal{E}(\bar{z})\big(\boldsymbol{P}_\rho(\bar{z}) + \boldsymbol{Q}_{*\rho}(\bar{z})\mathcal{E}(\bar{z})\big)^{-1} \in \bar{z}\overline{H^\infty}.$$

It follows that

$$z\boldsymbol{P}^*_{*\rho}(z)\big(\Xi_{\mathcal{E}}(z) - \Xi_{\mathbb{O}}(z)\big)\boldsymbol{P}_\rho(\bar{z}) = \rho\mathcal{E}(\bar{z})\big(I_{\mathcal{H}} + \mathcal{X}_\rho(\bar{z})\mathcal{E}(\bar{z})\big)^{-1} \in \overline{H^1}. \tag{4.56}$$

Therefore

$$\begin{aligned}&\big(I_{\mathcal{H}} - \mathcal{X}_\rho(\bar{z})\mathcal{E}(\bar{z})\big)\big(I_{\mathcal{H}} + \mathcal{X}_\rho(\bar{z})\mathcal{E}(\bar{z})\big)^{-1}\\ =&I_{\mathcal{H}} + \frac{2}{\rho}z\mathcal{X}_\rho(\bar{z})\boldsymbol{P}^*_{*\rho}(z)\big(\Xi_{\mathcal{E}}(z) - \Xi_{\mathbb{O}}(z)\big)\boldsymbol{P}_\rho(\bar{z}) \in \overline{H^1}.\end{aligned}$$

Consequently,

$$\big(I_{\mathcal{H}} + \mathcal{E}^*(\bar{z})\mathcal{X}^*_\rho(\bar{z})\big)^{-1}\big(I_{\mathcal{H}} - \mathcal{E}^*(\bar{z})\mathcal{X}^*_\rho(\bar{z})\big) \in H^1.$$

In the same way as in the proof of Theorem 4.11, we can deduce that $\mathcal{X}_\rho\mathcal{E} \in H^\infty$. It follows now from (4.56) that $\mathcal{E} \in H^\infty$. ■

Note that the proof of Lemma 4.14 yields that the transformation (4.51) maps the set of isometries from $\mathcal{H}$ to $\mathcal{K}$ onto itself. This implies the following result.

Corollary 4.15. *Let $\mathcal{E}$ be in the unit ball of $H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$. The function $\rho^{-1}\Xi_{\mathcal{E}}$ takes isometric values, i.e.,*

$$\rho^{-2}\Xi^*_{\mathcal{E}}(\zeta)\Xi_{\mathcal{E}}(\zeta) = I_{\mathcal{H}} \quad \text{a.e. on } \mathbb{T}$$

if and only if $\mathcal{E}$ is an inner function, i.e., $\mathcal{E}^*(\zeta)\mathcal{E}(\zeta) = I_{\mathcal{H}}$ for almost all $\zeta \in \mathbb{T}$.

To conclude this section, we consider the following version of the Nehari problem. Let $\Psi \in L^\infty(\mathcal{H},\mathcal{K}))$. Consider the Hankel operator $H_\Psi : H^2(\mathcal{H}) \to H^2_-(\mathcal{K})$. Let $\rho > \|H_\Psi\|$. We want to parametrize all functions $\Phi \in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ such that

$$H_\Phi = H_\Psi \quad \text{and} \quad \|\Phi\|_{L^\infty} \le \rho. \tag{4.57}$$

This problem easily reduces to the problem we have solved in this section. Let $\Omega_j = \hat{\Psi}(-j-1)$, $j \in \mathbb{Z}_+$. Consider the block Hankel matrix

$\Gamma = \{\Omega_{j+k}\}_{j,k\ge 0}$. Then $\|\Gamma\| = \|H_\Psi\|$. Clearly, a function Ξ in $L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ satisfies the conditions

$$\hat{\Xi}(j) = \Omega_j, \quad j \ge 0, \quad \text{and} \quad \|\Xi\|_{L^\infty} \le \rho$$

if and only if the function $\Phi = \bar{z}\Xi(\bar{z})$ satisfies (4.57). Starting from Γ we can construct the operator functions $\boldsymbol{P}_\rho$, $\boldsymbol{Q}_\rho$, $\boldsymbol{P}_{*\rho}$, and $\boldsymbol{Q}_{*\rho}$ as above and we can deduce from Theorem 4.13 the following parametrization formula for the solutions of the Nehari problem (4.57).

Theorem 4.16. *Suppose that* $\dim\mathcal{H} < \infty$, $\dim\mathcal{K} < \infty$, Ψ *is a function in* $L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$, *and* $\rho > \|H_\Psi\|$. *A function* $\Phi \in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ *satisfies* (4.57) *if and only if it has the form*

$$\Phi = \Phi_{\mathcal{E}} \stackrel{\text{def}}{=} \rho\big(\boldsymbol{Q}_\rho(\bar{z}) + \boldsymbol{P}_{*\rho}(\bar{z})\mathcal{E}(z)\big)\big(\boldsymbol{P}_\rho(z) + \boldsymbol{Q}_{*\rho}(z)\mathcal{E}(z)\big)^{-1} \tag{4.58}$$

for a function $\mathcal{E}$ *in the unit ball of* $H^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$.

Proof. This is an immediate consequence of Theorem 4.13. ■

Note that formula (4.58) can be rewritten in the following way:

$$\Phi_{\mathcal{E}} = \rho\boldsymbol{P}_{*\rho}^{-1}(\bar{z})\big(\mathcal{X}_\rho^*(z) + \mathcal{E}(z)\big)\big(I_{\mathcal{H}} + \mathcal{X}_\rho(z)\mathcal{E}(z)\big)^{-1}\boldsymbol{P}_\rho^{-1}(z).$$

This parametrization formula has the same form as the parametrization formula in the scalar case given in Theorem 1.13. Recall that in the scalar case Theorem 1.13 also covers the case $\rho = \|H_\Psi\|$.

5. Parametrization in the General Case

In this section we consider the most general Nehari problem and obtain a parametrization formula for its solutions.

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces and let Γ be a Hankel operator from $H^2(\mathcal{H})$ to $H^2_-(\mathcal{K})$, i.e., $\Gamma = H_\Psi$ for some $\Psi \in L^\infty(\mathcal{B}(\mathcal{H}, K))$. For $\rho \ge \|\Gamma\|$ we consider the Nehari problem of finding all functions $\Phi \in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ such that

$$H_\Phi = \Gamma \quad \text{and} \quad \|\Phi\|_{L^\infty} \le \rho. \tag{5.1}$$

If $\rho > 0$, we can multiply Γ by $1/\rho$ and reduce the general case to the case $\rho = 1$. Thus we assume that $\|\Gamma\| \le 1$ and we are going to describe operator functions Φ satisfying (5.1) with $\rho = 1$.

Recall that Γ satisfies the following commutation relation:

$$\Gamma(zf) = \mathbb{P}_- z\Gamma f, \quad f \in H^2(\mathcal{H}). \tag{5.2}$$

Consider the space $X = H^2_-(\mathcal{K}) \oplus H^2(\mathcal{H})$, which consists of pairs $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ such that $f_1 \in H^2_-(\mathcal{K})$ and $f_2 \in H^2(\mathcal{H})$. X is a Hilbert space with its natural inner product $(\cdot,\cdot)_2$. We define a new (semi-)inner product on X by

$$\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right)_X \stackrel{\text{def}}{=} \left(\begin{pmatrix} I & \Gamma \\ \Gamma^* & I \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right)_2.$$

Since $\|\Gamma\| \le 1$, it is easy to see that $(\cdot,\cdot)_X$ is a semi-inner product. We identify vectors f and g in X if $(f-g, f-g)_X = 0$. Then $(\cdot,\cdot)_X$ is an inner product on the space of equivalence classes. Finally, we consider the completion of this space with respect to this inner product and denote the resulting Hilbert space by $\boldsymbol{X}$. We use the notation $(\cdot,\cdot)_{\boldsymbol{X}}$ for the inner product in $\boldsymbol{X}$ and $\|\cdot\|_{\boldsymbol{X}}$ for the norm in $\boldsymbol{X}$: $\|f\|_{\boldsymbol{X}} \stackrel{\text{def}}{=} (f,f)_{\boldsymbol{X}}^{1/2}$.

We consider the operator V with domain

$$H^2_-(\mathcal{K}) \oplus zH^2(\mathcal{H})$$

and range

$$\bar{z}H^2_-(\mathcal{K}) \oplus H^2(\mathcal{H})$$

defined by

$$V\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \bar{z}f_1 \\ \bar{z}f_2 \end{pmatrix}.$$

Then V is an isometric operator in the norm $\|\cdot\|_{\boldsymbol{X}}$. Indeed, let $f_1,\ g_1 \in H^2_-(\mathcal{K})$ and $f_2,\ g_2 \in zH^2(\mathcal{H})$. We obtain from (5.2)

$$\begin{aligned} \left(V\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, V\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right)_X &= \left(\begin{pmatrix} \bar{z}f_1 + \Gamma\bar{z}f_2 \\ \Gamma^*\bar{z}f_1 + \bar{z}f_2 \end{pmatrix}, \begin{pmatrix} \bar{z}g_1 \\ \bar{z}g_2 \end{pmatrix}\right)_X \\ &= (\bar{z}f_1, \bar{z}g_1) + (\bar{z}f_2, \bar{z}g_2) \\ &+ (\Gamma\bar{z}f_2, \bar{z}g_1) + (\bar{z}f_1, \Gamma\bar{z}g_2) \\ &= (f_1, g_1) + (f_2, g_2) \\ &+ (\mathbb{P}_- z\Gamma\bar{z}f_2, g_1) + (f_1, \mathbb{P}_- z\Gamma\bar{z}g_2) \\ &= (f_1, g_1) + (f_2, g_2) + (\Gamma f_2, g_1) + (f_1, \Gamma g_2) \\ &= \left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right)_X. \end{aligned}$$

Thus V extends to an isometric operator in the norm of $\boldsymbol{X}$ with domain

$$\mathcal{D}_V \stackrel{\text{def}}{=} \operatorname{clos}_{\boldsymbol{X}} H^2_-(\mathcal{K}) \oplus zH^2(\mathcal{H})$$

and range

$$\mathcal{R}_V \stackrel{\text{def}}{=} \operatorname{clos}_{\boldsymbol{X}} \bar{z}H^2_-(\mathcal{K}) \oplus H^2(\mathcal{H}).$$

As in §1, we consider unitary extensions U of V on Hilbert spaces $\boldsymbol{Y} \supset \boldsymbol{X}$, i.e., U is a unitary operator on $\boldsymbol{Y}$ such that $U\big|\mathcal{D}_V = V$. We are interested only in *minimal* unitary extensions of V (see §1 for the definition).

Consider the natural imbedding $\mathcal{I}_1$ of $\mathcal{H}$ into $\boldsymbol{X}$ defined by

$$\mathcal{I}^{(1)}x \stackrel{\text{def}}{=} \begin{pmatrix} \mathbb{O} \\ \boldsymbol{x} \end{pmatrix}, \quad x \in \mathcal{H};$$

here $\boldsymbol{x}$ is the function identically equal to x. We also consider the isometric imbedding $\mathcal{I}^{(2)}$ of $\mathcal{K}$ into $\boldsymbol{X}$ defined by

$$\mathcal{I}^{(2)}x = \begin{pmatrix} \bar{z}\boldsymbol{x} \\ \mathbb{O} \end{pmatrix}, \quad x \in \mathcal{K}.$$

Definition. Given a unitary extension U of V, consider the harmonic operator function Φ_U on $\mathbb{D}$ defined by

$$\Phi_U(\zeta) : \mathcal{H} \to \mathcal{K}, \quad \Phi_U(\zeta) \stackrel{\text{def}}{=} \left(\mathcal{I}^{(2)}\right)^* U\big((I-\zeta U)^{-1} + (I-\bar{\zeta}U^*)^{-1} - I\big)\mathcal{I}^{(1)}.$$

The harmonic function Φ_U is called the *scattering function* of the unitary extension U of V.

Lemma 5.1. *Let U be a unitary extension of V. Then*

$$\|\Phi_U(\zeta)\|_{\mathcal{B}(\mathcal{H},\mathcal{K})} \le 1, \quad \zeta \in \mathbb{D}.$$

Proof. Let $x \in \mathcal{H}$. Let us show that

$$\|x\|^2_{\mathcal{H}} = \Big(\big((I-\zeta U)^{-1} + (I-\bar{\zeta}U^*)^{-1} - I\big)\mathcal{I}^{(1)}x, \mathcal{I}^{(1)}x\Big)_{\boldsymbol{Y}}, \quad \zeta \in \mathbb{D}. \tag{5.3}$$

Since

$$(I-\zeta U)^{-1} = \sum_{j\ge 0} \zeta^j U^j \quad \text{and} \quad (I-\bar{\zeta}U^*)^{-1} = \sum_{j\ge 0} \bar{\zeta}^j (U^*)^j, \quad \zeta \in \mathbb{D},$$

and $(\mathcal{I}^{(1)}x, \mathcal{I}^{(1)}x)_{\boldsymbol{Y}} = (x,x)_{\mathcal{H}}$, to prove (5.3), it suffices to show that

$$\Big((U^*)^j\mathcal{I}^{(1)}x, \mathcal{I}^{(1)}x\Big)_{\boldsymbol{Y}} = 0, \quad j > 0.$$

By definition, for $j > 0$

$$U^j \begin{pmatrix} \mathbb{O} \\ z^j\boldsymbol{x} \end{pmatrix} = V^j \begin{pmatrix} \mathbb{O} \\ z^j\boldsymbol{x} \end{pmatrix} = \begin{pmatrix} \mathbb{O} \\ \boldsymbol{x} \end{pmatrix} = \mathcal{I}^{(1)}x.$$

Hence,

$$(U^*)^j\mathcal{I}^{(1)}x = \begin{pmatrix} \mathbb{O} \\ z^j\boldsymbol{x} \end{pmatrix}.$$

Consequently, for $j > 0$ we obtain

$$\begin{aligned} \Big((U^*)^j\mathcal{I}^{(1)}x, \mathcal{I}^{(1)}x\Big)_{\boldsymbol{Y}} &= \left(\begin{pmatrix} I & \Gamma \\ \Gamma^* & I \end{pmatrix}\begin{pmatrix} \mathbb{O} \\ z^j\boldsymbol{x} \end{pmatrix}, \begin{pmatrix} \mathbb{O} \\ \boldsymbol{x} \end{pmatrix}\right)_2 \\ &= (z^j\boldsymbol{x}, \boldsymbol{x})_{H^2(\mathcal{H})} = 0, \end{aligned}$$

which proves (5.3).

Similarly, for $y \in \mathcal{K}$,

$$\|y\|^2_{\mathcal{K}} = \Big(\big((I-\zeta U)^{-1} + (I-\bar{\zeta}U^*)^{-1} - I\big)U^*\mathcal{I}^{(2)}y, U^*\mathcal{I}^{(2)}y\Big)_{\boldsymbol{Y}}, \quad \zeta \in \mathbb{D}. \tag{5.4}$$

Using the following easily verifiable identity

$$(I-\zeta U)^{-1}+(I-\bar{\zeta}U^*)^{-1}-I=\left(1-|\zeta|^2\right)(I-\bar{\zeta}U^*)^{-1}(I-\zeta U)^{-1},$$

we find from (5.3) and (5.4) that for $x \in \mathcal{H}$ and $y \in \mathcal{K}$

$$\begin{aligned}
&\left(\begin{pmatrix} I & \Phi_U(\zeta) \\ \Phi_U^*(\zeta) & I \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix}, \begin{pmatrix} y \\ x \end{pmatrix}\right)_2 \\
=&(y,y)+(\Phi_U(\zeta)x,y)+(\Phi_U^*(\zeta)y,x)+(x,x) \\
=&\Big(\big((I-\zeta U)^{-1}+(I-\bar{\zeta}U^*)^{-1}-I\big)\big(\mathcal{I}^{(1)}x+U^*\mathcal{I}^{(2)}y\big),\big(\mathcal{I}^{(1)}x+U^*\mathcal{I}^{(2)}y\big)\Big)_{\boldsymbol{Y}} \\
=&\left(1-|\zeta|^2\right)\Big((I-\zeta U)^{-1}\big(\mathcal{I}^{(1)}x+U^*\mathcal{I}^{(2)}y\big),(I-\zeta U)^{-1}\big(\mathcal{I}^{(1)}x+U^*\mathcal{I}^{(2)}y\big)\Big)_{\boldsymbol{Y}}\ge 0,
\end{aligned}$$

which proves the lemma. ■

Definition. If U is a unitary operator on Hilbert space, the function

$$\mathcal{P}_U(z)\stackrel{\mathrm{def}}{=}(I-zU)^{-1}+(I-\bar{z}U^*)^{-1}-I=\sum_{j<0}\bar{z}^j(U^*)^j+\sum_{j\ge 0}z^jU^j$$

is called the *Poisson transform* of U.

Clearly, $\mathcal{P}_U$ is an operator-valued harmonic function and $\mathcal{P}_U(\zeta)\ge\mathbb{O}$ for $\zeta\in\mathbb{D}$.

Since Φ_U is a bounded harmonic functions on $\mathbb{D}$, the limit $\lim\limits_{r\to 1-}\Phi_U(r\zeta)$ exists for almost all $\zeta\in\mathbb{T}$ in the weak operator topology. Indeed, it suffices to prove that the limit $\lim\limits_{r\to 1-}(\Phi_U(r\zeta)x,y)$ exists for countable dense subsets of vectors x in $\mathcal{H}$ and $y\in\mathcal{K}$, which follows from the corresponding fact for scalar functions. Thus we can associate with Φ_U the corresponding boundary-value function on $\mathbb{T}$, which we also denote by Φ_U.

Theorem 5.2. *A function $\Phi\in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ is a solution of the Nehari problem of norm at most 1 if and only if $\Phi=\Phi_U$ for a unitary extension U of V.*

Proof. Suppose that $\|\Phi\|_{L^\infty}\le 1$ and $H_\Phi=\Gamma$. Consider the space $Y\stackrel{\mathrm{def}}{=}L^2(\mathcal{K})\oplus L^2(\mathcal{H})$ and define a new (semi-)inner product in Y by

$$\begin{aligned}
&\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix},\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right)_Y \\
=&\int_{\mathbb{T}}\left(\begin{pmatrix} I & \Phi(\zeta) \\ \Phi^*(\zeta) & I \end{pmatrix}\begin{pmatrix} f_1(\zeta) \\ f_2(\zeta) \end{pmatrix},\begin{pmatrix} g_1(\zeta) \\ g_2(\zeta) \end{pmatrix}\right)_{\mathcal{K}\oplus\mathcal{H}}d\boldsymbol{m}(\zeta).
\end{aligned}$$

Again, we can consider the corresponding equivalence classes and take the completion, which we denote by $\boldsymbol{Y}$. Since $H_\Phi=\Gamma$, it is easy to see that $\boldsymbol{X}$ can be considered in a natural way as a subspace of $\boldsymbol{Y}$ with the same norm.

We define the operator U on $\boldsymbol{Y}$ by

$$U\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}=\begin{pmatrix} \bar{z}f_1 \\ \bar{z}f_2 \end{pmatrix}.$$

Again, it is easy to verify that U extends to a unitary operator on $\boldsymbol{Y}$ and U is an extension of V.

Let us show that $\Phi_U = \Phi$. We have

$$U^*\mathcal{I}^{(2)}y = U^*\begin{pmatrix} \bar{z}\boldsymbol{y} \\ \mathbb{O} \end{pmatrix} = \begin{pmatrix} \boldsymbol{y} \\ \mathbb{O} \end{pmatrix}.$$

It is easy to see that

$$(I - \zeta U)^{-1}f = (1 - \zeta\bar{z})^{-1}f, \quad f \in Y,$$

and

$$(I - \bar{\zeta}U^*)^{-1}f = (1 - \bar{\zeta}z)^{-1}f, \quad f \in Y.$$

Thus

$$\big((I - \zeta U)^{-1} + (I - \bar{\zeta}U^*)^{-1} - I\big)f = \frac{1 - |\zeta|^2}{|1 - \bar{\zeta}z|^2}f, \quad f \in Y.$$

Hence, for $x \in \mathcal{H}$ and $y \in \mathcal{K}$ we obtain

$$\begin{aligned} (\Phi_U(\zeta)x, y) &= \Big(\big((I - \zeta U)^{-1} + (I - \bar{\zeta}U^*)^{-1} - I\big)\mathcal{I}^{(1)}x, U^*\mathcal{I}^{(2)}x\Big)_{\boldsymbol{Y}} \\ &= \left(\frac{1 - |\zeta|^2}{|1 - \bar{\zeta}z|^2}\begin{pmatrix} \mathbb{O} \\ \boldsymbol{x} \end{pmatrix}, \begin{pmatrix} \boldsymbol{y} \\ \mathbb{O} \end{pmatrix}\right)_{\boldsymbol{Y}} \\ &= \int\limits_{\mathbb{T}} \frac{1 - |\zeta|^2}{|1 - \bar{\zeta}z|^2}(\Phi(\zeta)x, y)_{\mathcal{K}} d\boldsymbol{m}. \end{aligned}$$

However, this is just the Poisson integral of the function $(\Phi(\zeta)x, y)_{\mathcal{K}}$. Thus Φ_U is the harmonic extension of Φ to the unit disk.

Suppose now that U is a unitary extension of V. Let us show that Φ_U is a solution of the Nehari problem (the fact that $\|\Phi_U\|_{L^\infty} \le 1$ is just Lemma 5.1). For $j \ge 1$, $x \in \mathcal{H}$, $y \in \mathcal{K}$, we have

$$\begin{aligned} \left(\hat{\Phi}_U(-j)x, y\right)_{\mathcal{K}} &= \left((U^*)^j\mathcal{I}^{(1)}x, U^*\mathcal{I}^{(2)}y\right)_{\boldsymbol{Y}} \\ &= \left(\mathcal{I}^{(1)}x, U^{j-1}\mathcal{I}^{(2)}y\right)_{\boldsymbol{Y}} = \left(\begin{pmatrix} \mathbb{O} \\ \boldsymbol{x} \end{pmatrix}, U^{j-1}\begin{pmatrix} \bar{z}\boldsymbol{y} \\ \mathbb{O} \end{pmatrix}\right)_{\boldsymbol{Y}}. \end{aligned}$$

It is easy to see that for all $j \ge 1$

$$\begin{pmatrix} \bar{z}^j\boldsymbol{y} \\ \mathbb{O} \end{pmatrix} \in \mathcal{D}_V$$

and

$$U^{j-1}\begin{pmatrix} \bar{z}\boldsymbol{y} \\ \mathbb{O} \end{pmatrix} = V^{j-1}\begin{pmatrix} \bar{z}\boldsymbol{y} \\ \mathbb{O} \end{pmatrix} = \begin{pmatrix} \bar{z}^j\boldsymbol{y} \\ \mathbb{O} \end{pmatrix}.$$

Hence,

$$\begin{aligned}\left(\hat{\Phi}_U(-j)x, y\right)_{\mathcal{K}} &= \left(\begin{pmatrix} I & \Gamma \\ \Gamma^* & I \end{pmatrix}\begin{pmatrix} \mathbb{O} \\ \boldsymbol{x} \end{pmatrix}, \begin{pmatrix} \bar{z}^j \boldsymbol{y} \\ \mathbb{O} \end{pmatrix}\right)_X \\ &= (\Gamma\boldsymbol{x}, \bar{z}^j\boldsymbol{y})_2 = (\hat{\Psi}(-j)x, y)_{\mathcal{K}}\end{aligned}$$

(recall that Ψ is an arbitrary bounded symbol of Γ). Thus $H_\Phi = \Gamma$. ■

Now we proceed to the study of minimal unitary extensions of an isometric operator. Let $\boldsymbol{X}$ be a Hilbert space and let V be an isometric operator with domain $\mathcal{D}_V \subset \boldsymbol{X}$ and range $\mathcal{R}_V \subset \boldsymbol{X}$. Let $\mathcal{N}_1$ be a Hilbert space isomorphic to $\mathcal{D}_V^\perp$ and let $\mathcal{N}_2$ be a Hilbert space isomorphic to $\mathcal{R}_V^\perp$. We consider a unitary operator

$$A : \boldsymbol{X} \oplus \mathcal{N}_2 \to \boldsymbol{X} \oplus \mathcal{N}_1$$

such that $A\big|\mathcal{D}_V = V$, $A\big|\mathcal{D}_V^\perp$ is a unitary map of $\mathcal{D}_V^\perp$ onto $\mathcal{N}_1$, and $A\big|\mathcal{N}_2$ is a unitary map of $\mathcal{N}_2$ onto $\mathcal{R}_V^\perp$. Here we identify $\boldsymbol{X}$ with the subspace $\boldsymbol{X} \oplus \{\mathbb{O}\}$ of $\boldsymbol{X} \oplus \mathcal{N}_2$ and with the subspace $\boldsymbol{X} \oplus \{\mathbb{O}\}$ of $\boldsymbol{X} \oplus \mathcal{N}_1$, and we identify $\mathcal{N}_1$ with $\{\mathbb{O}\} \oplus \mathcal{N}_1$ and $\mathcal{N}_2$ with $\{\mathbb{O}\} \oplus \mathcal{N}_2$.

If $\boldsymbol{X}$, $\mathcal{N}_1$, and $\mathcal{N}_2$ are Hilbert spaces, we call operators from $\boldsymbol{X} \oplus \mathcal{N}_2$ to $\boldsymbol{X} \oplus \mathcal{N}_1$ *colligations.*

Consider another unitary colligation

$$B : \boldsymbol{Z} \oplus \mathcal{N}_1 \to \boldsymbol{Z} \oplus \mathcal{N}_2,$$

with the same spaces $\mathcal{N}_1$ and $\mathcal{N}_2$. The colligation B is called *simple* if B has no nonzero reducing subspace contained in $\boldsymbol{Z}$.

Let $U : \boldsymbol{X} \oplus \boldsymbol{Z} \to \boldsymbol{X} \oplus \boldsymbol{Z}$ be a unitary operator such that

$$U\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \tag{5.5}$$

where $x_2 \in \boldsymbol{X}$ and $y_2 \in \boldsymbol{Z}$ satisfy the following equalities:

$$A\begin{pmatrix} x_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ v_1 \end{pmatrix}, \quad B\begin{pmatrix} y_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} y_2 \\ v_2 \end{pmatrix} \tag{5.6}$$

for some vectors $v_1 \in \mathcal{N}_1$ and $v_2 \in \mathcal{N}_2$.

It follows from the definition of A that $P_{\mathcal{N}_1}A\big|\mathcal{N}_2 = \mathbb{O}$ ($P_{\mathcal{M}}$ is the orthogonal projection onto a subspace $\mathcal{M}$), and so it follows from (5.6) that

$$v_1 = P_{\mathcal{N}_1}A\begin{pmatrix} x_1 \\ \mathbb{O} \end{pmatrix}.$$

Hence, the vectors x_1 and y_1 uniquely determine v_1, v_2, x_2, and y_2. Thus the operator U is uniquely determined by (5.5) and (5.6). It is easy to see that U is an isometry. Indeed, since A and B are unitary operators, we have

$$\begin{aligned}\left\|U\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right\|^2 &= \|x_2\|^2 + \|y_2\|^2 \\ &= \left\|A\begin{pmatrix} x_1 \\ v_2 \end{pmatrix}\right\|^2 + \left\|B\begin{pmatrix} y_1 \\ v_1 \end{pmatrix}\right\|^2 - \|v_1\|^2 - \|v_2\|^2 \\ &= \|x_1\|^2 + \|y_1\|^2 = \left\|\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right\|^2.\end{aligned}$$

To prove that U is unitary, it remains to show that U maps $\boldsymbol{X} \oplus \boldsymbol{Z}$ onto itself. Suppose that $x_2 \in \boldsymbol{X}$ and $y_2 \in \boldsymbol{Z}$. Let us show that there exist $x_1 \in \boldsymbol{X}$ and $y_1 \in \boldsymbol{Z}$ such that (5.5) holds. Put

$$v_2 = P_{\mathcal{N}_2} A^* \begin{pmatrix} x_2 \\ \mathbb{O} \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ v_1 \end{pmatrix} = B^* \begin{pmatrix} y_2 \\ v_2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} x_1 \\ v_2 \end{pmatrix} = A^* \begin{pmatrix} x_2 \\ v_1 \end{pmatrix}.$$

It follows from the definition of A that v_1, v_2, x_1, and y_1 are well-defined, and so U is unitary.

The operator U is called the *coupling* of the unitary colligations A and B. As we have already observed, for our colligation A and an arbitrary unitary colligation $B : \boldsymbol{Z} \oplus \mathcal{N}_1 \to \boldsymbol{Z} \oplus \mathcal{N}_2$ there exists a unique coupling U.

Theorem 5.3. *Let A, B, U be as above. Then U is a unitary extension of V. If U is an arbitrary unitary extension of V, then U is the coupling of A and some unitary colligation $B : \boldsymbol{Z} \oplus \mathcal{N}_1 \to \boldsymbol{Z} \oplus \mathcal{N}_2$. Finally, U is a minimal unitary extension if and only if the colligation B is simple.*

Proof. Suppose that U is the coupling of A and B. Let $x_1 \in \mathcal{D}_V$. Put $y_1 = \mathbb{O}$, $y_2 = \mathbb{O}$, $x_2 = Vx_1$, $v_1 = \mathbb{O}$, and $v_2 = \mathbb{O}$ in (5.6). Then by definition,

$$U\begin{pmatrix} x_1 \\ \mathbb{O} \end{pmatrix} = \begin{pmatrix} Vx_1 \\ \mathbb{O} \end{pmatrix},$$

and so U is a unitary extension of V.

Suppose now that $\boldsymbol{Y}$ is a Hilbert space, $\boldsymbol{X} \subset \boldsymbol{Y}$, and U is a unitary extension of V on $\boldsymbol{Y}$. Put $\boldsymbol{Z} \stackrel{\text{def}}{=} \boldsymbol{Y} \ominus \boldsymbol{X}$. Then $\boldsymbol{Z} \oplus \mathcal{D}_V^\perp = \boldsymbol{Y} \ominus \mathcal{D}_V$ and $\boldsymbol{Z} \oplus \mathcal{R}_V^\perp = \boldsymbol{Y} \ominus \mathcal{R}_V$. Since U is an extension of V, U maps $\mathcal{D}_V$ onto $\mathcal{R}_V$. Since U is a unitary operator, it maps $\boldsymbol{Y} \ominus \mathcal{D}_V$ onto $\boldsymbol{Y} \ominus \mathcal{R}_V$. Thus we can consider the unitary colligation $B_\# : \boldsymbol{Z} \oplus \mathcal{D}_V^\perp \to \boldsymbol{Z} \oplus \mathcal{R}_V^\perp$ defined by $B_\# = U\big|\boldsymbol{Z} \oplus \mathcal{D}_V^\perp$. Finally, we define the unitary colligation

$$B : \boldsymbol{Z} \oplus \mathcal{N}_1 \to \boldsymbol{Z} \oplus \mathcal{N}_2$$

by

$$B = \begin{pmatrix} P_{\boldsymbol{Z}} B_\# \big| \boldsymbol{Z} & P_{\boldsymbol{Z}} B_\# A^* | \mathcal{N}_1 \\ A^* P_{\mathcal{R}_V^\perp} B_\# \big| \boldsymbol{Z} & A^* P_{\mathcal{R}_V^\perp} B_\# A^* | \mathcal{N}_1 \end{pmatrix}.$$

Clearly, B is a unitary colligation. Let

$$U\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Put $v_1 = AP_{\mathcal{D}_V^\perp} x_1$ and $v_2 = A^* P_{\mathcal{R}_V^\perp} x_2$. It is easy to see that (5.6) holds. Thus U is the coupling of A and B.

To complete the proof, it remains to describe the minimal unitary extensions. Put

$$\mathcal{L} \stackrel{\text{def}}{=} \{y \in \boldsymbol{Y} : \ y \perp U^j x, \ x \in \boldsymbol{X}, \ j \in \mathbb{Z}\}.$$

Clearly, $\mathcal{L} \subset \boldsymbol{Z}$. Obviously, $\mathcal{L}$ is a reducing subspace of B. Conversely, it is easy to see that if $\mathcal{L} \subset \boldsymbol{Z}$ is a reducing subspace of B, then $\mathcal{L} \perp U^j x$ for any $x \in \boldsymbol{X}$ and any $j \in \mathbb{Z}$. ■

We are going to construct the so-called scattering function of the colligation $A : \boldsymbol{X} \oplus \mathcal{N}_2 \to \boldsymbol{X} \oplus \mathcal{N}_1$ associated with our isometric operator V. Suppose we are given a vector $x^{(0)} \in \boldsymbol{X}$ (initial state) and sequences $\mathfrak{n}_2^{(k)} \in \mathcal{N}_2$, $k \ge 0$, and $\mathfrak{n}_1^{(k)} \in \mathcal{N}_1$, $k \le -1$. Then they uniquely determine sequences $x^{(k)} \in \boldsymbol{X}$, $k \in \mathbb{Z}$, $\mathfrak{n}_2^{(k)} \in \mathcal{N}_2$, $k \in \mathbb{Z}$, and $\mathfrak{n}_1^{(k)} \in \mathcal{N}_1$, $k \in \mathbb{Z}$, such that

$$A\begin{pmatrix} x^{(k)} \\ \mathfrak{n}_2^{(k)} \end{pmatrix} = \begin{pmatrix} x^{(k+1)} \\ \mathfrak{n}_1^{(k)} \end{pmatrix}, \quad k \in \mathbb{Z}. \tag{5.7}$$

Consider the following harmonic vector functions in $\mathbb{D}$:

$$\check{x}_- = \sum_{k=-\infty}^{-1} \bar{z}^k x^{(k)}, \quad \check{x}_+ = \sum_{k=0}^{\infty} z^k x^{(k)}, \quad \check{x} = \check{x}_- + \check{x}_+;$$

$$(\check{\mathfrak{n}}_1)_- = \sum_{k=-\infty}^{-1} \bar{z}^k \mathfrak{n}_1^{(k)}, \quad (\check{\mathfrak{n}}_1)_+ = \sum_{k=0}^{\infty} z^k \mathfrak{n}_1^{(k)}, \quad \check{\mathfrak{n}}_1 = (\check{\mathfrak{n}}_1)_- + (\check{\mathfrak{n}}_1)_+;$$

and

$$(\check{\mathfrak{n}}_2)_- = \sum_{k=-\infty}^{-1} \bar{z}^k \mathfrak{n}_2^{(k)}, \quad (\check{\mathfrak{n}}_2)_+(\zeta) = \sum_{k=0}^{\infty} z^k \mathfrak{n}_2^{(k)}, \quad \check{\mathfrak{n}}_2 = (\check{\mathfrak{n}}_2)_- + (\check{\mathfrak{n}}_2)_+.$$

Then (5.7) can be rewritten in the following form:

$$A\begin{pmatrix} \check{x}(\zeta) \\ \check{\mathfrak{n}}_2(\zeta) \end{pmatrix} = \begin{pmatrix} \zeta^{-1}\big(\check{x}_+(\zeta) - x^{(0)}\big) + \bar{\zeta}\big(\check{x}_-(\zeta) + x^{(0)}\big) \\ \check{\mathfrak{n}}_1(\zeta) \end{pmatrix}, \quad \zeta \in \mathbb{D}. \tag{5.8}$$

Here $x^{(0)}$, $(\check{\mathfrak{n}}_2)_+$, and $(\check{\mathfrak{n}}_1)_-$ are given, and they uniquely determine $\check{x}$, $\check{\mathfrak{n}}_2$, and $\check{\mathfrak{n}}_1$.

Now put

$$x^{(0)} = \mathcal{I}^{(1)} x = \begin{pmatrix} \mathbb{O} \\ \boldsymbol{x} \end{pmatrix}, \quad x \in \mathcal{H},$$

and

$$(\check{\mathfrak{n}}_1)_- = \mathbb{O}, \quad \text{i.e.,} \quad \mathfrak{n}_1^{(k)} = 0, \quad k \le -1.$$

Since obviously, $x^{(0)} \in \mathcal{R}_V$, we have

$$A^* \begin{pmatrix} \mathcal{I}^{(1)}x \\ \mathbb{O} \end{pmatrix} = \begin{pmatrix} V^*\mathcal{I}^{(1)}x \\ \mathbb{O} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \mathbb{O} \\ z\boldsymbol{x} \end{pmatrix} \\ \mathbb{O} \end{pmatrix} \in \mathcal{R}_V.$$

Hence, $x^{(-1)} \in \mathcal{R}_V$ and $\mathfrak{n}_2^{(-1)} = \mathbb{O}$. Iterating this process, we obtain

$$(A^*)^k \begin{pmatrix} \begin{pmatrix} \mathbb{O} \\ \boldsymbol{x} \end{pmatrix} \\ \mathbb{O} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \mathbb{O} \\ z^k\boldsymbol{x} \end{pmatrix} \\ \mathbb{O} \end{pmatrix}, \tag{5.9}$$

and so $x^{(-k)} \in \mathcal{R}_V$ and $\mathfrak{n}_2^{(-k)} = \mathbb{O}$ for $k \geq 1$. The latter means that $(\check{\mathfrak{n}}_2)_- = \mathbb{O}$. Thus with our choice, $\check{\mathfrak{n}}_2 = (\check{\mathfrak{n}}_2)_+$ and $\check{\mathfrak{n}}_1 = (\check{\mathfrak{n}}_1)_+$. Put

$$\psi(\zeta) \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{I}^{(2)} \\ \mathbb{O} \end{pmatrix}^* A \begin{pmatrix} \check{x}(\zeta) \\ \check{\mathfrak{n}}_2(\zeta) \end{pmatrix}.$$

Since $(\check{\mathfrak{n}}_1)_+$ is uniquely determined by $(\check{\mathfrak{n}}_2)_+$, there exists a matrix function

$$\mathfrak{S}(\zeta) = \begin{pmatrix} s_{11}(\zeta) & s_{12}(\zeta) \\ s_{21}(\zeta) & s_{22}(\zeta) \end{pmatrix} : \mathcal{N}_2 \oplus \mathcal{H} \to \mathcal{N}_1 \oplus \mathcal{K}$$

such that

$$\begin{pmatrix} (\check{\mathfrak{n}}_1)_+(\zeta) \\ \psi(\zeta) \end{pmatrix} = \mathfrak{S}(\zeta) \begin{pmatrix} (\check{\mathfrak{n}}_2)_+(\zeta) \\ x \end{pmatrix}.$$

The matrix function $\mathfrak{S}$ is called the *scattering function* of the colligation A. Let us compute the matrix entries of $\mathfrak{S}$.

We have

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : \boldsymbol{X} \oplus \mathcal{N}_2 \to \boldsymbol{X} \oplus \mathcal{N}_1.$$

We have already observed that $P_{\mathcal{N}_1} A \big| \mathcal{N}_2 = \mathbb{O}$, which means that $A_{22} = \mathbb{O}$.

Now we can rewrite (5.8) in the form

$$\begin{cases} A_{11}\check{x}(\zeta) + A_{12}(\check{\mathfrak{n}}_2)_+(\zeta) = \zeta^{-1}\left(\check{x}_+(\zeta) - x^{(0)}\right) + \bar{\zeta}\big(\check{x}_-(\zeta) + x^{(0)}\big), \\ A_{21}\check{x}(\zeta) = (\check{\mathfrak{n}}_1)_+(\zeta). \end{cases}$$

If we separate the analytic and antianalytic parts, we obtain the following:

$$\begin{cases} A_{11}\check{x}_-(\zeta) = \bar{\zeta}\big(\check{x}_-(\zeta) + x^{(0)}\big), \\ A_{11}\check{x}_+(\zeta) + A_{12}(\check{\mathfrak{n}}_2)_+(\zeta) = \zeta^{-1}\left(\check{x}_+(\zeta) - x^{(0)}\right), \\ A_{21}\check{x}_+(\zeta) = \check{\mathfrak{n}}_1(\zeta), \\ A_{21}\check{x}_-(\zeta) = \mathbb{O}. \end{cases} \tag{5.10}$$

We have now from (5.8)

$$A^* \begin{pmatrix} \zeta^{-1}(\check{x}_+(\zeta) - x^{(0)}) + \bar{\zeta}\big(\check{x}_-(\zeta) + x^{(0)}\big) \\ (\check{\mathfrak{n}}_1)_+(\zeta) \end{pmatrix} = \begin{pmatrix} \check{x}(\zeta) \\ (\check{\mathfrak{n}}_2)_+(\zeta) \end{pmatrix}, \quad \zeta \in \mathbb{D}.$$

Taking only the antianalytic parts of the functions, we obtain

$$A_{11}^*\bar{\zeta}\big(\check{x}_-(\zeta)+x^{(0)}\big)=\check{x}_-(\zeta).$$

Hence,

$$(I-\bar{\zeta}A_{11}^*)\check{x}_-(\zeta)=\bar{\zeta}A_{11}^*x^{(0)},$$

and so

$$\begin{aligned}\check{x}_-(\zeta) &= \bar{\zeta}(I-\bar{\zeta}A_{11}^*)^{-1}A_{11}^*x^{(0)}=\bar{\zeta}(I-\bar{\zeta}A_{11}^*)^{-1}A_{11}^*\mathcal{I}^{(1)}x\\ &= (I-\bar{\zeta}A_{11}^*)^{-1}\mathcal{I}^{(1)}x-\mathcal{I}^{(1)}x. \end{aligned}\tag{5.11}$$

Next, we obtain from the second equality in (5.10)

$$(I-\zeta A_{11})\check{x}_+(\zeta)=\zeta A_{12}(\check{\mathfrak{n}}_2)_+(\zeta)+x^{(0)},$$

and so

$$\begin{aligned}\check{x}_+(\zeta) &= (I-\zeta A_{11})^{-1}\Big(\zeta A_{12}(\check{\mathfrak{n}}_2)_+(\zeta)+x^{(0)}\Big)\\ &= \zeta(I-\zeta A_{11})^{-1}A_{12}(\check{\mathfrak{n}}_2)_+(\zeta)+(I-\zeta A_{11})^{-1}\mathcal{I}^{(1)}x. \end{aligned}\tag{5.12}$$

Finally, using this equality and the third equality in (5.10), we obtain

$$\begin{aligned}\check{\mathfrak{n}}_1(\zeta) &= (\zeta)+A_{21}\check{x}_+(\zeta)\\ &= \zeta A_{21}(I-\zeta A_{11})^{-1}A_{12}(\check{\mathfrak{n}}_2)_+(\zeta)+A_{21}(I-\zeta A_{11})^{-1}\mathcal{I}^{(1)}x\\ &= (\zeta A_{21}(I-\zeta A_{11})^{-1}A_{12})(\check{\mathfrak{n}}_2)_+(\zeta)+A_{21}(I-\zeta A_{11})^{-1}\mathcal{I}^{(1)}x. \end{aligned}\tag{5.13}$$

Combining (5.11) and (5.12), we get

$$\begin{aligned}\check{x}(\zeta) &= \zeta(I-\zeta A_{11})^{-1}A_{12}(\check{\mathfrak{n}}_2)_+(\zeta)\\ &+ \big((I-\zeta A_{11})^{-1}+(I-\bar{\zeta}A_{11}^*)^{-1}-I\big)\mathcal{I}^{(1)}x. \end{aligned}\tag{5.14}$$

Now we can find from (5.13) and (5.14) the following formulas for

$$\mathfrak{S}(\zeta)=\begin{pmatrix} s_{11}(\zeta) & s_{12}(\zeta)\\ s_{21}(\zeta) & s_{22}(\zeta)\end{pmatrix}:$$

$$s_{11}(\zeta)=\zeta A_{21}(I-\zeta A_{11})^{-1}A_{12};$$

$$s_{12}(\zeta)=A_{21}(I-\zeta A_{11})^{-1}\mathcal{I}^{(1)};$$

$$s_{21}(\zeta)=\begin{pmatrix}\mathcal{I}^{(2)}\\ \mathbb{O}\end{pmatrix}^*A\begin{pmatrix}\zeta(I-\zeta A_{11})^{-1}A_{12}\\ I\end{pmatrix};$$

$$s_{22}(\zeta)=\begin{pmatrix}\mathcal{I}^{(2)}\\ \mathbb{O}\end{pmatrix}^*A\begin{pmatrix}\big((I-\zeta A_{11})^{-1}+(I-\bar{\zeta}A_{11}^*)^{-1}-I\big)\mathcal{I}^{(1)}\\ \mathbb{O}\end{pmatrix}.$$

Lemma 5.4.

$$s_{11}(\zeta) = P_{\mathcal{N}_1} A(I - \zeta P_{\boldsymbol{X}} A)^{-1} \big| \mathcal{N}_2;$$

$$s_{12}(\zeta) = P_{\mathcal{N}_1} A(I - \zeta P_{\boldsymbol{X}} A)^{-1} \begin{pmatrix} \mathcal{I}^{(1)} \\ \mathbb{O} \end{pmatrix};$$

$$s_{21}(\zeta) = \begin{pmatrix} \mathcal{I}^{(2)} \\ \mathbb{O} \end{pmatrix}^* A(I - \zeta P_{\boldsymbol{X}} A)^{-1} \big| \mathcal{N}_2;$$

$$s_{22}(\zeta) = \begin{pmatrix} \mathcal{I}^{(2)} \\ \mathbb{O} \end{pmatrix}^* A\big((I - \zeta P_{\boldsymbol{X}} A)^{-1} + (I - \bar{\zeta} A^* P_{\boldsymbol{X}})^{-1} - I\big) \begin{pmatrix} \mathcal{I}^{(1)} \\ \mathbb{O} \end{pmatrix}.$$

Proof. We have

$$\begin{aligned}
(I - \zeta P_{\boldsymbol{X}} A)^{-1} &= \begin{pmatrix} I - \zeta A_{11} & -\zeta A_{12} \\ \mathbb{O} & I \end{pmatrix}^{-1} \\
&= \begin{pmatrix} (I - \zeta A_{11})^{-1} & \zeta (I - \zeta A_{11})^{-1} A_{12} \\ \mathbb{O} & I \end{pmatrix},
\end{aligned}$$

and so

$$\begin{aligned}
P_{\mathcal{N}_1} A(I - \zeta P_{\boldsymbol{X}} A)^{-1} \big| \mathcal{N}_2 &= \begin{pmatrix} A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \zeta (I - \zeta A_{11})^{-1} A_{12} \\ I \end{pmatrix} \\
&= A_{22} + \zeta A_{21} (I - \zeta A_{11})^{-1} A_{12}.
\end{aligned}$$

The corresponding formulas for $s_{12}(\zeta)$, $s_{21}(\zeta)$, and $s_{22}(\zeta)$ can be verified in a similar way. ■

Note that the matrix function $\mathfrak{S}$ is uniquely determined by our isometric operator V. Clearly, s_{11}, s_{12}, and s_{21} are analytic functions in $\mathbb{D}$ and $s_{11}(0) = \mathbb{O}$.

Now we can rewrite the conclusion of Lemma 5.4 in the following way:

$$\begin{aligned}
\mathfrak{S}(\zeta) = &\begin{pmatrix} P_{\mathcal{N}_1} \\ \begin{pmatrix} \mathcal{I}^{(2)} \\ \mathbb{O} \end{pmatrix}^* \end{pmatrix} A(I - \zeta P_{\boldsymbol{X}} A)^{-1} \begin{pmatrix} \mathbb{O} & \mathcal{I}^{(1)} \\ I & \mathbb{O} \end{pmatrix} \\
&+ \bar{\zeta} \begin{pmatrix} \mathbb{O} \\ \begin{pmatrix} \mathcal{I}^{(2)} \\ \mathbb{O} \end{pmatrix}^* \end{pmatrix} P_{\boldsymbol{X}} (I - \zeta P_{\boldsymbol{X}} A)^{-1} \begin{pmatrix} \mathbb{O} & \mathcal{I}^{(1)} \\ I & \mathbb{O} \end{pmatrix}.
\end{aligned} \tag{5.15}$$

Theorem 5.5. *For each $\zeta \in \mathbb{D}$ the matrix $\mathfrak{S}(\zeta)$ is a contractive operator from $\mathcal{N}_2 \oplus \mathcal{H}$ into $\mathcal{N}_1 \oplus \mathcal{K}$.*

Proof. Consider the Hilbert space

$$\mathcal{L}_A \stackrel{\text{def}}{=} \cdots \oplus \mathcal{N}_2^{(2)} \oplus \mathcal{N}_2^{(1)} \oplus \mathcal{N}_2^{(0)} \oplus \boldsymbol{X} \oplus \mathcal{N}_1^{(1)} \oplus \mathcal{N}_1^{(2)} \oplus \mathcal{N}_1^{(3)} \oplus \cdots \tag{5.16}$$

where the $\mathcal{N}_2^{(j)}$ are copies of $\mathcal{N}_2$ and the $\mathcal{N}_1^{(j)}$ are copies of $\mathcal{N}_1$. We define the unitary operator U_A on $\mathcal{L}_A$ by

$$\begin{array}{ll} U_A & (\cdots \oplus \xi_2 \oplus \xi_1 \oplus \xi_0 \oplus a \oplus \eta_1 \oplus \eta_2 \oplus \eta_3 \oplus \cdots) \\ = & (\cdots \oplus \xi_3 \oplus \xi_2 \oplus \xi_1 \oplus b \oplus \eta_0 \oplus \eta_1 \oplus \eta_2 \oplus \cdots), \end{array}$$

where $\xi_j \in \mathcal{N}_2^{(j)}$, $\eta_j \in \mathcal{N}_1^{(j)}$, $a \in \boldsymbol{X}$, and

$$\begin{pmatrix} b \\ \eta_0 \end{pmatrix} \stackrel{\text{def}}{=} A \begin{pmatrix} a \\ \xi_0 \end{pmatrix}.$$

Consider now the imbeddings $\mathcal{J} : \boldsymbol{X} \to \mathcal{L}_A$, $\mathcal{J}_1 : \mathcal{N}_1 \to \mathcal{L}_A$, and $\mathcal{J}_2 : \mathcal{N}_2 \to \mathcal{L}_A$ defined as follows. $\mathcal{J}$ maps naturally $\boldsymbol{X}$ onto the subspace

$$\cdots \oplus \{\mathbb{O}\} \oplus \{\mathbb{O}\} \oplus \boldsymbol{X} \oplus \{\mathbb{O}\} \oplus \{\mathbb{O}\} \oplus \cdots$$

of $\mathcal{L}_A$ in the orthogonal expansion (5.16),

$$\mathcal{J}a = \cdots \oplus \mathbb{O} \oplus \mathbb{O} \oplus a \oplus \mathbb{O} \oplus \mathbb{O} \oplus \cdots, \quad a \in \boldsymbol{X}.$$

Similarly, $\mathcal{J}_1$ is a natural imbedding of $\mathcal{N}_1$ onto the subspace

$$\cdots \oplus \{\mathbb{O}\} \oplus \{\mathbb{O}\} \oplus \{\mathbb{O}\} \oplus \mathcal{N}_1^{(1)} \oplus \{\mathbb{O}\} \oplus \{\mathbb{O}\} \oplus \cdots$$

of $\mathcal{L}_A$ and $\mathcal{J}_2$ is a natural imbedding of $\mathcal{N}_2$ onto the subspace

$$\cdots \oplus \{\mathbb{O}\} \oplus \{\mathbb{O}\} \oplus \mathcal{N}_2^{(0)} \oplus \{\mathbb{O}\} \oplus \{\mathbb{O}\} \oplus \{\mathbb{O}\} \oplus \cdots$$

of $\mathcal{L}_A$.

It is easy to verify that (5.15) can be rewritten in the following way:

$$\begin{aligned} \mathfrak{S}(\zeta) &= \begin{pmatrix} \mathcal{J}_1^* \\ \left(\mathcal{I}^{(2)}\right)^* \mathcal{J}^* \end{pmatrix} U_A \left((I - \zeta U_A)^{-1} + (I - \bar{\zeta} U_A^*)^{-1} - I\right) \begin{pmatrix} \mathcal{J}_2 & \mathcal{I}^{(1)} \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{J}_1^* \\ \left(\mathcal{I}^{(2)}\right)^* \mathcal{J}^* \end{pmatrix} U_A \mathcal{P}_{U_A}(\zeta) \begin{pmatrix} \mathcal{J}_2 & \mathcal{I}^{(1)} \end{pmatrix}, \end{aligned}$$

where $\mathcal{P}_{U_A}$ is the Poisson transform of U_A.

Now it can easily be verified that

$$\begin{aligned} &\begin{pmatrix} I & \mathfrak{S}(\zeta) \\ (\mathfrak{S}(\zeta))^* & I \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{J}_2^* \\ \left(\mathcal{I}^{(1)}\right)^* \\ \mathcal{J}_1^* U_A \\ \left(\mathcal{I}^{(2)}\right)^* \mathcal{J}^* U_A \end{pmatrix} \mathcal{P}_{U_A}(\zeta) \begin{pmatrix} \mathcal{J}_2 & \mathcal{I}^{(1)} & U_A^* \mathcal{J}_1 & U_A^* \mathcal{J} \mathcal{I}^{(2)} \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{pmatrix} I & \mathfrak{S}(\zeta) \\ (\mathfrak{S}(\zeta))^* & I \end{pmatrix} \geq \mathbb{O},$$

and so $\|\mathfrak{S}(\zeta)\| \leq 1$ for any $\zeta \in \mathbb{D}$. ■

Definition. The unitary operator U_A on the space $\mathcal{L}_A$ constructed in the proof of Theorem 5.5 is called the *unitary dilation* of the colligation A.

Consider now a unitary colligation $B : \boldsymbol{Z} \oplus \mathcal{N}_1 \to \boldsymbol{Z} \oplus \mathcal{N}_2$ and its unitary dilation U_B on the space

$$\mathcal{L}_B = \cdots \oplus \mathcal{N}_1^{(2)} \oplus \mathcal{N}_1^{(1)} \oplus \mathcal{N}_1^{(0)} \oplus \boldsymbol{X} \oplus \mathcal{N}_2^{(1)} \oplus \mathcal{N}_2^{(2)} \oplus \mathcal{N}_2^{(3)} \oplus \cdots .$$

With the unitary colligation B we associate the operator function $\mathcal{E}_B$ defined by

$$\mathcal{E}_B(\zeta) = \begin{pmatrix} \mathbb{O} & I \end{pmatrix} B(I - \zeta P_{\boldsymbol{Z}} B)^{-1} \begin{pmatrix} \mathbb{O} \\ I \end{pmatrix} : \mathcal{N}_1 \to \mathcal{N}_2.$$

Clearly, $\mathcal{E}_B$ is analytic in $\mathbb{D}$. The function $\mathcal{E}_B$ is called the *characteristic function* of the colligation B. If we consider the block matrix representation of B,

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

we can represent $\mathcal{E}_B$ as follows:

$$\mathcal{E}_B(\zeta) = B_{22} + \zeta B_{21}(I - \zeta B_{11})^{-1} B_{12}.$$

Theorem 5.6.

$$\|\mathcal{E}_B(\zeta)\| \leq 1$$

for any $\zeta \in \mathbb{D}$.

Proof. We denote by $\mathcal{J}_1 : \mathcal{N}_1 \to \mathcal{L}_B$ the natural imbedding of $\mathcal{N}_1$ onto

$$\cdots \oplus \mathbb{O} \oplus \mathbb{O} \oplus \mathcal{N}_1^{(0)} \oplus \mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O} \oplus \cdots \subset \mathcal{L}_B$$

and by $\mathcal{J}_2 : \mathcal{N}_2 \to \mathcal{L}_B$ the natural imbedding of $\mathcal{N}_2$ onto

$$\cdots \oplus \mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O} \oplus \mathcal{N}_2^{(1)} \oplus \mathbb{O} \oplus \mathbb{O} \oplus \cdots \subset \mathcal{L}_B.$$

Then it is easy to verify that

$$\mathcal{E}_B(\zeta) = \mathcal{J}_2^* U_B (I - \zeta U_B)^{-1} \mathcal{J}_1.$$

It is also easy to see that

$$\mathcal{J}_2^* U_B \big((I - \bar{\zeta} U_B^*)^{-1} - I \big) \mathcal{J}_1 = \mathbb{O}.$$

Thus

$$\mathcal{E}_B(\zeta) = \mathcal{J}_2^* U_B \big((I - \zeta U_B)^{-1} + (I - \bar{\zeta} U_B^*)^{-1} - I \big) \mathcal{J}_1 = \mathcal{J}_2^* U_B \mathcal{P}_{U_B} \mathcal{J}_1.$$

Using the following easily verifiable formulas:

$$\mathcal{J}_2^* U_B \mathcal{P}_{U_B}(\zeta) U_B^* \mathcal{J}_2 = I_{\mathcal{N}_2} \quad \text{and} \quad \mathcal{J}_1^* \mathcal{P}_{U_B}(\zeta) \mathcal{J}_1 = \mathcal{I}_{N_1}, \quad \zeta \in \mathbb{D},$$

we obtain

$$\begin{pmatrix} I & (\mathcal{E}_B(\zeta))^* \\ \mathcal{E}_B(\zeta) & \mathcal{I} \end{pmatrix} = \begin{pmatrix} \mathcal{J}_1^* \\ \mathcal{J}_2^* U_B \end{pmatrix} \mathcal{P}_{U_B}(\zeta) \begin{pmatrix} \mathcal{J}_1 & U_B^* \mathcal{J}_2 \end{pmatrix} \geq \mathbb{O}, \quad \zeta \in \mathbb{D}.$$

This implies the result. ■

We are going to prove now that an arbitrary contractive analytic $\mathcal{B}(\mathcal{N}_1, \mathcal{N}_2)$-valued function in $\mathbb{D}$ is the characteristic function of some unitary colligation $B : \boldsymbol{Z} \oplus \mathcal{N}_1 \to \boldsymbol{Z} \oplus \mathcal{N}_2$.

Theorem 5.7. *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be Hilbert spaces and let $\mathcal{E}$ be a function in the unit ball of $H^\infty(\mathcal{B}(\mathcal{N}_1, \mathcal{N}_2))$. Then there exist a Hilbert space $\boldsymbol{Z}$ and a unitary colligation $B : \boldsymbol{Z} \oplus \mathcal{N}_1 \to \boldsymbol{Z} \oplus \mathcal{N}_2$ such that $\mathcal{E}_B = \mathcal{E}$.*

Proof. Assume first that $\mathcal{E}$ is purely contractive, i.e., $\|\mathcal{E}(\zeta)x\| < \|x\|$, $\zeta \in \mathbb{D}$, $x \in \mathcal{N}_1$, $x \neq \mathbb{O}$. Then there exist a completely nonunitary contraction T on a Hilbert space $\boldsymbol{Z}$, unitary operators $\mathcal{U}_1 : \mathcal{N}_1 \to \mathfrak{D}_T$ and $\mathcal{U}_2 : \mathcal{N}_2 \to \mathfrak{D}_{T^*}$ such that

$$\Theta_T(\zeta) = \mathcal{U}_2 \mathcal{E}(\zeta) \mathcal{U}_1^*, \quad \zeta \in \mathbb{D},$$

where Θ_T is the characteristic function of T (see Appendix 1.6). Let us identify $\mathcal{N}_1$ with $\mathfrak{D}_T$ and $\mathcal{N}_2$ with $\mathfrak{D}_{T^*}$ with the help of the operators $\mathcal{U}_1$ and $\mathcal{U}_2$.

Consider the colligation $B : \boldsymbol{Z} \oplus \mathfrak{D}_T \to \boldsymbol{Z} \oplus \mathfrak{D}_{T^*}$ defined by

$$B = \begin{pmatrix} T^* & D_T \big| \mathfrak{D}_T \\ D_{T^*} & -T \big| \mathfrak{D}_T \end{pmatrix}.$$

Then B is a unitary colligation (cf. Lemma 8.2.1). By definition,

$$\mathcal{E}_B(\zeta) = -T + \zeta D_{T^*}(I - \zeta T^*)^{-1} D_T = \Theta_T(\zeta) = \mathcal{E}(\zeta).$$

If $\mathcal{E}$ is not purely contractive, consider the subspace

$$\mathcal{M} \stackrel{\text{def}}{=} \{x \in \mathcal{N}_1 : \ \|\mathcal{E}(0)x\| = \|x\|\}.$$

It follows easily from the maximum modulus principle that $\mathcal{E}\big|\mathcal{M}$ is a constant isometric-valued function and $\mathcal{E}\big|\mathcal{M}^\perp$ is purely contractive.

Consider the subspaces $\tilde{\mathcal{N}_1} \stackrel{\text{def}}{=} \mathcal{N}_1 \ominus \mathcal{M}$ and $\tilde{\mathcal{N}_2} \stackrel{\text{def}}{=} \mathcal{N}_2 \ominus \mathcal{E}(0)\mathcal{M}$ and the purely contractive operator function $\tilde{\mathcal{E}} \in H^\infty(\mathcal{B}(\mathcal{N}_1, \mathcal{N}_2))$ defined by $\tilde{\mathcal{E}}(\zeta) = \mathcal{E}(\zeta)\big|\tilde{\mathcal{N}_1}$. We have just proved that there exists a unitary colligation $\tilde{B} : \boldsymbol{Z} \oplus \tilde{\mathcal{N}_1} \to \boldsymbol{Z} \oplus \tilde{\mathcal{N}_2}$ such that $\mathcal{E}_{\tilde{B}} = \tilde{\mathcal{E}}$. We can now define the unitary colligation $B : \boldsymbol{Z} \oplus \mathcal{N}_1 \to \boldsymbol{Z} \oplus \mathcal{N}_2$ by

$$B(x_1 + x_2) = \tilde{B}x_1 + \mathcal{E}(0)x_2.$$

It is easy to see that B is a unitary colligation and $\mathcal{E}_B = \mathcal{E}$. ∎

Now we are ready to describe the solutions of the Nehari problem. Recall that $\mathfrak{S} = \{s_{jk}\}_{j,k=1,2}$ is the scattering matrix function of our colligation A.

Theorem 5.8. *The set of solutions of the Nehari problem can be parametrized by*

$$\Phi(\zeta) = s_{22}(\zeta) + s_{21}(\zeta)\mathcal{E}(\zeta)\big(I_{\mathcal{N}_1} - s_{11}(\zeta)\mathcal{E}(\zeta)\big)^{-1} s_{12}(\zeta), \quad \zeta \in \mathbb{D}, \quad (5.17)$$

where $\mathcal{E}$ ranges over the unit ball of the space $H^\infty(\mathcal{B}(\mathcal{N}_1, \mathcal{N}_2))$.

Note that here we identify the bounded operator functions on $\mathbb{T}$ with the bounded harmonic operator functions on $\mathbb{D}$.

Proof. By Theorem 5.2, we have to show that formula (5.17) parametrizes the scattering functions of all unitary extensions U of our isometric operator V. By Theorem 5.3, we have to describe the scattering functions of the unitary couplings U of our colligation $A : \boldsymbol{X} \oplus \mathcal{N}_2 \to \boldsymbol{X} \oplus \mathcal{N}_1$ with unitary colligations $B : \boldsymbol{Z} \oplus \mathcal{N}_1 \to \boldsymbol{Z} \oplus \mathcal{N}_2$.

For such a unitary coupling U the scattering function is defined by

$$\Phi_U(\zeta) = \left(\mathcal{I}^{(2)}\right)^* U \left(\sum_{k=-\infty}^{-1} \bar{\zeta}^{|k|} U^k + \sum_{k=0}^{\infty} \zeta^k U^k \right) \begin{pmatrix} \mathcal{I}^{(1)} \\ \mathbb{O} \end{pmatrix}.$$

Let $x^{(0)} \in \boldsymbol{X}$, $y^{(0)} \in \boldsymbol{Z}$. Put

$$\begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} = U^k \begin{pmatrix} x^{(0)} \\ y^{(0)} \end{pmatrix}.$$

Recall that $x^{(k)}$ and $y^{(k)}$ can be found from the system

$$A \begin{pmatrix} x^{(k)} \\ \mathfrak{n}_2^{(k)} \end{pmatrix} = \begin{pmatrix} x^{(k+1)} \\ \mathfrak{n}_1^{(k)} \end{pmatrix}; \tag{5.18}$$

and

$$B \begin{pmatrix} y^{(k)} \\ \mathfrak{n}_1^{(k)} \end{pmatrix} = \begin{pmatrix} y^{(k+1)} \\ \mathfrak{n}_2^{(k)} \end{pmatrix}. \tag{5.19}$$

Now everything is determined by $\begin{pmatrix} x^{(0)} \\ y^{(0)} \end{pmatrix}$. If we put $x^{(0)} = \mathcal{I}^{(1)} x$, $x \in \mathcal{H}$, and $y^{(0)} = \mathbb{O}$, then as we have already seen the equalities $\mathfrak{n}_1^{(k)} = \mathbb{O}$ and $\mathfrak{n}_2^{(k)} = \mathbb{O}$, $k \le -1$, are compatible with (5.18). Obviously, the conditions $y^{(k)} = 0$ for $k \le 0$ are compatible with (5.19). Since everything is uniquely determined by $x^{(0)}$ and $y^{(0)}$, we can make the conclusion that the conditions $x^{(0)} = \mathcal{I}^{(1)} x$ and $y^{(0)} = \mathbb{O}$ in (5.18) and (5.19) imply that $\mathfrak{n}_1(k) = 0$, $\mathfrak{n}_2^{(k)} = 0$, $y^{(k)} = \mathbb{O}$ for $k \le -1$.

We can now rewrite (5.18) and (5.19) in the following way:

$$A \begin{pmatrix} \check{x}(\zeta) \\ (\check{\mathfrak{n}}_2)_+(\zeta) \end{pmatrix} = \begin{pmatrix} \zeta^{-1}\bigl(\check{x}_+(\zeta) - x^{(0)}\bigr) + \bar{\zeta}\bigl(\check{x}_-(\zeta) + x^{(0)}\bigr) \\ (\check{\mathfrak{n}}_1)_+(\zeta) \end{pmatrix} \tag{5.20}$$

and

$$B \begin{pmatrix} \check{y}_+(\zeta) \\ (\check{\mathfrak{n}}_1)_+(\zeta) \end{pmatrix} = \begin{pmatrix} \zeta^{-1}\check{y}_+(\zeta) \\ (\check{\mathfrak{n}}_2)_+(\zeta) \end{pmatrix}, \quad \zeta \in \mathbb{D}, \tag{5.21}$$

where $\check{x}$, $\check{\mathfrak{n}}_1$, and $\check{\mathfrak{n}}_2$ are as above, and

$$\check{y} = \check{y}_+ = \sum_{k>0} z^k y^{(k)}.$$

Then the coupling U acts in the following way:

$$U \begin{pmatrix} \check{x}(\zeta) \\ \check{y}_+(\zeta) \end{pmatrix} = \begin{pmatrix} \zeta^{-1}\bigl(\check{x}_+(\zeta) - x^{(0)}\bigr) + \bar{\zeta}\bigl(\check{x}_-(\zeta) + x^{(0)}\bigr) \\ \zeta^{-1}\check{y}_+(\zeta) \end{pmatrix}.$$

As above, we put

$$\psi(\zeta) = \begin{pmatrix} \mathcal{I}^{(2)} \\ \mathbb{O} \end{pmatrix}^* A \begin{pmatrix} \check{x}(\zeta) \\ \check{\mathfrak{n}}_2(\zeta) \end{pmatrix}, \quad \zeta \in \mathbb{D},$$

and we have

$$\begin{pmatrix} (\check{\mathfrak{n}}_1)_+(\zeta) \\ \psi(\zeta) \end{pmatrix} = \begin{pmatrix} s_{11}(\zeta) & s_{12}(\zeta) \\ s_{21}(\zeta) & s_{22}(\zeta) \end{pmatrix} \begin{pmatrix} (\check{\mathfrak{n}}_2)_+(\zeta) \\ x \end{pmatrix}, \quad \zeta \in \mathbb{D}.$$

We have from (5.21)

$$\zeta \begin{pmatrix} I & \mathbb{O} \end{pmatrix} B \begin{pmatrix} \check{y}_+(\zeta) \\ (\check{\mathfrak{n}}_1)_+(\zeta) \end{pmatrix} = \check{y}_+(\zeta), \quad \zeta \in \mathbb{D}.$$

Thus

$$\zeta\big(B_{11}\check{y}_+(\zeta) + B_{12}(\check{\mathfrak{n}}_1)_+(\zeta)\big) = \check{y}_+(\zeta), \quad \zeta \in \mathbb{D},$$

and so

$$\check{y}_+(\zeta) = \zeta(I - \zeta B_{11})^{-1} B_{12}(\check{\mathfrak{n}}_1)_+(\zeta), \quad \zeta \in \mathbb{D}. \tag{5.22}$$

On the other hand, we have from (5.21)

$$\begin{pmatrix} \mathbb{O} & I \end{pmatrix} B \begin{pmatrix} \check{y}_+(\zeta) \\ (\check{\mathfrak{n}}_1)_+(\zeta) \end{pmatrix} = (\check{\mathfrak{n}}_2)_+(\zeta), \quad \zeta \in \mathbb{D}.$$

Thus

$$B_{21}\check{y}_+(\zeta) + B_{22}(\check{\mathfrak{n}}_1)_+(\zeta) = (\check{\mathfrak{n}}_2)_+(\zeta), \quad \zeta \in \mathbb{D}. \tag{5.23}$$

Substituting (5.22) in (5.23), we obtain

$$(\check{\mathfrak{n}}_2)_+(\zeta) = \big(\zeta B_{21}(I - \zeta B_{11})^{-1} B_{12} + B_{22}\big)(\check{\mathfrak{n}}_1)_+(\zeta), \quad \zeta \in \mathbb{D},$$

i.e.,

$$(\check{\mathfrak{n}}_2)_+(\zeta) = \mathcal{E}_B(\zeta)(\check{\mathfrak{n}}_1)_+(\zeta), \quad \zeta \in \mathbb{D}, \tag{5.24}$$

where $\mathcal{E}_B$ is the characteristic function of the colligation B, i.e.,

$$\mathcal{E}_B(\zeta) = P_{\mathcal{N}_2} B(I - \zeta P_{\boldsymbol{Z}} B)^{-1}\big|\mathcal{N}_1.$$

It follows from (5.24) that

$$\begin{pmatrix} (\check{\mathfrak{n}}_1)_+(\zeta) \\ \psi(\zeta) \end{pmatrix} = \begin{pmatrix} s_{11}(\zeta) & s_{12}(\zeta) \\ s_{21}(\zeta) & s_{22}(\zeta) \end{pmatrix} \begin{pmatrix} \mathcal{E}_B(\zeta)(\check{\mathfrak{n}}_1)_+(\zeta) \\ x \end{pmatrix}, \quad \zeta \in \mathbb{D}, \tag{5.25}$$

and so

$$s_{11}(\zeta)\mathcal{E}_B(\zeta)(\check{\mathfrak{n}}_1)_+(\zeta) + s_{12}(\zeta)x = (\check{\mathfrak{n}}_1)_+(\zeta), \quad \zeta \in \mathbb{D}.$$

Thus

$$(\check{\mathfrak{n}}_1)_+(\zeta) = (I - s_{11}(\zeta)\mathcal{E}_B(\zeta))^{-1} s_{12}(\zeta)x, \quad \zeta \in \mathbb{D}. \tag{5.26}$$

Next, we have

$$\begin{aligned}\psi(\zeta) &= \begin{pmatrix} \mathcal{I}^{(2)} \\ \mathbb{O} \end{pmatrix}^* A \begin{pmatrix} \check{x}(\zeta) \\ \check{\mathfrak{n}}_2(\zeta) \end{pmatrix} \\ &= \big(\mathcal{I}^{(2)}\big)^* \Big(\zeta^{-1}\big(\check{x}_+(\zeta) - x^{(0)}\big) + \bar{\zeta}\big(\check{x}_-(\zeta) + x^{(0)}\big)\Big) \\ &= \big(\mathcal{I}^{(2)}\big)^* U \begin{pmatrix} \check{x}(\zeta) \\ \check{y}_+(\zeta) \end{pmatrix}, \quad \zeta \in \mathbb{D}.\end{aligned}$$

Since $x^{(0)} = \mathcal{I}^{(1)}x$, this implies that

$$\psi(\zeta) = \Phi_U(\zeta)x, \quad \zeta \in \mathbb{D}.$$

Together with (5.25) and (5.26) this implies that

$$\Phi_U(\zeta)x = \big(s_{22}(\zeta) + s_{21}(\zeta)\mathcal{E}_B(\zeta)(I - s_{11}(\zeta)\mathcal{E}_B(\zeta))^{-1}\big)x, \quad \zeta \in \mathbb{D}.$$

It remains to show that for any $\mathcal{E}$ in the unit ball of $H^\infty(\mathcal{B}(\mathcal{N}_1, \mathcal{N}_2))$ the function Φ defined by (5.17) is the scattering function of some unitary extension U of V. This follows from Theorem 5.7. Indeed, given $\mathcal{E}$, we can find a unitary colligation $B : \boldsymbol{Z} \oplus \mathcal{N}_1 \to \boldsymbol{Z} \oplus \mathcal{N}_2$ such that $\mathcal{E} = \mathcal{E}_B$. Then $\Phi = \Phi_U$, where U is the unitary coupling of A and B. ■

Remark. Clearly, the Nehari problem has a unique solution if and only if $\mathcal{N}_1 = \{\mathbb{O}\}$ or $\mathcal{N}_2 = \{\mathbb{O}\}$.

Concluding Remarks

The results of §1 were obtained in Adamyan, Arov, and Krein [2]. Note that the proof of the fact that $\mathfrak{w} \in H^\infty$ (see the proof of Theorem 1.13) was not given in Adamyan, Arov, and Krein [2]. This proof can be found in Ando [1].

The results of §2 were obtained in Nevanlinna [1] by a different method. The example following Theorem 2.1 is taken from Garnett [1], Ch. IV, §4.

The results of §3 are due to Adamyan, Arov, and Krein [3]. Note also that in Adamyan, Arov, and Krein [3] the authors applied these results to the following interpolation problem (the Schur–Takagi problem) for the class $H^\infty_{(m)}$: given points $\zeta_1, \cdots, \zeta_n$ in $\mathbb{D}$ and complex numbers $w_1, \cdots, w_1$, parametrize all functions $f \in H^\infty_{(m)}$ such that $\|f\|_\infty \le 1$ and $f(\zeta_j) = w_j$.

The results of §4 are taken from Adamyan, Arov, and Krein [4].

Section 5 follows the paper Kheifets [1] where the author presents a method that allows him to parametrize the solutions of the Nehari problem in the general case. Note that Adamyan [1] parametrized all solutions of the Nehari problem for matrix functions in the completely indeterminate case, i.e., in the case when the subspaces $\mathcal{D}_V$ and $\mathcal{R}_V$ (see §5) have maximal codimensions. In the unpublished thesis Adamyan [2] the solutions of the Nehari problem were parametrized in the general case of matrix functions.

6

Hankel Operators and Schatten–von Neumann Classes

In this chapter we study Hankel operators that belong to the Schatten–von Neumann class $\boldsymbol{S}_p$, $0 < p < \infty$. The main result of the chapter says that $H_\varphi \in \boldsymbol{S}_p$ if and only if the function $\mathbb{P}_-\varphi$ belongs to the Besov class $B_p^{1/p}$ (see Appendix 2.6). We prove this result in §1 for $p = 1$. We give two different approaches. The first approach gives an explicit representation of a Hankel operator in terms of rank one operators while the second approach is less constructive but it allows one to represent a nuclear Hankel operator as an absolutely convergent series of rank one Hankel operators. We also characterize in §1 nuclear Hankel operators of the form $\Gamma[\mu]$ in terms of measures μ in $\mathbb{D}$. In §2 we prove the main result for $1 < p < \infty$. We use the result for $p = 1$ and the Marcinkiewicz interpolation theorem for linear operators. Finally, in §3 we treat the case $p < 1$. To prove the necessity of the condition $\varphi \in B_p^{1/p}$, we reduce the estimation of Hankel matrices to the estimation of certain special finite matrices that are normal and whose norms can be computed explicitly.

In §4 we use the real interpolation method to describe the Hankel operators of Schatten–Lorentz class $\boldsymbol{S}_{p,q}$. As an application we obtain a description of real interpolation spaces between $B_p^{1/p}$, $0 < p < \infty$, and the space VMO.

We study properties of projections onto the Hankel matrices in §5. We prove that the natural (averaging projection) $\mathcal{P}$ onto the set of Hankel matrices is bounded on $\boldsymbol{S}_p$ for $1 < p < \infty$ but is unbounded on $\boldsymbol{S}_1$ and on the set of bounded (or compact) operators. We also show that there are no bounded projections from the space of bounded (or compact) operators onto the set of bounded (or compact) Hankel operators. However, it turns out that there exist bounded (weighted averaging) projections from $\boldsymbol{S}_1$ onto the set of Hankel matrices in $\boldsymbol{S}_1$. Finally, we prove in §5 that $\mathcal{P}\boldsymbol{S}_1 \subset \boldsymbol{S}_{1,2}$ and if T belongs to the generalized Matsaev ideal $\boldsymbol{S}_{2,\omega}$, i.e.,

$$\sum_{j\geq 0} \frac{(s_j(T))^2}{1+j} < \infty,$$

then $\mathcal{P}T$ is a compact operator. On the other hand, we show in §5 that if $\{s_j\}$ is a nonincreasing sequence of positive numbers such that

$$\sum_{j\geq 0} \frac{s_j^2}{1+j} = \infty,$$

then there exists an operator T such that $s_j(T) = s_j$ and $\mathcal{P}T$ is not the matrix of a bounded operator.

In §6 we apply the main result of the chapter to the study of the rate of rational approximation in terms of properties of functions. We obtain analogs of the Jackson–Bernstein theorem for rational approximation in the norm of BMO. We also obtain some results on rational approximation in the L^∞ norm and in the norm of the Bloch space $(B^0_\infty)_+$. As an application of the nuclearity criterion for Hankel operators we obtain a sharp estimate of the principal part of a meromorphic function with at most n poles in $\mathbb{D}$ in terms of its norm in $L^\infty(\mathbb{T})$.

We consider in §7 several other applications of the main result of this chapter. Namely, we consider functions of the model operators, commutators, integral operators on $L^2(\mathbb{R}_+)$ with kernels depending on the sum of the variables, and Wiener–Hopf operators on finite intervals.

In §8 we consider generalized (weighted) Hankel matrices of the form $\{(1+j)^\alpha(1+k)^\beta\hat{\varphi}(j+k)\}_{j,k\geq 0}$ and we obtain boundedness, compactness, and $\boldsymbol{S}_p$ criteria for such matrices.

Finally, in §9 we study Hankel (and generalized Hankel) operators on spaces of vector functions.

1. Nuclearity of Hankel Operators

In this section we describe the trace class Hankel operators. For convenience we work with Hankel matrices $\Gamma_\varphi = \{\hat{\varphi}(j+k)\}_{j,k\geq 0}$, where $\varphi = \sum_{j\geq 0}\hat{\varphi}(j)z^j$ is a function analytic in the unit disk. We shall treat Hankel matrices Γ_φ as operators on the space ℓ^2.

The following theorem is the main result of this section.

Theorem 1.1. *Let φ be a function analytic in the unit disk. Then the Hankel operator Γ_φ belongs to the trace class $\boldsymbol{S}_1$ if and only if $\varphi \in B^1_1$.*

Recall that the analytic functions of Besov class B_1^1 admit the following descriptions (see Appendix 2.6):

(i) $\int\limits_{\mathbb{D}} |\varphi''(\zeta)| d\boldsymbol{m}_2(\zeta) < \infty$;

(ii) $\int\limits_{\mathbb{T}}\int\limits_{\mathbb{T}} \frac{|\varphi(\zeta\tau)-2\varphi(\zeta)+\varphi(\zeta\bar{\tau})|}{|\tau-1|^2} d\boldsymbol{m}(\zeta) d\boldsymbol{m}(\tau) < \infty$;

(iii) $\sum\limits_{n\geq 0} 2^n \|\varphi * W_n\|_{L^1} < \infty$ (the polynomials W_n are defined in Appendix 2.6).

The following description of trace class Hankel operators H_φ is an immediate consequence of Theorem 1.1.

Corollary 1.2. *Let φ be a function on $\mathbb{T}$ of class BMO. Then $H_\varphi \in \boldsymbol{S}_1$ if and only if $\mathbb{P}_-\varphi \in B_1^1$.*

There are several different proofs of Theorem 1.1. First we present a proof that gives sharp estimates for the norms $\|\Gamma_\varphi\|_{\boldsymbol{S}_1}$ from above and from below. Then we give another proof of the sufficiency of the condition $\varphi \in B_1^1$ that allows one to expand a trace class Hankel operator into an absolutely convergent series of rank one Hankel operators. In §5 we give another proof of the necessity of the condition $\varphi \in B_1^1$.

In the proof given below we use the duality between $\boldsymbol{S}_1$ and the space of bounded linear operators $\mathcal{B}$ (see Appendix 1.1). Recall that we consider the following pairing between $\boldsymbol{S}_1$ and $\mathcal{B}$:

$$\langle T, R\rangle = \operatorname{trace} TR^*, \quad T \in \boldsymbol{S}_1,\ R \in \mathcal{B}.$$

Proof of Theorem 1.1. Let us first prove that $\Gamma_\varphi \in \boldsymbol{S}_1$ if $\varphi \in B_1^1$. We have

$$\varphi = \sum_{n\geq 0} \varphi * W_n$$

and

$$\sum_{n\geq 0} 2^n \|\varphi * W_n\|_{L^1} < \infty$$

(see Appendix 2.6). Clearly, $\varphi * W_n$ is a polynomial of degree at most $2^{n+1}-1$. The following lemma gives a sharp estimate of the trace norm of a Hankel operator with polynomial symbol.

Lemma 1.3. *Let f be an analytic polynomial of degree m. Then*

$$\|\Gamma_f\|_{\boldsymbol{S}_1} \leq (m+1)\|f\|_1.$$

Proof of Lemma 1.3. Given $\zeta \in \mathbb{T}$, we define the elements x_ζ and y_ζ of ℓ^2 by

$$x_\zeta(j) = \begin{cases} \zeta^j, & 0 \le j \le m, \\ 0, & j > m; \end{cases}$$

$$y_\zeta(k) = \begin{cases} f(\zeta)\bar{\zeta}^k, & 0 \le k \le m, \\ 0, & k > m. \end{cases}$$

Define the rank one operator A_ζ on ℓ^2 by $A_\zeta x = (x, x_\zeta) y_\zeta$, $x \in \ell^2$. Then $A_\zeta \in \boldsymbol{S}_1$ and $\|A_\zeta\|_{\boldsymbol{S}_1} = \|x_\zeta\|_{\ell^2} \|y_\zeta\|_{\ell^2} = (m+1)|f(\zeta)|$. Let us prove that

$$\Gamma_f = \int_{\mathbb{T}} A_\zeta d\boldsymbol{m}(\zeta) \tag{1.1}$$

(the function $\zeta \mapsto A_\zeta$ is continuous and so the integral can be understood as the limit of integral sums). We have

$$(\Gamma_f e_j, e_k) = \hat{f}(j+k) = \int_{\mathbb{T}} f(\zeta) \bar{\zeta}^{j+k} d\boldsymbol{m}(\zeta),$$

$$(A_\zeta e_j, e_k) = f(\zeta) \bar{\zeta}^j \bar{\zeta}^k, \quad j, k \le m.$$

Since $\hat{f}(k) = 0$ for $k > m$, (1.1) holds and

$$\|\Gamma_f\|_{\boldsymbol{S}_1} \le \int_{\mathbb{T}} \|A_\zeta\|_{\boldsymbol{S}_1} d\boldsymbol{m}(\zeta) \le (m+1) \int_{\mathbb{T}} |f(\zeta)| d\boldsymbol{m}(\zeta). \quad \blacksquare$$

Let us now complete the proof of the sufficiency of the condition $f \in B_1^1$. It follows from Lemma 1.3 that

$$\|\Gamma_\varphi\|_{\boldsymbol{S}_1} \le \sum_{n \ge 0} \|\Gamma_{\varphi * W_n}\|_{\boldsymbol{S}_1} \le \sum_{n \ge 0} 2^{n+1} \|\varphi * W_n\|_{L^1}.$$

Suppose now that $\Gamma_\varphi \in \boldsymbol{S}_1$. Define the polynomials Q_n and R_n, $n \ge 1$, by

$$\hat{Q}_n(k) = \begin{cases} 0, & k \le 2^{n-1}, \\ 1 - |k - 2^n|/2^{n-1}, & 2^{n-1} \le k \le 2^n + 2^{n-1}, \\ 0 & k \ge 2^n + 2^{n-1}; \end{cases}$$

$$\hat{R}_n(k) = \begin{cases} 0, & k \le 2^n, \\ 1 - |k - 2^n - 2^{n-1}|/2^{n-1}, & 2^n \le k \le 2^{n+1}, \\ 0 & k \ge 2^{n+1}. \end{cases}$$

Clearly, $W_n = Q_n + \frac{1}{2} R_n$, $n \ge 1$.

Let us show that

$$\sum_{n \ge 0} 2^{2n+1} \|\varphi * Q_{2n+1}\|_{L^1} < \infty. \tag{1.2}$$

The inequalities $\sum_{n \ge 1} 2^{2n} \|\varphi * Q_{2n}\|_{L^1} < \infty$, $\sum_{n \ge 0} 2^{2n+1} \|\varphi * R_{2n+1}\|_{L^1} < \infty$, and $\sum_{n \ge 1} 2^{2n} \|\varphi * R_{2n}\| < \infty$ can be proved in exactly the same way.

To prove (1.2), we construct an operator B on ℓ^2 such that $\|B\| \le 1$ and $\langle \Gamma_\varphi, B\rangle = \sum_{n\ge 0} 2^{2n}\|f * Q_{2n+1}\|_{L^1}$.

Consider the squares

$$S_n = [2^{2n-1}, 2^{2n-1} + 2^{2n} - 1] \times [2^{2n-1} + 1, 2^{2n-1} + 2^{2n}], \quad n \ge 1,$$

on the plane.

Let $\{\psi_n\}_{n\ge 1}$ be a sequence of functions in L^∞ such that $\|\psi_n\|_{L^\infty} \le 1$. We define the matrix $\{b_{jk}\}_{j,k\ge 0}$ of B by

$$b_{jk} = \begin{cases} \hat{\psi}_n(j+k), & (j,k) \in S_n, \quad n \ge 1, \\ 0, & (j,k) \notin \bigcup_{n\ge 1} S_n. \end{cases}$$

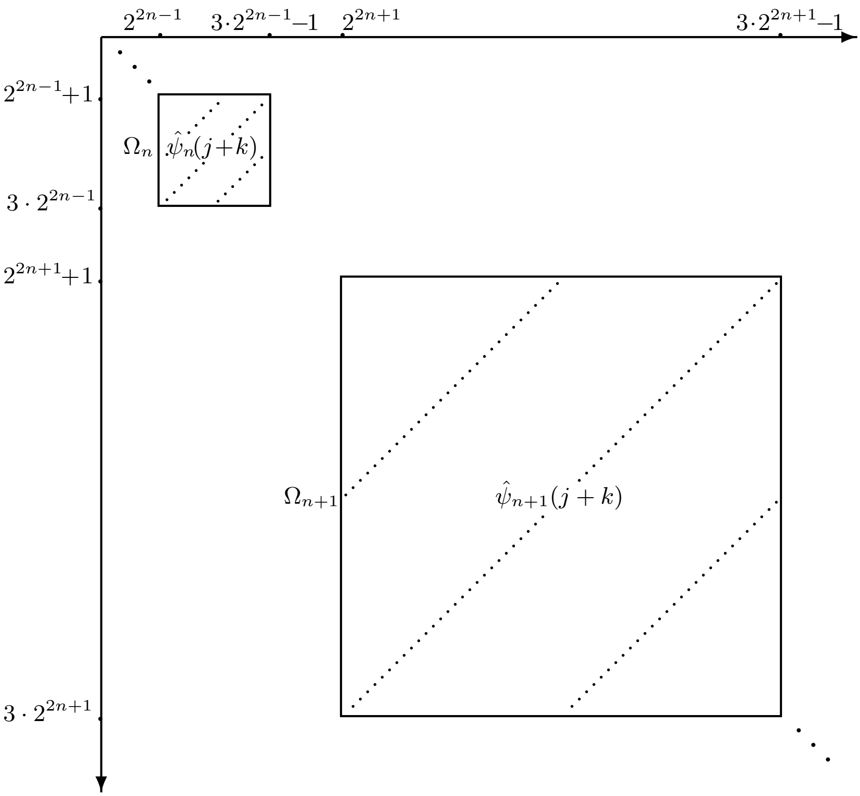

FIGURE 1.

Let us show that $\|B\| \le 1$. Consider the subspaces

$$\mathcal{H}_n = \operatorname{span}\{e_j : \ 2^{2n-1} \le j \le 2^{2n-1} + 2^{2n} - 1\},$$

$$\mathcal{H}'_n = \operatorname{span}\{e_j : \ 2^{2n-1} + 1 \le j \le 2^{2n-1} + 2^{2n}\}.$$

It is easy to see that

$$B = \sum_{n \ge 1} P'_n \Gamma_{\psi_n} P_n,$$

where P_n and P'_n are the orthogonal projection onto $\mathcal{H}_n$ and $\mathcal{H}'_n$, respectively. Since the spaces $\{\mathcal{H}_n\}_{n \ge 1}$ are pairwise orthogonal as well as the spaces $\{\mathcal{H}'_n\}_{n \ge 1}$, we have

$$\|B\| = \sup_n \|P'_n \Gamma_{\psi_n} P_n\| \le \sup_n \|\Gamma_{\psi_n}\| \le \sup_n \|\psi_n\|_{L^\infty} \le 1.$$

Let us show that

$$\langle \Gamma_\varphi, B \rangle = \sum_{n \ge 0} 2^{2n} (Q_{2n+1} * \varphi, \psi_n),$$

where as usual $(g,h) = \int\limits_{\mathbb{T}} f(\zeta)\overline{h(\zeta)} d\boldsymbol{m}(\zeta)$, $g \in L^1$, $h \in L^\infty$. We have

$$\begin{aligned}
\langle \Gamma_\varphi, B \rangle &= \sum_{n \ge 1} \langle \Gamma_\varphi, P'_n \Gamma_{\psi_n} P_n \rangle \\
&= \sum_{n \ge 1} \sum_{j = 2^{2n}}^{2^{2n} + 2^{2n+1}} (2^{2n} - |j - 2^{2n+1}|) \hat{\varphi}(j) \overline{\hat{\psi}_n(j)} \\
&= \sum_{n \ge 1} 2^{2n} (Q_{2n+1} * \varphi, \psi_n).
\end{aligned}$$

We can pick now a sequence $\{\psi_n\}_{n \ge 1}$ such that

$$(Q_{2n+1} * \varphi, \psi_n) = \|Q_{2n+1} * \varphi\|_{L^1}.$$

Then

$$\langle \Gamma_\varphi, B \rangle = \sum_{n \ge 1} 2^{2n} \|Q_{2n+1} * \varphi\|_{L^1}.$$

Hence, $\sum\limits_{n \ge 1} 2^{2n+1} \|Q_{2n+1} * \varphi\|_{L^1} = 2\langle \Gamma_\varphi, B \rangle \le 2\|\Gamma_\varphi\|_{\boldsymbol{S}_1} < \infty$. ■

Remark. It is easy to see that the above proof gives the following estimates

$$\frac{1}{6} \sum_{n \ge 1} 2^n \|\varphi * W_n\|_{L^1} \le \|\Gamma_\varphi\|_{\boldsymbol{S}_1} \le 2 \sum_{n \ge 0} 2^n \|\varphi * W_n\|_{L^1}.$$

To give an alternative proof of the sufficiency of the condition $\varphi \in B_1^1$ in Theorem 1.1, we need the following decomposition theorem for the class $(B_1^1)_+$.

Theorem 1.4. *Let $\varphi \in (B_1^1)_+$. Then there exist sequences of complex numbers $\{c_n\}_{n\ge1}$ and $\{\lambda_n\}_{n\ge1}$, $\lambda_n \in \mathbb{D}$, such that*

$$\varphi(\zeta) = \sum_{n\ge1} \frac{c_n}{1-\lambda_n\zeta}, \quad \zeta \in \mathbb{D}, \tag{1.3}$$

$$\sum_{n\ge1} \frac{|c_n|}{1-|\lambda_n|} < \infty. \tag{1.4}$$

Proof. Consider the space $(b_\infty^{-1})_+$ of functions analytic in $\mathbb{D}$ (see Appendix 2.6). Then its dual can naturally be identified with $(B_1^1)_+$ with respect to the following pairing:

$$(f,\varphi) = \sum_{n\ge0} \hat{f}(n)\overline{\hat{\varphi}(n)}, \tag{1.5}$$

where $\varphi \in (B_1^1)_+$ and f is a polynomial in $(b_\infty^{-1})_+$ (see Appendix 2.6). We consider here the following norm on $(b_\infty^{-1})_+$:

$$\|f\|_{\&} \stackrel{\text{def}}{=} \sup_{\zeta\in\mathbb{D}} |f(\zeta)|(1-|\zeta|).$$

Let X be the space of functions analytic in $\mathbb{D}$ that admit a representation (1.3) satisfying (1.4). The norm of a function φ in X is, by definition, the infimum of the right-hand side of (1.4) over all representations (1.3). Let us show that $(b_\infty^{-1})_+^* = X$ isometrically with respect to the pairing (1.5), which would imply that $X = (B_1^1)_+$ and the proof would be completed.

Let φ be as in (1.3), $f \in (b_\infty^{-1})_+$. We have

$$\begin{aligned} |(f,\varphi)| &= \left|\sum_{n\ge0} c_n \left(f, \frac{1}{1-\lambda_n z}\right)\right| = \left|\sum_{n\ge0} c_n f(\overline{\lambda_n})\right| \\ &\le \sum_{n\ge0} |c_n|\cdot|(f(\overline{\lambda_n}))| \le \sum_{n\ge0} \frac{|c_n|}{1-|\lambda_n|}\cdot\|f\|_{\&}. \end{aligned} \tag{1.6}$$

Hence, $\varphi \in (b_\infty^{-1})_+^*$ and $\|\varphi\|_{(b_\infty^{-1})_+^*} \le \|\varphi\|_X$. Let us now show that

$$\|f\|_{(b_\infty^{-1})_+} = \sup\{|(f,\varphi)| : \ \|\varphi\|_X \le 1\}. \tag{1.7}$$

It follows from (1.6) that the left-hand side of (1.7) is less that or equal to the right-hand side. Let us prove the opposite inequality. Let $\varphi(z) = \frac{1-|\lambda|}{1-\lambda z}$. We have

$$(f,\varphi) = (1-|\lambda|)f(\bar\lambda)$$

and since $\|\varphi\|_X \le 1$, this proves (1.7).

Therefore the norm $\|\cdot\|_X$ coincides on X with the norm $\|\cdot\|_{(b_\infty^{-1})_+^*}$ and so X is a closed subspace of B_1^1. The result follows from the fact that the linear combinations of functions of the form $(1-\lambda z)^{-1}$, $\lambda\in\mathbb{D}$, are dense in $(B_1^1)_+$. ■

Theorem 1.4 enables us to prove that any nuclear Hankel operator can be expanded into absolutely convergent series of rank one Hankel operators.

Theorem 1.5. *Let $\varphi \in (B_1^1)_+$. Then Γ_φ admits a representation*

$$\Gamma_\varphi = \sum_{n\geq 1} \Gamma_n,$$

where the Γ_n are Hankel operators of rank one and $\sum\limits_{n\geq 1} \|\Gamma_n\| < \infty$.

Proof. By Theorem 1.4, φ admits a representation (1.3) that satisfies (1.4). Let

$$\Gamma_n = \Gamma_{\frac{c_n}{1-\lambda_n z}}$$

Clearly, $\operatorname{rank}\Gamma_n = 1$ (see Theorem 3.1 of Chapter 1). The result follows from the obvious estimate

$$\left\|\Gamma_{\frac{1}{1-\lambda z}}\right\| \leq \frac{1}{1-|\lambda|}. \quad \blacksquare$$

As we have mentioned above, Theorem 1.5 gives an alternative proof of the necessity of the condition $f \in B_1^1$ for Γ_f to be nuclear.

In §1.7 we have considered the Hankel operators of the form $\Gamma[\mu]$, where μ is a finite complex measure on $\mathbb{D}$. The following theorem describes a class of measures that produce trace class Hankel operators.

Theorem 1.6. *Let μ be a finite complex measure on $\mathbb{D}$ such that*

$$\int_{\mathbb{D}} \frac{1}{(1-|\zeta|^2)} d\mu(\zeta) < \infty. \tag{1.8}$$

Then $\Gamma[\mu]$ is nuclear. Conversely, if Γ is a nuclear Hankel operator on ℓ^2, there exists a finite complex measure μ satisfying (1.8) *such that $\Gamma = \Gamma[\mu]$.*

Note that the proof that (1.8) implies the nuclearity of $\Gamma[\mu]$ is similar to the proof of the sufficiency of the condition $\varphi \in B_1^1$ in Theorem 1.1.

Proof. Suppose that μ satisfies (1.8). Let $\zeta \in \mathbb{D}$. Define the vector $f_\zeta \in \ell^2$ by

$$f_\zeta = (1-|\zeta|^2)^{1/2}\{\zeta^n\}_{n\geq 0}.$$

Consider the rank one operator A_ζ defined by $A_\zeta x = (x, f_{\bar\zeta}) f_\zeta$, $x \in \ell^2$. Clearly, $\|A_\zeta\| = 1$, $\zeta \in \mathbb{D}$. Let us show that $\Gamma[\mu] = \int\limits_{\mathbb{D}} (1-|\zeta|^2)^{-1} A_\zeta d\mu(\zeta)$. We have

$$\int_{\mathbb{D}} (A_\zeta e_j, e_k)(1-|\zeta|^2)^{-1} d\mu(\zeta) = \int_{\mathbb{D}} \zeta^{j+k} d\mu(\zeta) = (\Gamma[\mu]e_j, e_k),$$

where $\{e_j\}_{j\geq 0}$ is the standard orthonormal basis of the space ℓ^2. Therefore $\Gamma[\mu] = \int\limits_{\mathbb{D}} (1-|\zeta|^2)^{-1} A_\zeta d\mu(\zeta)$, and so

$$\|\Gamma[\mu]\|_{\boldsymbol{S}_1} \leq \int_{\mathbb{D}} \|A_\zeta\|(1-|\zeta|^2)^{-1} d\mu(\zeta) = \int_{\mathbb{D}} (1-|\zeta|^2)^{-1} d\mu(\zeta) < \infty.$$

Suppose now that Γ is a nuclear Hankel operator on ℓ^2. Then by Theorems 1.1 and 1.5, $\Gamma = \Gamma_\varphi$, where φ admits a representation (1.3) that satisfies (1.4). Put

$$\mu \stackrel{\text{def}}{=} \sum_{n\ge 1} c_n \delta_{\lambda_n},$$

where δ_λ is the unit mass at λ. It is easy to see that $\Gamma = \Gamma[\mu]$ and

$$\int_{\mathbb{D}} \frac{1}{(1-|\zeta|^2)} d\mu(\zeta) = \sum_{n\ge 1} \frac{|c_n|}{(1-|\lambda_n|^2)} < \infty,$$

which completes the proof. ■

Consider now positive Hankel matrices of trace class.

Theorem 1.7. *Let Γ be a positive Hankel operator on ℓ^2. Then $\Gamma \in \boldsymbol{S}_1$ if and only if $\Gamma = \Gamma[\mu]$ where μ is a positive measure on $(-1,1)$ such that*

$$\int_{-1}^{1} (1-t^2)^{-1} d\mu(t) < \infty. \tag{1.9}$$

Proof. If $\Gamma = \Gamma[\mu]$ and μ satisfies (1.9), then by Theorem 1.6, $\Gamma \in \boldsymbol{S}_1$. Suppose now that Γ is a positive nuclear operator. By Theorem 1.7.2, $\Gamma = \Gamma[\mu]$, where μ is a positive measure on $(-1,1)$. We have

$$\operatorname{trace} \Gamma[\mu] = \sum_{j\ge 0} (\Gamma[\mu] e_j, e_j) = \sum_{j\ge 0} \int_{-1}^{1} t^{2j} d\mu(t) = \int_{-1}^{1} (1-t^2)^{-1} d\mu(t) < \infty,$$

which completes the proof. ■

2. Hankel Operators of Class $\boldsymbol{S}_p$, $1 < p < \infty$

In this section we obtain a criterion for a Hankel operator to belong to $\boldsymbol{S}_p$ for $1 < p < \infty$. We are going to use the nuclearity criterion obtained in §1 and the Marcinkiewicz interpolation theorem. As in §1 we shall work with operators Γ_φ on ℓ^2 whose matrix in the standard basis is $\{\hat{\varphi}(n+k)\}_{n,k\ge 0}$. As usual we identify operators on ℓ^2 with their matrices in the standard basis of ℓ^2. The following theorem is the main result of the section.

Theorem 2.1. *Let $1 < p < \infty$ and let φ be a function analytic in $\mathbb{D}$. Then $\Gamma_\varphi \in \boldsymbol{S}_p$ if and only if $f \in B_p^{1/p}$.*

Corollary 2.2. *Let $1 < p < \infty$ and let φ be a function on $\mathbb{T}$ of class BMO. Then $H_\varphi \in \boldsymbol{S}_p$ if and only if $\mathbb{P}_-\varphi \in B_p^{1/p}$.*

We are going to use the following special case of the Marcinkiewicz interpolation theorem:

Let $(\mathcal{X}, \mu)$ be a measure space and let $\mathcal{A}$ be a bounded linear operator from ℓ^∞ to $L^\infty(\mathcal{X}, \mu)$. Suppose that $\mathcal{A}$ has weak type (1,1)*, i.e.,*

$$\mu\{x \in \mathcal{X} : |(\mathcal{A}\xi)(x)| > \delta\} \le \operatorname{const} \frac{\|\xi\|_{\ell^1}}{\delta}, \quad \xi \in \ell^1, \quad \delta > 0.$$

Then the restriction of $\mathcal{A}$ to ℓ^p is a bounded operator from ℓ^p to $L^p(\mathcal{X},\mu)$ for $1<p<\infty$.

Consider the closed subspace c_0 of ℓ^∞, which consists of sequences converging to zero. The conclusion of the Marcinkiewicz theorem holds if $\mathcal{A}$ is a bounded operator from c_0 to $L^\infty(\mathcal{X},\mu)$ and has weak type (1,1). We need the following consequence of the Marcinkiewicz interpolation theorem. Recall that $\boldsymbol{S}_\infty$ is the space of compact operators on Hilbert space.

Corollary 2.3. *Let $\mathcal{V}$ be a bounded linear operator from $\boldsymbol{S}_\infty$ to $L^\infty(\mathcal{X},\mu)$. Suppose that $\mathcal{A}$ has weak type* (1,1), *i.e.,*

$$\mu\{x\in\mathcal{X}:\ |(\mathcal{V}T)(x)|>\delta\}\le\operatorname{const}\frac{\|T\|_{\boldsymbol{S}_1}}{\delta},\quad T\in\boldsymbol{S}_1,\quad \delta>0.$$

Then the restriction of $\mathcal{V}$ to $\boldsymbol{S}_p$ is a bounded operator from $\boldsymbol{S}_p$ to $L^p(\mathcal{X},\mu)$ for $1<p<\infty$.

Proof. Suppose that $T\in\boldsymbol{S}_p$, $1<p<\infty$. Let us show that $\mathcal{V}T\in L^p(\mathcal{X},\mu)$. Consider the Schmidt expansion of T: $T=\sum\limits_{j\ge0}s_j(\cdot,f_j)g_j$. We define the operator $\mathcal{A}$ on c_0 by

$$\mathcal{A}\{x_j\}_{j\ge0}=\mathcal{V}\left(\sum_{j\ge0}x_j(\cdot,f_j)g_j\right).$$

Obviously, $\mathcal{A}$ is a bounded operator from c_0 to $L^\infty(\mathcal{X},\mu)$ and $\mathcal{A}$ has weak type (1,1). Consequently, by the Marcinkiewicz interpolation theorem $\mathcal{A}$ is a bounded operator from ℓ^p to $L^p(\mathcal{X},\mu)$. Clearly,

$$\mathcal{V}T=\mathcal{A}\{s_j\}_{j\ge0}\in L^p(\mathcal{X},\mu).\quad\blacksquare$$

To prove Theorem 2.1 we consider the dual spaces $\left(B_p^{1/p}\right)^*$, $1\le p<\infty$. Recall (see Appendix 2.6) that the dual space $\left(B_p^{1/p}\right)_+^*$, $1\le p<\infty$, can be identified with the space $\left(B_{p'}^{-1/p}\right)_+$ of functions analytic in $\mathbb{D}$ and we consider the following pairing

$$(f,g)=\sum_{j\ge0}\hat f(j)\overline{\hat g(j)},\quad f\in\boldsymbol{\mathcal{P}}_+,\quad g\in\left(B_{p'}^{-1/p}\right)_+.$$

For an operator T on ℓ^2 we define the analytic function $\mathcal{W}T$ by

$$(\mathcal{W}T)(z)=\sum_{n\ge0}t_nz^n,\quad t_n\stackrel{\text{def}}{=}\sum_{j=0}^n(Te_j,e_{n-j}),$$

where $\{e_j\}_{j\ge0}$ is the standard orthonormal basis of ℓ^2.

Lemma 2.4. *Let $1<p\le\infty$. Then $\mathcal{W}$ is a bounded operator from $\boldsymbol{S}_p$ to $B_p^{-1/p'}$.*

Proof. Consider the measure μ on the Borel subsets of $\mathbb{D}$ defined by

$$\mu(E) \stackrel{\text{def}}{=} \int_E \frac{1}{(1-|\zeta|)^2} d\boldsymbol{m}_2(\zeta).$$

Let $\mathcal{V}$ be the following transformation defined on the set of operators on ℓ^2:

$$(\mathcal{V}T)(z) \stackrel{\text{def}}{=} (1-|z|)^2(\mathcal{W}T)'(z).$$

To prove the lemma it suffices to show that $\mathcal{V}$ is a continuous operator from $\boldsymbol{S}_p$ to $L^p(\mathbb{D},\mu)$, $1 < p \le \infty$. Indeed, this would mean that

$$\int_{\mathbb{D}} |(\mathcal{W}T)'|^p(1-|\zeta|)^{2p-2} d\boldsymbol{m}_2(\zeta) < \infty, \quad 1 < p < \infty;$$

$$\sup_{\zeta\in\mathbb{D}} (1-|\zeta|)^2|(\mathcal{W}T)'(\zeta)| < \infty, \quad p = \infty.$$

The latter means exactly that $\mathcal{W}T \in B_p^{-1/p'}$.

Consider first the case $p = \infty$. We have to show that $\mathcal{W}T \in (B_1^1)_+^*$ whenever $T \in \boldsymbol{S}_\infty$. Let φ be a polynomial. Then

$$\langle \Gamma_\varphi, T\rangle = \operatorname{trace}\Gamma_\varphi T^* = \sum_{n\ge 0}\hat{\varphi}(n)\sum_{j=0}^{n}\overline{(Te_j, e_{n-j})} = (\varphi, \mathcal{W}T).$$

Therefore

$$|(\varphi,\mathcal{W}T)| \le \|T\|\cdot\|\Gamma_\varphi\|_{\boldsymbol{S}_1} \le \text{const}\cdot\|T\|\cdot\|\varphi\|_{B_1^1}.$$

It follows that $\mathcal{W}T$ determines a continuous linear functional on $(B_1^1)_+$, which means that $\mathcal{W}T \in B_\infty^{-1}$. Hence, $\mathcal{V}$ is a bounded operator from $\boldsymbol{S}_\infty$ to $L^\infty(\mathbb{D},\mu)$.

Let us show that $\mathcal{V}$ has weak type (1,1), i.e.,

$$\mu\{\zeta\in\mathbb{D} : |(\mathcal{V}T)(\zeta)| > \delta\} \le \text{const}\,\frac{\|T\|_{\boldsymbol{S}_1}}{\delta}, \quad \delta > 0.$$

Let $T \in \boldsymbol{S}_1$. Then $\mathcal{W}T \in H^1$. Indeed, it is sufficient to establish this only for rank one operator T. Let $T = (\cdot,\xi)\eta$, where $\xi = \{\xi_n\}_{n\ge 0}$, $y = \{y_n\}_{n\ge 0} \in \ell^2$. Consider the functions f and g in H^2 defined by $f(z) = \sum\limits_{n\ge 0}\bar{\xi}_n z^n$, $\sum\limits_{n\ge 0}\eta_n z^n \in H^2$. Clearly, $\mathcal{W}T = fg$ and, consequently, $\mathcal{W}T \in H^1$. (If we use the fact that every function in H^1 can be represented as a product of two functions in H^2, we get that $\mathcal{W}\boldsymbol{S}_1 = H^1$.)

We now use the Littlewood–Paley theorem (see Appendix 2.6):

$$\varphi \in H^1 \Rightarrow \int_{\mathbb{T}}\left(\int_0^1 |\varphi'(r\zeta)|^2(1-r)dr\right)^{1/2} d\boldsymbol{m}(\zeta) < \infty.$$

To prove that $\mathcal{V}$ has weak type (1,1), it suffices to prove the following assertion: if ψ is a measurable function on $[0,1]$ and

$$E_\delta = \{x\in[0,1] : |\psi(x)|(1-x)^2 > \delta\},$$

then

$$\int_{E_\delta} \frac{dx}{(1-x)^2} \le \frac{\text{const}}{\delta}\left(\int_0^1 |\psi(x)|^2(1-x)dx\right)^{1/2}. \tag{2.1}$$

Indeed, inequality (2.1) implies that $\mathcal{V}$ has weak type (1,1), since by (2.1)

$$\begin{aligned}&\mu\{\zeta\in\mathbb{D}:\ (1-|\zeta|)^2(\mathcal{W}T)'(\zeta)|>\delta\}\\ &\le \frac{\text{const}}{\delta}\int_{\mathbb{T}}\left(\int_0^1 |(\mathcal{W}T)'(r\zeta)|^2(1-r)dr\right)^{1/2} d\boldsymbol{m}(\zeta),\end{aligned}$$

and the integral on the right-hand side is finite by the Paley–Littlewood theorem.

Let us prove (2.1). We have

$$\int_0^1 |\psi(x)|^2(1-x)dx \ge \int_{E_\delta}|\psi(x)|^2(1-x)dx \ge \delta^2\int_{E_\delta}\frac{dx}{(1-x)^3}.$$

Therefore it is sufficient to show that

$$\int_E \frac{dx}{(1-x)^2} \le \text{const}\left(\int_E \frac{dx}{(1-x)^3}\right)^{1/2} \tag{2.2}$$

for an arbitrary measurable subset of [0,1]. Let us transfer the interval (0,1] onto the half-line $[1,\infty)$ by means of the substitution $s = 1/(1-x)$. It is easy to see that (2.2) is equivalent to the inequality

$$\int_F ds \le \text{const}\left(\int_F s\,ds\right)^{1/2}$$

for every measurable subset of $[1,\infty)$. It is easy to see that

$$\int_F s\,ds \ge \text{const}\int_1^{a+1} s\,ds = \frac{1}{2}((a+1)^2-1),$$

where $a = \int_F ds$ is the Lebesgue measure of F. Hence,

$$\left(\int_F s\,ds\right)^{1/2} \ge \left(\frac{a^2}{2}+a\right)^{1/2} \ge \frac{a}{\sqrt{2}} = \frac{1}{\sqrt{2}}\int_F ds.$$

Thus $\mathcal{V}$ has weak type (1,1), and so by Corollary 2.3, $\mathcal{V}$ is a bounded operator from $\boldsymbol{S}_p$ to $L^p(\mathbb{D},\mu)$ for $1<p\le\infty$. As we have already mentioned, it follows that $\mathcal{W}$ is a bounded operator from $\boldsymbol{S}_p$ to $B_p^{-1p'}$, $1<p\le\infty$. ■

Proof of Theorem 2.1. Suppose that $\Gamma_\varphi\in\boldsymbol{S}_p$. Let us show that $\varphi\in B_p^{1/p}$. By Lemma 2.4, $\mathcal{W}\Gamma_\varphi\in B_p^{-1/p'}$. This means that

$$\int_{\mathbb{D}}\left|\sum_{n\ge0}(n+1)\hat\varphi(n)\zeta^n\right|^p(1-|\zeta|)^{p-2}d\boldsymbol{m}_2(\zeta)<\infty.$$

Clearly, this is equivalent to the fact that $\int\limits_{\mathbb{D}}|\varphi'(\zeta)|(1-|\zeta|)^{p-2}d\boldsymbol{m}_2(\zeta)<\infty$. Hence, $\varphi\in B_p^{1/p}$.

Suppose now that $\varphi\in B_p^{1/p}$. Let us show that $\Gamma_\varphi\in\boldsymbol{S}_p$. Let T be an operator on ℓ^2 that has only finitely many nonzero matrix entries. We have

$$\langle T,\Gamma_\varphi\rangle=\sum_{n\ge0}\overline{\hat{\varphi}(n)}\sum_{j=0}^{n}(Te_j,e_{n-j})=(\mathcal{W}T,\varphi).$$

Therefore

$$|\langle T,\Gamma_\varphi\rangle|\le\text{const}\cdot\|\varphi\|_{B_p^{1/p}}\|\mathcal{W}T\|_{B_{p'}^{-1/p}}\le\text{const}\cdot\|\varphi\|_{B_p^{1/p}}\|T\|_{\boldsymbol{S}_p}$$

by Lemma 2.4. This implies that Γ_φ determines a continuous linear functional on $\boldsymbol{S}_{p'}$, that is, $\Gamma_\varphi\in\boldsymbol{S}_p$. ■

3. Hankel Operators of Class $\boldsymbol{S}_p$, $0<p<1$

In this section we describe the Hankel operators of class $\boldsymbol{S}_p$ with $0<p<1$. As in the previous sections for for a function φ analytic in the unit disk we denote by Γ_φ the Hankel matrix $\{\hat{\varphi}(j+k)\}_{j,k\ge0}$, which we identify with an operator on ℓ^2. The following theorem is the main result of the section.

Theorem 3.1. *Suppose that $0<p<1$ and let φ be a function analytic in $\mathbb{D}$. Then $\Gamma_\varphi\in\boldsymbol{S}_p$ if and only if $\varphi\in B_p^{1/p}$.*

Corollary 3.2. *Let $0<p<1$ and let φ be a function on $\mathbb{T}$ of class BMO. Then $H_\varphi\in\boldsymbol{S}_p$ if and only if $\mathbb{P}_-\varphi\in B_p^{1/p}$.*

The proof of the sufficiency is based on the same idea as in the case $p=1$ (see §1). The only difference is that instead of integrating over the unit circle $\mathbb{T}$ we integrate over its discrete subgroups. However, the proof of necessity is much trickier. Unlike the case $p\ge1$, we cannot use duality arguments. Instead we introduce certain operations on matrices that do not increase the $\boldsymbol{S}_p$ quasinorm. Using such operations we cut certain "pieces" of Γ_φ and then paste them to get a matrix whose singular values can be evaluated explicitly. This will allow us to get lower estimates for $\|\Gamma_\varphi\|_{\boldsymbol{S}_p}$.

In this section we use the following definition of the Besov space $B_p^{1/p}$ (see Appendix 2.6). Let $v\in C^\infty(\mathbb{R})$ such that $v\ge0$, $\operatorname{supp}v=[1/2,2]$, and $\sum\limits_{n\ge0}v(x/2^n)=1$ for $x\ge1$. The kernels V_n are defined by

$$V_n=\sum_{k\ge0}v\left(\frac{k}{2^n}\right)z^k,\quad n\ge1,\quad V_0(z)=\bar{z}+1+z.$$

A function φ analytic in $\mathbb{D}$ belongs to $B_p^{1/p}$ if and only if

$$\sum_{n\ge 0} 2^n \|\varphi * V_n\|_p^p < \infty.$$

Recall (see Appendix 1.1) that for $0<p<1$ the $\boldsymbol{S}_p$ quasinorm satisfies the following version of triangle inequality:

$$\|T_1+T_2\|_{\boldsymbol{S}_p}^p \le \|T_1\|_{\boldsymbol{S}_p}^p + \|T_2\|_{\boldsymbol{S}_p}^p, \quad T_1,\ T_2 \in \boldsymbol{S}_p. \tag{3.1}$$

We start with an auxiliary fact that we need to estimate certain "pieces" of Γ_φ. Let F be an infinitely differentiable function on $\mathbb{R}$ with compact support and let m be a positive integer. Consider the trigonometric polynomial

$$F_m(z) = \sum_{k\in\mathbb{Z}} F\left(\frac{k}{m}\right) z^k.$$

Recall that $\mathcal{F}F$ is the Fourier transform of F, $(\mathcal{F}F)(t) = \int\limits_{\mathbb{R}} F(x)e^{-2\pi \mathrm{i} xt}dx$.

Lemma 3.3. *Let*

$$G_m(t) = F_m(e^{2\pi \mathrm{i} t}) - m(\mathcal{F}F)(-mt), \quad t\in[-1/2,1/2].$$

Then

$$\lim_{m\to\infty} \|G_m\|_{L^\infty[-1/2,1/2]} m^N = 0$$

for all $N\in\mathbb{Z}_+$.

Proof. Let $\psi(x) = F(x/m)e^{2\pi \mathrm{i} xt}$. Then

$$(\mathcal{F}\psi)(y) = m(\mathcal{F}F)(m(y-t)).$$

Let us apply the Poisson summation formula for ψ (see Stein and Weiss [1], Ch. VII, §2):

$$\sum_{k\in\mathbb{Z}} \psi(k) = \sum_{k\in\mathbb{Z}} (\mathcal{F}\psi)(k).$$

It follows that

$$\sum_{k\in\mathbb{Z}} F\left(\frac{k}{m}\right) e^{2\pi \mathrm{i} kt} = m(\mathcal{F}F)(-mt) + \sum_{k\in\mathbb{Z},\, k\ne 0} m(\mathcal{F}F)(m(k-t)).$$

Then

$$G_m = \sum_{k\in\mathbb{Z},\, k\ne 0} m(\mathcal{F}F)(m(k-t)).$$

Since F is infinitely smooth and has compact support, we have

$$|(\mathcal{F}F)(x)| \le \frac{c_N}{(1+|x|)^N}$$

for all $N\in\mathbb{Z}_+$. Hence,

$$\begin{aligned}\|G_m\|_{L^\infty[-1/2,1/2]} &\le \sum_{k\ne 0} m \sup\{|\mathcal{F}F)(x)| :\ x\in[m(k-1/2), m(k+1/2)]\} \\ &\le 2c_N \sum_{k>0} \frac{1}{(1+m(k-1/2))^N} \le \text{const}\, \frac{m}{(1+m/2)^N},\end{aligned}$$

which implies the result. ■

Corollary 3.4. *$\|F_m\|_{L^p(\mathbb{T})} \le c_F\, m^{1-1/p}$ for some constant c_F.*

Proof. By Lemma 3.3 the result follows from the obvious inequality

$$\int_{-1/2}^{1/2} |m(\mathcal{F}F)(-mt)|^p dt \le m^{p-1}\int_{\mathbb{R}} |(\mathcal{F}F)(t)|^p dt. \quad ■$$

We identify operators on ℓ^2 with their matrices in the standard orthonormal basis $\{e_j\}_{j\ge0}$ of ℓ^2. We need the following notion.

Definition. Let $A=\{a_{jk}\}_{j,k\ge0}$, $B=\{b_{jk}\}_{j,k\ge0}$ be matrices. The *Schur product* of A and B is by definition the matrix

$$A\star B=\{a_{jk}b_{jk}\}_{j,k\ge0}.$$

If Ω is a bounded subset of $\mathbb{R}_+\times\mathbb{R}_+$, we denote by $\mathcal{P}_\Omega$ the operator whose matrix is given by

$$(\mathcal{P}_\Omega e_k, e_n)=\begin{cases}1, & (n,k)\in\Omega,\\ 0, & (n,k)\notin\Omega.\end{cases}$$

The following lemma is important both for upper and lower estimates.

Lemma 3.5. *Let ψ be a polynomial of degree $m-1$ and let $A\in\boldsymbol{S}_p$. Then*

$$\|\Gamma_\psi\star A\|_{\boldsymbol{S}_p}\le(2m)^{1/p-1}\|\psi\|_p\|A\|_{\boldsymbol{S}_p}. \tag{3.2}$$

Proof. It suffices to show that (3.2) holds for operators A of rank one. Indeed, if A is an arbitrary operator, then we can consider its Schmidt expansion and apply (3.2) to each term. Then inequality (3.2) for A follows from (3.1).

Let $Ax=(x,\alpha)\beta$, $x\in\ell^2$, where $\alpha=\{\alpha_n\}_{n\ge0}$, $\beta=\{\beta_n\}_{n\ge0}$. Let $\zeta_j=e^{2\pi\mathrm{i}j/(2m)}$, $0\le j\le 2m-1$. Define ℓ^2 vectors f_j and g_j, $0\le j\le 2m-1$, by

$$f_j(k)=\begin{cases}\overline{\psi(\zeta_j)}\alpha_k\zeta_j^k, & 0\le k<m,\\ 0, & k\ge m;\end{cases}\qquad g_j(n)=\begin{cases}\beta_n\bar\zeta_j^n, & 0\le n<m,\\ 0, & n\ge m.\end{cases}$$

We define the rank one operators A_j, $0\le j\le 2m-1$, by

$$A_jx=(x,f_j)g_j,\quad x\in\ell^2.$$

Let us show that

$$\Gamma_\psi\star A=\frac{1}{2m}\sum_{j=0}^{2m-1}A_j. \tag{3.3}$$

Indeed, if $k\ge m$ or $n\ge m$, then $(A_je_k,e_n)=((\Gamma_\psi\star A)e_k,e_n)=0$. On the other hand, if $n<m$ and $k<m$, then

$$(A_je_k,e_n)=\psi(\zeta_j)\overline{\alpha}_k\beta_n\bar\zeta_j^{n+k}$$

and

$$((\Gamma_\psi\star A)e_k,e_n)=\overline{\alpha}_k\beta_n\hat\psi(n+k).$$

Identity (3.3) now follows from the equality

$$\hat{\psi}(d) = \frac{1}{2m}\sum_{j=0}^{2m-1}\psi(\zeta_j)\bar{\zeta}_j^d, \quad 0 \le d \le 2m-1,$$

which is true for every polynomial ψ of degree at most $2m-1$.

We have

$$\|A_j\|_{\boldsymbol{S}_p} \le \|\alpha\|_{\ell^2}\|\beta\|_{\ell^2}|\psi(\zeta_j)| = |\psi(\zeta_j)|\cdot\|A\|_{\boldsymbol{S}_p}.$$

It now follows from (3.1) that

$$\|\Gamma_\psi \star A\|_{\boldsymbol{S}_p}^p \le \frac{1}{(2m)^p}\|A\|_{\boldsymbol{S}_p}^p \sum_{j=0}^{2m-1}|\psi(\zeta_j)|^p. \tag{3.4}$$

For $\tau \in \mathbb{T}$ we define the polynomial ψ_τ by $\psi_\tau(\zeta) = \psi(\tau\zeta)$. Let us show that $\|\Gamma_{\psi_\tau} \star A\|_{\boldsymbol{S}_p} = \|\Gamma_\psi \star A\|_{\boldsymbol{S}_p}$ for all $\tau \in \mathbb{T}$. Indeed, this follows from the obvious equality

$$\Gamma_{\psi_\tau} \star A = D_\tau(\Gamma_\psi \star A)D_\tau,$$

where D_t is the unitary operator on ℓ^2 defined by $D_\tau e_n = \tau^n e_n$ (here $\{e_n\}_{n\ge0}$ is the standard orthonormal basis of ℓ^2).

To complete the proof, we integrate inequality (3.4) in τ:

$$\begin{aligned}\|\Gamma_\psi \star A\|_{\boldsymbol{S}_p}^p &= \int_{\mathbb{T}}\|\Gamma_{\psi_\tau}\star A\|_{\boldsymbol{S}_p}^p d\boldsymbol{m}(\tau)\\ &\le \frac{1}{(2m)^p}\|A\|_{\boldsymbol{S}_p}^p\sum_{j=0}^{2m-1}\int_{\mathbb{T}}|\psi_\tau(\zeta_j)|^p d\boldsymbol{m}(\tau)\\ &= \frac{2m}{(2m)^p}\|A\|_{\boldsymbol{S}_p}^p\|\psi\|_p^p = (2m)^{1-p}\|A\|_{\boldsymbol{S}_p}^p\|\psi\|_p^p. \quad\blacksquare\end{aligned}$$

Remark. It can also be proved that

$$\|\Gamma_\psi \star A\|_{\boldsymbol{S}_p} \le (4m)^{1/p-1}\|\psi\|_p\|A\|_{\boldsymbol{S}_p}$$

for an arbitrary trigonometric polynomial ψ of degree at most $m-1$ (in this case $\Gamma_\psi = \{\hat{\psi}(n+k)\}_{n,k\ge0}$). The proof is exactly the same, except that it is necessary to consider the set of points $\{e^{2\pi i j/(4m)}\}_{0\le j\le 4m-1}$.

Corollary 3.6. *Let ψ be a polynomial of degree m and suppose that a, b, c, d are integers such that $0 \le a < b < m$, $0 \le c < d < m$. Put $\Omega = [a,b)\times[c,d)$. Then*

$$\|\mathcal{P}_\Omega \star \Gamma_\psi\|_{\boldsymbol{S}_p} \le ((b-a)(d-c))^{1/2}(2m)^{1/p-1}\|\psi\|_p.$$

Proof. We have $\mathcal{P}_\Omega x = (x,\alpha)\beta$, $x \in \ell^2$, where $\alpha = \sum\limits_{n=c}^{d-1} e_n$ and $\beta = \sum\limits_{n=a}^{b-1} e_n$. Therefore $\|\mathcal{P}_\Omega\|_{\boldsymbol{S}_p} = ((b-a)(d-c))^{1/2}$. The result now follows immediately from Lemma 3.6 by putting $A = \mathcal{P}_\Omega$. $\blacksquare$

Corollary 3.7. *Let ψ be a polynomial of degree $m-1$. Then*

$$\|\Gamma_\psi\|_{\boldsymbol{S}_p} \le 2^{1/p-1} m^{1/p} \|\psi\|_p.$$

Proof. It suffices to let $a=c=0$ and $b=d=m$. ■

To obtain a lower estimate for $\|\Gamma_\varphi\|_{\boldsymbol{S}_p}$ we evaluate explicitly the singular values of the following matrix:

$$A = \begin{pmatrix} c_0 & c_1 & \cdots & c_{N-2} & c_{N-1} \\ c_{N-1} & c_0 & \cdots & c_{N-3} & c_{N-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & \cdots & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{N-1} & c_0 \end{pmatrix}, \tag{3.5}$$

where the c_j, $0 \le j \le N-1$, are complex numbers.

Lemma 3.8. *Let $\psi(z) = \sum\limits_{k=0}^{N-1} c_k z^k$ and $\zeta_j = e^{2\pi \mathrm{i} j/N}$, $0 \le j \le N-1$. Then*

$$\|A\|_{\boldsymbol{S}_p} = \left(\sum_{j=0}^{N-1} |\psi(\zeta_j)|^p \right)^{1/p}. \tag{3.6}$$

Proof. Let x_j be the vector with coordinates $1, \zeta_j, \zeta_j^2, \cdots, \zeta_j^{N-1}$. It is easy to verify that the vectors x_j are pairwise orthogonal and $Ax_j = \psi(\zeta_j)x_j$. It follows that the matrix A is normal and so (3.6) holds. ■

Remark. Let

$$B = \begin{pmatrix} c_1 & c_2 & \cdots & c_{N-1} & c_0 \\ c_2 & c_3 & \cdots & c_0 & c_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{N-1} & c_0 & \cdots & c_{N-3} & c_{N-2} \\ c_0 & c_1 & \cdots & c_{N-2} & c_{N-1} \end{pmatrix}.$$

It is easy to see that $\|B\|_{\boldsymbol{S}_p} = \|A\|_{\boldsymbol{S}_p}$.

Consider now the squares $\Omega_n = [2^{n-3}, 2^{n+1}] \times [2^{n-3}, 2^{n+1}]$ in $\mathbb{R}_+ \times \mathbb{R}_+$. We are now in a position to get a lower estimate for $\|\mathcal{P}_{\Omega_n} * \Gamma_\varphi\|_{\boldsymbol{S}_p}$.

Lemma 3.9. $\|\mathcal{P}_{\Omega_n} \star \Gamma_\varphi\|_{\boldsymbol{S}_p} \ge \text{const} \cdot 2^{n/p} \|\varphi * V_n\|_p$.

Proof. It is not very convenient for us to deal with the polynomial $\varphi * V_n$ since the interval $[2^{n-1}+1, 2^{n+1}-1]$ on which the Fourier coefficients of V_n are nonzero is too wide for our purpose. To overcome this problem, we consider a representation of the function v, in terms of which the polynomials V_n are defined, in the form $v = \sum\limits_{s=0}^{10} r_s$, where $r_s \in C^\infty(\mathbb{R})$,

$\operatorname{supp} r_s = [1/2+s/8, 1/2+(s+1)/8]$. Consider the polynomials $R_n^{(s)}$ defined by

$$R_n^{(s)}(z) = \sum_{k\in\mathbb{Z}} r_s\left(\frac{k}{2^n}\right) z^k, \quad n \in \mathbb{Z}_+.$$

Clearly, $V_n = \sum_{s=0}^{10} R_n^{(s)}$. Obviously, it suffices to prove that

$$2^n \left\|\varphi * R_n^{(s)}\right\|_p^p \le \text{const} \|\mathcal{P}_{\Omega_n} \star \Gamma_\varphi\|_{\boldsymbol{S}_p}^p, \quad 0 \le s \le 10.$$

To be definite, assume that $s = 0$ (the proof is the same for the remaining values of s) and put $R_n \stackrel{\text{def}}{=} R_n^{(0)}$. Clearly,

$$\operatorname{supp} \hat{R}_n = \{k : \ 2^{n-1} < k < 2^{n-1} + 2^{n-2}\}.$$

We have

$$\Gamma_{R_n} \star \mathcal{P}_{\Omega_n} \star \Gamma_\varphi = \mathcal{P}_{\Omega_n} \star \Gamma_{\varphi * R_n}.$$

Consequently, by Lemma 3.5,

$$\|\mathcal{P}_{\Omega_n} \star \Gamma_{\varphi * R_n}\|_{\boldsymbol{S}_p} \le \text{const} \cdot 2^{n(1-1/p)} \|R_n\|_p \|\mathcal{P}_{\Omega_n} \star \Gamma_f\|_{\boldsymbol{S}_p}.$$

In view of Corollary 3.4, $\|R_n\|_p \le \text{const} \cdot 2^{n(1-1/p)}$, and so

$$\|\mathcal{P}_{\Omega_n} \star \Gamma_{\varphi * R_n}\|_{\boldsymbol{S}_p} \le \text{const} \|\mathcal{P}_{\Omega_n} \star \Gamma_\varphi\|_{\boldsymbol{S}_p}.$$

Therefore it is sufficient to get a lower estimate for $\|\mathcal{P}_{\Omega_n} \star \Gamma_{\varphi * R_n}\|_{\boldsymbol{S}_p}$.

Let $f \stackrel{\text{def}}{=} \varphi * R_n$, $N = 2^{n-2}$, $c_k = \hat{f}(2^{n-1} + k)$, $k = 0, 1, \cdots, N$, $I = [2^{n-3}, 2^{n-2} + 2^{n-3})$, $J_1 = (2^{n-3}, 2^{n-2} + 2^{n-3}]$, and $J_2 = (2^{n-2} + 2^{n-3}, 2^{n-1} + 2^{n-3}]$. Obviously,

$$\|\mathcal{P}_{I\times J_1} \star \Gamma_f\|_{\boldsymbol{S}_p} \le \|\mathcal{P}_{\Omega_n} \star \Gamma_f\|_{\boldsymbol{S}_p},$$
$$\|\mathcal{P}_{I\times J_2} \star \Gamma_f\|_{\boldsymbol{S}_p} \le \|\mathcal{P}_{\Omega_n} \star \Gamma_f\|_{\boldsymbol{S}_p}.$$

Consider the matrices

$$M_1 = \begin{pmatrix} c_1 & c_2 & \cdots & c_{N-1} & c_N \\ c_2 & c_3 & \cdots & c_N & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{N-1} & c_N & \cdots & 0 & 0 \\ c_N & 0 & \cdots & 0 & 0 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & \cdots & c_{N-1} & 0 \\ c_2 & c_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{N-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & c_0 \\ 0 & 0 & \cdots & c_0 & c_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c_0 & \cdots & c_{N-3} & c_{N-2} \\ c_0 & c_1 & \cdots & c_{N-2} & c_{N-1} \end{pmatrix}.$$

It is easy to see that

$$\|M_1\|_{\boldsymbol{S}_p} = \|\mathcal{P}_{I\times J_2} \star \Gamma_f\|_{\boldsymbol{S}_p} \le \|\mathcal{P}_{\Omega_n} \star \Gamma_f\|_{\boldsymbol{S}_p},$$

$$\|M_2\|_{\boldsymbol{S}_p} = \|\mathcal{P}_{I\times J_1} \star \Gamma_f\|_{\boldsymbol{S}_p} \le \|\mathcal{P}_{\Omega_n} \star \Gamma_f\|_{\boldsymbol{S}_p}.$$

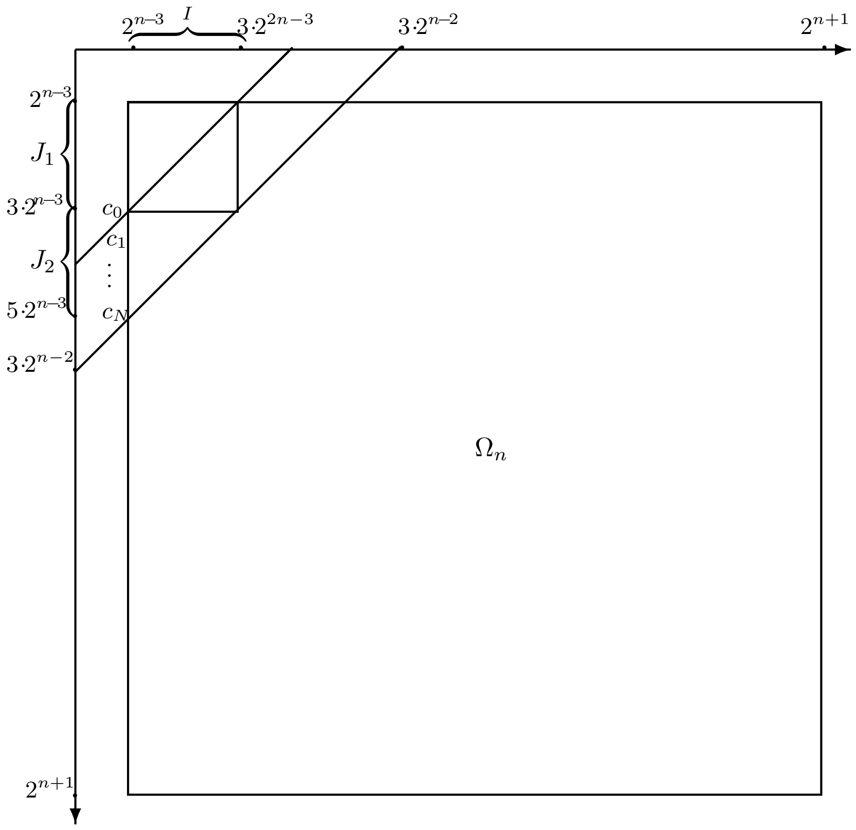

FIGURE 2.

Hence,

$$\left\| \begin{pmatrix} c_1 & c_2 & \cdots & c_{N-1} & c_0 \\ c_2 & c_3 & \cdots & c_0 & c_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{N-1} & c_0 & \cdots & c_{N-3} & c_{N-2} \\ c_0 & c_1 & \cdots & c_{N-2} & c_{N-1} \end{pmatrix} \right\|_{\boldsymbol{S}_p}^p = \|M_1+M_2\|_{\boldsymbol{S}_p}^p \le 2\|\mathcal{P}_{\Omega_n}\star\Gamma_f\|_{\boldsymbol{S}_p}^p.$$

Since $f = z^{2^{n-1}} \sum_{j=0}^{N-1} c_j z^j$, it follows from Lemma 3.8 and the remark after it that

$$\sum_{j=0}^{N-1} |f(\zeta_j)|^p \le 2\|\mathcal{P}_{\Omega_n} \star \Gamma_f\|_{\boldsymbol{S}_p}^p, \tag{3.7}$$

where $\zeta_j = e^{2\pi \mathrm{i} j/N}$, $0 \le j \le N-1$. To get a lower estimate of the right-hand side of (3.7) in terms of $\|f\|_p$, we argue as in the proof of Lemma 3.5. Let $\tau \in \mathbb{T}$, put $f_\tau(\zeta) \stackrel{\text{def}}{=} f(\tau\zeta)$, $\zeta \in \mathbb{T}$. It is easy to see that

$$\mathcal{P}_{\Omega_n} \star \Gamma_{f_\tau} = D_\tau(\mathcal{P}_{\Omega_n} \star \Gamma_f) D_\tau,$$

where the unitary operator D_τ is defined in the proof of Lemma 3.5. Hence,

$$\|\mathcal{P}_{\Omega_n} \star \Gamma_{f_\tau}\|_{\boldsymbol{S}_p} = \|\mathcal{P}_{\Omega_n} \star \Gamma_f\|_{\boldsymbol{S}_p}, \quad \tau \in \mathbb{T}.$$

Applying inequality (3.7) to f_τ and integrating with respect to τ, we obtain

$$2^{n-3}\|\varphi\|_p^p \le \|\mathcal{P}_{\Omega_n} \star \Gamma_f\|_{\boldsymbol{S}_p}^p. \quad \blacksquare$$

Proof of Theorem 3.1. Assume first that $\varphi \in B_p^{1/p}$. By Corollary 3.7 and inequality (3.1) we have

$$\|\Gamma_\varphi\|_{\boldsymbol{S}_p}^p \le \sum_{n\ge 0} \|\Gamma_{\varphi * V_n}\|_{\boldsymbol{S}_p}^p \le 2^{1-p} \sum_{n \ge 0} 2^n \|\varphi * V_n\|_p^p.$$

Consequently, $\Gamma_\varphi \in \boldsymbol{S}_p$.

Assume now that $\Gamma_\varphi \in \boldsymbol{S}_p$. We have to show that $\varphi \in B_p^{1/p}$. Fix an integer $M > 4$ whose choice will be specified later. Without loss of generality we can assume that $\hat\varphi(k) = 0$ for $k \le 2^M$, since Γ_φ is clearly in $\boldsymbol{S}_p$ for every polynomial φ.

Let us observe that it suffices to prove that

$$\|\Gamma_\psi\|_{\boldsymbol{S}_p}^p \ge \operatorname{const} \|\psi\|_{B_p^{1/p}} \tag{3.8}$$

for every polynomial ψ. Indeed, suppose that $q \in C^\infty(\mathbb{R})$, $\operatorname{supp} q = [-2, 2]$, and $q(t) = 1$ for $t \in [-1, 1]$. Let

$$Q_n = \sum_{k \in \mathbb{Z}} q\left(\frac{k}{2^n}\right) z^k,$$

and consider the polynomials $\varphi * Q_n$. It follows from (3.8) that

$$\|\varphi * Q_n\|_{B_p^{1/p}} \le \operatorname{const} \|\Gamma_{\varphi * Q_n}\|_{\boldsymbol{S}_p} = \operatorname{const} \|\Gamma_\varphi \star \Gamma_{Q_n}\|_{\boldsymbol{S}_p}.$$

In view of the remark after Lemma 3.5 we have

$$\|\Gamma_\varphi \star \Gamma_{Q_n}\|_{\boldsymbol{S}_p} \le \operatorname{const} \|Q_n\|_p 2^{n(1/p-1)} \|\Gamma_\varphi\|_{\boldsymbol{S}_p}$$

and, since $\|Q_n\|_p \le \operatorname{const} 2^{n(1-1/p)}$ (see Corollary 3.4), we obtain

$$\|\varphi * Q_n\|_{B_p^{1/p}} \le \operatorname{const}.$$

Therefore

$$\sum_{j=0}^{n-1} 2^j \|\varphi * V_j\|_p^p = \sum_{j=0}^{n-1} 2^j \|(\varphi * Q_n) * V_j\|_p^p \le \|\varphi * Q_n\|_{B_p^{1/p}}^p \le \text{const},$$

which implies that $\varphi \in B_p^{1/p}$.

To prove inequality (3.8), consider the orthogonal projection P_j onto

$$\operatorname{span}\left\{e_s:\ s \in \bigcup_{k\ge1}\left[2^{kM+j-3}, 2^{kM+j+1}\right]\right\}, \quad 0 \le j \le M-1.$$

Define the operator T by

$$T = \sum_{j=0}^{M-1} \oplus P_j \Gamma_\varphi P_j.$$

Obviously, $\|T\|_{\boldsymbol{S}_p}^p \le M\|\Gamma_\varphi\|_{\boldsymbol{S}_p}^p$. Let us estimate $\|T\|_{\boldsymbol{S}_p}$ from below. It is clear that $\|T\|_{\boldsymbol{S}_p}^p = \sum\limits_{j=0}^{M-1} \|P_j\Gamma_\varphi P_j\|_{\boldsymbol{S}_p}^p$.

Let

$$T_j^{(1)} \stackrel{\text{def}}{=} \sum_{k\ge1} \mathcal{P}_{\Omega_{kM+j}} \star \Gamma_\varphi \quad \text{and} \quad T_j^{(2)} = P_j\Gamma_\varphi P_j - T_j^{(1)}.$$

Since the intervals $[2^{kM+j-3}, 2^{kM+j+1}]$, $k \ge 1$, are pairwise disjoint, it follows that

$$\left\|T_j^{(1)}\right\|_{\boldsymbol{S}_p}^p = \sum_{k\ge1} \|\mathcal{P}_{\Omega_{kM+j}} \star \Gamma_\varphi\|_{\boldsymbol{S}_p}^p.$$

By Lemma 3.9,

$$\left\|T_j^{(1)}\right\|_{\boldsymbol{S}_p}^p \ge \text{const} \sum_{k\ge1} 2^{kM+j} \|\varphi * V_{kM+j}\|_p^p.$$

Let us get an upper estimate for $\left\|T_j^{(2)}\right\|_{\boldsymbol{S}_p}$. Put

$$\Delta_j^{(k)} = \left[2^{kM+j-3}, 2^{kM+j+1}\right] \times \left[0, 2^{(k-1)M+j+1}\right],$$

$$\Upsilon_j^{(k)} = \left[0, 2^{(k-1)M+j+1}\right] \times \left[2^{kM+j-3}, 2^{kM+j+1}\right].$$

It is easy to see that

$$T_j^{(2)} = \sum_{k\ge1} \mathcal{P}_{\Delta_j^{(k)}} \star T_j^{(2)} + \sum_{k\ge1} \mathcal{P}_{\Upsilon_j^{(k)}} \star T_j^{(2)}.$$

Consequently,

$$\left\|T_j^{(2)}\right\|_{\boldsymbol{S}_p}^p \le 2\sum_{k\ge1} \left\|\mathcal{P}_{\Delta_j^{(k)}} \star T_j^{(2)}\right\|_{\boldsymbol{S}_p}^p.$$

Obviously, $\left\|\mathcal{P}_{\Delta_j^{(k)}} \star T_j^{(2)}\right\|_{\boldsymbol{S}_p}^p \le \left\|\mathcal{P}_{\Delta_j^{(k)}} \star \Gamma_\varphi\right\|_{\boldsymbol{S}_p}^p$. It is also clear that

$$\mathcal{P}_{\Delta_j^{(k)}} \star \Gamma_\varphi = \mathcal{P}_{\Delta_j^{(k)}} \star \Gamma_\psi,$$

where

$$\psi = \varphi * \sum_{s=-3}^{2} V_{kM+j+s}.$$

Hence, by Corollary 3.6,

$$\begin{aligned}
\|\mathcal{P}_{\Delta_j^{(k)}} \star \Gamma_\psi\|_{\boldsymbol{S}_p}^p &\le \operatorname{const} 2^{(kM+j+1)p/2} 2^{((k-1)M+j+1)p/2} 2^{(kM+j+3)(1-p)} \|\psi\|_p^p \\
&\le \operatorname{const} 2^{-Mp/2} 2^{kM+j} \|\psi\|_p^p \\
&\le \operatorname{const} 2^{-Mp/2} 2^{kM+j} \sum_{s=-3}^{2} \|\varphi * V_{kM+j+s}\|_p^p.
\end{aligned}$$

Note that the constants are all independent of M. Therefore

$$\sum_{j=0}^{M-1} \left\|T_j^{(2)}\right\|_{\boldsymbol{S}_p}^p \le \operatorname{const} 2^{-Mp/2} \sum_{n \ge M} 2^n \|\varphi * V_n\|_p^p.$$

Finally, we have

$$\begin{aligned}
\|T\|_{\boldsymbol{S}_p}^p &\ge \sum_{j=0}^{M-1} \left\|T_j^{(1)}\right\|_{\boldsymbol{S}_p}^p - \sum_{j=0}^{M-1} \left\|T_j^{(2)}\right\|_{\boldsymbol{S}_p}^p \\
&\ge c_1 \sum_{n \ge M} 2^n \|\varphi * V_n\|_p^p - c_2 2^{-Mp/2} \sum_{n \ge M} 2^n \|\varphi * V_n\|_p^p.
\end{aligned}$$

We can choose now M large enough that $c_2 2^{-Mp/2} \le c_1/2$. Then we have

$$\|\Gamma_\varphi\|_{\boldsymbol{S}_p}^p \ge \frac{1}{M} \|T\|_{\boldsymbol{S}_p}^p \ge \frac{c_1}{2M} \sum_{n \ge M} 2^n \|\varphi * V_n\|_p^p. \quad \blacksquare$$

Remark. It is easy to see that the above proof also works for $p = 1$. It is also easy to modify the proof of necessity to work for $1 < p < \infty$.

We can now deduce the following property of polynomials.

Corollary 3.10. *Let n be a positive integer, and let $\zeta_j = e^{2\pi \mathrm{i} j/(4n)}$, $0 \le j \le 4n-1$. Then for $0 < p \le 1$*

$$c_1 \|\psi\|_p^p \le \frac{1}{4n} \sum_{j=0}^{4n-1} |\psi(\zeta_j)|^p \le c_2 \|\psi\|_p^p$$

for every polynomial ψ of degree $n-1$, where the positive constants c_1 and c_2 do not depend on n.

Proof. Put $\varphi = z^n\psi$. We have shown (see Lemma 3.5 and Corollary 3.7) that

$$\|\Gamma_\varphi\|^p_{\boldsymbol{S}_p} \le \text{const} \sum_{j=0}^{4n-1} |\psi(\zeta_j)|^p.$$

On the other hand, it follows from (3.7) that

$$\sum_{j=0}^{4n-1} |\psi(\zeta_j)|^p \le \text{const}\, \|\Gamma_\varphi\|^p_{\boldsymbol{S}_p}.$$

To complete the proof we note that for polynomials φ of the form $\varphi = z^n\psi$, $\deg\psi \le n-1$, we have

$$\|\varphi\|^p_{B_p^{1/p}} \le \text{const}\, n\|\varphi\|^p_p \le \text{const}\, \|\varphi\|^p_{B_p^{1/p}},$$

which follows easily from the definition of $B_p^{1/p}$ in terms of convolutions with the V_n. ■

4. Hankel Operators and Schatten–Lorentz Classes

In this section we study Hankel operators of Schatten–Lorentz classes $\boldsymbol{S}_{p,q}$. We shall see in the next section that such questions are important for the study of metric properties of the averaging projection onto the Hankel matrices.

For $0 < p < \infty$ and $0 < q \le \infty$ the *Schatten–Lorentz class* $\boldsymbol{S}_{p,q}$ is, by definition, the class of operators T on Hilbert space such that

$$\sum_{n\ge 0} (s_n(T))^q (1+n)^{q/p-1} < \infty, \quad q < \infty,$$

$$\sup_{n\in\mathbb{Z}_+} (1+n)^{1/p} s_n(T) < \infty, \quad q = \infty.$$

Clearly, $\boldsymbol{S}_{p,p} = \boldsymbol{S}_p$, $\boldsymbol{S}_{p_1,q_1} \subset \boldsymbol{S}_{p_2,q_2}$ if $p_1 < p_2$ and $\boldsymbol{S}_{p,q_1} \subset \boldsymbol{S}_{p,q_2}$ if $q_1 < q_2$.

The Schatten–Lorentz classes can be obtained from the Schatten–von Neumann classes $\boldsymbol{S}_p$ with the help of interpolation by the real method.

Definition. Let (X_0, X_1) be a *quasi-Banach pair*, i.e., X_0 and X_1 are imbedded in a quasi-Banach space X. We define on

$$X_0 + X_1 \stackrel{\text{def}}{=} \{x_0 + x_1 : x_0 \in X_0,\, x_1 \in X_1\}$$

the *Peetre K-functional* by

$$\begin{aligned} K(t, x, X_0, X_1) \quad &\stackrel{\text{def}}{=} \quad \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x_0 \in X_0,\, x_1 \in X_1\}, \\ & \qquad x \in X_0 + X_1, \quad t > 0. \end{aligned}$$

For $0<\theta<1,\ 0<q\le\infty$ the *real interpolation space* $(X_0,X_1)_{\theta,q}$ consists of the elements $x\in X_0+X_1$, for which

$$\|x\|_{\theta,q}\stackrel{\text{def}}{=}\left(\int_0^\infty\left(\frac{K(t,x,X_0,X_1)}{t^\theta}\right)^q\frac{dt}{t}\right)^{1/q}<\infty,\quad q<\infty,$$

$$\|x\|_{\theta,\infty}\stackrel{\text{def}}{=}\sup_{t>0}\frac{K(t,x,A_0,A_1)}{t^\theta}.$$

It is easy to see that if (X_0,X_1) and (Y_0,Y_1) are quasi-Banach pairs and $T:X_0\cap X_1\to Y_0+Y_1$ is a linear operator such that $T\big|X_0$ is a bounded operator from X_0 to X_1 and $T\big|Y_0$ is a bounded operator from Y_0 to Y_1, then $T\big|(X_0,X_1)_{\theta,q}$ is a bounded operator from $(X_0,X_1)_{\theta,q}$ to $(Y_0,Y_1)_{\theta,q}$ for $0<\theta<1$, $0<q<\infty$. We refer the reader to Bergh and Löfström [1] for the theory of interpolation spaces.

It is well known that for $0<p<\infty$ there are constants c_1 and c_2 such that

$$c_1\left(\sum_{j=0}^n(s_j(T))^p\right)^{1/p}\le K(t,T,\boldsymbol{S}_p,\boldsymbol{S}_\infty)\le c_2\left(\sum_{j=0}^n(s_j(T))^p\right)^{1/p},\tag{4.1}$$

$$n^{1/p}\le t\le(n+1)^{1/p},\quad T\in\boldsymbol{S}_\infty,$$

which implies that for $0<p_0<\infty$

$$(\boldsymbol{S}_{p_0},\boldsymbol{S}_\infty)_{\theta,q}=\boldsymbol{S}_{p,q},\quad 0<\theta<1,\ 0<q\le\infty,\ p=\frac{p_0}{1-\theta}\tag{4.2}$$

(see, e.g., Karadzhov [1]).

We denote by $\boldsymbol{\Gamma S}_{p,q}$ the space of Hankel matrices of class $\boldsymbol{S}_{p,q}$ and by $\boldsymbol{\Gamma S}_p$ the space of Hankel matrices of class $\boldsymbol{S}_p$.

Theorem 4.1. *Let $p_0>0$. Then*

$$(\boldsymbol{\Gamma S}_{p_0},\boldsymbol{\Gamma S}_\infty)_{\theta,q}=\boldsymbol{\Gamma S}_{p,q},\quad 0<\theta<1,\ 0<q\le\infty,\ p=\frac{p_0}{1-\theta}.$$

Proof. Clearly, it suffices to show that for $T\in\boldsymbol{\Gamma S}_\infty$

$$K(t,T,\boldsymbol{S}_{p_0},\boldsymbol{S}_\infty)\le K(t,T,\boldsymbol{\Gamma S}_{p_0},\boldsymbol{\Gamma S}_\infty)\le\operatorname{const}K(t,T,\boldsymbol{S}_{p_0},\boldsymbol{S}_\infty).$$

The left-hand inequality is obvious. Let us prove the right-hand one.

Let $n^{1/p}\le t\le(n+1)^{1/p}$. It follows from Theorem 4.1.1 that there exists a Hankel operator R of rank at most n such that $\|T-R\|=s_n(T)$. We have

$$K(t,T,\boldsymbol{\Gamma S}_{p_0},\boldsymbol{\Gamma S}_\infty)\le\|R\|_{\boldsymbol{S}_{p_0}}+t\|T-R\|=\left(\sum_{j=0}^{n-1}(s_j(R))^p\right)^{1/p}+ts_n(T).$$

It is easy to see that $s_j(R) \le \|T-R\| + s_j(T) = s_n(T) + s_j(T) \le 2s_j(T)$. Therefore

$$\begin{aligned} K(t,T,\boldsymbol{\Gamma S}_{p_0},\boldsymbol{\Gamma S}_\infty) &\le 2 \left(\sum_{j=0}^{n-1}(s_j(T))^p\right)^{1/p} + ts_n(T) \\ &\le \text{const}\left(\sum_{j=0}^{n}(s_j(T))^p\right)^{1/p}. \end{aligned}$$

It follows now from (4.1) that $K(t,T,\boldsymbol{\Gamma S}_{p_0},\boldsymbol{\Gamma S}_\infty) \le \text{const}\, K(t,T,\boldsymbol{S}_{p_0},\boldsymbol{S}_\infty)$. ■

Corollary 4.2. *Let $p_0 > 0$. Then*

$$(\boldsymbol{\Gamma S}_{p_0},\boldsymbol{\Gamma S}_\infty)_{\theta,q} = \boldsymbol{\Gamma S}_{p,q},$$

where $p = p_0/(1-\theta)$.

Proof. The result follows immediately from Theorem 4.1 and (4.2). ■

Corollary 4.2 implies the following description of the interpolation spaces between $B_p^{1/p}$ and VMO.

Theorem 4.3. *Let $p_0 > 0$. Then*

$$(B_{p_0}^{1/p_0}, VMO)_{\theta,p} = B_p^{1/p},$$

where $p = p_0/(1-\theta)$.

Proof. Since the Riesz projection $\mathbb{P}_+$ is bounded both on VMO and $B_p^{1/p}$, $0 < p < \infty$, it is easy to see that it is sufficient to prove that

$$\Big(\big(B_{p_0}^{1/p_0}\big)_+, VMOA\Big)_{\theta,p} = \big(B_p^{1/p}\big)_+. \tag{4.3}$$

Consider the map $\mathcal{J} : \varphi \mapsto \Gamma_\varphi$, $\varphi \in VMOA$. By Theorems 1.1, 2.1, and 3.1, $\mathcal{J}$ is an isomorphism of $\big(B_p^{1/p}\big)_+$ onto $\boldsymbol{\Gamma S}_p$, $0 < p < \infty$, and by Hartman's theorem $\mathcal{J}$ is an isomorphism of $VMOA$ onto $\boldsymbol{\Gamma S}_\infty$. Consequently, $\mathcal{J}$ maps $\Big(\big(B_{p_0}^{1/p_0}\big)_+, VMOA\Big)_{\theta,p}$ isomorphically onto $(\boldsymbol{\Gamma S}_{p_0},\boldsymbol{\Gamma S}_\infty)_{\theta,p}$. It now follows from Corollary 4.2 that

$$\mathcal{J}\Big(\big(B_{p_0}^{1/p_0}\big)_+, VMOA\Big)_{\theta,p} = \boldsymbol{\Gamma S}_p, \quad p = \frac{p_0}{1-\theta},$$

which implies (4.3). ■

Clearly, Corollary 4.2 implies that for $0 < p < \infty,\ 0 < q \le \infty$

$$\Gamma_\varphi \in \boldsymbol{S}_{p,q}, \ \Leftrightarrow\ \varphi \in \big(B_{p_0}^{1/p_0}, VMO\big)_{\theta,q}, \tag{4.4}$$

where p_0 is an arbitrary positive number less than p and $\vartheta = (p-p_0)/p$. To describe the space $\big(B_{p_0}^{1/p_0}, VMO\big)_{\theta,q}$ we make use of the reiteration theorem (see Bergh and Löfström [1], Ch. §3.5):

Let (X_0, X_1) *be a quasi-Banach pair,* $0 < \theta_0 < \theta_1 < 1$, $0 < q_0, q_1 < \infty$. *Put* $Y_0 = (X_0, X_1)_{\theta_0,q_0}$, $Y_1 = (X_0, X_1)_{\theta_1,q_1}$. *Then*

$$(Y_0, Y_1)_{\theta,q} = (X_0, X_1)_{(1-\theta)\theta_0+\theta\theta_1,q}.$$

It follows from the reiteration theorem and from (4.4) that

$$\Gamma_\varphi \in \boldsymbol{S}_{p,q}, \quad \Longleftrightarrow \quad \varphi \in \left(B_{p_0}^{1/p_0}, B_{p_1}^{1/p_1}\right)_{\theta,q},$$

where $p_0 < p < p_1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$.

To describe the space $\left(B_{p_0}^{1/p_0}, B_{p_1}^{1/p_1}\right)_{\theta,q}$, we have to introduce the *Lorentz spaces.* Let $(\mathcal{X}, \mu)$ be a measure space. The Lorentz space $L^{p,q}(\mathcal{X}, \mu)$, $0 < p < \infty$, $0 < q \le \infty$, consists of measurable functions f on $\mathcal{X}$ such that

$$\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} < \infty, \quad q < \infty,$$

$$\sup_{t>0} t^{1/p} f^*(t) < \infty, \quad q = \infty,$$

where f^* is the nonincreasing rearrangement of f, i.e., f^* is the measurable function on $(0, \infty)$ defined by

$$f^*(t) \stackrel{\text{def}}{=} \inf\{\sigma : \ \mu\{x \in \mathcal{X} : \ |f(x)| > \sigma\} \ge t\}, \quad t > 0.$$

It is well known that for $f \in L^{p_0}(\mathcal{X}, \mu) + L^\infty(\mathcal{X}, \mu)$

$$c_1 \left(\int_0^{t^{p_0}} (f^*(s))^{p_0} ds \right)^{1/p_0} \le K(t, f, L^{p_0}, L^\infty) \le c_2 \left(\int_0^{t^{p_0}} (f^*(s))^{p_0} ds \right)^{1/p_0},$$

which by the reiteration theorem implies that

$$(L^{p_0}(\mathcal{X}, \mu), L^{p_1}(\mathcal{X}, \mu))_{\theta,q} = L^{p,q}(\mathcal{X}, \mu),$$

$1/p = (1-\theta)/p_0 + \theta/p_1$ (see Bergh and Löfström [1], §5.2).

We describe the spaces $\left(\left(B_{p_0}^{1/p_0}\right)_+, \left(B_{p_1}^{1/p_1}\right)_+\right)_{\theta,q}$. It will be clear that the spaces $\left(B_{p_0}^{1/p_0}, B_{p_1}^{1/p_1}\right)_{\theta,q}$ admit a similar description. Consider the measure space

$$(\mathcal{X}, \mu) = \sum_{n=0}^\infty \oplus (\mathbb{T}, 2^n \boldsymbol{m}),$$

i.e., $\mathcal{X}$ is an infinite union of disjoint copies of $\mathbb{T}$ and the restriction of μ to the nth copy is $2^n \boldsymbol{m}$. Consider the space $\mathfrak{B}_{p,q}^{1/p}$, $0 < p < \infty$, $0 < q \le \infty$, which consists of functions φ, which are analytic in $\mathbb{D}$ and such that

$$\sum_{n=0}^\infty \oplus \ \varphi * V_n \in L^{p,q}(\mathcal{X}, \mu).$$

Theorem 4.4. *Let* φ *be a function analytic in* $\mathbb{D}$ *and let* $0 < p < \infty$, $0 < q \le \infty$. *Then* $\Gamma_\varphi \in \boldsymbol{S}_{p,q}$ *if and only if* $\varphi \in \mathfrak{B}_{p,q}^{1/p}$.

Proof. Clearly, to prove the theorem, it suffices to show that

$$\left(\left(B_{p_0}^{1/p_0}\right)_+, \left(B_{p_1}^{1/p_1}\right)_+\right)_{\theta,q} = \mathfrak{B}_{p,q}^{1/p}, \tag{4.5}$$

where $p_0 < p < p_1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$.

To prove (4.5), we use a "retract argument". Consider the linear operator J defined on the space of functions analytic in $\mathbb{D}$ by

$$Jf \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \oplus\, \varphi * V_n.$$

Clearly, J is a bounded operator from $\left(B_{p_0}^{1/p_0}\right)_+$ to $L^{p_0}(\mathcal{X},\mu)$ and a bounded operator from $\left(B_{p_1}^{1/p_1}\right)_+$ to $L^{p_1}(\mathcal{X},\mu)$. Hence, J is a bounded operator from $\left(\left(B_{p_0}^{1/p_0}\right)_+, \left(B_{p_1}^{1/p_1}\right)_+\right)_{\theta,q}$ to $(L^{p_0}(\mathcal{X},\mu), L^{p_1}(\mathcal{X},\mu))_{\theta,q} = L^{p,q}(\mathcal{X},\mu)$, and so $\left(\left(B_{p_0}^{1/p_0}\right)_+, \left(B_{p_1}^{1/p_1}\right)_+\right)_{\theta,q} \subset \mathfrak{B}_{p,q}^{1/p}$.

Let us show that $\mathfrak{B}_{p,q}^{1/p} \subset \left(\left(B_{p_0}^{1/p_0}\right)_+, \left(B_{p_1}^{1/p_1}\right)_+\right)_{\theta,q}$. Define the polynomials $\breve{V}_n$, $n \ge 0$, by

$$\breve{V}_n \stackrel{\text{def}}{=} \begin{cases} V_{n-1} + V_n + V_{n+1}, & n > 0, \\ V_0 + V_1, & n = 0. \end{cases}$$

Consider the linear operator ρ on $L^{p_0}(\mathcal{X},\mu) + L^{p_1}(\mathcal{X},\mu)$ defined by

$$\rho \sum_{n\ge 0} \oplus\, \varphi_n = \sum_{n \ge 0} \varphi_n * \breve{V}_n.$$

Clearly, ρ is a continuous linear operator from $L^{p_0}(\mathcal{X},\mu)$ to $\left(B_{p_0}^{1/p_0}\right)_+$ and from $L^{p_1}(\mathcal{X},\mu)$ to $\left(B_{p_1}^{1/p_1}\right)_+$. Hence, ρ is a continuous linear operator from $(L^{p_0}(\mathcal{X},\mu), L^{p_1}(\mathcal{X},\mu))_{\theta,q} = L^{p,q}(\mathcal{X},\mu)$ to $\left(\left(B_{p_0}^{1/p_0}\right)_+, \left(B_{p_1}^{1/p_1}\right)_+\right)_{\theta,q}$.

It is easy to see that $\rho J \varphi = \varphi$ for any analytic function φ. Suppose that $\varphi \in \mathfrak{B}_{p,q}^{1/p}$, which, by definition, means that $Jf \in L^{p,q}(\mathcal{X},\mu)$. As we have already observed, it follows that $\varphi = \rho J\varphi \in \left(\left(B_{p_0}^{1/p_0}\right)_+, \left(B_{p_1}^{1/p_1}\right)_+\right)_{\theta,q}$. ■

Remark. It is easy to see that in a similar way one can describe the interpolation spaces $\left(B_{p_0}^{1/p_0}, B_{p_1}^{1/p_1}\right)_{\theta,q}$. The only difference is that we have to consider the measure space $(\mathcal{Y},\nu) = \sum_{n\in\mathbb{Z}} \oplus (\mathbb{T}, 2^{|n|}\boldsymbol{m})$. Then $\varphi \in (B_{p_0}^{1/p_0}, B_{p_1}^{1/p_1})_{\theta,q}$ if and only if $\sum_{n\in\mathbb{Z}} \oplus\, \varphi * V_n \in L^{p,q}(\mathcal{Y},\nu)$.

5. Projecting onto the Hankel Matrices

In this section we study continuity properties of the averaging projection $\mathcal{P}$ onto the Hankel matrices. We show that it is bounded on $\boldsymbol{S}_p$ for

$1 < p < \infty$ and unbounded on $\boldsymbol{S}_1$, $\boldsymbol{S}_\infty = \boldsymbol{\mathcal{C}}$, and $\mathcal{B}(\ell^2)$. Then we study the behavior of the averaging projection on $\boldsymbol{S}_1$ and we show that it maps $\boldsymbol{S}_1$ to $\boldsymbol{S}_{1,2}$. This allows us to find the best possible condition on the singular values of a linear operator T under which the operator $\mathcal{P}T$ is bounded (or compact). However, it turns out that there are projections onto the set of Hankel matrices which are bounded on $\boldsymbol{S}_1$ while there are no bounded projections on $\mathcal{B}(\ell^2)$ onto the set of Hankel matrices. As before we identify operators on ℓ^2 with their matrices in the standard orthonormal basis $\{e_j\}_{j\ge 0}$.

Let us define the averaging projection $\mathcal{P}$ onto the space of Hankel matrices. Let $A = \{a_{jk}\}_{j,k\ge 0}$ be an arbitrary matrix. The Hankel matrix $\mathcal{P}A$ is given by

$$\mathcal{P}A \stackrel{\text{def}}{=} \{\gamma_{j+k}\}_{j,k\ge 0}, \quad \gamma_m = \frac{1}{m+1}\sum_{j+k=m} a_{jk}.$$

It is easy to see that if A belongs to the Hilbert–Schmidt class $\boldsymbol{S}_2$, then $\mathcal{P}A$ is the orthogonal projection of A onto the space of Hankel matrices in the Hilbert–Schmidt norm.

Theorem 5.1. *$\mathcal{P}$ is a bounded operator on $\boldsymbol{S}_p$ for $1 < p < \infty$.*

Proof. In §2 we have defined the operator $\mathcal{W}$ on the space of matrices to the space of functions analytic in $\mathbb{D}$ by

$$(\mathcal{W}T)(z) = \sum_{m\ge 0} t_m z^m, \quad t_m \stackrel{\text{def}}{=} \sum_{j+k=m} t_{jk},$$

where $T = \{t_{jk}\}_{j,k\ge 0}$. We have shown in Lemma 2.4 that $\mathcal{W}$ is a bounded operator from $\boldsymbol{S}_p$ to $B_p^{-1/p'}$, $1 < p < \infty$. Therefore for $T \in \boldsymbol{S}_p$

$$\int_{\mathbb{D}} |(\mathcal{W}T)(\zeta)|^p (1-|\zeta|)^{p-2} d\boldsymbol{m}_2(\zeta) < \infty. \tag{5.1}$$

We have

$$\mathcal{P}T = \Gamma_\varphi, \quad \varphi = \sum_{n\ge 0} \frac{1}{n+1}\widehat{\mathcal{W}T}(n) z^n.$$

It now follows from (5.1) that

$$\int_{\mathbb{D}} |\varphi'(\zeta)|^p (1-|\zeta|)^{p-2} d\boldsymbol{m}_2(\zeta) < \infty,$$

i.e., $\varphi \in B_p^{1/p}$. By Theorem 3.1, $\mathcal{P}T = \Gamma_\varphi \in \boldsymbol{S}_p$. ■

Theorem 5.2. *The operator $\mathcal{P}$ is unbounded on $\boldsymbol{S}_1$, $\boldsymbol{\mathcal{C}}$, and $\mathcal{B}(\ell^2)$.*

Proof. Let us first prove that $\mathcal{P}$ is unbounded on $\boldsymbol{S}_1$. It has been shown in the proof of Lemma 2.4 that $\mathcal{W}\boldsymbol{S}_1 = H^1$. It follows that

$$\mathcal{P}\boldsymbol{S}_1 = \{\varphi : \ \varphi' \in H^1\}. \tag{5.2}$$

Let us show that $\{\varphi : \varphi' \in H^1\} \not\subset B_1^1$, which would imply that $\mathcal{P}\boldsymbol{S}_1 \not\subset \boldsymbol{S}_1$. Let $\{c_n\}_{n\ge0} \in \ell^2$ and let

$$\varphi(z) = \sum_{n\ge0} 2^{-n} c_n z^{2^n}.$$

Clearly, $\varphi' \in H^2 \subset H^1$. On the other hand, it follows immediately from the description of B_1^1 in terms of convolutions with the polynomials W_n (see Appendix 2.6) that $\varphi \in B_1^1$ if and only if $\{c_n\}_{n\ge0} \in \ell^1$.

Since $(\boldsymbol{\mathcal{C}})^* = \boldsymbol{S}_1$ and $(\boldsymbol{S}_1)^* = \mathcal{B}(\ell^2)$ (see Appendix 1.1), it now follows by duality that $\mathcal{P}$ is unbounded on $\boldsymbol{\mathcal{C}}$ and $\mathcal{B}(\ell^2)$. ■

Theorem 5.2 asserts that for an operator $T \in \boldsymbol{S}_1$ the operator $\mathcal{P}T$ need not be in $\boldsymbol{S}_1$. We are now going to obtain a sharp estimate of the singular values of $\mathcal{P}T$ for an arbitrary operator $T \in \boldsymbol{S}_1$.

Theorem 5.3. *Let $T \in \boldsymbol{S}_1$. Then $\mathcal{P}T \in \boldsymbol{S}_{1,2}$.*

Recall (see §4) that the ideal $\boldsymbol{S}_{1,2}$ consists of the operators A, for which

$$\sum_{j\ge0} (s_j(A))^2(1+j) < \infty.$$

To prove Theorem 5.3 we need an inequality for functions in the Lorentz class $L^{1,2}$.

Lemma 5.4. *Let $1 \le q \le \infty$ and let f_1 and f_2 be functions in $L^{1,q}(\mathcal{X}, \mu)$ such that $f_1(x)f_2(x) = 0$, μ-a.e. Then*

$$\|f_1 + f_2\|_{1,q} \le \|f_1\|_{1,q} + \|f_2\|_{1,q}. \tag{5.3}$$

Note that for $1 < q \le \infty$ the quasinorm $\|\cdot\|_{1,q}$ does not satisfy the triangle inequality. However, the lemma asserts that the triangle inequality holds for disjoint functions.

Proof of Lemma 5.4. Let us prove (5.3) for $1 < q < \infty$ (for $q = \infty$ the inequality is obvious). Put

$$\lambda_f(t) \stackrel{\text{def}}{=} \mu\{x : |f(x)| > t\}.$$

It is not hard to see that

$$\|f\|_{1,q} = \left(\int_0^\infty (\lambda_f(t))^q t^{q-1} dt\right)^{1/q}, \quad f \in L^{1,q},$$

(it is sufficient to verify this equality for step functions f).

Since the functions f_1 and f_2 are disjoint, it follows that

$$\lambda_{f_1+f_2} = \lambda_{f_1} + \lambda_{f_2}.$$

Consequently,

$$\|f_1 + f_2\|_{1,q} \le \left(\int_0^\infty (\lambda_{f_1}(t))^q t^{q-1} dt\right)^{1/q} + \left(\int_0^\infty (\lambda_{f_2}(t))^q t^{q-1} dt\right)^{1/q}. \quad ■$$

Proof of Theorem 5.3. Let

$$(\mathcal{X},\mu)=\sum_{n=0}^{\infty}\oplus(\mathbb{T},2^n\boldsymbol{m})$$

be the measure space, for which $\mathcal{X}$ is the infinite union of disjoint copies of $\mathbb{T}$, the nth copy equipped with the measure $2^n\boldsymbol{m}$. Theorem 4.4 asserts that $\Gamma_\varphi\in\boldsymbol{S}_{1,2}$ if and only if $\varphi\in\mathfrak{B}^1_{1,2}$. Therefore in view of (5.2) it remains to show that $\sum\limits_{n=0}^{\infty}\oplus\,\varphi*V_n\in L^{1,2}(\mathcal{X},\mu)$ for any function φ such that $\varphi'\in H^1$.

Note that $\varphi\in\mathfrak{B}^1_{1,2}$ if and only if

$$\sum_{n=0}^{\infty}2^{-n}(\varphi'*V_n)\in\mathfrak{B}^1_{1,2}.$$

Indeed, this follows from the fact that a function φ analytic in $\mathbb{D}$ belongs to $B_p^{1/p}$, $0<p<\infty$, if and only if

$$\sum_{n=0}^{\infty}2^{-n}(\varphi'*V_n)\in B_p^{1/p}$$

(see Appendix 2.6) and the fact that $\mathfrak{B}^1_{1,2}=\Big(\big(B^2_{1/2}\big)_+,\big(B_2^{1/2}\big)_+\Big)_{\frac{1}{2},2}$ (see Theorem 4.4).

Therefore to prove the theorem it is sufficient to show that for every function ψ in H^1

$$\sum_{n=0}^{\infty}\oplus\,2^{-n}(\psi*V_n)\in L^{1,2}(\mathcal{X},\mu).$$

We are going to use the following characterization of the space H^1:

$$\psi\in H^1\quad\Longleftrightarrow\quad\int_{\mathbb{T}}\left(\sum_{n=0}^{\infty}|(\psi*V_n)(\zeta)|^2\right)^{1/2}d\boldsymbol{m}(\zeta)<\infty$$

(see Stein [1]). In fact, we need only the implication $\Rightarrow$.

Therefore we can reduce Theorem 5.3 to the following assertion:

Let $\{\psi_n\}_{n\ge0}$ be a sequence of polynomials on $\mathbb{T}$ such that

$$\int_{\mathbb{T}}\left(\sum_{n=0}^{\infty}|\psi_n(\zeta)|^2\right)^{1/2}d\boldsymbol{m}(\zeta)<\infty.$$

Then

$$\sum_{n=0}^{\infty}\oplus\,2^{-n}\psi_n\in L^{1,2}(\mathcal{X},\mu).$$

Clearly, it is sufficient to assume that among the functions ψ_n there are only finitely many nonzero functions and to prove the inequality

$$\left\|\sum_{n=0}^{\infty}\oplus 2^{-n}\psi_n\right\|_{L^{1,2}(\mathcal{X},\mu)} \le \text{const}\int_{\mathbb{T}}\left(\sum_{n=0}^{\infty}|\psi_n(\zeta)|^2\right)^{1/2} d\boldsymbol{m}(\zeta). \tag{5.4}$$

Let N be a positive integer. It is easy to see that it is sufficient to establish inequality (5.4) for functions ψ_n that are constant on each of the arcs

$$I_j = \left\{e^{\mathrm{i}\vartheta} : \frac{2\pi j}{N} \le \vartheta \le \frac{2\pi(j+1)}{N}\right\}, \quad 0 \le j \le N-1,$$

since we can always approximate polynomials by such step function.

Let $0 \le j \le N-1$ and let $\psi_n = \sum\limits_{j=0}^{N-1}\psi_{n,j}$, where $\psi_{n,j}$ is constant on I_j and $\psi_{n,j}$ is identically equal to 0 on $\mathbb{T}\setminus I_j$. By Lemma 5.4,

$$\left\|\sum_{n=0}^{\infty}\oplus 2^{-n}\psi_n\right\|_{L^{1,2}} \le \sum_{j=0}^{N-1}\left\|\sum_{n=0}^{\infty}\oplus 2^{-n}\psi_{n,j}\right\|_{L^{1,2}}.$$

Therefore to prove inequality (5.4) it is sufficient to assume that for some j all functions ψ_n are constant on I_j and vanish on $\mathbb{T}\setminus I_j$. Let $\psi_n(\zeta) = c_n$ for $\zeta \in I_j$. Without loss of generality we may assume that $c_n \ge 0$, $n \ge 0$, and the sequence $\{2^{-n}c_n\}_{n\ge 0}$ is nonincreasing. Put $\delta \stackrel{\text{def}}{=} \boldsymbol{m}I_j$. We have

$$\begin{aligned}\left\|\sum_{n=0}^{\infty}\oplus 2^{-n}\psi_n\right\|_{L^{1,2}(\mathcal{X},\mu)} &= \left(\sum_{n=0}^{\infty}2^{-2n}c_n^2\delta^2\int_{(2^n-1)\delta}^{(2^{n+1}-1)\delta} t\,dt\right)^{1/2}\\ &\le \text{const}\cdot\delta\left(\sum_{n=0}^{\infty}c_n^2\right)^{1/2}\\ &= \text{const}\int_{\mathbb{T}}\left(\sum_{n=0}^{\infty}|\psi_n(\zeta)|^2\right)^{1/2} d\boldsymbol{m}(\zeta),\end{aligned}$$

which completes the proof. ■

To obtain a sharp sufficient condition for the boundedness of $\mathcal{P}T$ we introduce the sequence spaces $d(q,\omega)$ and the generalized Matsaev ideals $\boldsymbol{S}_{q,\omega}$.

Let $1 \le q < \infty$. We define the space $d(q,\omega)$ of sequences $\{a_n\}_{n\ge 0}$ such that

$$\|\{a_n\}_{n\ge 0}\|_{d(q,\omega)} \stackrel{\text{def}}{=} \left(\sum_{n\ge 0}\frac{(a_n^*)^q}{1+n}\right)^{1/q} < \infty,$$

where $\{a_n^*\}_{n\ge0}$ is the nonincreasing rearrangement of $\{|a_n|\}_{n\ge0}$. Consider now the ideal $\boldsymbol{S}_{q,\omega}$ of operators T on Hilbert space such that

$$\|T\|_{\boldsymbol{S}_{q,\omega}} \stackrel{\text{def}}{=} \left(\sum_{n\ge0}\frac{(s_n(T))^q}{1+n}\right)^{1/q} < \infty.$$

Then $d(q,\omega)$ and $\boldsymbol{S}_{q,\omega}$ are Banach spaces (see Appendix 1.1).

It is easy to see that $\boldsymbol{S}_p \subset \boldsymbol{S}_{q_1,\omega} \subset \boldsymbol{S}_{q_2,\omega}$, $0<p<\infty$, $1\le q_1\le q_2$.

Recall that the dual space $(\boldsymbol{S}_{q,\omega})^*$ can be identified with respect to the Hermitian pairing

$$\langle T,R\rangle = \operatorname{trace} TR^*, \quad T\in\boldsymbol{S}_{q,\omega}, \quad R\in(\boldsymbol{S}_{q,\omega})^*$$

with the ideal of operators whose singular values belong to $(d(q,\omega))^*$ (see Appendix 1.1). Here we identify $(d(q,\omega))^*$ with a sequence space with respect to the pairing

$$(a,b)=\sum_{n\ge0}a_n\bar b_n, \quad a=\{a_n\}_{n\ge0}\in d(q,\omega), \quad b=\{b_n\}_{n\ge0}\in(d(q,\omega))^*.$$

Theorem 5.5. *If $T\in\boldsymbol{S}_{2,\omega}$, then $\mathcal{P}T\in\boldsymbol{S}_\infty$.*

We need the following fact.

Lemma 5.6. $\boldsymbol{S}_{1,2}\subset(\boldsymbol{S}_{2,\omega})^*$.

Proof of Lemma 5.6. Since

$$T\in(\boldsymbol{S}_{2,\omega})^* \quad \text{if and only if} \quad \{s_n(T)\}_{n\ge0}\in(d(2,\omega))^*,$$

it suffices to show that

$$\sum_{n\ge0}(b_n^*)^2(1+n)<\infty \Rightarrow \{b_n\}_{n\ge0}\in(d(2,\omega))^*.$$

Let $a=\{a_n\}_{n\ge0}$ be a finitely supported sequence in $d(2,\omega)$. We have

$$|(a,b)|\le\sum_{n\ge0}|a_n\bar b_n| = \sum_{k\ge0}|a_{n_k}|b_k^*$$

for some permutation $\{n_k\}_{k\ge0}$ of the set $\mathbb{Z}_+$. It follows that

$$\begin{aligned}|(a,b)| &\le \sum_{k\ge0}|a_k|b_k^* \le \left(\sum_{k\ge0}(b_k^*)^2(1+k)\right)^{1/2}\left(\sum_{k\ge0}\frac{|a_{n_k}|^2}{1+k}\right)^{1/2}\\ &\le \left(\sum_{k\ge0}(b_k^*)^2(1+k)\right)^{1/2}\left(\sum_{k\ge0}\frac{|a_k^*|^2}{1+k}\right)^{1/2}.\end{aligned}$$

Hence, b determines a continuous linear functional on the space $d(2,\omega)$, and so $b\in(d(2,\omega))^*$. ■

Remark. It can be shown in the same way that $\boldsymbol{S}_{1,q}\subset(\boldsymbol{S}_{q',\omega})^*$, $1<q<\infty$, where $q'\stackrel{\text{def}}{=}q/(q-1)$.

Proof of Theorem 5.5. It follows from Theorem 5.3 and Lemma 5.6 that $\mathcal{P}\boldsymbol{S}_1 \subset (\boldsymbol{S}_{2,\omega})^*$. By duality, $\mathcal{P}\boldsymbol{S}_{2,\omega} \subset \mathcal{B}(\ell^2)$. Since the set of operators with all but finitely many matrix entries are zero is dense in $\boldsymbol{S}_{2,\omega}$, it follows that $\mathcal{P}\boldsymbol{S}_{2,\omega} \subset \boldsymbol{\mathcal{C}}$. ■

Let us now show that Theorem 5.5 is the best possible result.

Theorem 5.7. *Let $\{s_n\}_{n\ge0}$ be a nonincreasing sequence of positive numbers such that*

$$\sum_{n\ge0} s_n^2(1+n)^{-1} = \infty.$$

Then there exists an operator T on ℓ^2 such that $s_n(T) = s_n$ and $\mathcal{P}T$ is unbounded.

Proof. We define the matrix $\{t_{jk}\}_{j,k\ge0}$ of T as follows:

$$t_{jk} = \begin{cases} s_0, & j=0,\ k=0, \\ s_j, & 2^n \le j \le 2^{n+1}-1,\ k = 2^n + 2^{n+1} - j,\ n \ge 0, \\ 0, & \text{otherwise}, \end{cases}$$

(see Fig. 3).

Clearly, $s_j(T) = s_j$. Assume that $\mathcal{P}T$ is bounded and let $\mathcal{P}T = \Gamma_\varphi$. Then

$$\hat\varphi(0) = s_0, \quad \hat\varphi(2^n + 2^{n+1} - 1) = \frac{\sum\limits_{j=2^n}^{2^{n+1}-1} s_j}{2^n + 2^{n+1}}, \quad n \ge 0,$$

and $\hat\varphi(m) = 0$ if $m > 0$ and $m \ne 2^n + 2^{n+1} - 1$, $n \in \mathbb{Z}_+$. If Γ_f is bounded, then by the Nehari theorem, $\varphi \in BMOA$, and so $\sum\limits_{m\ge0} |\hat\varphi(m)|^2 < \infty$. We have

$$\begin{aligned} \sum_{m\ge0} |\hat\varphi(m)|^2 &= s_0^2 + \sum_{n\ge0} |\hat\varphi(2^n + 2^{n+1} - 1)|^2 \ge \frac{1}{9} \sum_{n\ge0} s_{2^{n+1}}^2 \\ &\ge \frac{1}{9} \sum_{n\ge0} \frac{1}{2^{n+1}} \sum_{k=0}^{2^{n+1}-1} s_{2^{n+1}+k}^2 \ge \frac{1}{9} \sum_{k\ge2} \frac{s_k^2}{k} = \infty. \end{aligned}$$

Hence, Γ_φ is unbounded. ■

Now we are in a position to show that in a sense Theorem 5.3 cannot be improved.

Corollary 5.8. *Let $q < 2$. Then $\mathcal{P}\boldsymbol{S}_1 \not\subset \boldsymbol{S}_{1,q}$.*

Proof. Assume that $\mathcal{P}\boldsymbol{S}_1 \subset \boldsymbol{S}_{1,q}$. Clearly, it is sufficient to consider the case $1 < q < 2$. Then as in Theorem 5.5 it would follow from the Remark after Lemma 5.6 that $\mathcal{P}T$ is bounded whenever $T \in \boldsymbol{S}_{q',\omega}$. However, this would contradict Theorem 5.7. ■

Although the natural averaging projection $\mathcal{P}$ is unbounded on $\boldsymbol{S}_1$, it turns out that there are bounded projections from $\boldsymbol{S}_1$ onto the subspace of

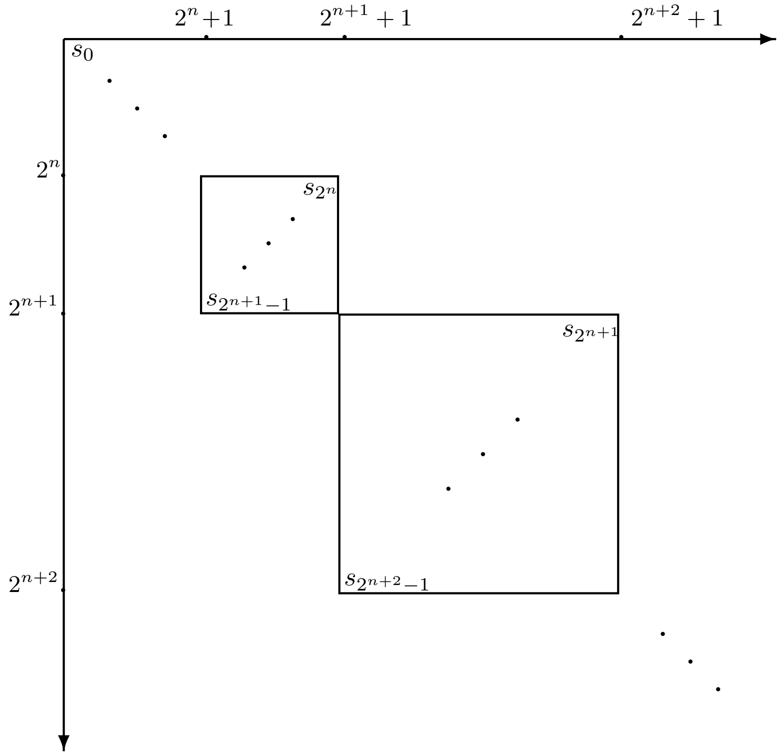

FIGURE 3.

nuclear Hankel matrices. Recall that

$$D_n^{(\alpha)} = \begin{pmatrix} n+\alpha \\ n \end{pmatrix} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} \sim \frac{n^\alpha}{\Gamma(\alpha+1)}$$

(see Appendix 2.6; here Γ is the gamma function, not a Hankel operator!).

For α, $\beta > 0$ we define the projection $\widetilde{\mathcal{P}}_{\alpha\beta}$ onto the set of Hankel matrices by

$$\widetilde{\mathcal{P}}_{\alpha\beta} A \stackrel{\text{def}}{=} \{\gamma_{j+k}\}_{j,k\ge 0}, \quad \gamma_m = \frac{1}{D_m^{\alpha+\beta+1}} \sum_{j+k=m} D_j^{(\alpha)} D_k^{(\beta)} a_{jk},$$

where $A = \{a_{jk}\}_{j,k\ge 0}$.

Theorem 5.9. *Let α, $\beta > 0$. Then $\widetilde{\mathcal{P}}_{\alpha\beta}$ is a bounded projection on $\boldsymbol{S}_1$ onto the set of Hankel matrices in $\boldsymbol{S}_1$.*

Proof. It is easy to see that $\sum\limits_{j=0}^{m} D_j^{(\alpha)} D_{m-j}^{(\beta)} = D_m^{(\alpha+\beta+1)}$. It follows that $\widetilde{\mathcal{P}}_{\alpha\beta}$ is a projection onto the Hankel matrices. Let us show that $\widetilde{\mathcal{P}}_{\alpha,\beta}$ is bounded on $\boldsymbol{S}_1$.

Clearly, it suffices to prove the inequality $\|\widetilde{\mathcal{P}}_{\alpha\beta}R\|_{\boldsymbol{S}_1} \le \text{const}\,\|f\|_2\|g\|_2$ for an arbitrary rank one operator $R = (\cdot, f)g$, where $f = \{f_n\}_{n\ge 0} \in \ell^2$, $g = \{g_n\}_{n\ge 0} \in \ell^2$. Consider the analytic functions $F \stackrel{\text{def}}{=} \sum\limits_{n\ge 0} D_n^{(\alpha)} \bar{f}_n z^n$ and $G \stackrel{\text{def}}{=} \sum\limits_{n\ge 0} D_n^{(\beta)} g_n z^n$. Then $F \in B_2^{-\alpha}$ and $G \in B_2^{-\beta}$ (see Appendix 2.6), and so

$$\int_{\mathbb{D}} |F(\zeta)|^2 (1-|\zeta|)^{2\alpha-1} d\boldsymbol{m}_2(\zeta) \le \text{const}\,\|f\|_{\ell^2}$$

and

$$\int_{\mathbb{D}} |G(\zeta)|^2 (1-|\zeta|)^{2\beta-1} d\boldsymbol{m}_2(\zeta) \le \text{const}\,\|g\|_{\ell^2}.$$

By Hölder's inequality this implies that

$$\int_{\mathbb{D}} |(FG)(\zeta)| (1-|\zeta|)^{\alpha+\beta-1} d\boldsymbol{m}_2(\zeta) \le \text{const}\,\|f\|_{\ell^2}\|g\|_{\ell^2},$$

and so $FG \in B_1^{-(\alpha+\beta)}$ (see Appendix 2.6).

It is easy to see that

$$\widetilde{\mathcal{P}}_{\alpha\beta}R = \Gamma_\varphi, \quad \varphi(z) = \sum_{n\ge 0} D_n^{(\alpha+\beta+1)} \widehat{FG}(n) z^n = \tilde{I}_{-(\alpha+\beta+1)}(FG)$$

(see Appendix 2.6). Therefore $\varphi \in B_1^1$ (see Appendix 2.6) and by Theorem 1.1, $\widetilde{\mathcal{P}}_{\alpha\beta}R = \Gamma_\varphi \in \boldsymbol{S}_1$. ■

Theorem 5.9 implies the following result.

Corollary 5.10. *Let α, $\beta > 0$ and let F be an analytic in $\mathbb{D}$ function of class $B_1^{-(\alpha+\beta)}$. Then there exist sequences of analytic in $\mathbb{D}$ functions $\{\varphi_k\}_{k\ge 0}$ and $\{\psi_k\}_{k\ge 0}$ such that $\varphi_k \in B_2^{-\alpha}$, $\psi_k \in B_2^{-\beta}$, and*

$$F = \sum_{k\ge 0} \varphi_k \psi_k, \quad \sum_{k\ge 0} \|\varphi_k\|_{B_2^{\alpha}} \|\psi\|_{B_2^{-\beta}} \le \text{const}\,\|F\|_{B_1^{-(\alpha+\beta)}}.$$

Proof. Let $G = \tilde{I}_{\alpha+\beta+1}F$. Then $G \in B_1^1$ (see Appendix 2.6). By Theorem 1.1,

$$\Gamma_G = \sum_{k\ge 0} (\cdot, f^{(k)}) g^{(k)}, \quad f^{(k)} = \{f_n^{(k)}\}_{n\ge 0} \in \ell^2, \quad g^{(k)} = \{g_n^{(k)}\}_{n\ge 0} \in \ell^2,$$

and

$$\sum_{k\ge 0} \|f^{(k)}\|_{\ell^2} \|g^{(k)}\|_{\ell^2} = \|\Gamma_G\|_{\boldsymbol{S}_1} \le \text{const}\,\|G\|_{B_1^1} \le \text{const}\,\|F\|_{B_1^{-(\alpha+\beta)}}.$$

Put

$$\varphi_k = \sum_{n\ge 0} D_n^{(\alpha)} \bar{f}_n^{(k)} \quad \text{and} \quad \psi_k = \sum_{n\ge 0} D_n^{(\alpha)} g_n^{(k)}.$$

Clearly,

$$\sum_{k\geq 0}\|\varphi_k\|_{B_2^\alpha}\|\psi\|_{B_2^{-\beta}}\leq \text{const}\,\|F\|_{B_1^{-(\alpha+\beta)}}.$$

We have

$$\Gamma_G=\widetilde{\mathcal{P}}_{\alpha\beta}\Gamma_G=\sum_{k\geq 0}\widetilde{\mathcal{P}}_{\alpha\beta}(\cdot,f^{(k)}g^{(k)})=\sum_{k\geq 0}\Gamma_{\tilde{I}_{\alpha+\beta+1}(\varphi_k\psi_k)},$$

which implies that $F=\sum\limits_{k\geq 0}\varphi_k\psi_k$. ■

Remark. We can slightly modify the definition of $\widetilde{\mathcal{P}}_{\alpha\beta}$. For α, $\beta>0$ consider the projection $\mathcal{P}_{\alpha\beta}$:

$$\mathcal{P}_{\alpha\beta}A\stackrel{\text{def}}{=}\{\gamma_{j+k}\}_{j,k\geq 0},\quad \gamma_m=\frac{\sum\limits_{0\leq j+k\leq m}(1+j)^\alpha(1+k)^\beta a_{jk}}{\sum\limits_{0\leq j+k\leq m}(1+j)^\alpha(1+k)^\beta},$$

where $A=\{a_{jk}\}_{j,k\geq 0}$. If we replace in the proof of Theorem 5.9 the fractional derivatives $\tilde{I}_s$ by the fractional derivatives I_s (see Appendix 2.6), we can prove that $\mathcal{P}_{\alpha\beta}$ is a bounded projection on $\boldsymbol{S}_1$ onto the nuclear Hankel matrices.

However, it turns out that in the case of the space of compact operators or bounded operators the situation is quite different: there are no bounded projections onto the set of Hankel matrices. To prove this we use some results from Banach space geometry.

Theorem 5.11. *There are no bounded projections from $\boldsymbol{\mathcal{C}}$ onto the compact Hankel matrices.*

Proof. Assume that P is a bounded projection from $\boldsymbol{\mathcal{C}}$ onto the compact Hankel matrices. Since the space of compact Hankel matrices is isomorphic to $VMOA$, it follows that there exists a continuous linear operator A from $\boldsymbol{\mathcal{C}}$ onto $VMOA$. Hence, the operator T^* is an isomorphic imbedding of H^1 into $\boldsymbol{S}_1$. But this is impossible since every subspace of $\boldsymbol{S}_1$ either is isomorphic to a Hilbert space or contains a subspace isomorphic to ℓ^1 (see Arazy and Lindenstrauss [1]), but H^1 contains subspaces isomorphic to ℓ^p, $1<p<2$, (see Kwapień and Pełczyński [1]).

Clearly, ℓ^p for $1<p<2$ does not have a subspace isomorphic to ℓ^1 since ℓ^p is reflexive, while ℓ^1 is not. It remains to show that ℓ^p is not isomorphic to a Hilbert space for $1<p<2$. To prove this we observe the following. Let $x_1,x_2,\cdots,x_n$ be vectors in a Hilbert space. Then it follows by induction from the parallelogram identity that

$$2^{-n}\sum\|\pm x_1\pm x_2\pm\cdots\pm x_n\|^2=\|x_1\|^2+\|x_2\|^2+\cdots+\|x_n\|^2,$$

where the sum is taken over all combinations of signs. On the other hand, if $e_1,e_2,\cdots,e_n$ are standard basis vectors of ℓ^p, then

$$2^{-n}\sum\|\pm e_1\pm e_2\pm\cdots\pm e_n\|_{\ell^p}^2=n^{2/p}=n^{2/p-1}(\|e_1\|_{\ell^p}^2+\|e_2\|_{\ell^p}^2+\cdots+\|e_n\|_{\ell^p}^2),$$

which implies that there is no isomorphism from ℓ^p onto a Hilbert space. ■

Theorem 5.12. *There are no bounded projections from $\mathcal{B}(\ell^2)$ onto the bounded Hankel matrices.*

Proof. Assume that P is a bounded projection from $\mathcal{B}(\ell^2)$ onto the space of bounded Hankel matrices $\mathcal{B}(\Gamma)$. Let M be the symbol operator from $\mathcal{B}(\Gamma)$ onto $BMOA$: $M\Gamma_\varphi \stackrel{\text{def}}{=} \varphi$. Put $R \stackrel{\text{def}}{=} MP|\boldsymbol{\mathcal{C}}$. Consider the operator $R^* : (BMOA)^* \to \boldsymbol{S}_1$ and its restriction $R^*\big|H^1$ to $H^1 \subset (BMOA)^*$. Let us show that $R^*\big|H^1$ is an isomorphic imbedding of H^1 into $\boldsymbol{S}_1$. Suppose that $f \in H^1$. Let $\varphi \in VMOA$, $\|\varphi\|_{VMOA} = 1$, and $|(f,\varphi)| \ge \text{const}\,\|f\|_{H^1}$. Since $R\Gamma_\varphi = \varphi$, we have

$$|\langle R^* f, \Gamma_\varphi\rangle| = |(f,\varphi)| \ge \text{const}\,\|f\|_{H^1},$$

and since $\|\Gamma_\varphi\| \le \text{const}$, it follows that $\|R^* f\| \ge \text{const}\,\|f\|_{H^1}$. The rest of the proof repeats the proof of Theorem 5.11. ■

6. Rational Approximation

Classical theorems on polynomial approximation describe classes of smooth functions in terms of the rate of polynomial approximation (in one or another norm). The smoother the function is, the more rapidly its deviations to the set of polynomials of degree n decay. For example, an L^∞ function φ belongs to Λ_α, $\alpha > 0$, if and only if

$$\operatorname{dist}_{L^\infty}\{\varphi, \boldsymbol{\mathcal{P}}_n\} \le \text{const}\cdot(1+n)^{-\alpha},$$

where $\boldsymbol{\mathcal{P}}_n$ is the set of trigonometric polynomials of degree at most n. This is a classical result due to Bernstein, Jackson, and de la Vallée Poussin; see Akhiezer [2].

However, it turned out that in the case of rational approximation the corresponding problems are considerably more complicated. For a long time there were no sharp results describing classes of functions in terms of the rate of rational approximation. The first sharp result was deduced from the $\boldsymbol{S}_p$ criterion for Hankel operators (Theorems 1.1, 2.1). In earlier results there were gaps between “direct” and “inverse” theorems (see the Concluding Remarks to this chapter).

In this section we describe the Besov spaces $B_p^{1/p}$ in terms of the rate of rational approximation in the norm of BMO. Then we obtain an improvement of Grigoryan’s theorem, which estimates the L^∞ norm of $\mathbb{P}_- f$ in terms of $\|f\|_{L^\infty}$ for functions f such that $\mathbb{P}_- f$ is a rational function of degree n. As a consequence we obtain a sharp result about rational approximation in the L^∞ norm.

Using the $\boldsymbol{S}_p$ criterion obtained in §§1–3 we obtain sharp Bernstein–S.M. Nikol’skii type inequalities for rational functions of degree n. We also obtain estimates of the $B_p^{1/p}$ norms of rational functions in terms of their norms in B_∞^0. This will allow us to conclude this section with a sharp result on rational approximation in the norm of the Bloch space $\mathfrak{B} = \big(B_\infty^0\big)_+$.

Denote by $\mathcal{R}_n$, $n \ge 0$, the set of rational functions of degree at most n with poles outside $\mathbb{T}$. For $f \in BMO$ put

$$\rho_n(f) \stackrel{\text{def}}{=} \operatorname{dist}_{BMO}\{f, \mathcal{R}_n\}.$$

Theorem 6.1. *Let $\varphi \in BMO$ and $0 < p < \infty$. Then $\{\rho_n(\varphi)\}_{n\ge 0} \in \ell^p$ if and only if $\varphi \in B_p^{1/p}$.*

Proof. We have $\mathbb{P}_+BMO \subset BMO$ and $\mathbb{P}_+B_p^{1/p} \subset B_p^{1/p}$ (see Appendix 2.6). Clearly, $\mathbb{P}_+\mathcal{R}_n \subset \mathcal{R}_n$. Therefore it is sufficient to prove the theorem for functions φ in $BMOA$ and for functions $\varphi \in \mathbb{P}_-BMO$. Let us assume that $\varphi = \mathbb{P}_-\varphi$, the corresponding result in the case $\varphi = \mathbb{P}_+\varphi$ follows by passing to the complex conjugate.

It follows from Theorem 4.1.1 that

$$s_n(H_\varphi) = \inf\{\|H_\varphi - H_r\| : \ \operatorname{rank} H_r \le n\}.$$

Without loss of generality we may assume that $r = \mathbb{P}_-r$. By Corollary 1.3.2, $\operatorname{rank} H_r \le n$ if and only if $r \in \mathcal{R}_n$. Together with Theorem 1.1.3 this yields

$$c_1 s_n(H_\varphi) \le \inf\{\|\varphi - r\|_{BMO} : \ r \in \mathcal{R}_n\} \le c_2 s_n(H_\varphi)$$

for some positive constants c_1 and c_2.

The result now follows from Theorems 1.1, 2.1, and 3.1. ■

Denote now by $\mathcal{R}_n^+$ the set of rational functions of degree at most n with poles outside the closed unit disk and put

$$\rho_n^+(\varphi) \stackrel{\text{def}}{=} \operatorname{dist}_{BMOA}\{\varphi, \mathcal{R}_n^+\}.$$

Theorem 6.2. *Suppose that $\varphi \in BMOA$ and $0 < p < \infty$. Then $\{\rho_n^+(\varphi)\}_{n\ge 0} \in \ell^p$ if and only if $\varphi \in \big(B_p^{1/p}\big)_+$.*

Proof. This is an immediate consequence of Theorem 6.1 ■.

We proceed now to an improvement of a theorem of Grigoryan that estimates the $\|\mathbb{P}_-\varphi\|_{L^\infty}$ in terms of $\|\varphi\|_{L^\infty}$ in the case $\mathbb{P}_-\varphi \in \mathcal{R}_n$. Clearly, the last condition is equivalent to the fact that φ is a boundary-value function of a meromorphic function in $\mathbb{D}$ which has at most n poles (counted with multiplicities) and is bounded near $\mathbb{T}$. It is not quite obvious that such an estimate exists. If we consider the same question in the case $\mathbb{P}_-\varphi$ is a polynomial of degree n, it is well known that $\|\mathbb{P}_-\varphi\|_{L^\infty} \le \operatorname{const}\log(1+n)$; this follows immediately from the fact that

$$\left\|\sum_{j=0}^{n} z^j\right\|_{L^1} \le \operatorname{const}\log(1+n)$$

(see Zygmund [1], Ch. II, §12). Grigoryan's theorem says that if $\mathbb{P}_-\varphi \in \mathcal{R}_n$, then

$$\|\mathbb{P}_-\varphi\|_{L^\infty} \le \text{const} \cdot n \, \|\varphi\|_{L^\infty}. \tag{6.1}$$

The following result improves the above estimate. The proof is based on the $\boldsymbol{S}_1$ criterion for Hankel operators (Theorem 1.1).

Theorem 6.3. *Let n be a positive integer and let φ be a function in L^∞ such that $\mathbb{P}_-\varphi \in \mathcal{R}_n$. Then*

$$\|\mathbb{P}_-\varphi\|_{B_1^1} \le \text{const} \cdot n \, \|\varphi\|_{L^\infty}. \tag{6.2}$$

Let us first observe that inequality (6.2) implies (6.1). Indeed, if $f \in B_1^1$, then $\sum\limits_{n\ge 0} 2^n \|f * W_n\|_{L^1} \le \text{const} \, \|f\|_{B_1^1}$ (see Appendix 2.6). It is easy to show that

$$\|\varphi\|_{L^\infty} \le \sum_{j\ge 0} |\hat{f}(j)| \le \text{const} \sum_{n\ge 0} 2^n \|f * W_n\|_{L^1},$$

and so (6.2) implies (6.1).

Proof of Theorem 6.3. Consider the Hankel operator H_φ. By the Nehari theorem, $\|H_\varphi\| \le \|\varphi\|_{L^\infty}$. By Kronecker's theorem, $\operatorname{rank} H_\varphi \le n$. Therefore $\|H_\varphi\|_{\boldsymbol{S}_1} \le n\|H_\varphi\|$. The result now follows from Theorem 1.1 which says that $\|\mathbb{P}_-\varphi\|_{B_1^1} \le \text{const} \, \|H_\varphi\|_{\boldsymbol{S}_1}$. ■

We can now obtain a result on rational approximation in the L^∞ norm. For $\varphi \in L^\infty$ we put

$$d_n^\infty(\varphi) \stackrel{\text{def}}{=} \operatorname{dist}_{L^\infty}\{\varphi, \mathcal{R}_n\}, \quad n \in \mathbb{Z}_+.$$

Theorem 6.4. *Let $\varphi \in L^\infty$. Then the $d_n^\infty(\varphi)$ decay more rapidly than any power of n if and only if $\varphi \in \bigcap\limits_{p>0} B_p^{1/p}$.*

Lemma 6.5. *Let $r \in \mathcal{R}_n$. Then*

$$\|r\|_{L^\infty} \le \text{const} \cdot n \, \|r\|_{BMO}.$$

Proof. Clearly, it is sufficient to prove the inequality in the case $r = \mathbb{P}_- r$ and in the case $r = \mathbb{P}_+ r$. Assume that $r = \mathbb{P}_- r$. Let f be the symbol of H_r of minimal norm, i.e., $\mathbb{P}_- r = \mathbb{P}_- f$ and $\|f\|_{L^\infty} = \|H_r\|$ (see Corollary 1.1.6). We have

$$\begin{aligned} \|\mathbb{P}_- r\|_{L^\infty} &= \|\mathbb{P}_- f\|_{L^\infty} \le \text{const} \cdot n\|f\|_{L^\infty} \\ &= \text{const} \cdot n\|H_r\| \le \text{const} \cdot n\|\mathbb{P}_- r\|_{BMO} \end{aligned}$$

by (6.1) and Theorem 1.1.3. ■

It is easy to see that Theorem 6.4 is a consequence of the following lemma.

Lemma 6.6. *Let $\lambda > 1$ and let φ be a function in L^∞ such that $\rho_n(\varphi) \le \text{const} \cdot n^{-\lambda}$, $n \ge 0$. Then*

$$d_n^\infty(\varphi) \le \text{const} \cdot n^{-\lambda+1}, \quad n \ge 0.$$

Proof. Suppose that $r_n \in \mathcal{R}_{2^n}$ and $\|\varphi - r_n\|_{BMO} \le \text{const}\, 2^{-n\lambda}$. We have

$$\varphi - r_n = \sum_{j\ge 0}\big((\varphi - r_{n+j}) - (\varphi - r_{n+j+1})\big) = \sum_{j\ge 0}(r_{n+j+1} - r_{n+j}).$$

Under the hypotheses of the lemma

$$\|r_{n+j+1} - r_{n+j}\|_{BMO} \le \text{const}\, 2^{-(n+j)\lambda},$$

and since $r_{n+j+1} - r_{n+j} \in \mathcal{R}_{2^{n+j+2}}$, it follows from Lemma 6.5 that

$$\|r_{n+j+1} - r_{n+j}\|_{L^\infty} \le \text{const}\, 2^{-(n+j)(\lambda-1)}.$$

Therefore

$$d^\infty_{2^n}(\varphi) \le \|\varphi - r_n\|_{L^\infty}\, \text{const}\, 2^{-n(\lambda-1)},$$

and since the sequence $\rho^\infty_n(\varphi)$ is nonincreasing, this implies the conclusion of the lemma. ∎

Let us proceed now to Bernstein–Nikol'skii type inequalities for rational functions.

Recall that the space B^0_∞ consists of distributions (see Appendix 2.6). Its subspace $\big(B^0_\infty\big)_+$ can be identified with the Bloch space $\mathfrak{B}$ of functions f analytic in $\mathbb{D}$ and satisfying

$$\sup_{\zeta\in\mathbb{D}} |f'(\zeta)|(1 - |\zeta|) < \infty.$$

Theorem 6.7. *Let r be a rational function of degree $n \ge 1$ with poles outside $\mathbb{T}$ and let $0 < p \le q \le \infty$. Then*

$$\|r\|_{B_p^{1/p}} \le \text{const}\cdot n^{1/p}\|r\|_{BMO} \tag{6.3}$$

and

$$\|r\|_{B_p^{1/p}} \le \text{const}\cdot n^{1/p-1/q}\|r\|_{B_q^{1/q}}. \tag{6.4}$$

Proof. Let us first prove (6.3). Clearly, it is sufficient to prove it for rational functions r in H^2_- of degree n. Consider the Hankel operator H_r. By Corollary 1.3.2, $\operatorname{rank} H_r = n$. Hence,

$$\|H_r\|_{\boldsymbol{S}_p} = \left(\sum_{j=0}^{n-1} s_j(H_r)^p\right)^{1/p} \le n^{1/p} s_0(H_r) = n^{1/p}\|H_r\|. \tag{6.5}$$

To prove (6.3), it is sufficient to observe that by Theorems 1.1, 2.1, and 3.1, the left-hand side of (6.5) is equivalent to $\|r\|_{B_p^{1/p}}$ and by Theorem 1.1.2, the right-hand side of (6.5) is equivalent to $\|r\|_{BMO}$.

Similarly, for $q < \infty$ and $r \in H^2_-$, (6.4) follows from Theorems 1.1, 2.1, 3.1, and Hölder's inequality:

$$\left(\sum_{j=0}^{n-1} s_j(H_r)^p\right)^{1/p} \le n^{1/p-1/q}\left(\sum_{j=0}^{n-1} s_j(H_r)^q\right)^{1/q}.$$

It remains to prove (6.4) for $q = \infty$. This time it is more convenient to assume that $r \in H^2$. We may also assume that $r(0) = 0$. Consider first the case $p = 2$. For $f, g \in B_2^{1/2}$ we put

$$(f, g)_* = \int_{\mathbb{D}} f'(\zeta)\overline{g'(\zeta)}d\boldsymbol{m}_2(\zeta) \quad \text{and} \quad \|f\|_* = (f, f)_*^{1/2}.$$

Clearly, $(\cdot, \cdot)_*$ is a semi-inner product on $B_2^{1/2}$ and $\|f\|_* = 0$ if and only if f is a constant function. It is easy to see that

$$\|r\|_{B_2^{1/2}} \le \text{const} \cdot \sup\{|(r, \rho)_*| : \ \rho \in \mathcal{R}_n^+, \ \|\rho\|_* = 1\}. \tag{6.6}$$

It follows easily from the description of the dual space to B_1^1 (see Appendix 2.6) that for a rational function $\rho \in H^2$

$$|(r, \rho)_*| \le \text{const}\, \|r\|_{B_\infty^0} \|\rho\|_{B_1^1}.$$

Using (6.4) with $p = 1$ and $q = 2$, we obtain

$$|(r, \rho)_*| \le \text{const} \cdot n^{1/2} \|r\|_{B_\infty^0} \|\rho\|_{B_2^{1/2}}, \quad \rho \in \mathcal{R}_n^+,$$

and together with (6.6) this yields

$$\|r\|_{B_2^{1/2}} \le \text{const} \cdot n^{1/2} \|r\|_{B_\infty^0}.$$

It is easy to see now that inequality (6.4) for $p < 2$ and $q = \infty$ follows from inequality (6.4) for $q = 2$ and inequality (6.4) for $p = 2$ and $q = \infty$.

It remains to prove inequality (6.4) for $2 < p < \infty$ and $q = \infty$. We have

$$\begin{aligned}
\|r\|_{B_p^{1/p}} &\le \text{const} \left(\int_{\mathbb{D}} |r'(\zeta)|^p (1 - |\zeta|)^{p-2} d\boldsymbol{m}_2(\zeta) \right)^{1/p} \\
&= \text{const} \left(\int_{\mathbb{D}} |r'(\zeta)|^2 (|r'(\zeta)|(1 - |\zeta|))^{p-2} d\boldsymbol{m}_2(\zeta) \right)^{1/p} \\
&\le \left(\int_{\mathbb{D}} |r'(\zeta)|^2 d\boldsymbol{m}_2(\zeta) \right)^{1/p} (\sup\{|r'(\zeta)|(1 - |\zeta|) : \zeta \in \mathbb{D}\})^{\frac{p-2}{p}} \\
&= \|r\|_{B_2^{1/2}}^{2/p} \|r\|_{B_\infty^0}^{(p-2)/p} \le n^{1/p} \|r\|_{B_\infty^0}
\end{aligned}$$

by inequality (6.4) for $p = 2$ and $q = \infty$. ■

Remark. Note that inequalities (6.3) and (6.4) are sharp. Indeed, we can consider the function $z^n \in \mathcal{R}_n$, $n \ge 0$. Since $L^\infty \subset BMO \subset L^2$,

$$c_1 \le \|z^n\|_{BMO} \le c_2$$

for some constants c_1 and c_2. On the other hand, it follows from the description of the classes $B_p^{1/p}$ in terms of kernels V_n (see Appendix 2.6) that for $0 < p \le \infty$

$$C_1 n^{1/p} \le \|z^n\|_{B_p^{1/p}} \le C_2 n^{1/p},$$

where C_1 and C_2 do not depend on n.

We conclude this section with an analog of Theorem 6.1 for rational approximation in the norm of B_∞^0. For a function $\varphi \in \mathfrak{B}$ we put

$$\delta_n(\varphi) \stackrel{\text{def}}{=} \operatorname{dist}_{\mathfrak{B}}\{\varphi, \mathcal{R}_n^+\}, \quad n \geq 0.$$

Theorem 6.8. *Let $\varphi \in \mathfrak{B}$ and let $0 < p < \infty$. Then $\{\delta_n(\varphi)\}_{n\geq 0} \in \ell^p$ if and only if $\varphi \in B_p^{1/p}$.*

Proof of Theorem 6.8. If $\varphi \in B_p^{1/p}$, it follows from Theorem 6.2 that $\{d_n^+(\varphi)\}_{n\geq 0} \in \ell^p$, and since $BMOA \subset \mathfrak{B}$, it follows that $\{\delta_n(\varphi)\}_{n\geq 0} \in \ell^p$.

Suppose now that $\{\delta_n(\varphi)\}_{n\geq 0} \in \ell^p$. Let us show that $\varphi \in B_p^{1/p}$. Consider first the case $0 < p \leq 1$.

For $j \in \mathbb{Z}_+$, consider a rational function r_j in $\mathcal{R}_{2^j}^+$ such that $\|\varphi - r_j\|_{\mathfrak{B}} \leq 2\delta_{2^j}(\varphi)$. We have

$$\varphi = r_1 + \sum_{j=1}^{\infty}(r_{j+1} - r_j),$$

and it follows from inequality (6.4) with $q = \infty$ that

$$\begin{aligned}
\|\varphi\|_{B_p^{1/p}}^p &\leq \|r_1\|_{B_p^{1/p}}^p + \sum_{j=1}^{\infty} \|r_{j+1} - r_j\|_{B_p^{1/p}}^p \\
&\leq \|r_1\|_{B_p^{1/p}}^p + \text{const}\sum_{j=1}^{\infty} 2^j \|r_{j+1} - r_j\|_{\mathfrak{B}}^p \\
&\leq \text{const}\left(\|r_1\|_{\mathfrak{B}}^p + \sum_{j=1}^{\infty} 2^j \left(\|\varphi - r_j\|_{\mathfrak{B}}^p + \|\varphi - r_{j+1}\|_{\mathfrak{B}}^p\right)\right) \\
&\leq \text{const}\left(\|r_1\|_{\mathfrak{B}}^p + \sum_{j=1}^{\infty} 2^j \left(\delta_{2^j}(\varphi)\right)^p\right) \\
&\leq \text{const}\left(\|r_1\|_{\mathfrak{B}}^p + \sum_{m=1}^{\infty} \left(\delta_m(\varphi)\right)^p\right).
\end{aligned}$$

The last inequality easily follows from the fact that the sequence $\{\delta_n(\varphi)\}_{n\geq 0}$ is nonincreasing.

Suppose now that $1 \leq p < \infty$. We use the following seminorm on $\left(B_p^{1/p}\right)_+$:

$$\|\varphi\|_{(p)} \stackrel{\text{def}}{=} \int_{\mathbb{D}} \left(|\varphi''(\zeta)|(1 - |\zeta|)^2\right)^p (1 - |\zeta|)^{-2} d\boldsymbol{m}_2(\zeta).$$

A function φ analytic in $\mathbb{D}$ belongs to $B_p^{1/p}$, $1 \leq p < \infty$, if and only if $\|\varphi\|_{(p)} < \infty$ (see Appendix 2.6).

Suppose that $r_j \in \mathcal{R}_{2^j}$ and $\|\varphi - r_j\|_{\mathfrak{B}} \le 2\delta_{2^j}(\varphi)$. We have

$$r_m = r_0 + \sum_{j=0}^{m-1} (r_{j+1} - r_j).$$

By inequality (6.4) with $q = \infty$ we obtain

$$\begin{aligned} \|r_m\|_{(1)} &\le \|r_0\|_{(1)} + \text{const} \sum_{j=0}^{m-1} 2^j \|r_j - r_{j+1}\|_{\mathfrak{B}} \\ &\le \text{const} \sum_{j=0}^{m-1} 2^j \left(\delta_{2^j}(\varphi) + \delta_{2^{j+1}}(\varphi)\right) \qquad (6.7) \\ &\le \text{const} \sum_{k=0}^{2^m} \delta_k(\varphi), \end{aligned}$$

since the sequence $\{\delta_j(\varphi)\}_{j\ge 0}$ is nonincreasing.

For a function f analytic in $\mathbb{D}$ and $t > 0$ we put

$$N(f,t) \stackrel{\text{def}}{=} \mu\{\zeta \in \mathbb{D} : \ |f''(\zeta)|(1 - |\zeta|)^2 > t\},$$

where the measure μ on $\mathbb{D}$ is defined by $d\mu(\zeta) \stackrel{\text{def}}{=} (1 - |\zeta|)^{-2} d\boldsymbol{m}_2(\zeta)$.

Recall that for a function $f \in \mathfrak{B}$

$$\sup_{\zeta \in \mathbb{D}} |f''(\zeta)|(1 - |\zeta|)^2 \le \text{const}\, \|f\|_{\mathfrak{B}},$$

and so there exists a positive constant M such that $N(f,t) = 0$ whenever $t \ge \frac{1}{2} M \|f\|_{\mathfrak{B}}$.

To estimate $\|\varphi\|_{(p)}$ we use the following well-known formula (see, e.g., Stein [2], Ch. 1, 1.5):

$$\int_{\mathbb{D}} \left|\varphi''(\zeta)(1 - |\zeta|)^2\right|^p d\mu(\zeta) = p \int_0^\infty t^{p-1} N(\varphi, t) dt. \qquad (6.8)$$

Put

$$f_m \stackrel{\text{def}}{=} \varphi - r_m.$$

We have by (6.8)

$$\begin{aligned} \|\varphi\|_{(p)}^p &= p \int_0^\infty t^{p-1} N(\varphi, t) dt \\ &\le \text{const} \int_0^\infty t^{p-1} N(\varphi, (1+M)t) dt. \end{aligned}$$

To simplify notation we put $\gamma_m = \delta_{2^m}(\varphi)$. If $t \ge \gamma_{m+1}$, then

$$\begin{aligned} N(\varphi, (1+M)t) &\le N(r_m, t) + N(f_m, Mt) \\ &\le N(r_m, t) + N(f_m, M\gamma_{m+1}) \\ &= N(r_m, t), \end{aligned}$$

since $\gamma_{m+1} \ge \frac{1}{2} \|f_m\|_{\mathfrak{B}}$.

Hence,

$$\|\varphi\|_{(p)}^p \le \text{const} \sum_{m=0}^{\infty} \int_{\gamma_{m+1}}^{\gamma_m} t^{p-1} N(r_m, t)\,dt.$$

Using (6.7), we obtain

$$\begin{aligned} N(r_m, t) &\le \frac{1}{t} \int_{\mathbb{D}} |r_m''(\zeta)|(1-|\zeta|)^2 d\mu(\zeta) \\ &\le \frac{1}{t}\|r_m\|_{(1)} \le \text{const}\, \frac{1}{t} \sum_{j=0}^{2^m} \delta_j(\varphi). \end{aligned}$$

Therefore

$$\begin{aligned} \int_{\gamma_{m+1}}^{\gamma_m} t^{p-1} N(r_m, t)\,dt &\le \text{const} \left(\sum_{j=0}^{2^m} \delta_j(\varphi) \right) \int_{\gamma_{m+1}}^{\gamma_m} t^{p-2}\,dt \\ &\le \text{const} \left(\sum_{j=0}^{2^m} \delta_j(\varphi) \right) \left(\gamma_m^{p-1} - \gamma_{m+1}^{p-1} \right). \end{aligned}$$

Finally,

$$\begin{aligned} \|\varphi\|_{(p)}^p &\le \text{const} \sum_{m=0}^{\infty} \sum_{j=0}^{2^m} \delta_j(\varphi) \left(\gamma_m^{p-1} - \gamma_{m+1}^{p-1} \right) \\ &= \text{const} \sum_{j=0}^{\infty} \delta_j(\varphi) \sum_{\{m: 2^m \ge j\}} \left(\gamma_m^{p-1} - \gamma_{m+1}^{p-1} \right) \\ &\le \text{const} \sum_{j=0}^{\infty} \delta_j(\varphi) \left(\delta_j(\varphi)\right)^{p-1} = \text{const} \sum_{j=0}^{\infty} \left(\delta_j(\varphi)\right)^p, \end{aligned}$$

which completes the proof. ■

We conclude this section with the inverse problem for rational approximation in $BMOA$. By the classical theorem of Bernstein (see Bernstein [1]), for any nonincreasing sequence $\{c_n\}_{n\ge 0}$ tending to 0 there exists a continuous function f on $[0,1]$ such that the distance in $C([0,1])$ from f to the set of polynomials of degree at most n is equal to c_n. We can ask a similar question about rational approximation in $BMOA$: is it true that for any nonincreasing sequence $\{c_n\}_{n\ge 0}$ there exists $\varphi \in BMOA$ such that

$$\rho_n^+(\varphi) = c_n, \quad n \in \mathbb{Z}_+?$$

By the Kronecker theorem and the Adamyan, Arov, and Krein theorem, this question is equivalent to the problem of whether for a given nonincreasing sequence $\{c_n\}_{n\ge 0}$ there exists a bounded Hankel operator Γ such that

$$s_n(\Gamma) = c_n, \quad n \in \mathbb{Z}_+.$$

This problem was posed in Khrushchëv and Peller [1]. It will be solved in §12.8 as a consequence of the solution of the problem to describe the nonnegative operators on Hilbert space that are unitarily equivalent to the moduli of Hankel operators.

7. Other Applications of the $\boldsymbol{S}_p$ Criterion

In this section we consider other applications of the $\boldsymbol{S}_p$ criterion obtained in Sections 1–3.

Functions of the Model Operator

We consider functions $\varphi(S_{[\vartheta]})$ of the model operator $S_{[\vartheta]}$ on the space $K_\vartheta = H^2 \ominus H^2$ (see §1.2). The following theorem gives a necessary and sufficient condition for $\varphi(S_{[\vartheta]})$ to be in $\boldsymbol{S}_p$.

Theorem 7.1. *Let $0 < p < \infty$, $\varphi \in H^\infty$ and let ϑ be an inner function. The following statements are equivalent:*

(i) $\varphi(S_{[\vartheta]}) \in \boldsymbol{S}_p$;

(ii) $\mathbb{P}_-\bar{\vartheta}\varphi \in B_p^{1/p}$.

Proof. By formula (2.9) of Chapter 1, the operator $\varphi(S_{[\vartheta]})$ belongs to $\boldsymbol{S}_p$ if and only if $H_{\bar{\vartheta}\varphi} \in \boldsymbol{S}_p$. Thus the result follows from Theorems 1.1, 2.1, and 3.1. ■

As in §1.5 we consider the special case when the inner function is an interpolating Blaschke product, which we denote by B. As we have mentioned in the discussion before Theorem 1.5.11, the functions f_j defined by formula (5.5) of Chapter 1 form an unconditional basis in K_B and $\varphi(S_{[B]})f_j = \varphi(\zeta_j)f_j$, $\varphi \in H^\infty$. Again, as we have mentioned before Theorem 1.5.11 the basis $\{f_j\}_{j\ge1}$ is equivalent to an orthogonal basis. Together with Theorem 7.1, this implies the following result.

Theorem 7.2. *Let $0 < p < \infty$, $\varphi \in H^p$, and suppose that B is an interpolating Blaschke product with zeros $\{\zeta_j\}_{j\ge1}$. The following statements are equivalent:*

(i) $\varphi(S_{[B]}) \in \boldsymbol{S}_p$;

(ii) $\mathbb{P}_-\bar{B}\varphi \in B_p^{1/p}$;

(iii) $\{\varphi(\zeta_j)\}_{j\ge0} \in \ell^p$.

Commutators

We consider here commutators $\mathcal{C}_\varphi$ on $L^2(\mathbb{T})$ defined for $\varphi \in BMO$ by

$$\mathcal{C}_\varphi f = \text{p.v.} \int_{\mathbb{T}} \frac{\varphi(\zeta) - \varphi(\tau)}{1 - \bar{\tau}\zeta} f(\tau) d\boldsymbol{m}(\tau), \quad f \in L^2,$$

(see §1.1). By Theorem 1.1.10, $\mathcal{C}_\varphi$ is bounded on L^2 if and only if $\varphi \in BMO$. The following result describes the commutators $\mathcal{C}_\varphi$ of class $\boldsymbol{S}_p$, $0 < p < \infty$.

Theorem 7.3. *Let $\varphi \in BMO$ and let $0 < p < \infty$. Then $\mathcal{C}_\varphi \in \boldsymbol{S}_p$ if and only if $\varphi \in B_p^{1/p}$.*

Proof. It has been shown in the proof of Theorem 1.1.10 that

$$\mathcal{C}_\varphi = H_\varphi(\mathbb{P}_+ f) + H^*_{\bar{\varphi}}(\mathbb{P}_- f), \quad f \in L^2.$$

Thus $\mathcal{C}_\varphi \in \boldsymbol{S}_p$ if and only if $H_\varphi \in \boldsymbol{S}_p$ and $H_{\bar{\varphi}} \in \boldsymbol{S}_p$. The result follows now from Theorems 1.1, 2.1, and 3.1. ■

Integral Operators on $L^2(\mathbb{R}_+)$

We consider here integral operators $\boldsymbol{\Gamma}_k$ on $L^2(\mathbb{R}_+)$ defined by

$$(\boldsymbol{\Gamma}_k f)(t) = \int_0^\infty k(s+t) f(s) ds.$$

Here k is a distribution on $(0, \infty)$ such that the operator $\boldsymbol{\Gamma}_k$ is bounded on $L^2(\mathbb{R}_+)$ (see §1.8). In §1.8 we have characterized the bounded, compact, and finite rank operators $\boldsymbol{\Gamma}_k$. Together with operators $\boldsymbol{\Gamma}_k$ we have considered in §1.8 integral operators $G_q : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_-)$ defined by

$$(G_q f)(t) = \int_0^\infty q(t-s) f(s) ds, \quad t < 0.$$

Then

$$(\boldsymbol{\Gamma}_k f)(t) = (G_q f)(-t), \quad k(t) = q(-t), \quad t \in (0, \infty)$$

(see §1.8).

We have proved in §1.8 that if G_q is bounded, then there exists a function ψ in $L^\infty(\mathbb{R})$ such that $q = \mathcal{F}\psi \big| (-\infty, 0)$. Moreover,

$$G_q = \mathcal{F} \mathcal{H}_\psi \mathcal{F}^*,$$

where $\mathcal{H}_\psi$ is the Hankel operator from $H^2(\mathbb{C}_+)$ to $H^2(\mathbb{C}_-)$ defined by $\mathcal{H}_\psi F = \boldsymbol{P}_- \psi F$ (see §1.8). Next, if ω is the conformal map of $\mathbb{D}$ onto $\mathbb{C}_+$ defined by

$$\omega(\zeta) = \mathrm{i}\frac{1+\zeta}{1-\zeta}, \quad \zeta \in \mathbb{D},$$

then $\mathcal{U}^* \mathcal{H}_\psi \mathcal{U} = H_\varphi$, where $\varphi = \psi \circ \omega$ and $\mathcal{U}$ is the unitary operator from L^2 onto $L^2(\mathbb{R})$ defined by

$$(\mathcal{U} f)(t) = \frac{1}{\sqrt{\pi}} \frac{(f \circ \omega^{-1})(t)}{t + \mathrm{i}}.$$

Thus it follows from Theorems 1.1, 2.1, and 3.1 that $\boldsymbol{\Gamma}_k \in \boldsymbol{S}_p$ if and only if $\mathbb{P}_- \varphi \in B_p^{1/p}$, $0 < p < \infty$. Let us rewrite this condition in terms of the function ψ.

We define the Besov space $B_p^{1/p}(\mathbb{R})$, $0 < p < \infty$, as the space of distributions ξ on $\mathbb{R}$ such that

$$\sum_{j \in \mathbb{Z}} 2^j \left(\|\xi * \mathcal{V}_j\|^p_{L^p(\mathbb{R})} + \|\xi * \overline{\mathcal{V}}_j\|^p_{L^p(\mathbb{R})} \right) < \infty. \tag{7.1}$$

Here the $\mathcal{V}_j$ are functions such that their Fourier transforms $v_j \stackrel{\text{def}}{=} \mathcal{F}\mathcal{V}_j$ have the following properties:

$$v_j \in C^\infty, \quad v_j \geq 0, \quad \operatorname{supp} v_j \subset \left[2^{j-1}, 2^{j+1}\right],$$
$$(v_j)(x) = (v_{j-1})\left(\frac{x}{2}\right), \quad x > 0,$$

and

$$\sum_{j \in \mathbb{Z}} (v_j)(x) = 1, \quad x > 0.$$

We refer the reader to Peetre [1] for more detailed information on Besov classes. Note that the class $B_p^{1/p}(\mathbb{R})$ does not depend on the choice of the functions $\mathcal{V}_j$. Note also that if a distribution ξ satisfies (7.1), and r is an arbitrary polynomial, then $r + \xi$ also satisfies (7.1). If $\xi \in B_p^{1/p}(\mathbb{R})$, we have the convergence of the series

$$\sum_{j \in \mathbb{Z}} (\xi * \mathcal{V}_j + \xi * \overline{\mathcal{V}}_j)$$

to ξ modulo polynomials. In fact, it can be shown that the series converges modulo constants (see Peetre [1], Ch. 3 and 11).

It is convenient to identify $B_p^{1/p}$ with a space of functions on $\mathbb{R}$ that is contained in VMO. It follows from Dyn'kin's description of Besov classes (see Appendix 2.6) that if $\psi \in BMO(\mathbb{R})$, then $\boldsymbol{P}^- \psi \in B_p^{1/p}(\mathbb{R})$ if and only if $\mathbb{P}_-(\psi \circ \omega) \in B_p^{1/p}$, where

$$\boldsymbol{P}^- \psi \stackrel{\text{def}}{=} (\mathbb{P}_-(\psi \circ \omega)) \circ \omega^{-1}, \quad \psi \in L^\infty(\mathbb{R})$$

(see §1.8).

Thus the following theorem follows from Theorems 1.1, 2.1, and 3.1.

Theorem 7.4. *Let* $0 < p < \infty$. *Suppose that* k *is a distribution on* $(0, \infty)$. *The following are equivalent:*

(i) $\boldsymbol{\Gamma}_k \in \boldsymbol{S}_p$;

(ii) $k = \mathcal{F}\psi \big| (0, \infty)$ *for a function* $\psi \in B_p^{1/p}(\mathbb{R})$.

Note that (ii) can be reformulated in the following way:

$$\sum_{j \in \mathbb{Z}} |\operatorname{supp} \omega_j| \cdot \|\mathcal{F}(\omega_j k)\|_{L^p(\mathbb{R})}^p < \infty, \tag{7.2}$$

where the ω_j are defined above and $|\operatorname{supp} \omega_j|$ means the length of the interval $\operatorname{supp} \omega_j$. Here we can consider the function $\omega_j k$ as a function in $C^\infty(\mathbb{R})$ with compact support.

Wiener–Hopf Operators on a Finite Interval

Let k be a function or a distribution on $\mathbb{R}$. For $\sigma > 0$ we consider the integral operator $W_{\sigma,k}$ on $L^2[-\sigma,\sigma]$ defined by

$$(W_{\sigma,k}f)(x) = \int_{-\sigma}^{\sigma} k(x-y)f(y)dy, \quad x \in [-\sigma,\sigma].$$

Such operators are called *Wiener–Hopf operators on a finite interval.* We obtain here a necessary and sufficient condition for $W_{\sigma,k} \in \boldsymbol{S}_p$, $0 < p < \infty$.

Together with the Wiener–Hopf operator $W_{\sigma,k}$ on $L^2[-\sigma,\sigma]$ we consider here truncated "Hankel operators" $\boldsymbol{\Gamma}_{\sigma,k}$ on $L^2[-\sigma,\sigma]$ defined by

$$(\boldsymbol{\Gamma}_{\sigma,k}f)(x) = \int_{-\sigma}^{\sigma} k(x+y)f(y)dy, \quad x \in [-\sigma,\sigma].$$

It is easy to see that $\boldsymbol{\Gamma}_{\sigma,k} = W_{\sigma,k}U$, where U is the unitary operator on $L^2[-\sigma,\sigma]$ defined by

$$(Uf)(x) = f(-x), \quad x \in [-\sigma,\sigma].$$

To state the result, we introduce the functions ν_j associated with an interval $[\alpha,\beta]$. We start with the system of functions ω_j defined above. Put

$$\nu_j(x) = \omega_j\left(2\frac{x-\alpha}{\beta-\alpha}\right), \quad j < 0,$$

$$\nu_j(x) = \nu_j(\alpha+\beta-x), \quad j > 0,$$

$$\nu_0(x) = 1 - \sum_{j \neq 0} \nu_j(x).$$

Clearly, $0 \le \nu_j(x) \le 1$, $j \in \mathbb{Z}$, $\operatorname{supp}\nu_j \subset [\alpha,(\alpha+\beta)/2]$ for $j < 0$ and $\operatorname{supp}\nu_j \subset [(\alpha+\beta)/2,\beta]$ for $j > 0$.

Theorem 7.5. *Let $0 < p < \infty$, $\sigma > 0$, and let $\{\nu_j\}_{j\in\mathbb{Z}}$ be the system of functions associated with $[-2\sigma,2\sigma]$. Then $W_{\sigma,k} \in \boldsymbol{S}_p$ if and only if*

$$\sum_{j\in\mathbb{Z}} 2^{-|j|}\|\mathcal{F}(\nu_j k)\|_{L^p}^p < \infty. \tag{7.3}$$

Note that (7.3) is equivalent to the condition

$$\sum_{j\in\mathbb{Z}} |\operatorname{supp}\nu_j| \cdot \|\mathcal{F}(\nu_j k)\|_{L^p}^p < \infty,$$

which is similar to condition (7.2).

It is more convenient to work with operators $\boldsymbol{\Gamma}_k^a$ on $L^2[0,a]$ defined by

$$(\boldsymbol{\Gamma}_k^a f)(x) = \int_0^a k(x+y)f(y)dy, \quad x \in [0,a].$$

Clearly, the operator $\boldsymbol{\Gamma}_k^a$ depends only on the restriction of k to $[0,2a]$. It is easy to see that Theorem 7.5 is equivalent to the following one.

Theorem 7.6. *Let $0 < p < \infty$, $a > 0$, and let $\{\nu_j\}_{j\in\mathbb{Z}}$ be the system of functions associated with $[0, 2a]$. Then $\boldsymbol{\Gamma}_k^a \in \boldsymbol{S}_p$ if and only if*

$$\sum_{j\in\mathbb{Z}} 2^{-|j|} \|\mathcal{F}(\nu_j k)\|_{L^p}^p < \infty.$$

Proof. To prove Theorem 7.6, we reduce the study of the operators $\boldsymbol{\Gamma}_k^a$ to the case of operators $\boldsymbol{\Gamma}_k$ on $L^2(\mathbb{R}_+)$. If $\operatorname{supp} k \subset [0, a]$, it is easy to see that $\boldsymbol{\Gamma}_k^a \in \boldsymbol{S}_p$ if and only if $\boldsymbol{\Gamma}_k \in \boldsymbol{S}_p$, since in this case $\boldsymbol{\Gamma}_k^a$ is the restriction of $\boldsymbol{\Gamma}_k$ to $L^2[0, a]$ while $\boldsymbol{\Gamma}_k \big| L^2[0, \infty) = \mathbb{O}$. (We identify in a natural way $L^2[0, a]$ with a subspace of $L^2(\mathbb{R}_+)$.) Thus by Theorem 7.4, if $\operatorname{supp} k \subset [0, a]$, then $\boldsymbol{\Gamma}_k^a \in \boldsymbol{S}_p$ if and only if

$$\sum_{j\le 0} 2^{-|j|} \|\mathcal{F}(\nu_j k)\|_{L^p}^p < \infty.$$

The same can be done if $\operatorname{supp} k \subset [a, 2a]$. Indeed, in this case $\boldsymbol{\Gamma}_k^a$ is unitarily equivalent to the operator $\boldsymbol{\Gamma}_{k_\#}^a$, where $k_\#(x) = k(2a - x)$ (unitary equivalence is provided by the unitary operator V on $L^2[0, a]$ defined by $(Vf)(x) = f(a - x)$, $f \in L^2[0, a]$). Clearly, $\operatorname{supp} k_\# \subset [0, a]$, and so $\boldsymbol{\Gamma}_k^a \in \boldsymbol{S}_p$ if and only if

$$\sum_{j\ge 0} 2^{-|j|} \|\mathcal{F}(\nu_j k)\|_{L^p}^p < \infty.$$

To treat the general case, we cut the function k into three pieces. Consider C^∞ functions v_1, v_2, and v_3 with compact supports such that $v_1(x) + v_2(x) + v_3(x) = 1$ for $x \in [0, 2a]$, $0 \le v_j(x) \le 1$, for $j = 1, 2, 3$ and $x \in \mathbb{R}$, $\operatorname{supp} v_2 = [3a/4, 5a/4]$, $\operatorname{supp} v_3 = [a, 3a]$, and $v_1(x) = v_3(2a - x)$. Put $k_j \stackrel{\text{def}}{=} v_j k$. We claim that $\boldsymbol{\Gamma}_k^a \in \boldsymbol{S}_p$ if and only if $\boldsymbol{\Gamma}_{k_j}^a \in \boldsymbol{S}_p$ for $j = 1, 2, 3$. This is a consequence of the following more general fact.

Lemma 7.7. *Let φ be a C^∞ function with compact support in $(0, \infty)$ and let $0 < p < \infty$. Suppose that R is an integral operator on $L^2(\mathbb{R}_+)$,*

$$(Rf)(x) = \int_{\mathbb{R}_+} u(x, y) f(u)\,dy, \quad f \in L^2(\mathbb{R}_+).$$

Consider the integral operator T on $L^2(\mathbb{R}_+)$ defined by

$$(Tf)(x) = \int_{\mathbb{R}_+} \varphi(x + y) u(x, y) f(u)\,dy, \quad f \in L^2(\mathbb{R}_+).$$

If $R \in \boldsymbol{S}_p$, then $T \in \boldsymbol{S}_p$.

Let us first complete the proof of Theorem 7.6. By Lemma 7.7, $\boldsymbol{\Gamma}_k^a \in \boldsymbol{S}_p$ if and only if $\boldsymbol{\Gamma}_{k_j}^a \in \boldsymbol{S}_p$ for $j = 1, 2, 3$. Clearly, $\operatorname{supp} k_1 \subset [0, a]$ and $\operatorname{supp} k_3 \subset [a, 2a]$. Thus as we have already observed

$$\boldsymbol{\Gamma}_{k_1}^a \in \boldsymbol{S}_p \iff \sum_{j\le 0} 2^{-|j|} \|\mathcal{F}(\nu_j k_1)\|_{L^p}^p < \infty \tag{7.4}$$

and

$$\boldsymbol{\Gamma}^a_{k_3} \in \boldsymbol{S}_p \quad \Longleftrightarrow \quad \sum_{j\ge 0} 2^{-|j|}\|\mathcal{F}(\nu_j k_3)\|^p_{L^p} < \infty. \tag{7.5}$$

To estimate $\boldsymbol{\Gamma}^a_{k_2}$, we need the following fact.

Lemma 7.8. *Let $0 < p < \infty$ and let k be a function on $\mathbb{R}$ with support in $[3a/4, 5a/4]$. Then $\boldsymbol{\Gamma}^a_k \in \boldsymbol{S}_p$ if and only if $\mathcal{F}k \in L^p(\mathbb{R})$.*

Let us first complete the proof of the theorem. By Lemma 7.8, $\boldsymbol{\Gamma}^a_k \in \boldsymbol{S}_p$ if and only if $\mathcal{F}(k_2) \in L^p(\mathbb{R})$ and and the right-hand sides of (7.4) and (7.5) are finite. Note that for $j \ge 2$ we have $\nu_j k_3 = \nu_j k$ and for $k \le -2$ we have $\nu_j k_1 = \nu_j k$. It remains to show that

$$\sum_{j=-1}^{1} \|\mathcal{F}(\nu_j k_1)\|^p_{L^p} < \infty \tag{7.6}$$

if and only if

$$\begin{aligned} \|\mathcal{F}(\nu_{-1}k_1)\|^p_{L^p} \quad &+ \quad \|\mathcal{F}(\nu_0 k_1)\|^p_{L^p} + \|\mathcal{F}k_2\|^p_{L^p} \\ &+ \quad \|\mathcal{F}(\nu_0 k_3)\|^p_{L^p} + \|\mathcal{F}(\nu_1 k_3)\|^p_{L^p} < \infty. \end{aligned} \tag{7.7}$$

This is a consequence of the following well-known fact (see Peetre [1]): let ψ be a C^∞ function with compact support in $(0,\infty)$, $0 < \alpha < \beta < \infty$, and let g be a function with support in $[\alpha,\beta]$; then

$$\|\mathcal{F}(\psi g)\|_{L^p} \le \text{const}\,\|\mathcal{F}g\|_{L^p}; \tag{7.8}$$

the constant may depend on ψ, α, and β but does not depend on g. (Note that (7.8) also follows easily from Theorem 7.4 and Lemma 7.7.)

Indeed, it follows from (7.8) that both (7.6) and (7.7) are equivalent to the fact that

$$\mathcal{F}\left(k\sum_{j=-1}^{1}\nu_j\right) \in L^p(\mathbb{R}). \quad \blacksquare$$

Proof of Lemma 7.7. Suppose that $\operatorname{supp}\varphi \subset [0, T/2]$. Put

$$v(x,y) \stackrel{\text{def}}{=} \varphi(x+y)u(x,y).$$

Without loss of generality we may assume that $\operatorname{supp} u \subset [0,T/2]\times[0,T/2]$. Consider the function ϕ on $\mathbb{T}$ defined by

$$\phi\left(e^{2\pi \mathrm{i} x/T}\right) = \varphi(x), \quad 0 \le x \le T.$$

Then

$$\varphi(x+y) = \sum_{n\in\mathbb{Z}} \hat{\phi}(n) e^{2\pi \mathrm{i} n(x+y)/T}.$$

Thus

$$T = \sum_{n\in\mathbb{Z}} \hat{\phi}(n) M_{\xi_n} R M_{\eta_n}, \tag{7.9}$$

where $\xi_n(x) = e^{2\pi i n x/T}$, $\eta_n(y) = e^{2\pi i n y/T}$, and M_h is multiplication by h on L^2. Since $\phi \in C^\infty$, it follows that

$$\sum_{n\in\mathbb{Z}} |\hat{\phi}(n)|^p < \infty$$

for any $p > 0$. Hence, the series (7.9) converges absolutely, and so $T \in \boldsymbol{S}_p$. ■

Proof of Lemma 7.8. We consider three integral operators with the same kernel $k(x+y)$. The first one is our $\boldsymbol{\Gamma}_k^a$ on $L^2[0,a]$. The second operator is the operator $\boldsymbol{\Gamma}_k$ on $L^2(\mathbb{R}_+)$. The third one is the integral operator on $L^2[a/4,a]$, which we denote by $\boldsymbol{\Gamma}_k^\#$. It is easy to see that

$$\|\boldsymbol{\Gamma}_k^\#\|_{\boldsymbol{S}_p} \le \|\boldsymbol{\Gamma}_k^a\|_{\boldsymbol{S}_p} \le \|\boldsymbol{\Gamma}_k\|_{\boldsymbol{S}_p}.$$

Indeed, obviously,

$$\boldsymbol{\Gamma}_k^a = P_0^a \boldsymbol{\Gamma}_k \big| L^2[0,a] \quad \text{and} \quad \boldsymbol{\Gamma}_k^\# = P_{a/4}^a \boldsymbol{\Gamma}_k^a \big| L^2[a/4,a],$$

where P_α^β is the orthogonal projection onto $L^2[\alpha,\beta]$.

Since $\operatorname{supp}\mathcal{F}k$ is separated away from zero and infinity, it follows from Theorem 7.4 that $\|\boldsymbol{\Gamma}_k\|_{\boldsymbol{S}_p}$ is equivalent to $\|\mathcal{F}k\|_{L^p}$. On the other hand, $\|\boldsymbol{\Gamma}_k^\#\|_{\boldsymbol{S}_p} = \|\boldsymbol{\Gamma}_{k_\#}\|_{\boldsymbol{S}_p}$, where $k_\#(x) = k(x+a/4)$. Hence, for the same reason, $\|\boldsymbol{\Gamma}_{k_\#}\|_{\boldsymbol{S}_p}$ is equivalent to $\|\mathcal{F}k\|_{L^p}$. ■

8. Generalized Hankel Matrices

In this section we study matrices of the form

$$\Gamma_\varphi^{\alpha,\beta} = \{(1+j)^\alpha(1+k)^\beta\hat{\varphi}(j+k)\}_{j,k\ge 0},$$

where φ is a function analytic in $\mathbb{D}$, and α and β are real numbers. We obtain boundedness and compactness criteria for certain values of α and β. Then we study the question of when $\Gamma_\varphi^{\alpha,\beta} \in \boldsymbol{S}_p$. As usual, here we identify operators on ℓ^2 with their matrices in the standard basis $\{e_j\}_{j\ge 0}$ of ℓ^2. It is easy to see that $\Gamma_\varphi^{\alpha,\beta}$ is bounded (compact, or belongs to $\boldsymbol{S}_p$) if and only if the same is true for $\Gamma_\varphi^{\beta,\alpha}$.

For $\alpha > -1$ and $\beta > -1$ we also consider the matrices $\widetilde{\Gamma}_\varphi^{\alpha,\beta}$ defined by

$$\widetilde{\Gamma}_\varphi^{\alpha,\beta} = \{D_j^{(\alpha)} D_k^{(\beta)} \hat{\varphi}(j+k)\}_{j,k\ge 0},$$

where

$$D_n^{(\alpha)} = \begin{pmatrix} n+\alpha \\ n \end{pmatrix} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}$$

(see Appendix 2.6). Since

$$c < \frac{K_n^{(\alpha)}}{(1+n)^\alpha} < C$$

for some $c, C \in (0,\infty)$ (see Appendix 2.6), it is easy to see that if $\min\{\alpha,\beta\} > -1$, then $\Gamma_\varphi^{\alpha,\beta}$ is bounded (compact) if and only if $\widetilde{\Gamma}_\varphi^{\alpha,\beta}$ is, and $\Gamma_\varphi^{\alpha,\beta} \in \boldsymbol{S}_p$ if and only if $\widetilde{\Gamma}_\varphi^{\alpha,\beta} \in \boldsymbol{S}_p$, $0 < p < \infty$.

Theorem 8.1. *Let α, $\beta > 0$. Then $\Gamma_\varphi^{\alpha,\beta}$ is the matrix of a bounded operator if and only if $\varphi \in \Lambda_{\alpha+\beta}$. The operator norm of $\Gamma_\varphi^{\alpha,\beta}$ is equivalent to $\|\varphi\|_{\Lambda_{\alpha+\beta}}$.*

Proof. Suppose that $\varphi \in \Lambda_{\alpha+\beta}$. Let $a = \{a_j\}_{j\ge0}$ and $b = \{b_j\}_{j\ge0}$ be finitely supported sequences in of ℓ^2. We have

$$\begin{aligned}(\Gamma_\varphi^{\alpha,\beta}a,b) &= \sum_{j,k\ge0}(1+j)^\alpha(1+k)^\beta\hat\varphi(j+k)a_j\bar b_k\\ &= \sum_{m\ge0}\hat\varphi(m)\sum_{j=0}^m(1+j)^\alpha a_j(1+m-j)^\beta\bar b_{m-j}\\ &= \sum_{m\ge0}\hat\varphi(m)\sum_{j=0}^m\hat f(j)\hat g(m-j) = \sum_{m\ge0}\hat\varphi(m)\hat h(m) = (\varphi,h),\end{aligned}$$

where $f(z) \stackrel{\text{def}}{=} \sum_{j\ge0}(1+j)^\alpha a_j z^j$, $g(z) \stackrel{\text{def}}{=} \sum_{j\ge0}(1+j)^\beta \bar b_j z^j$, and $h \stackrel{\text{def}}{=} fg$.

We have $f = I_{-\alpha}\sum_{j\ge0}a_j z^j$, $g = I_{-\beta}\sum_{j\ge0}\bar b_j z^j$, and so

$$\|f\|_{B_2^{-\alpha}} \le \text{const}\,\|a\|_{\ell^2}, \quad \|g\|_{B_2^{-\beta}} \le \text{const}\,\|b\|_{\ell^2}$$

(see Appendix 2.6). Therefore

$$\int_{\mathbb{D}}|f(\zeta)|^2(1-|\zeta|)^{2\alpha-1}d\boldsymbol{m}_2(\zeta) \le \text{const}\,\|a\|_{\ell^2},$$

$$\int_{\mathbb{D}}|g(\zeta)|^2(1-|\zeta|)^{2\beta-1}d\boldsymbol{m}_2(\zeta) \le \text{const}\,\|b\|_{\ell^2}$$

(see Appendix 2.6). It follows that

$$\|h\|_{B_1^{-(\alpha+\beta)}} \le \text{const}\int_{\mathbb{D}}|h(\zeta)|(1-|\zeta|)^{\alpha+\beta-1}d\boldsymbol{m}_2(\zeta) \le \text{const}\,\|a\|_{\ell^2}\|b\|_{\ell^2}.$$

Taking into account the fact that $\left(B_1^{-(\alpha+\beta)}\right)^* = \Lambda_{\alpha+\beta}$, we obtain

$$|(\Gamma_\varphi^{\alpha,\beta}a,b)| \le \text{const}\,\|\varphi\|_{\Lambda_{\alpha+\beta}}\|a\|_{\ell^2}\|b\|_{\ell^2},$$

and so $\Gamma_\varphi^{\alpha,\beta}$ is bounded.

Suppose now that φ is a function analytic in $\mathbb{D}$ such that the operator $\Gamma_\varphi^{\alpha,\beta}$ is bounded on ℓ^2. We have to show that $\varphi \in \Lambda_{\alpha+\beta}$. We may assume that $\widetilde{\Gamma}_\varphi^{\alpha,\beta}$ is bounded.

Let ψ be an analytic in $\mathbb{D}$ function of class B_1^1. By Theorem 1.1, $\Gamma_\psi \in \boldsymbol{S}_1$. We have (see Appendix 1.1)

$$\begin{aligned}\langle \Gamma_\psi, \widetilde{\Gamma}_\varphi^{\alpha,\beta}\rangle &= \sum_{m=0}^\infty \sum_{j=1}^m D_j^{(\alpha)} D_{m-j}^{(\beta)} \hat{\psi}(m)\overline{\hat{\varphi}(m)} \\ &= \sum_{m=0}^\infty D_m^{(\alpha+\beta+1)} \hat{\psi}(m)\overline{\hat{\varphi}(m)} = (\psi, \widetilde{I}_{-(\alpha+\beta+1)}\varphi).\end{aligned}$$

It follows that $\widetilde{I}_{-(\alpha+\beta+1)}\varphi$ determines a continuous linear functional on $\left(B_1^1\right)_+$, and so $\widetilde{I}_{-(\alpha+\beta+1)}\varphi \in \Lambda_{-1}$ (see Appendix 2.6). Hence, $\varphi \in \Lambda_{\alpha+\beta}$ (see Appendix 2.6). ■

The following theorem establishes the compactness criterion in the case when $\alpha > 0$ and $\beta > 0$.

Theorem 8.2. *Let α, $\beta > 0$. Then $\Gamma_\varphi^{\alpha,\beta}$ is compact if and only if $\varphi \in \lambda_{\alpha+\beta}$.*

To prove the necessity of the condition $\varphi \in \lambda_{\alpha+\beta}$ we use the Schur product of matrices.

Recall that in §3 for matrices $A = \{a_{jk}\}_{j,k\ge 0}$, $B = \{b_{jk}\}_{j,k\ge 0}$, we have defined the *Schur product* of A and B by

$$A \star B = \{a_{jk}b_{jk}\}_{j,k\ge 0}.$$

A matrix A is called a *Schur multiplier* of the space of bounded operators if Schur multiplication by A is a bounded operator on the space $\mathcal{B}(\ell^2)$. The norm of A in the space of Schur multipliers is denoted by $\|A\|_\star$,

$$\|A\|_\star \stackrel{\text{def}}{=} \sup\{\|A\star T\| : \|T\| \le 1\}.$$

Let μ belong to the space $\mathfrak{M}$ of finite complex Borel measure on $\mathbb{T}$. Denote by Γ_μ the Hankel matrix $\{\hat{\mu}(j+k)\}_{j,k\ge 0}$. We need the following fact.

Theorem 8.3. *Let μ be a finite complex Borel measure μ on $\mathbb{T}$. Then Γ_μ is a Schur multiplier of $\mathcal{B}(\ell^2)$ and*

$$\|\Gamma_\mu\|_\star \le \|\mu\|_{\mathfrak{M}}.$$

Proof. Consider first the case when $\mu = \delta_\tau$, $\tau \in \mathbb{T}$, i.e., $\mu(\{\tau\}) = 1$, and $\mu(E) = 0$ if $E \subset \mathbb{T}$ and $\tau \notin E$. Then $\hat{\mu}(j) = \bar{\tau}^j$, $j \in \mathbb{Z}$.

Let T be a bounded operator on ℓ^2 with matrix $\{t_{jk}\}_{j,k\ge 0}$. Clearly, $\Gamma_\mu \star T = D_{\bar{\tau}} T D_{\bar{\tau}}$, where $D_{\bar{\tau}}$ is the unitary operator on ℓ^2 defined by $D_{\bar{\tau}} e_j = \bar{\tau}^j e_j$, $j \ge 0$. Hence, $\|\Gamma_\mu \star T\| = \|T\|$.

Now let μ be an arbitrary measure in the unit ball of $\mathfrak{M}$. By the Krein–Milman theorem, μ is the limit in the weak topology $\sigma(\mathfrak{M}, C(T))$ of a sequence of measures of the form

$$\sum_{j=1}^N \lambda_j \delta_{\tau_j}, \tag{8.1}$$

where $\tau_j \in \mathbb{T}$ and $\sum_{j=1}^{N} |\lambda_j| \le 1$. Clearly, $\|\Gamma_\nu\|_\star \le 1$ for any measure ν of the form (8.1).

It is also easy to see that if $\{\nu_j\}_{j\ge0}$ is a sequence of measure converging to μ in the weak topology $\sigma(\mathfrak{M}, C(T))$, then $\{\Gamma_{\nu_j}\}_{j\ge0}$ converges to Γ_μ in the weak operator topology, which implies that $\|\Gamma_\mu\| \le 1$. ■

Recall that for a natural number N, K_N stands for the Fejér kernel, and $\|K_N\|_{L^1} = 1$ (see Appendix 2.1).

Corollary 8.4. *Let T be a compact operator on ℓ^2. Then*

$$\lim_{N\to\infty} \|T - \Gamma_{K_N} \star T\| = 0. \tag{8.2}$$

Proof. Clearly, (8.2) holds for operators T, which can have only finitely many nonzero matrix entries. The result now follows from Theorem 8.3 and the fact that $\|K_N\|_{L^1} = 1$. ■

Proof of Theorem 8.2. If $\varphi \in \lambda_{\alpha+\beta}$, we can approximate φ by polynomials in $\Lambda_{\alpha+\beta}$. Since the operator $\Gamma_\psi^{\alpha,\beta}$ has finite rank for any polynomial ψ, it follows now from Theorem 8.1 that $\Gamma_\varphi^{\alpha,\beta}$ is compact.

Suppose now that $\Gamma_\varphi^{\alpha,\beta}$ is compact. By Corollary 8.4,

$$\|\Gamma_\varphi^{\alpha,\beta} - \Gamma_{K_N} \star \Gamma_\varphi^{\alpha,\beta}\| = \|\Gamma_\varphi^{\alpha,\beta} - \Gamma_{\varphi * K_N}^{\alpha,\beta}\| = \|\Gamma_{\varphi - \varphi * K_N}^{\alpha,\beta}\| \to 0 \quad \text{as} \quad N \to \infty.$$

It follows from Theorem 8.1 that $\|\varphi - \varphi * K_N\|_{\Lambda_\alpha} \to 0$ as $N \to \infty$. Hence, $\varphi \in \lambda_{\alpha+\beta}$. ■

Let us characterize the bounded operators $\Gamma_\varphi^{\alpha,0}$ for $\alpha > 0$.

Theorem 8.5. *Let $\alpha > 0$. Then $\Gamma_\varphi^{\alpha,0}$ is the matrix of a bounded operator if and only if $I_{-\alpha}\varphi \in BMO$. The operator norm of $\Gamma_\varphi^{\alpha,0}$ is equivalent to $\|I_{-\alpha}\varphi\|_{BMO}$.*

Proof. Suppose that $I_{-\alpha}\varphi \in BMO$. Let $a = \{a_j\}_{j\ge0}$ and $b = \{b_j\}_{j\ge0}$ be finitely supported sequences in of ℓ^2. We have

$$\begin{aligned}
(\Gamma_\varphi^{\alpha,0} a, b) &= \sum_{j,k\ge0} (1+j)^\alpha \hat{\varphi}(j+k) a_j \bar{b}_k \\
&= \sum_{m\ge0} \hat{\varphi}(m) \sum_{j=0}^{m} (1+j)^\alpha a_j \bar{b}_{m-j} \\
&= \sum_{m\ge0} \hat{\varphi}(m) \sum_{j=0}^{m} \hat{f}(j)\hat{g}(m-j) = \sum_{m\ge0} \hat{\varphi}(m)\hat{h}(m) = (\varphi, h),
\end{aligned}$$

where $f(z) \stackrel{\text{def}}{=} \sum_{j\ge0} (1+j)^\alpha a_j z^j$, $g(z) \stackrel{\text{def}}{=} \sum_{j\ge0} \bar{b}_j z^j$, and $h \stackrel{\text{def}}{=} fg$.

We have

$$\left(\int_{\mathbb{D}} |f(\zeta)|^2 (1 - |\zeta|)^{2\alpha-1} d\boldsymbol{m}_2(\zeta) \right)^{1/2} \le \text{const}\,\|a\|_{\ell^2} \le \text{const}\,\|a\|_{\ell^2}$$

(see Appendix 2.6). Clearly, $\|g\|_{H^2} = \|b\|_{\ell^2}$.

Consider the radial maximal function $g^{(*)}$ on $\mathbb{T}$ defined by

$$g^{(*)}(\zeta) \stackrel{\text{def}}{=} \sup_{0<r<1} |g(r\zeta)|, \quad \zeta \in \mathbb{T}.$$

Then $\|g^{(*)}\|_{L^2} \le \text{const}\,\|g\|_{H^2} = \text{const}\,\|a\|_{\ell^2}$ (see Appendix 2.1). Thus

$$\begin{aligned}
\|I_\alpha h\|_{H^1} &\le \text{const}\int_{\mathbb{T}}\left(\int_0^1 |h(r\zeta)|^2(1-r)^{2\alpha-1}dr\right)^{1/2} d\boldsymbol{m}(\zeta)\\
&= \text{const}\int_{\mathbb{T}}\left(\int_0^1 |f(r\zeta)g(r\zeta)|^2(1-r)^{2\alpha-1}dr\right)^{1/2} d\boldsymbol{m}(\zeta)\\
&\le \text{const}\int_{\mathbb{T}}\left(\int_0^1 |f(r\zeta)g^{(*)}(\zeta)|^2(1-r)^{2\alpha-1}dr\right)^{1/2} d\boldsymbol{m}(\zeta)\\
&= \text{const}\int_{\mathbb{T}}|g^{(*)}(\zeta)|\left(\int_0^1 |f(r\zeta)|^2(1-r)^{2\alpha-1}dr\right)^{1/2} d\boldsymbol{m}(\zeta)\\
&\le \text{const}\,\|g^{(*)}\|_{L^2}\int_{\mathbb{T}}\left(\int_0^1 |f(r\zeta)|^2(1-r)^{2\alpha-1}dr\right)^{1/2} d\boldsymbol{m}(\zeta)\\
&\le \text{const}\,\|a\|_{\ell^2}\|b\|_{\ell^2}
\end{aligned}$$

(see Appendix 2.6), and so

$$|(\Gamma_\varphi^{\alpha,0}a,b)| \le \|I_{-\alpha}\varphi\|_{BMO}\|I_\alpha h\|_{H^1} \le \text{const}\,\|I_{-\alpha}\varphi\|_{BMO}\|a\|_{\ell^2}\|b\|_{\ell^2},$$

which proves that $\Gamma_\varphi^{\alpha,0}$ is bounded.

Suppose now that the matrix $\Gamma_\varphi^{\alpha,0}$ is bounded. It is easy to see that in this case the matrix $\{j^\alpha\hat\varphi(j+k)\}_{j,k\ge0}$ is bounded. Clearly, this implies that the matrix $\{k^\alpha\hat\varphi(j+k)\}_{j,k\ge0}$, and so their sum $\{(j^\alpha+k^\alpha)\hat\varphi(j+k)\}_{j,k\ge0}$ is a bounded matrix. Consider the matrix $M=\{m_{jk}\}_{j,k\ge0}$ defined by

$$m_{jk} = \begin{cases} \frac{(j+k)^\alpha}{j^\alpha+k^\alpha}, & j+k>0,\\ 0, & j=k=0.\end{cases}$$

Lemma 8.6. *The matrix M is a Schur multiplier of the space $\mathcal{B}(\ell^2)$.*

Let us first complete the proof of Theorem 8.5. As we have already noticed, the matrix $\{(j^\alpha+k^\alpha)\hat\varphi(j+k)\}_{j,k\ge0}$ is bounded. Its Schur product with M is equal to the Hankel matrix $\{(j+k)^\alpha\hat\varphi(j+k)\}_{j,k\ge0}$. It follows easily from the Nehari theorem that $I_{-\alpha}\varphi\in BMO$. ■

To prove Lemma 8.6 we need the following fact.

Lemma 8.7. *Let μ be a finite complex Borel measure on $\mathbb{R}$. Let $A=\{a_{jk}\}_{j,k\ge0}$ be the matrix defined by*

$$a_{jk} = \begin{cases} \int_{\mathbb{R}} k^{2\pi i\xi}\, j^{-2\pi i\xi}d\mu(\xi), & j>0,\ k>0,\\ 0, & \textit{otherwise.}\end{cases}$$

Then A is a Schur multiplier of $\mathcal{B}(\ell^2)$ and $\|A\|_\star \le \|\mu\|$, where $\|\mu\|$ is the total variation of μ.

Proof. It is easy to see that as in the proof of Theorem 8.3 it is sufficient to consider the case when μ is a point mass at a point $v \in \mathbb{R}$, i.e., $\mu(\{v\}) = 1$ and $\mu(E) = 0$ if $v \notin E$. In this case

$$a_{jk} = k^{2\pi \mathrm{i} v} j^{-2\pi \mathrm{i} v}, \quad j, k > 0.$$

It is easy to see that $A \star T = PDTD^{\#}P$, where P is the orthogonal projection onto the subspace $\{\mathcal{H} \in \ell^2 : (h, e_0) = 0\}$, D and $D^{\#}$ are multiplication operators defined by

$$De_k = k^{2\pi \mathrm{i} v}, \quad k > 0,$$
$$D^{\#}e_j = j^{-2\pi \mathrm{i} v}, \quad j > 0.$$

It follows that $\|A \star T\| \le \|T\|$. ■

Proof of Lemma 8.6. Since the entries of M are bounded, it is sufficient to show that the matrix $\widetilde{M} = \{\tilde{m}_{jk}\}_{j,k \ge 0}$,

$$\tilde{m}_{jk} = \begin{cases} m_{jk}, & j, k > 0, \\ 0, & \text{otherwise}, \end{cases}$$

is a Schur multiplier of $\mathcal{B}(\ell^2)$. We have

$$m_{jk} - 1 = F(j, k), \quad j > 0,\ k > 0,$$

where F is a function on $\mathbb{R}_+ \times \mathbb{R}_+$ defined by

$$F(x, y) = \frac{(x + y)^\alpha - x^\alpha - y^\alpha}{x^\alpha + y^\alpha}, \quad x, y > 0.$$

Clearly,

$$F(e^s, e^t) = f(s - t), \quad s, t \in \mathbb{R},$$

where

$$f(t) \stackrel{\text{def}}{=} \frac{(e^t + 1)^\alpha - e^{t\alpha} - 1}{e^{t\alpha} + 1}, \quad t \in \mathbb{R}.$$

It is easy to see that f belongs to the class $\boldsymbol{\mathcal{S}}$ (see §1.8). Therefore f is a Fourier transform of an L^1 function on $\mathbb{R}$. It follows that there exists a finite complex Borel measure μ on $\mathbb{R}$ such that

$$f(t) + 1 = \int_{\mathbb{R}} e^{-2\pi \mathrm{i} t\xi} d\mu(\xi), \quad t \in \mathbb{R}.$$

We have

$$\begin{aligned} m_{jk} &= F(j, k) + 1 = f(\log j - \log k) + 1 \\ &= \int_{\mathbb{R}} e^{2\pi \mathrm{i} (\log j)\xi} e^{-2\pi \mathrm{i} (\log k)\xi} d\mu(\xi) \\ &= \int_{\mathbb{R}} k^{2\pi \mathrm{i}\xi} j^{-2\pi \mathrm{i}\xi} d\mu(\xi), \quad j, k > 0. \end{aligned}$$

The result now follows from Lemma 8.7. ■

The proof of the following result is similar to the proof of Theorem 8.2.

Theorem 8.8. *Let $\alpha > 0$. Then $\Gamma_\varphi^{\alpha,0}$ is the matrix of a compact operator if and only if $I_{-\alpha}\varphi \in VMO$.*

Let us now characterize the $\Gamma_\varphi^{\alpha,\beta}$ of class $\boldsymbol{S}_p$ for $\alpha,\,\beta > \max\{-\frac{1}{2}, -\frac{1}{p}\}$ (see Appendix 2.6 for the definition of the Besov classes B_p^s).

Theorem 8.9. *Let* $0 < p < \infty$, $\min\{\alpha,\,\beta\} > \max\{-\frac{1}{2}, -\frac{1}{p}\}$, *and let* φ *be a function analytic in* $\mathbb{D}$. *Then* $\Gamma_\varphi^{\alpha,\beta} \in \boldsymbol{S}_p$ *if and only if* $\varphi \in B_p^{1/p+\alpha+\beta}$.

One can easily verify that in the case $0 < p < 1$, the proof of Theorem 3.1 also works for operators $\Gamma_\varphi^{\alpha,\beta}$ with $\min\{\alpha,\,\beta\} > -\frac{1}{2}$. The same proof also works for $p = 1$ (see the Remark after the proof of Theorem 3.1). Here we present another proof of necessity in the case $p = 1$ that is of independent interest.

Proof of necessity for $p = 1$. As before, instead of $\Gamma_\varphi^{\alpha,\beta}$ we can consider the operator $\widetilde{\Gamma}_\varphi^{\alpha,\beta}$. Suppose that $\widetilde{\Gamma}_\varphi^{\alpha,\beta} \in \boldsymbol{S}_1$. Let us show that $\varphi \in B_1^{1+\alpha+\beta}$.

Let $\varepsilon > 0$ and let ψ be an analytic polynomial. Consider the operator

$$\widetilde{\widetilde{\Gamma}} = \left\{ \frac{D_j^{\alpha+\varepsilon} D_k^{\beta+\varepsilon}}{D_j^\alpha D_k^\beta} \hat{\psi}(j+k) \right\}_{j,k\ge 0}.$$

It follows easily from Theorem 8.1 that

$$\big\|\widetilde{\widetilde{\Gamma}}\big\| \le \operatorname{const} \|\psi\|_{\Lambda_{2\varepsilon}}.$$

We have

$$\begin{aligned}
\langle \widetilde{\widetilde{\Gamma}}, \widetilde{\Gamma}_\varphi^{\alpha,\beta} \rangle &= \sum_{m\ge 0} \left(\sum_{j=0}^m D_j^{\alpha+\varepsilon} D_{m-j}^{\beta+\varepsilon} \right) \hat{\psi}(m) \overline{\hat{\varphi}(m)} \\
&= \sum_{m \ge 0} D_m^{\alpha+\beta+2\varepsilon+1} \hat{\psi}(m) \overline{\hat{\varphi}(m)} = (\psi, \tilde{I}_{-(\alpha+\beta+2\varepsilon+1)}\varphi).
\end{aligned}$$

It follows that

$$|(\psi, \tilde{I}_{-(\alpha+\beta+2\varepsilon+1)}\varphi)| \le \operatorname{const} \|\widetilde{\Gamma}_\varphi^{\alpha,\beta}\|_{\boldsymbol{S}_1} \big\|\widetilde{\widetilde{\Gamma}}\big\| \le \operatorname{const} \|\widetilde{\Gamma}_\varphi^{\alpha,\beta}\|_{\boldsymbol{S}_1} \|\psi\|_{\Lambda_{2\varepsilon}}.$$

Hence, $\tilde{I}_{-(\alpha+\beta+2\varepsilon+1)}\varphi$ determines a continuous linear functional on $\mathbb{P}_+\lambda_{2\varepsilon}$, and so $\tilde{I}_{-(\alpha+\beta+2\varepsilon+1)}\varphi \in B_1^{-2\varepsilon}$ (see Appendix 2.6). Therefore $\varphi \in B_1^{1+\alpha+\beta}$ (see Appendix 2.6). ■

We start the proof of Theorem 8.9 for $1 < p < \infty$ with sufficiency in the case $p = 2$.

Lemma 8.10. *Let* φ *be an analytic in* $\mathbb{D}$ *function of class* $B_2^{1/2+\alpha+\beta}$. *Then* $\widetilde{\Gamma}_\varphi^{\alpha,\beta} \in \boldsymbol{S}_2$.

Proof. Obviously,

$$\begin{aligned}\|\widetilde{\Gamma}_{\varphi}^{\alpha,\beta}\|_{\boldsymbol{S}_2}^2 &= \sum_{j,k\ge 0}\left(D_j^{(\alpha)}\right)^2\left(D_k^{(\beta)}\right)^2|\hat{\varphi}(j+k)|^2\\ &\le \text{const}\sum_{m\ge 0}\left(|\hat{\varphi}(m)|^2\sum_{j+k=m}(1+j)^{2\alpha}(1+k)^{2\beta}\right)\\ &\le \text{const}\sum_{m\ge 0}(1+m)^{2(\alpha+\beta)+1}|\hat{\varphi}(m)|^2\le \text{const}\,\|\varphi\|_{B_2^{1/2+\alpha+\beta}}. \quad \blacksquare\end{aligned}$$

To prove Theorem 8.9 for $1 < p < \infty$, we need an interpolation theorem for analytic families of operators. For a Banach pair (X_0, X_1) (see §4) one can associate the spaces

$$X_0 + X_1 \stackrel{\text{def}}{=} \{x = x_0 + x_1 : x_0 \in X_0,\ x_1 \in X_1\}$$

endowed with the norm

$$\|x\|_{X_0+X_1} \stackrel{\text{def}}{=} \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x_0 \in X_0,\ x_1 \in X_1,\ x_0 + x_1 = x\},$$

and $X_0 \cap X_1$ endowed with the norm

$$\|x\|_{X_0\cap X_1} \stackrel{\text{def}}{=} \|x\|_{X_0} + \|x\|_{X_1}.$$

For $0 < \vartheta < 1$ the complex interpolation space $(X_0, X_1)_{[\theta]} \subset X_0 + X_1$ can be defined as follows. Consider the class $\mathfrak{F}(X_0, X_1)$ of continuous $(X_0 + X_1)$-valued functions f on $\{\zeta \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\}$ that are bounded and analytic in the open strip $\{\zeta \in \mathbb{C} : 0 < \operatorname{Re}\zeta < 1\}$ and such that

$$\|f\|_{\mathfrak{F}} \stackrel{\text{def}}{=} \sup\{\|f(j + \mathrm{i}t)\|_{X_j} : j = 0, 1,\ t \in \mathbb{R}\} < \infty.$$

We define the complex interpolation space $(X_0, X_1)_{[\theta]}$ to be $\{f(\theta) : f \in \mathfrak{F}(X_0, X_1)\}$ and endow it with the natural quotient norm.

We need the following result (see Cwikel and Janson [1]).

Interpolation theorem for an analytic family of operators. *Let (X_0, X_1) and (Y_0, Y_1) be Banach pairs. Suppose that at least one of the spaces Y_0 or Y_1 is separable. Let $\zeta \mapsto \mathcal{T}_\zeta$ be a bounded function that is defined and continuous in the closed strip $\{0 \le \operatorname{Re}\zeta \le 1\}$, takes values in the space of bounded linear operators from $X_0 \cap X_1$ to $Y_0 + Y_1$, and is analytic in the open strip $\{0 < \operatorname{Re}\zeta < 1\}$. If*

$$\|T_{j+\mathrm{i}t}x\| \le M_j\|x\|_{X_j}, \quad j = 0, 1, \quad t \in \mathbb{R}, \quad x \in X_0 \cap X_1,$$

then T_θ extends to a continuous linear operator from X_θ to Y_θ, $0 < \theta < 1$, of norm at most $M_0^{1-\theta}M_1^{\theta}$.

In the special case when the function $\zeta \mapsto \mathcal{T}_\zeta$ is constant this result holds without the separability assumption (Calderón's theorem; see Bergh and Löfström [1]).

Let us introduce spaces of vector sequences, which will be used in the proof of Theorem 8.9.

Definition. Let B be a Banach space. For $1 \le p < \infty$ and $s > 0$ we denote by $\ell^{p,s}(B)$ the space of sequences $\{f_n\}_{n\ge 0}$, $f_n \in B$, satisfying

$$\sum_{n\ge 0} (2^{ns}\|f_n\|_B)^p < \infty,$$

and by $c_0^s(B)$ the space of sequences $\{f_n\}_{n\ge 0}$, $f_n \in B$, satisfying

$$\lim_{n\to\infty} 2^{ns}\|f_n\|_B = 0.$$

Before we proceed to the proof of Theorem 8.9 in the case $1 < p < \infty$ we make the following observation. We can consider the generalized Hankel matrices $\widetilde{\Gamma}_\varphi^{\alpha,\beta}$ also for complex α and β. It is easy to see that $\widetilde{\Gamma}_\varphi^{\alpha,\beta} = M_1\widetilde{\Gamma}_\varphi^{\operatorname{Re}\alpha,\operatorname{Re}\beta}M_2$, where M_1 and M_2 are the unitary operators on ℓ^2 defined by $M_1e_j = (1+j)^{\mathrm{i}\operatorname{Im}\alpha}e_j$, $M_2e_j = (1+j)^{\mathrm{i}\operatorname{Im}\beta}e_j$. Hence, $\widetilde{\Gamma}_\varphi^{\alpha,\beta} \in \boldsymbol{S}_p$ if and only if $\widetilde{\Gamma}_\varphi^{\operatorname{Re}\alpha,\operatorname{Re}\beta} \in \boldsymbol{S}_p$.

Proof of Theorem 8.9 for $1 < p < \infty$. Let us first prove that the condition $\varphi \in B_p^{1/p+\alpha+\beta}$ is sufficient for $\widetilde{\Gamma}_\varphi^{\alpha,\beta} \in \boldsymbol{S}_p$. Consider first the case $2 < p < \infty$. Put $\alpha_0 \stackrel{\text{def}}{=} \alpha - \frac{p-2}{2p}$, $\alpha_1 \stackrel{\text{def}}{=} \alpha + 1/p$, $\beta_0 \stackrel{\text{def}}{=} \beta - \frac{p-2}{2p}$, $\beta_1 \stackrel{\text{def}}{=} \beta + 1/p$. Clearly, $\alpha_0 > -1/2$, $\alpha_1 > 0$, $\beta_0 > -1/2$, $\beta_1 > 0$.

For $0 \le \operatorname{Re}\zeta \le 1$ we define the operator $\mathcal{T}_\zeta$ on the set of L^1 sequences $\{f_n\}_{n\ge 0}$ by

$$\mathcal{T}_\zeta\{f_n\}_{n\ge 0} \stackrel{\text{def}}{=} \widetilde{\Gamma}_\psi^{\alpha_\zeta,\beta_\zeta},$$

where $\alpha_\zeta \stackrel{\text{def}}{=} \alpha_0 + \zeta(\alpha_1 - \alpha_0)$, $\beta_\zeta \stackrel{\text{def}}{=} \beta_0 + \zeta(\beta_1 - \beta_0)$, and $\psi = \sum_{n\ge 0} f_n * W_n$ (see Appendix 2.6 for the definition of the kernels W_n).

Let us verify the hypotheses of the interpolation theorem for analytic families of operators with $X_0 = \ell^{2,1/2+\alpha_0+\beta_0}(L^2)$, $X_1 = c_0^{\alpha_1+\beta_1}(L^\infty)$, $Y_0 = \boldsymbol{S}_2$, and $Y_1 = \boldsymbol{S}_\infty$.

If $\operatorname{Re}\zeta = 0$ and $\{f_n\}_{n\ge 0} \in X_0$, then $\psi \in B_2^{1/2+\alpha_0+\beta_0}$, and so by Lemma 8.10, $\mathcal{T}_\zeta\{f_n\}_{n\ge 0} \in \boldsymbol{S}_2$. If $\operatorname{Re}\zeta = 1$ and $\{f_n\}_{n\ge 0} \in X_1$, then $\psi \in \lambda_0^{\alpha_1+\beta_1}$, and so by Theorem 8.2, we have $\mathcal{T}_\zeta\{f_n\}_{n\ge 0} \in \boldsymbol{S}_\infty$.

It is easy to see that

$$\|\mathcal{T}_\zeta\{f_n\}_{n\ge 0}\|_{\boldsymbol{S}_\infty} = \|\mathcal{T}_{\operatorname{Re}\zeta}\{f_n\}_{n\ge 0}\|_{\boldsymbol{S}_\infty} \le \|\mathcal{T}_1\{f_n\}_{n\ge 0}\|_{\boldsymbol{S}_\infty} \le \text{const}$$

for $0 \le \operatorname{Re}\zeta \le 1$ and $\{f_n\}_{n\ge 0} \in c_0^{\alpha_1+\beta_1}$, since obviously,

$$\mathcal{T}_\zeta\{f_n\}_{n\ge 0} = A_1\mathcal{T}_1\{f_n\}_{n\ge 0}A_2$$

for some operators A_1, A_2 satisfying $\|A_1\| \le 1$, $\|A_2\| \le 1$.

Clearly, the function $\zeta \mapsto \mathcal{T}_\zeta$ is analytic in $\{\zeta : 0 < \operatorname{Re}\zeta < 1\}$ and the hypotheses of the interpolation theorem for analytic families of operators are satisfied.

We need now the following well-known facts:
$(\boldsymbol{S}_2, \boldsymbol{S}_\infty)_{[\theta]} = \boldsymbol{S}_p$, $\theta = \frac{p-2}{p}$ (see Reed and Simon [1], Ch. IX, §4);

$$\begin{aligned}(\ell^{2,1/2+\alpha_0+\beta_0}(L^2), c_0^{\alpha_1+\beta_1}(L^\infty))_{[\theta]} &= \ell^{p,1/p+\alpha+\beta}((L^2, L^\infty)_{[\theta]}) \\ &= \ell^{p,1/p+\alpha+\beta}(L^p), \quad \theta = \frac{p-2}{p}\end{aligned}$$

(see Bergh and Löfström [1], §5.3 and §5.1).

It follows that $\widetilde{\Gamma}_\psi^{\alpha,\beta} \in \boldsymbol{S}_P$ whenever

$$\psi = \sum_{n\ge 0} f_n * W_n \quad \text{and} \quad \{f_n\}_{n\ge 0} \in \ell^{p,1/p+\alpha+\beta}.$$

Now let $\varphi \in B_p^{1/p+\alpha+\beta}$. Put $f_n = \varphi * (W_{n-1} + W_n + W_{n+1})$. It follows easily from the definition of Besov classes in terms of convolutions with W_n that $\{f_n\}_{n\ge 0} \in \ell^{p,1/p+\alpha+\beta}$. It remains to observe that $\sum_{n\ge 0} f_n * W_n = \varphi$.

The case $1 < p < 2$ is easier since we can apply the complex interpolation method for one operator (Calderón's theorem). We define the operator $\mathcal{T}$ on the space of L^1 sequences $\{f_n\}_{n\ge 0}$ by

$$\mathcal{T}\{f_n\}_{n\ge 0} = \widetilde{\Gamma}_\psi^{\alpha,\beta},$$

where $\psi = \sum_{n\ge 0} f_n * W_n$. Since Theorem 8.9 has been already proved for $p = 1$, it follows that $\mathcal{T}$ is a bounded operator from $\ell^{1,1+\alpha+\beta}(L_1)$ to $\boldsymbol{S}_1$. Lemma 8.10 implies that $\mathcal{T}$ is a bounded operator from $\ell^{2,1/2+\alpha+\beta}(L^2)$ to $\boldsymbol{S}_2$. Applying the interpolation theorem for $\mathcal{T}$ and $\theta = \frac{2p-2}{p}$, we find that $\mathcal{T}$ is a bounded operator from $(\ell^{1,1+\alpha+\beta}(L_1), \ell^{2,1/2+\alpha+\beta}(L^2))_{[\theta]}$ to $(\boldsymbol{S}_1, \boldsymbol{S}_2)_{[\theta]}$. It is well known that $(\ell^{1,1+\alpha+\beta}(L_1), \ell^{2,1/2+\alpha+\beta}(L^2))_{[\theta]} = \ell^{p,1/p+\alpha+\beta}(L^p)$ (see Bergh and Löfström [1], §5.3 and §5.4) and $(\boldsymbol{S}_1, \boldsymbol{S}_2)_{[\theta]} = \boldsymbol{S}_p$ (see Reed and Simon [1], Ch. IX, §4). The rest of the proof is the same as in the case $p > 2$.

It remains to prove the necessity of the condition $\varphi \in B_p^{1/p+\alpha+\beta}$ for $\widetilde{\Gamma}_\varphi^{\alpha,\beta} \in \boldsymbol{S}_p$. Let ψ be a polynomial. It follows from Theorem 2.1 that $\|\Gamma_\psi\|_{\boldsymbol{S}_{p'}} \le \text{const}\, \|\psi\|_{B_{p'}^{1/p'}}$. We have

$$\begin{aligned}|(\psi, \tilde{I}_{-(1+\alpha+\beta)}\varphi)| &= \left|\sum_{m\ge 0} D_m^{1+\alpha+\beta}\hat{\psi}(m)\overline{\hat{\varphi}(m)}\right| = \left|\langle \Gamma_\psi, \widetilde{\Gamma}_\varphi^{\alpha,\beta}\rangle\right| \\ &\le \left\|\widetilde{\Gamma}_\varphi^{\alpha,\beta}\right\|_{\boldsymbol{S}_p} \|\Gamma_\psi\|_{\boldsymbol{S}_{p'}} \le \left\|\widetilde{\Gamma}_\varphi^{\alpha,\beta}\right\|_{\boldsymbol{S}_p} \|\psi\|_{B_{p'}^{1/p'}}.\end{aligned}$$

Hence, $\tilde{I}_{-(1+\alpha+\beta)}\varphi$ determines a continuous linear functional on $B_{p'}^{1/p'}$, and so $\tilde{I}_{-(1+\alpha+\beta)}\varphi \in B_p^{-1/p'}$, which is equivalent to the fact that $\varphi \in B_p^{1/p+\alpha+\beta}$ (see Appendix 2.6). ■

Remark. The restriction $\min\{\alpha,\beta\} > \max\{-\frac{1}{2}, -\frac{1}{p}\}$ in the statement of Theorem 8.9 is essential. Indeed, suppose that $2 \le p < \infty$. Without loss of generality we may assume that $\alpha \le \beta$. It follows from the definition of the Besov spaces (see Appendix 2.6) that

$$\|z^m\|_{B_p^{1/p+\alpha+\beta}} \le \text{const}\cdot m^{1/p+\alpha+\beta}.$$

On the other hand, if $\alpha \le -1/p$, it is easy to verify that

$$\begin{aligned}\|\Gamma_{z^m}^{\alpha,\beta}\|_{\boldsymbol{S}_p} &= \left(\sum_{j=0}^{m}(1+j)^{\alpha p}(m-j+1)^{\beta p}\right)^{1/p}\\ &\ge \text{const}\begin{cases} m^\beta, & \alpha < -1/p,\\ m^\beta(\log(1+m))^{1/p}, & \alpha = -1/p,\end{cases}\end{aligned}$$

and so the condition $\varphi \in B_p^{1/p+\alpha+\beta}$ does not imply that $\Gamma_\varphi^{\alpha,\beta} \in \boldsymbol{S}_p$.

Suppose now that $0 < p < 2$, $\alpha \le \beta$, and $\alpha \le -1/2$. Let us show that the condition $\varphi \in B_p^{1/p+\alpha+\beta}$ does not imply that $\Gamma_\varphi^{\alpha,\beta} \in \boldsymbol{S}_2$. We have

$$\begin{aligned}\|\Gamma_\varphi^{\alpha,\beta}\|_{\boldsymbol{S}_2} &= \left(\sum_{m\ge 0}\sum_{j=0}^{m}(1+j)^{2\alpha}(m-j+1)^{2\beta}|\hat\varphi(m)|^2\right)^{1/2}\\ &\ge \text{const}\begin{cases} \left(\sum\limits_{j=0}^{m}(1+m)^{2\beta}|\hat\varphi(m)|^2\right)^{1/2}, & \alpha < -1/2,\\ \left(\sum\limits_{j=0}^{m}(1+m)^{2\beta}\log(1+m)|\hat\varphi(m)|^2\right)^{1/2}, & \alpha = -1/2.\end{cases}\end{aligned}$$

Suppose now that φ is a polynomial of the form $\varphi = \sum\limits_{j=N+1}^{2N-1}\hat\varphi(m)z^m$. Then

$$\|\Gamma_\varphi^{\alpha,\beta}\|_{\boldsymbol{S}_2} \ge \text{const}\begin{cases} N^\beta\|\varphi\|_2, & \alpha < -1/2,\\ N^\beta(\log N)^{1/2}\|\varphi\|_2, & \alpha = -1/2.\end{cases}$$

On the other hand, it follows from the definition of the Besov spaces (see Appendix 2.6) that

$$\|\varphi\|_{B_p^{1/p+\alpha+\beta}} \le \text{const}\cdot N^{1/p+\alpha+\beta}\|\varphi\|_p.$$

To prove that the condition $\varphi \in B_p^{1/p+\alpha+\beta}$ does not imply that $\Gamma_\varphi^{\alpha,\beta} \in \boldsymbol{S}_2$, it remains to take a nonzero C^∞ function F with support in $[1,2]$, put

$$\varphi = \sum_{k\in\mathbb{Z}} F\left(\frac{k}{N}\right)z^k,$$

and apply Lemma 3.4.

9. Generalized Block Hankel Matrices and Vectorial Hankel Operators

In this section we study generalized block Hankel matrices of the form $\Gamma_\Phi^{\alpha,\beta} = \{(1+j)^\alpha(1+k)^\beta\hat{\Phi}(j+k)\}_{j,k\ge 0}$, where Φ is an analytic function in $\mathbb{D}$ that takes values in the space $\mathcal{B}(\mathcal{H},\mathcal{K})$ of bounded linear operators from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$. In Chapter 2 we have obtained boundedness and compactness criteria for $\Gamma_\Phi = \Gamma_\Phi^{0,0}$. Here we consider the more general case $\alpha, \beta > 0$ and we also study conditions for $\Gamma_\Phi^{\alpha,\beta}$ to belong to $\boldsymbol{S}_p$, $0 < p < \infty$. Then we obtain some applications.

To state the results we need some information about Besov classes of vector functions.

Consider first Besov spaces $B_p^s(X)$, where $1 \le p \le \infty$, $s \in \mathbb{R}$, and X is a Banach space. The space $B_p^s(X)$ can be defined as the space of X-valued distributions f (or X-valued formal trigonometric series $f = \sum_{j=-\infty}^{\infty} c_j z^j$, $\hat{f}(j) \stackrel{\text{def}}{=} c_j \in X$) such that

$$\left\{2^{s|n|}\|W_n * f\|_{L^p(X)}\right\}_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$$

(see Appendix 2.6 for the definition of the W_n). As in the scalar case for a scalar trigonometric polynomial q and a formal X-valued trigonometric series f

$$q * f \stackrel{\text{def}}{=} \sum_{j=-\infty}^{\infty} \hat{q}(j)\hat{f}(j).$$

We shall use the notation $\Lambda_s(X)$ for $B_\infty^s(X)$. The closure of the X-valued trigonometric polynomials in $\Lambda_s(X)$ will be denoted by $\lambda_s(X)$.

As in the scalar case it can be proved that for $1 \le p < \infty$ the dual space $\big(B_p^s(X)\big)^*$ can be identified with the space $B_{p'}^{-s}(X^*)$ with respect to the following pairing:

$$(f, g) = \sum_{j=-\infty}^{\infty} (\hat{f}(j), \hat{g}(j)),$$

where f is a trigonometric polynomial in $B_p^s(X)$ and $g \in B_{p'}^{-s}(X^*)$. The space $(\lambda_s(X))^*$ can be identified with $B_1^{-s}(X^*)$ with respect to the same pairing.

For $s > 0$ the space $B_p^s(X)$ can also be described as the space of X-valued functions f in $L^p(X)$ for which

$$\int_{\mathbb{T}} \frac{\|\Delta_\tau^n \varphi\|_{L^p(X)}}{|1-\tau|^{1+ps}} d\boldsymbol{m}(\tau) < \infty, \quad p < \infty, \tag{9.1}$$

$$\sup_{\tau\in\mathbb{T},\tau\ne 1} \frac{\|\Delta_\tau^n f\|_{L^\infty(X)}}{|1-\tau|^s} < \infty, \quad p = \infty,$$

where n is an integer such that $n > s$, $(\Delta_t f)(\zeta) \stackrel{\text{def}}{=} f(\zeta\tau) - f(\zeta)$, and $\Delta_t^{n+1} \stackrel{\text{def}}{=} \Delta_t \Delta_t^n$. The proof is the same as in the scalar case.

The subspace $\left(B_p^s\right)_+(X) = \{f \in B_p^s(X) : \hat{f}(j) = \mathbb{O} \text{ for } j < 0\}$ can be identified in a natural way with the space of X-valued functions analytic in $\mathbb{D}$ and satisfying

$$\begin{gathered}\int_{\mathbb{D}} \|f^{(n)}(\zeta)\|_{L^p(X)}^p (1-|\zeta|)^{(n-s)p-1} d\boldsymbol{m}_2(\zeta) < \infty, \quad p < \infty, \\ \sup_{\zeta \in \mathbb{D}} \|f^{(n)}(\zeta)\|_{L^\infty(X)} (1-|\zeta|)^{n-s} < \infty, \quad p = \infty,\end{gathered} \tag{9.2}$$

where n is an integer such that $n > s$. This can be proved in the same way as in the scalar case.

We also need spaces $B_p^s(X)$ with $0 < p < 1$. In this case we assume that $s > 1/p - 1$. The space X here does have to be a Banach space. We assume that X is a *p-Banach space*, which means that X is a complete quasinormed space with quasinorm $\|\cdot\|_X$ satisfying

$$\|x+y\|_X^p \le \|x\|_X^p + \|y\|_X^p, \quad x, y \in X.$$

Clearly, any Banach space is p-Banach. Other examples of p-Banach spaces are $L^p(\mu)$, $\boldsymbol{S}_p$.

For a p-Banach space X and $s > 1/p - 1$ the Besov space $B_p^s(X)$ can be defined as the space of X-valued formal trigonometric series such that

$$\left\{2^{s|n|}\|V_n * f\|_{L^p(X)}\right\}_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$$

(see Appendix 2.6 for the definition of the polynomials V_n).

It can be proved that the space $B_p^s(X)$ consists of the functions $f \in L^p(X)$ satisfying (9.1) and the subspace $\left(B_p^s\right)_+(X)$ consists of the X-functions f analytic in $\mathbb{D}$ and satisfying (9.2).

Let us now state the main results of this section.

Theorem 9.1. *Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces and let Φ be a function analytic in $\mathbb{D}$ that takes values in the space $\mathcal{B}(\mathcal{H},\mathcal{K})$. Suppose that $\alpha > 0$ and $\beta > 0$. Then $\Gamma_\Phi^{\alpha,\beta}$ is a bounded operator if and only if $\Phi \in \Lambda_{\alpha+\beta}(\mathcal{B}(\mathcal{H},\mathcal{K}))$.*

Theorem 9.2. *Suppose that $\mathcal{H}$, $\mathcal{K}$ are Hilbert spaces and let Φ be a $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued function analytic in $\mathbb{D}$. Suppose that $\alpha > 0$ and $\beta > 0$. Then $\Gamma_\Phi^{\alpha,\beta}$ is a compact operator if and only if $\Phi \in \lambda_{\alpha+\beta}(\boldsymbol{S}_\infty(\mathcal{H},\mathcal{K}))$.*

Theorem 9.3. *Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces, $0 < p < \infty$, and let Φ be a $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued function analytic in $\mathbb{D}$. Suppose that $\min\{\alpha,\beta\} > \max\{-1/2, -1/p\}$. Then $\Gamma_\Phi^{\alpha,\beta} \in \boldsymbol{S}_p$ if and only if $\Phi \in B_p^{1/p+\alpha+\beta}(\boldsymbol{S}_p(\mathcal{H},\mathcal{K}))$.*

Corollary 9.4. *Let $\Phi \in L^\infty(\mathcal{B}(\mathcal{H},\mathcal{K}))$ and let $0 < p < \infty$. The Hankel operator $H_\Phi : H^2(\mathcal{H}) \to H_-^2(\mathcal{K})$ belongs to $\boldsymbol{S}_p$ if and only if $\mathbb{P}_-\Phi \in B_p^{1/p}(\boldsymbol{S}_p)$.*

The proofs of Theorems 9.1–9.3 are similar to the proofs of the corresponding scalar results given in Sections 1, 3, and 8. We give some indications here. Note that the proof of Theorem 2.1 given in §2 does not work in the case of operator-valued functions; we should use the approach given in §8.

To prove Theorems 9.1–9.3 we can start with the sufficiency of the condition $\Phi \in B_1^{1+\alpha+\beta}(\boldsymbol{S}_1)$ for $\Gamma_\Phi^{\alpha,\beta}$ to belong to $\boldsymbol{S}_1$. As in the scalar case this follows immediately from the following lemma.

Lemma 9.5. *Let $\alpha > -1/2$, $\beta > -1/2$, and let Ψ be an analytic polynomial of degree m with coefficients in $\boldsymbol{S}_1(\mathcal{H},\mathcal{K})$. Then*

$$\left\|\Gamma_\Psi^{\alpha,\beta}\right\|_{\boldsymbol{S}_1} \le \text{const} \cdot m^{1+\alpha+\beta} \, \|\Psi\|_{L^1(\boldsymbol{S}_1(\mathcal{H},\mathcal{K}))},$$

the constant being independent of Ψ.

The proof of the lemma follows the proof of Lemma 1.3. We can consider the operators $A_\zeta : \ell^2(\mathcal{H}) \to \ell^2(\mathcal{K})$, $\zeta \in \mathbb{T}$, defined by

$$(A_\zeta\{x_j\}_{j\ge 0}, \{y_k\}_{k\ge 0}) \stackrel{\text{def}}{=} \left(\Psi(\zeta)\sum_{j=0}^m (1+j)^\alpha \bar{\zeta}^j x_j, \sum_{k=0}^m (1+k)^\beta \zeta^k y_k\right).$$

It can be shown easily that

$$\begin{aligned}\|A_\zeta\|_{\boldsymbol{S}_1} &\le \|\Psi(\zeta)\|_{\boldsymbol{S}_1}\left(\sum_{j=0}^m (1+j)^{2\alpha}\right)^{1/2}\left(\sum_{k=0}^m (1+k)^{2\beta}\right)^{1/2}\\ &\le \text{const}\,\|\Psi(\zeta)\|_{\boldsymbol{S}_1}(1+m)^{1+\alpha+\beta}\end{aligned}$$

and

$$\Gamma_\Psi^{\alpha,\beta} = \int_{\mathbb{T}} A_\zeta d\boldsymbol{m}(\zeta),$$

which implies the desired inequality.

The next step is to prove Theorem 9.1. The proof is exactly the same as that of Theorem 8.1 in the scalar case.

Then we can prove the necessity of the condition $\Phi \in B_1^{1+\alpha+\beta}$ for $\Gamma_\Phi^{\alpha,\beta} \in \boldsymbol{S}_1$. This can be done in exactly the same way as in Theorem 8.9.

To prove Theorem 9.2 we proceed in the same way as in the proof of Theorem 8.2. The sufficiency of the condition $\Phi \in \lambda_{\alpha+\beta}(\boldsymbol{S}_\infty)$ follows immediately from Theorem 9.1. To prove necessity, as in the scalar case we can show that

$$\lim_{N\to\infty} \|\Phi - \Phi * K_N\|_{\Lambda\alpha+\beta} = 0.$$

It remains to observe that the fact that $\Gamma_\Phi^{\alpha,\beta}$ is compact implies that the Fourier coefficients $\hat{\Phi}(j)$ are compact for all $j \in \mathbb{Z}_+$. Therefore the polynomial $\Phi * K_N$ has compact Fourier coefficients, and so $\Phi * K_N \in \lambda_{\alpha+\beta}(\boldsymbol{S}_\infty)$, which implies the result.

To prove Theorem 9.3 for $1 < p < \infty$ we use the method of interpolation of an analytic family of operators. The proof is the same as in the scalar case (Theorem 8.9) if we use the following facts (see Bergh and Löfström, [1], §5.1 and §5.6):

$$\begin{aligned}&\left(\ell^{2,1/2+\alpha_1+\beta_1}\left(L^2(\boldsymbol{S}_2)\right), c_0^{\alpha_2+\beta_2}\left(L^\infty(\boldsymbol{S}_\infty)\right)\right)_{[\theta]}\\ =\;& \ell^{p,1/p+\alpha+\beta}\left((L^2(\boldsymbol{S}_2), L^\infty(\boldsymbol{S}_\infty))_{[\theta]}\right)\end{aligned}$$

and

$$\left(L^2(\boldsymbol{S}_2), L^\infty(\boldsymbol{S}_p)\right)_{[\theta]} = L^p\left((\boldsymbol{S}_2, \boldsymbol{S}_\infty)_{[\theta]}\right) = L^p(\boldsymbol{S}_p)$$

(see the proof of Theorem 8.9, where α_1, α_2, β_1, β_2, and θ are defined).

It remains to prove Theorem 9.3 for $0 < p < 1$. The sufficiency of the condition $\Phi \in B_p^{1/p+\alpha+\beta}(\boldsymbol{S}_p)$ can be proved in the same way as in the scalar case (see Theorems 3.1 and 8.9). The proof of necessity is practically the same as in the scalar case. The only point we should mention here is that in the operator case we should consider matrices

$$A = \begin{pmatrix} C_0 & C_1 & \cdots & C_{N-2} & C_{N-1}\\ C_{N-1} & C_0 & \cdots & C_{N-3} & C_{N-2}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ C_1 & C_2 & \cdots & C_{N-1} & C_0\end{pmatrix},$$

where $C_j \in \boldsymbol{S}_p(\mathcal{H}, \mathcal{K})$, $0 \le j \le N-1$. As in the scalar case we can introduce the polynomial $\Psi(z) = \sum\limits_{j=0}^{N-1} C_j z^j$. We have

$$\|A\|_{\boldsymbol{S}_p}^p = \sum_{j=o}^{N-1} \|\Psi(\zeta_j)\|_{\boldsymbol{S}_p}^p,$$

where $\zeta_j = e^{2\pi \mathrm{i} j/N}$, $0 \le j \le N-1$. This is a consequence of the following identity, which can easily be verified:

$$A\begin{pmatrix}1\\ \zeta_j\\ \vdots\\ \zeta_j^{N-1}\end{pmatrix} x = \begin{pmatrix}1\\ \zeta_j\\ \vdots\\ \zeta_j^{N-1}\end{pmatrix}\Psi(\zeta_j)x, \quad x \in \mathcal{H}.$$

Let us proceed now to applications of Theorem 9.3.

First we study continuity properties of the *averaging projection* onto the space of block Hankel matrices. Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces. We can identify bounded linear operators from $\ell^2(\mathcal{H})$ to $\ell^2(\mathcal{K})$ with block matrices $\{A_{jk}\}_{j,k\ge 0}$, where $A_{jk} \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. On the space of such block matrices we define the averaging projection $\mathcal{P}$ onto the space of block Hankel matrices.

For $A = \{\Omega_{jk}\}_{j,k\ge 0}$ the block Hankel matrix $\mathcal{P}A$ is defined by

$$\mathcal{P}A \stackrel{\text{def}}{=} \{\Xi_{j+k}\}_{j,k\ge 0}, \quad \Xi_m = \frac{1}{m+1}\sum_{j+k=m}\Omega_{jk}.$$

Theorem 9.6. *Let $1 < p < \infty$. The averaging projection $\mathcal{P}$ is bounded on $\boldsymbol{S}_p(\mathcal{H},\mathcal{K})$.*

The proof is based on Theorem 9.3 for $\alpha = \beta = 0$ and $1 < p < \infty$ and it is exactly the same as the proof of Theorem 5.1 in the scalar case.

We conclude this section with a theorem on operators commuting with a contraction of class C_{00} (see §2.3). Recall that a C_{00}-contraction is unitarily equivalent to a Sz.-Nagy–Foias model operator $S_{[\Theta]}$ on K_Θ, where Θ is a unitary-valued inner operator function in $H^\infty(\mathcal{B}(\mathcal{H}))$, $\mathcal{H}$ is a separable Hilbert space, and $K_\Theta = H^2(\mathcal{H}) \ominus \Theta H^2(\mathcal{H})$. The model operator $S_{[\Theta]}$ is defined on K_Θ by

$$S_{[\Theta]}f = P_\Theta z f, \quad f \in H^2(\mathcal{H}),$$

where P_Θ is the orthogonal projection from $H^2(\mathcal{H})$ onto K_Θ.

By Theorem 2.3.1 and formula (2.3.8), each operator R on K_Θ that commutes with $S_{[\Theta]}$ has the form

$$Rf = P_\Theta \Phi f, \quad f \in K_\Theta,$$

for some $\Phi \in H^\infty(\mathcal{B}(\mathcal{H}))$. Moreover,

$$R = M_\Theta H_{\Theta^*\Phi}\big|K_\Theta$$

and

$$H_{\Theta^*\Phi}\big|\Theta H^2(\mathcal{H}) = \mathbb{O}$$

(see the Remark after Theorem 2.3.1).

Theorem 9.7. *Let Θ, Φ, R be as above. Then $R \in \boldsymbol{S}_p$, $0 < p < \infty$, if and only if $\mathbb{P}_-\Theta^*\Phi \in B_p^{1/p}(\boldsymbol{S}_p)$.*

The result follows immediately from Theorem 2.3.1 and the remark after it.

Concluding Remarks

The problem to describe the trace class Hankel operators was well known for a long time. It was explicitly stated by Rosenblum [2], Krein [3], and Holland (see Anderson, Barth and Brannan [1]). The problem was solved in Peller [1]; see also Peller [2] and [3]. (Let us mention here some earlier results by Rosenblum and Howland, see Howland [1].) After the paper Peller [1] had appeared several other approaches were found. The approach based on the decomposition of functions in B_1^1 in a series of reproducing kernels was given in Coifman and Rochberg [1] (note that the proof of Theorem 1.4 given in this book differs from the proof given in Coifman and Rochberg

[1]). Theorem 1.5 is due to Coifman and Rochberg [1]. Theorem 1.6 is taken from Power [2]. Theorem 1.7 is due to Howland [1].

We mention here another approach to the problem of nuclearity of Hankel operators. It is based on the notion of Möbius invariant spaces (see Arazy, Fisher, and Peetre [1]). A Banach space X of functions analytic in $\mathbb{D}$ is called *Möbius invariant* if

$$\sup\left\{\inf_{c\in\mathbb{C}}\|\varphi\circ\omega - c\|_X\right\} \le \text{const}\,.$$

It turns out that $\left(B_1^1\right)_+$ is the minimal Möbius invariant space which implies easily that the condition $\varphi \in \left(B_1^1\right)_+$ is sufficient for $\Gamma_\varphi \in \boldsymbol{S}_1$ (see Arazy, Fisher, and Peetre [1]).

The results of §2 had been obtained in Peller [2] and [3]. Later other proofs of the $\boldsymbol{S}_p$ criterion for $1 < p < \infty$ were found, see Rochberg [1], Peetre and Svensson [1], Arazy, Fisher, and Peetre [1].

The sufficiency of the condition $\varphi \in B_p^{1/p}$ for $\Gamma_\varphi \in \boldsymbol{S}_p$ in the case $0 < p < 1$ was established in Peller [3]. The necessity of the condition $\varphi \in B_p^{1/p}$ was proved in Peller [8] and Semmes [1] by different methods. The proof given in §3 follows the paper Peller [8]. Another proof is contained in the paper Pekarskii [2], which deals with rational approximation in the norm of BMO (see the discussion of results on rational approximation below). Note that Lemma 3.3 and Corollary 3.4 were obtained in Aleksandrov [1].

Theorem 4.1 was obtained in Peller [3] for $p_0 \ge 1$ and in Peller [8] for arbitrary $p_0 > 0$. We mention here the paper Janson [1] for further results in this direction. The spaces $\mathfrak{B}_{pq}^{1/p}$ were described in Peller [8] in different terms.

Theorems 5.1 and 5.2 were obtained in Peller [2] and [3]. In Peller [9] it was shown that the averaging projection $\mathcal{P}$ has weak type (1,1), i.e., $\mathcal{P}\boldsymbol{S}_1 \subset \boldsymbol{S}_{1,\infty}$ and $\mathcal{P}$ maps the Matsaev ideal $\boldsymbol{S}_\omega = \boldsymbol{S}_{1,\omega}$ into the ideal of compact operators. Then in Peller [10] those results were improved and Theorems 5.3, 5.5, and 5.7 were proved. A bounded projection on the set of Hankel matrices in $\boldsymbol{S}_1$ was constructed by A.B. Aleksandrov (see Peller [3]). Theorems 5.11 and 5.12 are due to Kislyakov. Note that Theorems 5.11 and 5.12 are consequences of stronger results found in the paper Kislyakov [1] in which the author obtained lower and upper estimates for the norms of projections onto the subspace of bounded Hankel matrices with $\{a_{j+k}\}_{j,k\ge 0}$ with $a_m = 0$ for $m > n$. Corollary 5.10 was found in Peller [3]. Note that in the case $\alpha = \beta > 0$ this was proved earlier in Horowitz [1]. Moreover, it was shown in Horowitz [1] that for any $F \in \left(B_1^{-2\alpha}\right)_+$ there exist $\varphi, \psi \in \left(B_2^{-\alpha}\right)_+$ such that $F = \varphi\psi$. However, the technique of Horowitz does not work in the case $\beta \ne \alpha$.

Theorem 6.1 was obtained in Peller [3] for $1 \le p < \infty$. This was the first result in rational approximation in which the “direct theorems” match the "inverse theorems". There were many earlier results in which there were gaps between the "direct theorems" and the "inverse theorems" (i.e., between necessary conditions and sufficient conditions). We mention here the results of Gonchar [1], Dolzhenko [1] and [2], and Brudnyi [1]. In particular, in Dolzhenko [1] it was shown that if

$$\sum_{n\geq 0} d_n^\infty(\varphi) < \infty,$$

then $\varphi' \in L^1$. Clearly, Theorem 6.1 for $p = 1$ improves the Dolzhenko theorem, since $L^\infty \subset BMO$ and $B_1^1 \subset \{\varphi:\ \varphi' \in L^1\}$.

Theorem 6.1 for $0 < p < 1$ was obtained in Peller [8], Pekarskii [2], and Semmes [1]; all three papers used quite different methods. Later Pekarskii [3] obtained similar results on rational approximation in the norm of L^p, $0 < p \leq \infty$. Note that similar results for rational approximation of functions on a finite interval were obtained in Peller [6]. Theorem 6.4 is taken from Peller [8]. Theorem 6.7 for $q < \infty$ is an immediate consequence of the above results by Peller, Semmes, and Pekarskii. We also mention here the paper Dyn'kin [3] in which a new approach to such Bernstein–Nikol'skii type inequalities is given as well as new Bernstein–Nikol'skii type inequalities were found. Theorem 6.7 for $q = \infty$ is due to Semmes [1]. Theorem 6.8 was obtained by Semmes [1]. Grigoryan's theorem (inequality (6.1)) was established in Grigoryan [1]. Its improvement (Theorem 6.3) was found in Peller [8]. Note that the best possible constant in (6.1) was given by Pekarskii [1]. Theorems 7.1 and 7.2 were proved in Peller [3] and [8] (the equivalence (ii)$\Leftrightarrow$(iii) is due to Clark [2]). Theorem 7.3 was found in Peller [4]. Theorem 7.4 is due to Peller [3] for $1 \leq p < \infty$, and Peller [8] and Semmes [1] for $0 < p < 1$. Theorem 7.5 was established by Rochberg [2] for $p \geq 1$ and later by Peller [16] for $0 < p < \infty$ (Rochberg's method works only for $p \geq 1$). Theorems 8.1 and 8.2 are taken from Peller [4]. A version of Theorem 8.5 for integral operators on $L^2(\mathbb{R})$ can be found in Janson and Peetre [2]. Volberg informed the author that he also proved Theorem 8.5 but never published his proof. The proof given in §8 was found by the author. Theorem 8.9 for $p \geq 1$ was obtained in Rochberg [2] and Peller [4], for $p < 1$ in Semmes [1] and Peller [8]. The idea to use the interpolation theorem for analytic families of operators is due to Rochberg.

The results of §9 can be found in Peller [3], [4], and [8].

We mention here several related results and several applications of the results of this chapter.

Jawerth and Milman [1] used extrapolation technique to deduce from Theorem 2.1 a description of Hankel operators in the Matsaev ideal $\boldsymbol{S}_\omega = \boldsymbol{S}_{1,\omega}$.

Recently Gheorghe [1] studied the problem of when a Hankel operator H_φ belongs to operator ideals $\boldsymbol{S}_E$, where E is a monotone Riesz–Fischer space (see the definition in Gheorghe [1]). Such ideals $\boldsymbol{S}_E$ include the Schatten–von Neumann classes $\boldsymbol{S}_p$ as well as the Schatten–Lorentz classes $\boldsymbol{S}_{p,q}$. Using an approach different from the approach given in §4, Gheorghe shows that $H_\varphi \in \boldsymbol{S}_E$ if and only if $\mathbb{P}_-\varphi$ belongs to the generalized Besov space B_E defined in terms of the monotone Riesz–Fischer space E.

Bonsall [3] found the following sufficient conditions for $\{a_{j+k}\}_{j,k\ge 0}$ to belong to $\boldsymbol{S}_1$:

If $\lim\limits_{n\to\infty} a_n = 0$ *and* $\sum\limits_{n=2}^{\infty} |a_{n-1} - a_n| \log n < \infty$, *then* $\{a_{j+k}\}_{j,k\ge 0} \in \boldsymbol{S}_1$;

if $\lim\limits_{n\to\infty} a_n = 0$ *and* $\sum\limits_{n=1}^{\infty} |a_{n-1} - 2a_n + a_{n+1}| n^2 < \infty$, *then* $\{a_{j+k}\}_{j,k\ge 0} \in \boldsymbol{S}_1$.

It was also proved in Bonsall [3] that if $\{a_n\}_{n\ge 0}$ is a convex nonincreasing sequence of nonnegative numbers, then $\{a_{j+k}\}_{j,k\ge 0} \in \boldsymbol{S}_1$ if and only if $\sum\limits_{n\ge 0} a_n < \infty$. On the other hand, it was shown in Bonsall [3] that there exists a nonincreasing sequence of nonnegative numbers $\{a_n\}_{n\ge 0}$ such that $\sum\limits_{n\ge 0} a_n < \infty$ but $\{a_{j+k}\}_{j,k\ge 0} \notin \boldsymbol{S}_1$.

Exercise. *Deduce the above results of Bonsall from Theorem 1.1.*

We also mention here the paper Bonsall and Walsh [1] in which the following sharp estimates were obtained for a function φ analytic in $\mathbb{D}$:

$$\frac{\pi}{8}\|(z^2\varphi)''\|_{L^1(\boldsymbol{m}_2)} \le \|\Gamma_\varphi\|_{\boldsymbol{S}_1} \le \|(z^2\varphi)''\|_{L^1(\boldsymbol{m}_2)}.$$

An interesting generalization of Hankel operators was given in Janson and Peetre [1]. Later a new approach to this generalization was suggested by Rochberg [3]. In fact, in those papers the authors considered generalized Hankel forms rather than generalized Hankel operators but it is easy to reduce the study of the forms to the study of the operators, and vice versa. Recall that an operator T on H^2 is a Hankel operator (i.e., an operator of the form Γ_φ) if and only if it satisfies the relation $S^*T = TS$, i.e., T belongs to the kernel of the transformer $\Delta : \mathcal{B}(H^2) \to \mathcal{B}(H^2)$, $\Delta T = S^*T - TS$. An operator T is called a *Hankel operator of order* n if $T \in \operatorname{Ker} \Delta^n$. It is easy to verify that the operator T defined on the set of analytic polynomials by $\Gamma_\varphi^{[n]} f = \Gamma_\varphi f^{(n)}$ is a Hankel operator of order $n+1$. It can also be verified that $\Gamma_\varphi^{[n]} \in \boldsymbol{S}_p$ if and only if the generalized Hankel matrix $\Gamma_\varphi^{n,0}$ (see §8) belongs to $\boldsymbol{S}_p$. Let Hank_n be the set of Hankel operators of order n. In the paper Janson and Peetre [1] the authors considered the following decomposition of the class of Hilbert–Schmidt operators on H^2:

$$\mathrm{Hank}_1 \oplus \bigoplus_{n\ge 1} (\mathrm{Hank}_{n+1} \ominus \mathrm{Hank}_n).$$

They characterized the operators in $\mathrm{Hank}_{n+1} \ominus \mathrm{Hank}_n$ as orbits under the action of Möbius group and they described operators in $\mathrm{Hank}_{n+1} \ominus \mathrm{Hank}_n$ of class $\boldsymbol{S}_p$.

Note that interesting estimates of singular values of Hankel operators were found in Parfënov [1] and [2]. In particular, in Parfënov [1] such estimates were used to solve the following problem on rational approximation posed by Gonchar. Let K be a compact subset of $\mathbb{D}$ such that $\mathbb{C} \setminus K$ is connected. Suppose that φ is a function analytic in $\hat{\mathbb{C}} \setminus K$, where $\hat{\mathbb{C}}$ is the extended complex plane. Then

$$\liminf_{n\to\infty} \left(d_n^\infty(\varphi)\right)^{1/n} \le e^{-2\mathrm{cap}^{-1}(K,\mathbb{T})},$$

where $\mathrm{cap}(K, \mathbb{T})$ is the condenser capacity of the pair F, T.

Other applications of Hankel operators to rational approximation can be found in Prokhorov [1] and [2].

In Peller [12] a description of nuclear Hankel operators from H^p to H^q was obtained for $1 < q \le p < \infty$: the Hankel operator Γ_φ is nuclear if and only if $\varphi \in B_1^{1/p+1/q'}$, where $q' = p/(p-1)$.

By Corollary 2.2, the Hankel operator H_φ belongs to $\boldsymbol{S}_p$, $1 < p < \infty$, if and only if

$$\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|(\mathbb{P}_-\varphi)(\zeta) - (\mathbb{P}_-\varphi)(\tau)|^p}{|\zeta-\tau|^2} d\boldsymbol{m}(\zeta) d\boldsymbol{m}(\tau) < \infty.$$

In Janson, Upmeier, and Wallstén [1] the authors studied the question for which p there exists a constant c_p such that

$$\|H_\varphi\|_{\boldsymbol{S}_p}^p = c_p \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|(\mathbb{P}_-\varphi)(\zeta) - (\mathbb{P}_-\varphi)(\tau)|^p}{|\zeta-\tau|^2} d\boldsymbol{m}(\zeta) d\boldsymbol{m}(\tau) < \infty$$

for any $\varphi \in B_p^{1/p}$. Obviously, this is true for $p = 2$ with $c_2 = 1$. It was shown in Janson, Upmeier, and Wallstén [1] that this is true if and only if $p = 2$, $p = 4$, or $p = 6$ with $c_4 = 1/2$ and $c_6 = 1/6$.

Note that Lemma 3.5 gives an upper estimate for the norm of Hankel matrices in the space of Schur multipliers of $\boldsymbol{S}_p$. We refer the reader to Aleksandrov and Peller [2] for further results on Hankel and Toeplitz Schur multipliers of $\boldsymbol{S}_p$ for $0 < p < 1$.

The description of Hankel operators of class $\boldsymbol{S}_p$ was used in Peller [3] to solve the problem on majorization properties of $\boldsymbol{S}_p$ that had been posed in Simon [2]. The space $\boldsymbol{S}_p$ is said to possess the *majorization property* if for any operators T and R on $L^2(\mu)$ the conditions $R \in \boldsymbol{S}_p$ and $|(Tf)(x)| \le (R|f|)(x)$ a.e. for every $f \in L^2(\mu)$ imply that $T \in \boldsymbol{S}_p$. It is easy to show that the space $\boldsymbol{S}_p$ possesses the majorization property if p is an even integer. Pitt [1] showed that the space of compact operators has the majorization property. Simon [2] conjectured that $\boldsymbol{S}_p$ has the majorization property for $p \in [2, \infty)$. It was shown in Peller [3] that $\boldsymbol{S}_p$ does not have the majorization property unless p is an even integer. Namely, it was shown in Peller [3] that if $p < \infty$ is not an even integer, then there exist functions φ and ψ analytic in $\mathbb{D}$ such that $|\hat{\varphi}(n)| \le \hat{\psi}(n)$, $n \in \mathbb{Z}_+$, $\Gamma_\psi \in \boldsymbol{S}_p$ but $\Gamma_\varphi \notin \boldsymbol{S}_p$. Later Simon [3] offered another counterexample based on finite matrices of the form (3.5).

The nuclearity criterion for Hankel operators was used in Peller [13] and [18] in the perturbation theory of self-adjoint and unitary operators. Consider here for simplicity the case of unitary operators. Let $\varphi \in L^\infty(\mathbb{T})$. It is called *operator Lipschitz* if

$$\|\varphi(U) - \varphi(V)\| \le \text{const}\,\|U - V\|$$

for any unitary operators U and V on Hilbert space. It was shown in Peller [13] that if φ is operator Lipschitz, then $\varphi \in B_1^1$. Indeed, if φ is operator Lipschitz, then it can be shown that the function $\check{\varphi}$ on $\mathbb{T} \times \mathbb{T}$,

$$\check{\varphi}(\zeta, \tau) \stackrel{\text{def}}{=} \frac{\varphi(\zeta) - \varphi(\tau)}{1 - \bar{\tau}\zeta},$$

is a Schur multiplier of the space $\boldsymbol{S}_1$, i.e., if T is an integral operator with kernel k of class $\boldsymbol{S}_1$, then the integral operator with kernel $k\check{\varphi}$ also belongs to $\boldsymbol{S}_1$ (see Birman and Solomyak [2]). Thus the commutator $\mathcal{C}_\varphi$ belongs to $\boldsymbol{S}_1$, and so by Theorem 7.3, $\varphi \in B_1^1$. In particular, this shows that the condition $\varphi \in C^1(\mathbb{T})$ does not imply that φ is operator Lipschitz. Note that in Peller [13] and [18] another necessary condition stronger that $\varphi \in B_1^1$ was also found with the help of Hankel operators. It was also shown in Peller [13] that if $\varphi \in B_{\infty 1}^1$, then φ is operator Lipschitz (see Peller [18], where similar results for functions on $\mathbb{R}$ and for self-adjoint operators were obtained). We mention here the paper Arazy, Barton, and Friedman [1] in which a better sufficient condition was found.

Note also that the $\boldsymbol{S}_p$ criterion for Hankel operators is used in the theory of Toeplitz determinants. We refer the reader to Bötcher and Silbermann [1] for a detailed presentation of this theory.

An interesting and important problem is to describe functions φ and ψ in L^2 for which $H_\varphi^* H_\psi \in \boldsymbol{S}_p$; this problem is still open. We mention here the paper Volberg and Ivanov [1] in which a necessary condition for $H_\varphi^* H_\psi \in \boldsymbol{S}_p$ was obtained for $p \ge 2$.

Finally, we mention that the $\boldsymbol{S}_p$ criterion for Hankel operators is applied in noncommutative geometry; see Connes [1] and [2].

7

Best Approximation by Analytic and Meromorphic Functions

Let φ be a function on $\mathbb{T}$ of class BMO. As we have already discussed in Chapter 1, there exists a function $f \in BMOA$ such that $\varphi - f \in L^\infty(\mathbb{T})$ and

$$\|\varphi - f\|_{L^\infty} = \|H_\varphi\|. \tag{0.1}$$

Such a function f is called a *best approximation* to φ by analytic functions in L^∞. We have already seen in §1.1 that in general a best approximation is not unique (see Theorem 5.1.5, which gives a necessary and sufficient condition for uniqueness in terms of the Hankel operator H_φ).

If we assume that $\varphi \in VMO$, then there exists a unique best approximation f (see Corollary 1.1.6). This allows us to define on the class VMO the (nonlinear) *operator $\mathcal{A}$ of best approximation by analytic functions*: for $\varphi \in VMO$ we put $\mathcal{A}\varphi = f$, where f is the unique function in $BMOA$ satisfying (0.1). Note however, that the operator $\mathcal{A}$ is *homogeneous*: $\mathcal{A}(\lambda\varphi) = \lambda\mathcal{A}\varphi$, $\lambda \in \mathbb{C}$. Obviously, if $\varphi \in C_A$, then $\mathcal{A}\varphi$ is the function in H^∞ closest to φ in the L^∞-norm.

In §§1–3 of this chapter we study heredity properties of the operator $\mathcal{A}$. Namely, for a function space X, $X \subset VMO$, we consider the *heredity* problem of whether

$$\varphi \in X \quad \Longrightarrow \quad \mathcal{A}\varphi \in X.$$

In this case we say that X is *hereditary for the operator $\mathcal{A}$*.

Together with this problem we study the *recovery problem for unimodular functions*. Let $X \subset VMO$. The recovery problem is whether under certain natural conditions on a unimodular function u

$$\mathbb{P}_- u \in X \quad \Longrightarrow \quad u \in X.$$

Certainly, without any additional assumption on u the above implication cannot be true. For example, if u is an inner function that is not a finite Blaschke product, then u cannot belong to X. We consider one of the following conditions on u:

the Toeplitz operator T_u has dense range in H^2;

the Toeplitz operator T_u on H^2 is Fredholm.

Let us explain without entering into detail why these two problems are related to each other. Suppose that $\mathbb{P}_- X \subset X$ and let $\varphi \in X$. Then by Theorem 1.1.4, $\varphi - \mathcal{A}\varphi = cu$, where $c \in \mathbb{C}$ and u is a unimodular function. Clearly, $\mathbb{P}_- \varphi = c\mathbb{P}_- u \in X$, and so $\varphi \in X$ if and only if $u \in X$.

Conversely, if we assume that $T_u H^2 = H^2$, then $T_u f = \mathbf{1}$ for some $f \in H^2$. Hence, $T_{\bar{z}u} f = \mathbb{O}$. By Theorem 3.1.11, $\operatorname{dist}_{L^\infty}(\bar{z}u, H^\infty) = 1$. But this means exactly that $-\mathbb{P}_+ \bar{z}u = \mathcal{A}\mathbb{P}_- \bar{z}u$. Thus in this special case the recovery problem reduces to the heredity problem for $\mathcal{A}$.

Together with the operator $\mathcal{A}$ we consider the nonlinear operators $\mathcal{A}_m$, $m \geq 0$, of *best approximation by meromorphic functions with at most m poles*. We denote by $BMOA_{(m)}$ the set of functions $\psi \in BMO$ such that $\mathbb{P}_- \psi \in \mathcal{R}_m$, i.e., $\mathbb{P}_- \psi$ is a rational function of degree at most m. Clearly, we can identify functions in $BMOA_{(m)}$ with meromorphic functions in $\mathbb{D}$ with at most m poles (counted with multiplicities). For a function $\varphi \in VMO$ there exists a unique function $\psi \in BMOA_{(m)}$ such that $\varphi - \psi \in L^\infty(\mathbb{T})$ and

$$\|\varphi - \psi\|_\infty = \inf\left\{g \in BMOA_{(m)} : \|\varphi - g\|_{L^\infty(\mathbb{T})}\right\} = s_m(H_\varphi)$$

(see Theorems 4.1.2 and 4.1.3). We define the operator $\mathcal{A}_m$ by $\mathcal{A}_m \varphi = \psi$. Clearly, $\mathcal{A}_0 = \mathcal{A}$.

In §1 we study the above problems in so-called $\mathcal{R}$-spaces, i.e., spaces of functions that can be defined in terms of rational approximation in the norm of BMO. This class of function spaces includes the Besov spaces $B_p^{1/p}$, $0 < p < \infty$, the space VMO, the space of rational functions. For Banach (and quasi-Banach) $\mathcal{R}$-spaces X we prove that the operators $\mathcal{A}$ and $\mathcal{A}_m$ not only map X into itself but also they are bounded on X, i.e.,

$$\|\mathcal{A}_m f\|_X \leq \text{const}\, \|f\|_X, \quad f \in X.$$

In §2 we consider a huge class of Banach algebra satisfying certain natural axioms (we call such spaces *decent spaces*) and we prove that they have hereditary properties, which also leads to corresponding results on the recovery problem for unimodular functions.

The third class of function spaces for which we solve the above problems is considered in §3. It includes many spaces without a norm; in particular, it includes Carleman classes under certain very mild conditions (see §4).

In §4 we give many concrete examples of function spaces that have hereditary properties and we also give some counter-examples.

We study in §5 *badly approximable functions*, i.e., L^∞ functions φ such that $\operatorname{dist}_{L^\infty}(\varphi, H^\infty) = \|\varphi\|_{L^\infty}$. We describe the badly approximable functions φ satisfying the condition $\operatorname{dist}_{L^\infty}(\varphi, H^\infty + C) < \operatorname{dist}_{L^\infty}(\varphi, H^\infty)$. We also consider a similar problem of describing the error functions $\varphi - f$, where f is a best approximant of φ by functions in $BMOA_{(m)}$.

In §6 we investigate the problem of the behavior of multiple singular values of Hankel operators under perturbation of symbols by rational fractions and under multiplication of the symbol by z and $\bar{z}$.

We consider in §7 the boundedness problem for the operators $\mathcal{A}$ and $\mathcal{A}_m$. We prove in §1 that these operators are bounded on $B_p^{1/p}$ and VMO. We show in §7 that these operators are unbounded on Λ_α, $\alpha > 0$, B_p^s, $0 < p < \infty$, $s > 1/p$, and on the space $\mathcal{F}\ell^1$ of functions with absolutely converging Fourier series.

In §8 we describe the arguments of unimodular functions in a broad class of function spaces X. We show that under certain mild conditions on X a unimodular function u belongs to X if and only if it can be represented in the form $u = z^n e^{\mathrm{i}\varphi}$, where $n \in \mathbb{Z}$ and φ is a real function in X.

In §9 we study properties of Schmidt functions of Hankel operators with symbols in a function space X. We prove that under certain natural conditions on X all Schmidt functions must also belong to the same space X. We also describe the Schmidt functions of the compact Hankel operators and the Schmidt functions of the Hankel operators of class $\boldsymbol{S}_p$, $0 < p < \infty$.

In Sections 10 and 11 we study the continuity problem for the operators $\mathcal{A}$ and $\mathcal{A}_m$. We prove in §10 that unless $\varphi \in H^\infty$, φ is a discontinuity point of the operator $\mathcal{A}$ in the sup-norm. We also obtain a similar result for the operators $\mathcal{A}_m$. In §11 we consider the continuity problem in the norm of a decent function space X (see §2). We show that $\mathcal{A}$ is continuous at $\varphi \in X$ in the norm of X if and only if $\|H_\varphi\|$ is a singular value of H_φ of multiplicity 1. We also show that in the case of $\mathcal{A}_m$ (under certain restrictions) φ is a continuity point in the norm of X if and only if the singular value $s_m(H_\varphi)$ of H_φ has multiplicity 1. Finally, we show that independently of the multiplicity of $s_m(H_\varphi)$ the operator $\mathcal{A}_m$ is continuous at φ as a function from X to L^q, $q < \infty$.

In §12 we consider a recovery problem in spaces of measures. We use the results of §1 to prove that if μ is a complex measure on $\mathbb{T}$ such that

$$\sum_{j<0} \frac{|\hat{\mu}(j)|^2}{|j|} < \infty,$$

then

$$\sum_{j>0} \frac{|\hat{\mu}(j)|^2}{j} < \infty.$$

We obtain similar results for some other spaces of measures. As a corollary we describe closed subsets E of $\mathbb{T}$ such that $\big(B_p^{1/p}\big)_+\big|E = C(E)$, where $1 < p < \infty$.

We conclude this chapter with §13 in which we obtain an analog of the Fefferman–Stein decomposition for the Besov spaces $B_p^{1/p}$.

1. Function Spaces That Can Be Described in Terms of Rational Approximation in *BMO*

In this section we introduce the so-called $\mathcal{R}$-spaces, i.e., function spaces that can be described in terms of rational approximation in BMO. We prove that if X is an $\mathcal{R}$-space, then the operator $\mathcal{A}$ of best approximation by analytic functions maps X into itself. Moreover, we show that if X is a Banach (or quasi-Banach) $\mathcal{R}$-space, then the operator $\mathcal{A}$ is bounded on X.

We consider here the following seminorm on BMO:

$$\|\varphi\|_\star \stackrel{\text{def}}{=} \inf_{f\in BMOA}\|\varphi - f\|_{L^\infty} + \inf_{f\in BMOA}\|\bar\varphi - f\|_{L^\infty} = \|H_\varphi\| + \|H_{\bar\varphi}\|. \tag{1.1}$$

To define $\mathcal{R}$-spaces, we need several definitions.

Recall that $\mathcal{R}_n$ is the set of rational functions with poles outside $\mathbb{T}$ of degree at most n. For $\varphi \in BMO$ we put

$$\rho_n(\varphi) \stackrel{\text{def}}{=} \inf\{\|\varphi - r\|_\star :\ r \in \mathcal{R}_n\}.$$

A linear space E of sequences $\{x_j\}_{j\ge0}$ is called a *Köthe space* if with any sequence $\{x_j\}_{j\ge0}$ it contains all sequences $\{y_j\}_{j\ge0}$ satisfying $|y_j| \le |x_j|$, $j \ge 0$. We say that E is a *Banach Köthe space* (quasi-Banach Köthe space) if in addition to that E is a Banach (quasi-Banach) space equipped with a norm (quasinorm) $\|\cdot\|_E$ that has the following property:

$$\{x_j\}_{j\ge0} \in E, \quad |y_j| \le |x_j|,\ j \ge 0, \quad \Longrightarrow \quad \|\{y_j\}_{j\ge0}\|_E \le \|\{x_j\}_{j\ge0}\|_E.$$

Definition. A class X of functions on $\mathbb{T}$ is said to *admit a description in terms of rational approximation in the BMO norm* if $X \subset VMO$ and there exists a Köthe sequence space E such that

$$f \in X \quad \Longleftrightarrow \quad \{\rho_n(f)\}_{n\ge0} \in E.$$

For brevity we call such spaces $\mathcal{R}$-*spaces*. Finally, we say X is a Banach (quasi-Banach) $\mathcal{R}$-space if the corresponding space E is a Banach (quasi-Banach) ideal space and

$$C_1 \inf_{c\in\mathbb{C}}\|f - c\|_X \le \|\{\rho_n(f)\}_{n\ge0}\|_E \le C_2 \inf_{c\in\mathbb{C}}\|f - c\|_X, \quad f \in X,$$

for some positive constants C_1, C_2.

It is easy to see that VMO is an $\mathcal{R}$-space. Indeed, it corresponds to the space $E = c_0 \stackrel{\text{def}}{=} \big\{\{x_n\}_{n\ge 0} : \lim_{n\to\infty} x_n = 0\big\}$. Other important examples of $\mathcal{R}$-spaces are the Besov spaces $B_p^{1/p}$, $0 < p < \infty$. In this case $E = \ell^p$. This follows from Corollaries 6.1.2, 6.2.2, and 6.3.2, since by Theorem 4.1.1, for any $\varphi \in BMO$

$$\rho_n(\mathbb{P}_-\varphi) = s_n(H_\varphi), \quad n \ge 0. \tag{1.2}$$

The (nonlinear) set $\mathcal{R}_n$, $n \ge 0$, is also an $\mathcal{R}$-space. The corresponding sequence space E is the space of sequences $\{x_j\}_{j\ge 0}$ such that $x_j = 0$ for $j \ge m$. The space $\mathcal{R}$ of rational functions with poles outside $\mathbb{T}$ is also an $\mathcal{R}$-space in which case E is the space of finitely supported sequences.

Let us briefly mention some properties of $\mathcal{R}$-spaces. Let X be an $\mathcal{R}$-space. Clearly, $\varphi \in X$ if and only if $\bar\varphi \in X$. It is obvious that $\mathbb{P}_+X \subset X$.

If X is an $\mathcal{R}$-space, E is the corresponding Köthe sequence space, $\varphi \in BMO$, and $\mathbb{P}_+f = \mathbb{O}$, then it follows from (1.2) that $\varphi \in X$ if and only if $\{s_n(H_\varphi)\}_{n\ge 0} \in E$. Hence, if X is a linear $\mathcal{R}$-space, then a function $\varphi \in BMO$ belongs to X if and only if $\{s_n(H_\varphi)\}_{n\ge 0} \in E$ and $\{s_n(H_{\bar\varphi})\}_{n\ge 0} \in E$.

Lemma 1.1. *Suppose that X is an $\mathcal{R}$-space. The following assertions hold:*
(i) *if $f \in H^\infty$ and $\varphi \in X$, then $\mathbb{P}_+\bar f\varphi \in X$;*
(ii) *if X is a linear $\mathcal{R}$-space, then $X \cap L^\infty$ is an algebra.*

If X is a (quasi-)Banach $\mathcal{R}$-space, then

$$\|\mathbb{P}_+\bar f\varphi\| \le \text{const}\,\|f\|_\infty \cdot \|\varphi\|_X, \quad f \in H^\infty,\ \varphi \in X,$$

and

$$\|\varphi\psi\|_X \le \text{const}(\|\varphi\|_X \cdot \|\psi\|_\infty + \|\psi\|_X \cdot \|\varphi\|_\infty), \quad \varphi, \psi \in X \cap L^\infty.$$

Proof. (i). Since obviously $\mathbb{P}_+\bar f\varphi = \mathbb{P}_+\bar f\mathbb{P}_+\varphi$, we may assume that $\varphi = \mathbb{P}_+\varphi$. Let $r \in \mathcal{R}_n$. It is easy to see that the function $r_\# \stackrel{\text{def}}{=} \mathbb{P}_+\bar f r$ is also a rational function of class $\mathcal{R}_n$ (this function can be computed explicitly in the same way as in the Remark at the end of §3 of Ch. 1). It follows that

$$\|\mathbb{P}_+\bar f\varphi - r_\#\|_\star = \|\mathbb{P}_+\bar f(\varphi - r)\|_\star = \|\mathbb{P}_+\bar f(\varphi - \mathbb{P}_+r)\|_\star \le \|f\|_\infty\|\varphi - \mathbb{P}_+r\|_\star.$$

The last inequality is an obvious consequence of the following trivial inequality for the seminorm $\|\cdot\|_\star$ defined in (1.1):

$$\|\mathbb{P}_+\bar f g\|_\star \le \|f\|_\infty\|g\|_\star, \quad f \in H^\infty, \quad g \in BMOA.$$

Therefore $\rho_n(\mathbb{P}_+\bar f\varphi) \le \|f\|_\infty\|\varphi - \mathbb{P}_+r\|_\star \le \|f\|_\infty\|\varphi - r\|_\star$, and so $\rho_n(\mathbb{P}_+\bar f\varphi) \le \|f\|_\infty\rho_n(\varphi)$, which implies that $\mathbb{P}_+\bar f\varphi \in X$.

(ii) Let E be the corresponding Köthe sequence space. Let $\varphi, \psi \in X \cap L^\infty$. Clearly, it is sufficient to prove that $\mathbb{P}_-(\varphi\psi) \in X$. Let $f \in H^2$. We have

$$H_{\varphi\psi}f = \mathbb{P}_-\varphi\psi f = \mathbb{P}_-\varphi\mathbb{P}_+\psi f + \mathbb{P}_-\varphi\mathbb{P}_-\psi f = H_\varphi T_\psi f + \breve{T}_\varphi H_\psi f,$$

where $\breve{T}_\varphi : H^2_- \to H^2_-$, $\breve{T}_\varphi g = \mathbb{P}_-\varphi g$, $g \in H^2_-$. Clearly, T_ψ and $\breve{T}_\varphi$ are bounded operators, and so $s_n(H_\varphi T_\psi) \le \|T_\psi\| s_n(H_\varphi)$ and $s_n(\breve{T}_\varphi H_\psi) \le \|\breve{T}_\varphi\| s_n(H_\psi)$. This implies that $\mathbb{P}_-(\varphi\psi) \in X$.

It is easy to see that if X is a (quasi-)Banach $\mathcal{R}$-space, then the above reasonings yield the desired norm estimates. ■

Another important property of Banach (or quasi-Banach) $\mathcal{R}$-spaces is that they must be Möbius invariant, i.e., conformally invariant modulo constants.

Lemma 1.2. *Let X be a Banach (or quasi-Banach) $\mathcal{R}$-space. Then there is a positive number K such that*

$$\inf_{c\in\mathbb{C}} \|\varphi \circ \gamma - c\|_X \le \inf_{c\in\mathbb{C}} \|\varphi - c\|_X \tag{1.3}$$

for any $\varphi \in X$ and any conformal mapping $\gamma : \mathbb{C} \to \mathbb{C}$ such that $\gamma(\mathbb{T}) = \mathbb{T}$.

Proof. The result follows from the fact that (1.3) holds for $X = BMO$ (this follows immediately from the definition (1.1) of the semi-norm $\|\cdot\|_\star$), and from the obvious facts that $\mathcal{R}_n + \text{const} = \mathcal{R}_n$ and $\gamma\mathcal{R}_n = \mathcal{R}_n$. ■

Note, however, that the above conditions are not sufficient for a function space X to be an $\mathcal{R}$-space. We shall see in §4 that the space

$$\{\varphi :\ (\mathbb{P}_+\varphi)' \in H^1,\ (\mathbb{P}_+\bar{\varphi})' \in H^1\},$$

which has all the above properties, is not an $\mathcal{R}$-space.

Remark. It can easily be deduced from Lemma 1.2 that if X is a Banach (or quasi-Banach) $\mathcal{R}$-space, then $X \not\subset \text{Lip}\omega$ for any modulus of continuity ω. To prove this it is sufficient to consider conformal mappings γ, $\gamma(\zeta) = (\zeta - \alpha)(1 - \bar{\alpha}\zeta)^{-1}$, $\alpha \in \mathbb{D}$, and take α close to $\mathbb{T}$.

Now we are ready to solve the best approximation problem and the recovery problem for unimodular functions.

Theorem 1.3. *Let u be a unimodular function on $\mathbb{T}$ such that $\mathbb{P}_-u \in VMO$. Suppose that T_u has dense range in H^2. We have*

(1) $\rho_n(\mathbb{P}_+u) \le \rho_n(\mathbb{P}_-u)$, $n \in \mathbb{Z}_+$;

(2) *if X is an $\mathcal{R}$-space and $\mathbb{P}_-u \in X$, then $\mathbb{P}_+u \in X$;*

(3) *if X is a (quasi-)Banach $\mathcal{R}$-space and $\mathbb{P}_-u \in X$, then*

$$\|\mathbb{P}_+u\|_X \le \text{const}\, \|\mathbb{P}_-u\|_X.$$

Proof. Clearly, $\mathbb{P}_-u \in VMO$, and so the Hankel operator H_u is compact. It follows from Corollary 4.4.4 that $s_n(H_{\bar{u}}) \le s_n(H_u)$, $n \in \mathbb{Z}_+$. Therefore by (1.2)

$$\rho_n(\mathbb{P}_+u) = \rho_n(\mathbb{P}_-\bar{u}) \le \rho_n(\mathbb{P}_-u).$$

Clearly, (2) and (3) follow immediately from (1). ■

Remark. It follows immediately from Corollary 4.4.4 that if the operator T_u is invertible on H^2, then

$$\rho_n(\mathbb{P}_-u) = \rho_n(\mathbb{P}_+u), \quad n \in \mathbb{Z}_+.$$

For linear $\mathcal{R}$-spaces we can solve another version of the recovery problem.

Theorem 1.4. *Let X be a linear $\mathcal{R}$-space and let u be a unimodular function on $\mathbb{T}$ such that $\mathbb{P}_- u \in X$. Suppose that the Toeplitz operator T_u on H^2 is Fredholm. Then $\mathbb{P}_+ u \in X$.*

Proof. Let n be an integer such that the Toeplitz operator $T_{z^n u}$ is invertible. It follows from Lemma 1.1 that $\mathbb{P}_- z^n u \in X$. By Theorem 1.3, we have $\mathbb{P}_+ z^n u \in X$, and so $z^n u \in X$. Again, by Lemma 1.1, $u \in X$. ■

Recall that T_u is Fredholm if u is a unimodular function in VMO.

Theorem 1.5. *Let $\varphi \in VMO$. Then $\rho_n(\mathcal{A}\varphi) \le \rho_n(\varphi)$, $n \in \mathbb{Z}_+$. Moreover, if $\mathbb{P}_-\varphi \ne \mathbb{O}$, then $\rho_n(\mathcal{A}\varphi) \le \rho_{n+1}(\varphi)$, $n \in \mathbb{Z}_+$.*

Proof. It is easy to see that to prove the desired inequality we may assume that $\mathbb{P}_+\varphi = \mathbb{O}$. By Corollary 1.1.5, the function $\varphi - \mathcal{A}\varphi$ has the form

$$\varphi - \mathcal{A}\varphi = c\bar{z}\bar{\vartheta}\bar{h}/h,$$

where $c \in \mathbb{C}$, ϑ is an inner function, and h is an outer function in H^2. Let $u \stackrel{\text{def}}{=} \bar{z}\bar{\vartheta}\bar{h}/h$. Clearly, $\varphi = c\mathbb{P}_- u$ and $\mathcal{A}\varphi = c\mathbb{P}_+ u$. By Theorem 4.4.10, the operator T_u has dense range in H^2. Clearly, $\operatorname{Ker} T_u \ne \{\mathbb{O}\}$ since $h \in \operatorname{Ker} T_u$. It now follows from Corollary 4.4.4 and (1.2) that

$$\rho_n(\mathbb{P}_+\bar{u}) = s_n(H_{\bar{u}}) \le s_{n+1}(H_u) = \rho_{n+1}(\mathbb{P}_- u), \quad n \in \mathbb{Z}_+,$$

which implies the result. ■

Theorem 1.6. *If X is an $\mathcal{R}$-space, then $\mathcal{A}X \subset X$. If X is a Banach (quasi-Banach) $\mathcal{R}$-space, then $\mathcal{A}$ is a bounded operator on X, i.e.,*

$$\|\mathcal{A}\varphi\|_X \le \operatorname{const} \|\varphi\|_X, \quad \varphi \in X.$$

Proof. If X is a $\mathcal{R}$-space, then $\mathcal{A}X \subset X$ by Theorem 1.5. Let us show that if X is a Banach (quasi-Banach) $\mathcal{R}$-space, then $\mathcal{A}$ is a bounded operator on X. To estimate, $\|\mathcal{A}\varphi\|_X$ it is sufficient to assume that $\mathbb{P}_+\varphi = \mathbb{O}$. It is easy to see that

$$\inf_{c \in \mathbb{C}} \|\varphi - c\|_X + \|\hat{\varphi}(0)\mathbf{1}\|_X, \quad \varphi \in X,$$

is an equivalent (quasi-)norm on X. (Recall that $\mathbf{1}$ is the constant function identically equal to 1.) It follows immediately from Theorem 1.5 and the definition of a Banach (quasi-Banach) $\mathcal{R}$-space that

$$\inf_{c \in \mathbb{C}} \|\mathcal{A}\varphi - c\|_X \le \operatorname{const} \|\varphi\|_X.$$

On the other hand,

$$|\widehat{\mathcal{A}\varphi}(0)| = |\hat{\varphi}(0) - \widehat{\mathcal{A}\varphi}(0)| \le \|\varphi - \mathcal{A}\varphi\|_{L^\infty} = \|H_\varphi\| = s_0(H_\varphi).$$

Hence,

$$\|\widehat{\mathcal{A}\varphi}(0)\mathbf{1}\|_X \le \operatorname{const} \|s_0(H_\varphi)\mathbf{1}\|_X \le \operatorname{const} \|\varphi\|_X. \quad ■$$

As we have already noticed, the spaces $B_p^{1/p}$, $0 < p < \infty$, and VMO are $\mathcal{R}$-spaces, and so Theorems 1.3, 1.4, and 1.6 hold for these spaces. We can obtain the following interesting consequence of the above results.

Corollary 1.7. *Let $X = B_p^{1/p}$, $1 < p < \infty$, or $X = VMO$. Then there exists a function $g \in X \bigcap H^2$ such that $f - g \in L^\infty$.*

Proof. Clearly, it is sufficient to put $g = \mathcal{A}f$. ■

Note that for $p \le 1$ the class $B_p^{1/p}$ is contained in L^∞, and so the above corollary is trivial for such spaces.

Let us obtain a corollary of the above results for rational functions.

Corollary 1.8. *Let $f \in \mathcal{R}_n$, $n > 0$. Then $\mathcal{A}f \in \mathcal{R}_n$. If, in addition, $\mathbb{P}_- f \ne \mathbb{O}$, then $\mathcal{A}f \in \mathcal{R}_{n-1}$.*

Proof. The result follows immediately from Theorem 1.5. ■

Corollary 1.8 allows us to obtain an application to the Nevanlinna–Pick interpolation problem. Suppose that $\{\lambda_1, \cdots, \lambda_m\}$ are distinct points of $\mathbb{D}$, $\{d_1, \cdots, d_m\}$ are positive integers, and $\{\alpha_{jk} : \; 1 \le j \le m, \; 0 \le k < d_j\}$ is a family of complex numbers. We are interested in finding a function $f \in H^\infty$ of minimal norm such that

$$f^{(k)}(\lambda_j) = \alpha_{jk}, \quad 1 \le j \le m, \quad 0 \le k < d_j. \tag{1.4}$$

Corollary 1.9. *Let f be the solution of the interpolation problem* (1.4) *of minimal norm in H^∞. Then $f = cB$, where $c \in \mathbb{C}$ and B is a Blaschke product of class $\mathcal{R}_{d_1+\cdots+d_m-1}$.*

Proof. Lagrange's interpolation formula gives us a polynomial P of degree at most $d_1 + \cdots + d_m - 1$ that solves the interpolation problem (1.4). Let B_0 be the finite Blaschke product that has at λ_j a zero of multiplicity d_j, $1 \le j \le m$. Then $B_0 \in \mathcal{R}_{d_1+\cdots+d_m}$. The general solution of (1.4) is $f = P + B_0 h$, $h \in H^\infty$. Clearly,

$$\inf_{h \in H^\infty} \|P - B_0 h\|_\infty = \inf_{h \in H^\infty} \|\overline{B_0} P - h\|_\infty. \tag{1.5}$$

We have $\overline{B_0} P \in \mathcal{R}$. Let $\boldsymbol{h}$ be an H^∞ function that minimizes the infimum in (1.5). By Corollary 1.8, $\boldsymbol{h} = \mathcal{A}(\overline{B_0} P) \in \mathcal{R}$. It is easy to see that $P - B_0 \boldsymbol{h} = cB$, where $c \in \mathbb{C}$ and B is a finite Blaschke product. Consider

the Toeplitz operator $T_{B\overline{B_0}}$ on H^2. It follows from (1.5) that $\operatorname{dist}_{L^\infty}(B\overline{B_0}, H^\infty) = 1$, and so the kernel of $T_{B\overline{B_0}}$ is nontrivial (see Theorem 3.1.11). Let μ be the degree of B (the number of zeros of B counted with multiplicities). We have

$$\dim \operatorname{Ker} T_{B\overline{B_0}} = \operatorname{ind} T_{\overline{B_0}} T_B = \operatorname{ind} T_{\overline{B_0}} + \operatorname{ind} T_B = d_1 + \cdots + d_m - \mu > 0.$$

It follows that $B \in \mathcal{R}_{d_1+\cdots+d_m-1}$. ■

We can consider now the behavior of the operators $\mathcal{A}_m$, $m \ge 0$, on $\mathcal{R}$-spaces. First, we prove that $\mathcal{A}_m \mathcal{R}_k \subset \mathcal{R}_k$.

Theorem 1.10. *Let $m, k \in \mathbb{Z}_+$ and $m < k$. If $\varphi \in \mathcal{R}_k \setminus H^\infty_{(m)}$, then $\mathcal{A}_m \varphi \in \mathcal{R}_{k-1}$.*

Proof. Suppose that the singular value $s_k(H_\varphi)$ of H_φ has multiplicity μ and

$$s_d(H_\varphi) = \cdots = s_{d+\mu-1}(H_\varphi) > s_{d+\mu}(H_\varphi), \quad d \le m \le d+\mu-1.$$

Clearly, $\mathcal{A}_m\varphi = \mathcal{A}_d\varphi$ and $r \stackrel{\text{def}}{=} \mathbb{P}_-\mathcal{A}_m\varphi$ is a rational function of degree d. Put $g = \mathbb{P}_+\mathcal{A}_m\varphi$. It is easy to see that $g = \mathcal{A}(\varphi - r)$, and so by Corollary 1.8, g is rational.

Consider the function $u \stackrel{\text{def}}{=} \varphi - \mathcal{A}_m\varphi = \varphi - r - \mathcal{A}(\varphi - r)$. Then u has constant modulus and $\dim \operatorname{Ker} T_u = \operatorname{ind} T_u = 2d + \mu$ (see Theorem 4.1.7). Without loss of generality we may assume that the function u is unimodular. Since u is rational, u has the form $\overline{B_1}B_2$, where B_1 and B_2 are finite Blaschke products with disjoint zeros. It is easy to see that $\deg \mathbb{P}_-u = \deg B_1$ and $\deg \mathbb{P}_+u = \deg B_2$. We have

$$\operatorname{ind} T_u = \operatorname{ind} T_{\overline{B_1}} + \operatorname{ind} T_{B_2} = \deg B_1 - \deg B_2 = \deg \mathbb{P}_-u - \deg \mathbb{P}_+u = 2d+\mu.$$

Clearly, $\deg \mathbb{P}_-u = \deg \mathbb{P}_-(\varphi - r) \le k + d$. Hence,

$$\begin{aligned} \deg g &= \deg \mathbb{P}_+u = \deg \mathbb{P}_-u - 2d - \mu \\ &\le \deg \mathbb{P}_-\varphi + \deg r - 2d - \mu \\ &\le k + d - 2d - \mu = k - d - \mu, \end{aligned}$$

and so

$$\deg \mathcal{A}_m\varphi = \deg r + \deg g \le d + k - d - \mu = k - \mu \le k - 1. \quad \blacksquare$$

Theorem 1.11. *Let X be a linear $\mathcal{R}$-space and let $m \in \mathbb{Z}_+$. Then $\mathcal{A}_mX \subset X$. If X is a Banach (or quasi-Banach) $\mathcal{R}$-space, then the operators $\mathcal{A}_m$ are uniformly bounded on X, i.e.,*

$$\|\mathcal{A}_m\varphi\|_X \le C\|\varphi\|_X, \quad \varphi \in X,$$

for a constant C, which does not depend on m.

Proof. Let $\varphi \in X$ and let r_m be the unique function in $\mathcal{R}_m$ such that $\mathbb{P}_+r_m = \mathbb{O}$ and $\|H_{\varphi - r_m}\| = s_m(H_\varphi)$ (see Theorems 4.1.2 and 4.1.3). It is easy to see that

$$\mathcal{A}_m\varphi = r_m + \mathcal{A}(\varphi - r_m). \tag{1.6}$$

It is also easy to see that if X is a linear $\mathcal{R}$-space, then $\mathcal{R}_n \subset X$ for every $n \in \mathbb{Z}_+$. It now follows from Theorem 1.6 that $\mathcal{A}_m\varphi \in X$.

Suppose now that X is a Banach (or quasi-Banach) $\mathcal{R}$-space. By (1.6) and Theorem 1.6, it is sufficient to show that

$$\|r_m\|_X \le \text{const}\,\|\varphi\|_X.$$

For $0 \le k \le m-1$ we have

$$\begin{aligned} s_k(H_{r_m}) &= s_k(H_{r_m - \varphi} + H_\varphi) \le \|H_{r_m-\varphi}\| + s_k(H_\varphi) \\ &= s_m(H_\varphi) + s_k(H_\varphi) \le 2s_k(H_\varphi). \end{aligned}$$

Clearly, for $k \ge m$ we have $s_k(H_{r_m}) = 0$.

Therefore

$$\|r_m\|_X \le \text{const}\,\|\varphi\|_X. \quad \blacksquare$$

2. Best Approximation in Decent Banach Algebras

In this section we consider a class of so-called decent function spaces, i.e., a class of Banach algebras of functions on $\mathbb{T}$ that satisfy certain natural conditions (axioms (A1)–(A4)). We prove that the operator $\mathcal{A}$ of best approximation by analytic functions acts on decent function spaces and we solve the recovery problem for such Banach algebras. Note, however, that unlike the case of $\mathcal{R}$-spaces the operator $\mathcal{A}$ does not have to be bounded on decent function spaces (see §5). In §4 we see examples of many classical decent Banach algebras.

Let us introduce here the following notation. For a space $X \subset L^1$ of functions on $\mathbb{T}$ we put

$$X_+ \stackrel{\text{def}}{=} \{f \in X : \ \hat{f}(j) = 0,\ j < 0\}, \quad X_- \stackrel{\text{def}}{=} \{f \in X : \ \hat{f}(j) = 0,\ j \ge 0\}.$$

Definition. A space of functions X on $\mathbb{T}$ is called *decent* if X contains the set of trigonometric polynomials $\boldsymbol{\mathcal{P}}$, consists of continuous functions, and satisfies the following axioms:

(A1) *if $f \in X$, then $\bar{f} \in X$ and $\mathbb{P}_+ f \in X$;*

(A2) *X is a Banach algebra with respect to pointwise multiplication;*

(A3) *for every $\varphi \in X$ the Hankel operator H_φ is a compact operator from X_+ to X_-;*

(A4) *if $f \in X$ and f does not vanish on $\mathbb{T}$, then $1/f \in X$.*

Note that we do not assume here that the norm in X satisfies the inequality $\|fg\|_X \le \|f\|_X \cdot \|g\|_X$. It is sufficient to assume that $\|fg\|_X \le c\|f\|_X \cdot \|g\|_X$ for some constant c. In this case one can introduce an equivalent norm in X for which $c = 1$.

It is easy to see that if X satisfies (A1) and (A2), then for every $\varphi \in X$ the Hankel operator H_φ maps boundedly X_+ to X_-. Axiom (A3) says that this operator must be a compact operator from X_+ to X_-. Let us establish two sufficient conditions for (A3). The first condition is almost obvious.

Theorem 2.1. *Let X be a space of functions on $\mathbb{T}$ that satisfies* (A1), (A2). *Suppose that the trigonometric polynomials are dense in X. Then X satisfies* (A3).

Proof. Let $\varphi \in X$ and let $\{\varphi_n\}$ be a sequence of trigonometric polynomials that converges to φ. It follows immediately from (A1) and (A2) that the sequence H_{φ_n} converges to H_φ in the norm of the space of bounded operators from X_+ to X_-. The result follows from the fact that the operators H_{φ_n} have finite rank. ■

The second sufficient condition also works for nonseparable Banach algebras.

Theorem 2.2. *Let X be a Banach space of functions on $\mathbb{T}$ that satisfies* (A1) *and* (A2). *Suppose that there exists a Banach space Y such that $X_+ \subset Y \subset H^\infty$, the inclusion $X_+ \to Y$ is compact, and $T_{\bar{f}}X_+ \subset X_+$ for every $f \in Y$. Then X satisfies* (A3).

Proof. Let $\varphi \in X$. We want to show that the Hankel operator H_φ is a compact operator from X_+ to X_-. Since $H_\varphi = H_{\mathbb{P}_-\varphi}$, we can assume without loss of generality that $\varphi \in X_-$. Since $T_{\bar{f}}X_+ \subset X_+$ for every $f \in Y$, it follows that $H_\varphi f \in X_-$ for every $f \in Y$. By the closed graph theorem, H_φ is a bounded operator from Y to X_-. The result follows now from the fact that the inclusion $X_+ \to Y$ is compact. ■

In §4 we shall see examples of many classical nonseparable Banach spaces that satisfy the hypotheses of Theorem 2.2.

Theorem 2.3. *Let X be a decent space and let u be a unimodular function on $\mathbb{T}$ such that $\mathbb{P}_-u \in X$. Suppose that at least one of the following two conditions is satisfied:*

(1) *T_u is Fredholm;*

(2) $\operatorname{clos}(T_uH^2) = H^2$.

Then $u \in X$.

Theorem 2.4. *Let X be a decent space. Then $\mathcal{A}X \subset X$.*

Proof of Theorem 2.3. Since $X \subset C(\mathbb{T})$, we have $\mathbb{P}_-u \in VMO$. We have already solved in the previous section the recovery problem in the space VMO, and so each of the conditions (1) and (2) implies that $u \in VMO$. Then by Theorem 3.3.10 and Corollary 3.2.6, u admits a representation $u = z^n\bar{h}/h$, where $n \in \mathbb{Z}$ and h is an outer function in H^2 such that the Toeplitz operator $T_{\bar{h}/h}$ is invertible on H^2. By (A2) and (A3), $zX \subset X$, and so without loss of generality we may assume that $n = -1$. Then the space

$$\operatorname{Ker} T_u = \{f \in H^2 : \ \|H_uf\|_2 = \|f\|_2\} = \{\lambda h : \ \lambda \in \mathbb{C}\}$$

is one-dimensional.

Since $H_u = H_{\mathbb{P}_-u}$, it follows from (A3) that the Hankel operator H_u is a compact operator from X_+ to X_-. Similarly, the operator $H_u^*\big|X_-$ is a compact operator from X_- to X_+. Consider the operator R on X_+ defined by $Rf = H_u^*H_uf$, $f \in X_+$. Then R is a compact operator on X_+.

We have $X_+ \subset H^2$. We can naturally imbed the space H^2 in the dual space X_+^* as follows. Let $g \in H^2$. We associate with it the linear functional $\mathcal{J}(g)$ on X_+ defined by

$$f \mapsto (f, g) = \int_{\mathbb{T}} f(\zeta)\overline{g(\zeta)}d\boldsymbol{m}(\zeta).$$

Note that $\mathcal{J}(\lambda_1 g_1 + \lambda_2 g_2) = \overline{\lambda_1}\mathcal{J}(g_1) + \overline{\lambda_2}\mathcal{J}(g_2)$, g_1, $g_2 \in H^2$, λ_1, $\lambda_2 \in \mathbb{C}$.

The imbedding $\mathcal{J}$ allows us to consider H^2 as a subset of X_+^*. It is easy to see that $Rf = H_u^* H_u f$ for $f \in X_+$ and $R^* g = H_u^* H_u g$ for any $g \in H^2$. Hence,

$$\operatorname{Ker}(I - R) \subset \operatorname{Ker}(I - H_u^* H_u) \subset \operatorname{Ker}(I - R^*).$$

Since R is a compact operator, it follows from the Riesz–Schauder theorem (see Yosida [1], Ch. X, §5) that $\dim \operatorname{Ker}(I - R) = \dim \operatorname{Ker}(I - R^*)$. Therefore

$$\operatorname{Ker}(I - R) = \operatorname{Ker}(I - H_u^* H_u) = \operatorname{Ker}(I - R^*).$$

Since we know that

$$\operatorname{Ker}(I - H_u^* H_u) = \{\lambda h : \ \lambda \in \mathbb{C}\},$$

it follows that $h \in X_+$. To complete the proof we have to show that h has no zeros on $\mathbb{T}$, which would imply by (A1), (A2), and (A4) that $u = \bar{z}\bar{h}/h \in X$.

Suppose that $h(\tau) = 0$ for some $\tau \in \mathbb{T}$. Consider the linear functional l on X_+ defined by

$$l(f) = (\mathbb{P}_+ \bar{h} f)(\tau), \quad f \in X_+. \tag{2.1}$$

Lemma 2.5. *$R^* l = l$.*

Proof of Lemma 2.5. Let $f \in X_+$. We have

$$(R^* l)(f) = l(Rf) = l(H_u^* H_u f) = (\mathbb{P}_+(\bar{h} H_u^* H_u f))(\tau)$$

by (2.1). Therefore

$$\begin{aligned}(R^* l)(f) &= (\mathbb{P}_+(\bar{h}\mathbb{P}_+(\bar{u} H_u f)))(\tau) \\ &= (\mathbb{P}_+(\bar{h}\bar{u} H_u f))(\tau) - (\mathbb{P}_+(\bar{h}\mathbb{P}_-(\bar{u} H_u f)))(\tau).\end{aligned}$$

Since $\bar{h}\mathbb{P}_- \bar{u} H_u f \in H_-^2$, it follows that $\mathbb{P}_+(\bar{h}\mathbb{P}_- \bar{u} H_u f) = \mathbb{O}$. Hence,

$$\begin{aligned}(R^* l)(f) &= (\mathbb{P}_+(\overline{hu} H_u f))(\tau) \\ &= (\mathbb{P}_+(\overline{\mathbb{P}_-(hu)} H_u f))(\tau) + (\mathbb{P}_+(\overline{\mathbb{P}_+(hu)} H_u f))(\tau).\end{aligned}$$

Clearly, $\overline{\mathbb{P}_+(hu)} H_u f \in H_-^2$, and so $\mathbb{P}_+(\overline{\mathbb{P}_+(hu)} H_u f) = \mathbb{O}$. We have

$$\begin{aligned}(R^* l)(f) &= (\mathbb{P}_+(\overline{H_u h} H_u f))(\tau) = (\mathbb{P}_+(\overline{H_u h}\mathbb{P}_-(uf)))(\tau) \\ &= (\mathbb{P}_+(\overline{H_u h}(uf)))(\tau) - (\mathbb{P}_+(\overline{H_u h}\mathbb{P}_+(uf)))(\tau).\end{aligned}$$

Since $\overline{H_u h}\mathbb{P}_+(uf) \in H^2$, it follows that

$$(\mathbb{P}_+(\overline{H_u h}\mathbb{P}_+(uf)))(\tau) = \overline{(H_u h)(\tau)}(\mathbb{P}_+(uf))(\tau) = 0$$

since $(H_u h)(\tau) = (\mathbb{P}_- \bar{z}\bar{h})(\tau) = \bar{\tau}\overline{h(\tau)} = 0$ by the assumption. Since $u = \bar{z}\bar{h}/h$, we have

$$(R^* l)(f) = (\mathbb{P}_+(\overline{H_u h}(uf)))(\tau) = (\mathbb{P}_+(zhuf))(\tau) = (\mathbb{P}_+(\bar{h}f))(\tau) = l(f). \quad \blacksquare$$

Let us now complete the proof of Theorem 2.3. We prove that under the assumption $h(\tau) = 0$ the function $h(\mathbf{1} - \bar{\tau}z)^{-1}$, which is analytic in $\mathbb{D}$, must belong to X_+ and

$$R\frac{h}{\mathbf{1} - \bar{\tau}z} = \frac{h}{\mathbf{1} - \bar{\tau}z}.$$

We have already proved that $\operatorname{Ker}(I - R^*) \subset X_+$ (here we identify H^2 with a subset of X^*). Therefore the functional l defined by (2.1) must be of the form $l(f) = (f, \psi)$, where $\psi \in X_+$ and $R\psi = \psi$. Let us show that ψ and $h(\mathbf{1} - \bar{\tau}z)^{-1}$ have the same Taylor coefficients. Let $n \in \mathbb{Z}_+$. We have

$$\hat{\psi}(n) = \overline{l(z^n)} = \overline{(\mathbb{P}_+ z^n \bar{h})(\tau)} = \sum_{j=0}^{n} \hat{h}(j)\bar{\tau}^{n-j}.$$

On the other hand,

$$\frac{h}{\mathbf{1} - \bar{\tau}z} = h \sum_{n \geq 0} \bar{\tau}^n z^n = \sum_{n \geq 0} \left(\sum_{j=0}^{n} \hat{h}(j)\bar{\tau}^{n-j} \right) z^n.$$

Clearly, the functions h and $h(\mathbf{1} - \bar{\tau}z)^{-1}$ are linearly independent, which contradicts the fact that $\dim \operatorname{Ker}(I - R) = 1$. ■

Proof of Theorem 2.4. Suppose that $\varphi \in X$ and $\mathbb{P}_-\varphi \neq \mathbb{O}$. Then the function $u = \|H_\varphi\|^{-1}(\varphi - \mathcal{A}\varphi)$ is unimodular and $\mathbb{P}_- u \in X$. Since the Toeplitz operator T_u has dense range in H^2 (see the proof of Theorem 1.5), it follows from Theorem 2.3 that $u \in X$. Hence, $\mathcal{A}\varphi \in X$. ■

Remark. As in §1 it is easy to prove that if X is a decent function space, then $\mathcal{A}_m X \subset X$ for every $m \in \mathbb{Z}_+$.

We complete this section with the description of the space of maximal ideals of the Banach algebras X_+ and X for decent function spaces X. In other words, we describe all complex homomorphisms on X and X_+. Recall that a *complex homomorphism* on a Banach algebra is, by definition, a nonzero multiplicative linear functional. If the trigonometric polynomials are dense in X, it is almost obvious that the maximal ideal space of X can be identified naturally with $\mathbb{T}$ while the maximal ideal space of X_+ can be identified with the closed unit disk. We prove the same result for decent Banach algebras X.

Theorem 2.6. *Let X be a decent space. Then each complex homomorphism on X has the form $f \mapsto f(\zeta)$ for some $\zeta \in \mathbb{T}$, while each complex homomorphism on X_+ has the form $f \mapsto f(\zeta)$ for some $\zeta \in \operatorname{clos}\mathbb{D}$.*

Proof. Let ω be a complex homomorphism on X that is not a point evaluation at a point of $\mathbb{T}$. Then there are functions $f_1, \cdots, f_n$ in X such that

$$\inf_{\zeta \in \mathbb{T}} \sum_{k=1}^{n} |f_k(\zeta)|^2 > 0, \quad \omega(f_k) = 0, \ k = 1, \cdots, n.$$

Put $f = \sum_{k=1}^{n} |f_k|^2 \in X$. Clearly, $\omega(f) = \sum_{k=1}^{n} \omega(\bar{f}_k)\omega(f_k) = 0$. Hence, f is not invertible in X, which contradicts (A4).

Now let ω be a complex homomorphism on X_+ that is not of the form $f \mapsto f(\zeta)$, $\zeta \in \operatorname{clos}\mathbb{D}$. Then there are functions $f_1, \cdots, f_n$ in X_+ such that

$$\inf_{\zeta\in\operatorname{clos}\mathbb{D}} \sum_{k=1}^{n} |f_k(\zeta)|^2 > 0, \quad \omega(f_k) = 0,\ k = 1, \cdots, n.$$

Since $X_+ \subset C_A \stackrel{\text{def}}{=} C(\mathbb{T}) \cap H^2$ and all complex homomorphisms on C_A are point evaluations at points of $\operatorname{clos}\mathbb{D}$, there exist $g_1, \cdots, g_n$ in C_A such that $\sum\limits_{k=1}^{n} f_k g_k = \mathbf{1}$. Clearly, we can find analytic polynomials $q_1, \cdots, q_n$ such that

$$\inf_{\zeta\in\operatorname{clos}\mathbb{D}} \left|\sum_{k=1}^{n} f_k(\zeta) q_k(\zeta)\right| > 0.$$

By (A4), the function $h \stackrel{\text{def}}{=} \left(\sum\limits_{k=1}^{n} f_k q_k\right)^{-1}$ belongs to X_+. Put $\varphi_k \stackrel{\text{def}}{=} h q_k \in X_+$. Clearly, $\sum\limits_{k=1}^{n} \varphi_k f_k = \mathbf{1}$, and so $\omega(\mathbf{1}) = \sum\limits_{k=1}^{n} \omega(\varphi_k)\omega(f_k) = 0$, which is impossible since $\omega(\mathbf{1}) = 1$. ■

3. Best Approximation in Spaces without a Norm

The class of decent spaces introduced in the previous section covers a broad class of function spaces (see §4, where many examples of classical functions spaces satisfying (A1)–(A4) are given). However, there are many locally convex function spaces for which it is important to study hereditary properties. In this section we give an approach that allows us to treat such spaces. We introduce the axioms (B1)–(B3), which do not assume any norm and we show that if X is a function space satisfying (B1)–(B3), then $\mathcal{A}X \subset X$. In §4 we shall see that many Carleman classes satisfy (B1)–(B3).

We consider here linear spaces X (at this point we do not assume any topology on X) that satisfy the following system of axioms:

(B1) *X is an algebra with respect to pointwise multiplication which contains the trigonometric polynomials and such that if $f \in X$, then $\bar{f} \in X$ and $\mathbb{P}_+ f \in X$;*

(B2) *there exists a decent function space Y such that $X \subset Y$ and $T_{\bar{f}} X_+ \subset X_+$ for every $f \in Y_+$;*

(B3) *if $f \in X$, and $\inf\limits_{\tau\in\mathbb{T}} |f(\tau)| > 0$, then $f^{-1} \in X$.*

Lemma 3.1. *Suppose that X satisfies* (B1) *and* (B2). *Let h be an outer function such that $h, h^{-1} \in H^2$. Then $\mathbb{P}_- \bar{h}/h \in X$ if and only if $|h|^2 \in X$ and $\inf\limits_{\tau\in\mathbb{T}} |h(\tau)| > 0$.*

Proof. Suppose that $\mathbb{P}_-\bar{h}/h \in X$. Since h, $h^{-1} \in H^2$, it follows that $\operatorname{clos} T_{\bar{h}/h}H^2 = H^2$ (see Theorem 4.4.10). Let Y be the space from (B2). Since $\mathbb{P}_-\bar{h}/h \in X \subset Y$, it follows from Theorem 2.2 that $\bar{h}/h \in Y$. Moreover, it was shown in the proof of Theorem 2.2 that $\inf\limits_{\tau\in\mathbb{T}} |h(\tau)| > 0$, and so $h^{-1} \in Y$. Let us prove that $|h|^2 \in X$.

Since $z\mathbb{P}_-\bar{z}g = \overline{\mathbb{P}_+\bar{g}}$ for any $g \in L^2$, it follows that $\mathbb{P}_+h/\bar{h} \in X$. We have

$$\mathbb{P}_+|h|^2 = \mathbb{P}_+\frac{h}{\bar{h}}\bar{h}^2 = \mathbb{P}_+\left(\bar{h}^2\mathbb{P}_+\frac{h}{\bar{h}}\right) = T_{\bar{h}^2}\left(\mathbb{P}_+\frac{h}{\bar{h}}\right) \in X$$

by (B2). It follows easily from (B1) that $\mathbb{P}_+z|h|^2 \in X$. Since $|h|^2$ is real, we have $\mathbb{P}_-|h|^2 = \bar{z}\overline{\mathbb{P}_+z|h|^2} \in X$.

Suppose now that $|h|^2 \in X$ and $\inf\limits_{\tau\in\mathbb{T}} |h(\tau)| > 0$. Then

$$\mathbb{P}_+h/\bar{h} = \mathbb{P}_+\frac{1}{\bar{h}^2}|h|^2 = T_{\bar{h}^{-2}}\mathbb{P}_+|h|^2 \in X,$$

since $h^{-2} = c|h|^-2\exp(-\mathrm{i}\,\widetilde{\log|h|^2})$, $|h|^2 \in Y$, and Y satisfies (A1), (A2), and (A4). ■

Now we can obtain analogs of the results of §2 for spaces satisfying the axioms (B1)–(B3).

Theorem 3.2. *Let X be a space satisfying* (B1)–(B3) *and let u be a unimodular function on $\mathbb{T}$ such that $\mathbb{P}_-u \in X$. Suppose that at least one of the following two conditions is satisfied:*

(1) *T_u is Fredholm;*

(2) $\operatorname{clos}(T_uH^2) = H^2$.

Then $u \in X$.

Theorem 3.3. *Let X be a space satisfying* (B1)–(B3). *Then $\mathcal{A}X \subset X$.*

Proof of Theorem 3.2. As in Theorem 2.2 each of the conditions (1) and (2) implies that $u \in VMO$. Again by Theorem 3.3.10 and Corollary 3.2.6, u admits a representation $u = z^n\bar{h}/h$, where $n \in \mathbb{Z}$ and h is an outer function in H^2 such that $1/h \in H^2$ and the Toeplitz operator $T_{\bar{h}/h}$ is invertible on H^2. Clearly, $\mathbb{P}_-\bar{h}/h \in X$. By Lemma 3.1, we conclude that $|h|^2 \in X$ and $\inf_{\tau\in\mathbb{T}} |h(t)| > 0$. It follows from (B3) that $|h|^{-2} \in X$. Since $X \subset C(\mathbb{T})$, we have $\inf_{\tau\in\mathbb{T}} |h(t)|^{-2} > 0$. We can now apply Lemma 3.1 to h^{-1} and we find that

$$\mathbb{P}_-h/\bar{h} = \mathbb{P}_-\overline{h^{-1}}/h^{-1} \in X.$$

It follows that $\mathbb{P}_+\bar{h}/h \in X$, and so $u = z^n\bar{h}/h \in X$. ■.

Theorem 3.3 can be deduced from Theorem 3.2 in the same way as Theorem 2.4 has been deduced from Theorem 2.3 in §2.

Remark. As in §1 and §2 it is easy to see that if X satisfies (B1)–(B3), then $\mathcal{A}_mX \subset X$ for any $m \in \mathbb{Z}_+$.

4. Examples and Counterexamples

In this section we give examples of many classical function spaces that are hereditary for the operator $\mathcal{A}$ of best approximation by analytic functions as well as examples of spaces that are not hereditary.

$\mathcal{R}$-Spaces

As we have already noticed in §1, the Besov spaces $B_p^{1/p}$, $0 < p < \infty$, and the space VMO are $\mathcal{R}$-spaces. Moreover, the spaces VMO and $B_p^{1/p}$, $1 \le p < \infty$, are Banach $\mathcal{R}$-spaces, while the spaces $B_p^{1/p}$, $0 < p < 1$, are quasi-Banach $\mathcal{R}$-spaces. It is also easy to see that the spaces $\left(B_{p_0}^{1/p_0}, B_{p_1}^{1/p_1}\right)_{\theta,q}$ considered in §6.4 are also $\mathcal{R}$-spaces. Hence, the operators $\mathcal{A}_m$ are uniformly bounded on such spaces (see Theorem 1.11).

Consider now the space $\{f \in L^1 : \ (\mathbb{P}_+ f)' \in H^1, \ (\mathbb{P}_+ \bar{f})' \in H^1\}$. As we have mentioned in §1, this space satisfies the necessary conditions to be an $\mathcal{R}$-space, which are given in §1. Let us show now that it is not an $\mathcal{R}$-space. Clearly, it suffices to establish the following result.

Theorem 4.1. *There is no Köthe sequence space E such that*

$$\varphi' \in H^1 \quad \Longleftrightarrow \quad \{s_n(\Gamma_\varphi)\}_{n\ge 0} \in E.$$

Recall that for a function φ analytic in $\mathbb{D}$ the Hankel operator Γ_φ on ℓ^2 is the operator with Hankel matrix $\{\hat{\varphi}(j+k)\}_{j,k\ge 0}$.

We are going to use results of §6.5 on properties of the averaging projection $\mathcal{P}$ onto the space of Hankel matrices. Recall that

$$\{\varphi : \ \varphi' \in H^1\} = \mathcal{P}\boldsymbol{S}_1. \tag{4.1}$$

Let us introduce the following notation. For a sequence $\{a_n\}_{n\ge 0}$ of complex numbers we denote by $\{a_n^*\}_{n\ge 0}$ the nonincreasing rearrangement of the sequence $\{|a_n|\}_{n\ge 0}$. We need the following lemma.

Lemma 4.2. *There exist a subspace $\mathcal{L}$ of the space of bounded Hankel operators and a one-to-one mapping $\mathcal{V}$ of $\mathcal{L}$ onto ℓ^∞ such that*

$$c_1(\mathcal{V}T)_n^* \le s_n(T) \le c_2(\mathcal{V}T)_n^*, \quad T \in \boldsymbol{S}_\infty \cap \mathcal{L},$$

for some positive constants c_1 and c_2.

Proof. It is more convenient to work with the Hankel operators H_ψ. Let B be an interpolating Blaschke product with zeros $\{\zeta_n\}_{n\ge 0}$. Define

$$\mathcal{L} = \{H_{f\overline{B}} : \ f \in H^\infty\}.$$

It is easy to see that $H_{f_1\overline{B}} = H_{f_2\overline{B}}$ for $f_1, f_2 \in H^\infty$ if and only if $f_1(\zeta_j) = f_2(\zeta_j)$, $j \ge 0$. We define the operator $\mathcal{V} : \mathcal{L} \to \ell^\infty$ by

$$\mathcal{V}H_{f\overline{B}} = \{f(\zeta_n)\}_{n\ge 0}, \quad f \in H^\infty.$$

Clearly, $\mathcal{V}$ is well defined. Since B is an interpolating Blaschke product, the map $\mathcal{V}$ is one-to-one and onto.

Consider now the model operator S_B (see §1.2). Put

$$g_n(z) = (1 - |\zeta_n|^2)^{1/2} \frac{B(z)}{z - \zeta_n}, \quad n \geq 0.$$

As we have noticed in the discussion preceding Theorem 1.5.11, the functions g_n form an unconditional basis in the space $K_B = H^2 \ominus BH^2$ and $f(S_B)g_n = f(\zeta_n)g_n$, $f \in H^\infty$, $n \geq 0$. The result follows now from the equality $s_n(H_{f\overline{B}}) = s_n(f(S_B))$, $n \geq 0$, (see (1.2.9)). ■

Proof of Theorem 4.1. Suppose that such a Köthe space E exists. Consider the space $\{\Gamma_\varphi \in \mathcal{L} : \ \varphi' \in H^1\}$. By Lemma 4.2, the operator $\mathcal{V}$ maps this space one-to-one onto the space

$$\{\{a_n\}_{n\geq 0} : \ \{a_n^*\}_{n\geq 0} \in E\}.$$

This allows us to introduce on this sequence space a norm that makes it a separable Banach space isomorphic to $\{\Gamma_\varphi \in \mathcal{L} : \ \varphi' \in H^1\}$. Hence, the space

$$\boldsymbol{S} \stackrel{\text{def}}{=} \{T \in \mathcal{B}(\ell^2) : \ \{s_n(T)\}_{n\geq 0} \in E\}$$

is also a separable Banach space (see Gohberg and Krein [2], Ch. III, §3).

Let us show that $\boldsymbol{S} \subset \boldsymbol{S}_{1,2}$. By Lemma 4.2 it is sufficient to show that

$$T \in \mathcal{L} \cap \boldsymbol{S} \quad \Longrightarrow \quad T \in \boldsymbol{S}_{1,2}.$$

However, this follows immediately from Theorem 6.5.3 and from (4.1).

We have $\boldsymbol{S} \subset \boldsymbol{S}_{1,2} \subset (\boldsymbol{S}_{2,\omega})^*$ and $\boldsymbol{S} \neq \boldsymbol{S}_{2,\omega}$, since $\boldsymbol{S}_{1,2}$ is not a Banach space (see Appendix 1.1). Consider the dual space $\boldsymbol{S}^*$, which can be identified with an ideal of compact operators on $\mathcal{B}(\ell^2)$ (see Gohberg and Krein [2], Ch. III, §12). Since $\mathcal{P}\boldsymbol{S}_1 \subset \boldsymbol{S}$, it follows by duality that $\mathcal{P}\boldsymbol{S}^* \subset \mathcal{B}(\ell^2)$. However, $\boldsymbol{S}_{2,\omega} \subsetneq \boldsymbol{S}^*$, which contradicts Theorem 6.5.7. ■

We show in this section (Theorem 4.5) that

$$\{f \in L^1 : \ (\mathbb{P}_+ f)', (\mathbb{P}_+ \bar{f})' \in H^1\}$$

is a decent space, and so the operators $\mathcal{A}_m$, $m \in \mathbb{Z}_+$, leave it invariant.

Hölder Classes

It is easy to see that the separable Hölder–Zygmund classes λ_α, $0 < \alpha < \infty$, are decent spaces (see Appendix 2.6). Hence, $\mathcal{A}\lambda_\alpha \subset \lambda_a$.

The fact that the nonseparable Hölder–Zygmund classes Λ_α, $0 < \alpha < \infty$, are also decent spaces is more complicated. It is obvious that Λ_α satisfies (A1), (A2), and (A4). To prove that Λ_α satisfies (A3) we make use of Theorem 2.2.

Theorem 4.3. *Let* $0 < \alpha < \infty$. *Then* Λ_α *satisfies the axiom* (A3).

Proof. Let $X = \Lambda_\alpha$. By Theorem 2.2, it is sufficient to show that $T_{\bar{f}} X_+ \subset X_+$ for any $f \in H^\infty$ and the identical inclusion $X_+ \to H^\infty$ is compact.

Let us first show that for any function f in H^∞ the Toeplitz operator $T_{\bar f}$ maps $(\Lambda_\alpha)_+$ into itself. Consider the space $(B_1^{-\alpha})_+$ that consists of functions analytic in $\mathbb{D}$ and satisfying

$$\int_{\mathbb{D}} |\varphi(\zeta)|(1-|\zeta|)^{\alpha-1}d\boldsymbol{m}_2(\zeta) < \infty.$$

The dual space $(B_1^{-\alpha})_+^*$ can be identified with the space $(\Lambda_\alpha)_+$ with respect to the pairing

$$(\varphi,\psi) = \sum_{n\ge 0}\hat\varphi(n)\overline{\hat\psi(n)}, \quad \varphi\in\boldsymbol{\mathcal{P}}_A, \quad \psi\in(\Lambda_\alpha)_+, \tag{4.2}$$

(see Appendix 2.6). It is easy to see that the operator $T_{\bar f}$ is the adjoint of multiplication by f on $(B_1^{-\alpha})_+^*$, which is obviously bounded. Hence, $T_{\bar f}$ is bounded on $(\Lambda_\alpha)_+$.

It remains to prove that the identical inclusion of $(\Lambda_\alpha)_+$ in H^∞ is compact. We prove a stronger result, which we will need later. Let $0<\beta<\alpha$. We show that the identical inclusion of Λ_α in Λ_β is compact. Let W_n, $n\in\mathbb{Z}$, be the trigonometric polynomials defined in Appendix 2.6. Define the finite rank operator $R_m:\Lambda_\alpha\to\Lambda_\beta$, $m\in\mathbb{Z}_+$, by

$$R_m f = \sum_{n=-m}^{m} f * W_n.$$

It is easy to see that

$$\begin{aligned}\|f-R_mf\|_{\Lambda_\beta} &\le \sum_{|n|\ge m}\|f*W_n\|_{\Lambda_\beta}\\ &\le \sum_{|n|\ge m}2^{|n|\beta}\|f*W_n\|_{L^\infty}\le \text{const}\,\|f\|_{\Lambda_\alpha}\sum_{|n|\ge m}2^{|n|(\beta-\alpha)}\end{aligned}$$

(see Appendix 2.6). The result follows from the fact that $\sum_{|n|\ge m} 2^{|n|(\beta-\alpha)}\to 0$ as $m\to\infty$. ■

It follows from Theorem 4.3 that $\mathcal{A}_m\Lambda_\alpha\subset\Lambda_\alpha$, $0<\alpha<\infty$.

The Space C^∞

The space C^∞ of infinitely differentiable functions is not a Banach space. However, it is easy to see that it is hereditary: $\mathcal{A}_mC^\infty\subset C^\infty$, $m\in\mathbb{Z}_+$. This follows immediately from the equality $C^\infty = \bigcap_{\alpha<\infty}\Lambda_\alpha$.

Besov Classes $B^\alpha_{p,q}$

We assume here that $1\le p\le\infty$, $1\le q\le\infty$, and $\alpha\ge 1/p$.

Consider first the case $\alpha>1/p$ and $q<\infty$. Then the trigonometric polynomials are dense in $B^\alpha_{p,q}$ and $B^\alpha_{p,q}$ satisfies the axioms (A1)–(A4) (see

Appendix 2.6). The same is true if $\alpha = 1/p$ and $q = 1$ (see Appendix 2.6). Therefore for such values of α, p, q the Besov classes $B^\alpha_{p,q}$ have the heredity property: $\mathcal{A}_m B^\alpha_{p,q} \subset B^\alpha_{p,q}$.

Consider now the case $\alpha > 1/p$ and $q = \infty$. The classes $B^\alpha_{p,\infty}$ are non-separable. We are going to show that they are still decent spaces. It is easy to see that the spaces $B^\alpha_{p,\infty}$ satisfy (A1), (A2), and (A4) (see Appendix 2.6). It remains to verify the axiom (A3).

Theorem 4.4. *Let $1 \le p < \infty$ and $\alpha > 1/p$. The spaces $B^\alpha_{p,\infty}$ satisfy the axiom* (A3).

Proof. First of all, $B^\alpha_{p,\infty} \subset \Lambda_\beta$ for $0 < \beta < \alpha - 1/p$. (This follows immediately from the fact that $B^s_{p,\infty} \subset L^\infty$ for $s > 1/p$; see Appendix 2.6.) As in the proof of Theorem 4.3, we prove that the Toeplitz operators $T_{\bar f}$ are bounded on $(B^\alpha_{p,\infty})_+$ for $f \in H^\infty$.

Consider the space $(B^{-\alpha}_{p',1})_+$ that consists of functions φ analytic in $\mathbb{D}$ and such that

$$\int_0^1 (1-r)^{\alpha-1}\left(\int_{\mathbb{T}} |\varphi(r\zeta)|^{p'} d\boldsymbol{m}(\zeta)\right)^{1/p'} dr < \infty.$$

Its dual space $(B^{-\alpha}_{p',1})^*_+$ can be identified with our space $B^\alpha_{p,\infty}$ with respect to the pairing (4.2) (see Appendix 2.6). Since obviously multiplication by f, $f \in H^\infty$, is bounded on $(B^{-\alpha}_{p',1})_+$, it follows by duality that $T_{\bar f}$ is bounded on $(B^\alpha_{p,\infty})_+$.

By Theorem 2.2, it remains to show that the identical inclusion of $(B^\alpha_{p,\infty})_+$ in H^∞ is compact. Since $B^\alpha_{p,\infty} \subset \Lambda_\beta$ for $0 < \beta < \alpha - 1/p$, the result follows from the compactness of the identical inclusion of $(\Lambda_\beta)_+$ in H^∞, which has been established in the proof of Theorem 4.3. ■

Note that here we have examined all Besov classes B^α_p, $1 \le p \le \infty$, $\alpha > 0$. If $\alpha = 1/p$ and $1 \le p < \infty$, the space B^α_p is an $\mathcal{R}$-space. If $\alpha > 1/p$ and $1 \le p \le \infty$, the space B^α_p is decent. However, if $p < \infty$ and $\alpha < 1/p$, the space B^α_p is not contained in BMO and the operator $\mathcal{A}$ is not defined on B^α_p. Let us explain briefly why $B^\alpha_p \not\subset BMO$ for $\alpha < 1/p$. Let $r > 1$ be a number satisfying $1/r < \delta \stackrel{\text{def}}{=} 1/p - \alpha$. Suppose that $B^\alpha_p \subset BMO$. Then $B^\alpha_p \subset L^r$. By the Hardy–Littlewood theorem (see Zygmund [1], Ch. XII, §9) the operator I_δ of fractional integration maps L^r to the space $\lambda_{\delta-1/r}$. Since $I_\delta B^\alpha_p = B^{\alpha+\delta}_p$ (see Appendix 2.6), we have

$$B^{1/p}_p = I_\delta B^\alpha_p \subset I_\delta L^r \subset \lambda_{\delta-1/r}.$$

However, this contradicts the fact that the space $B^{1/p}_p$ is Möbius invariant (see Lemma 1.2 and the Remark after it).

Note that the space B^1_1 is an $\mathcal{R}$-space and at the same time it is a decent space.

Spaces of Bessel Potentials

The spaces $\mathcal{L}_p^s$ for $1 < p < \infty$ and $s > 1/p$ are decent. Indeed, $\mathcal{L}_p^s$ forms an algebra for the above choice of indices (see Appendix 2.6). The remaining axioms (A1), (A3), and (A4) are obvious. Therefore $\mathcal{A}_m \mathcal{L}_p^s \subset \mathcal{L}_p^s$ for $1 < p < \infty$, $s > 1/p$, and $m \in \mathbb{Z}_+$.

Spaces of Functions Whose nth Derivatives Belong to VMO, BMO, $C_A + \overline{C_A}$, $H^\infty + \overline{H^\infty}$, $H^1 + \overline{H^1}$, or $L^1 + \widetilde{L^1}$

Let Z be one of the spaces VMO, BMO, $C_A + \overline{C_A}$, $H^\infty + \overline{H^\infty}$, $H^1 + \overline{H^1}$, or $L^1 + \widetilde{L^1}$. Consider the following spaces:

$$Z^{(n)} \stackrel{\text{def}}{=} \{f : \ f^{(n)} \in Z\}, \quad n = 1, 2, \cdots .$$

We can understand the nth derivative in the distributional sense.

Theorem 4.5. *Let Z be VMO, BMO, $C_A + \overline{C_A}$, $H^\infty + \overline{H^\infty}$, or $H^1 + \overline{H^1}$ and let n be a positive integer. Then $Z^{(n)}$ is a decent space. If $n \ge 2$, then the space $\big(L^1 + \widetilde{L^1}\big)^{(n)}$ is also a decent space.*

Note, however, that the space $\big(L^1 + \widetilde{L^1}\big)^{(1)}$ is not an algebra. Moreover, this space is not contained in L^∞. Indeed, suppose that $\big(L^1 + \widetilde{L^1}\big)^{(1)} \subset L^\infty$. By the closed graph theorem this inclusion must be bounded. Consider the de la Vallée Poussin kernel Υ_n, $n \in \mathbb{Z}_+$, i.e., Υ_n is the trigonometric polynomial defined by

$$\hat{\Upsilon}_n(j) = \begin{cases} 1, & |j| \le n, \\ 1 - \frac{|j| - n}{n}, & n \le |j| \le 2n, \\ 0, & |j| \ge 2n. \end{cases} \tag{4.3}$$

It is easy to see that $\Upsilon_n = 2K_{2n} - K_n$, where K_n is the Fejér kernel. Hence, $\|\Upsilon_n\|_{L^1} \le \text{const}$, and so $\|\mathbb{P}_+ \Upsilon_n\|_{L^1 + \widetilde{L^1}} \le \text{const}$. It follows that

$$\left\| \sum_{j \ge 1} \frac{\hat{\Upsilon}(j)}{j} z^j \right\|_{\left(L^1 + \widetilde{L^1}\right)^{(1)}} \le \text{const}\,.$$

However, it is easy to see that

$$\left\| \sum_{j \ge 1} \frac{\hat{\Upsilon}(j)}{j} z^j \right\|_\infty \ge \sum_{j=1}^n \frac{1}{j} \ge \text{const} \cdot \log(1 + n),$$

which proves that $\big(L^1 + \widetilde{L^1}\big)^{(1)} \not\subset L^\infty$.

To prove Theorem 4.5 we need the following lemma.

Lemma 4.6. *Let $n \in \mathbb{Z}_+$. The following assertions hold:*
(i) *if $f^{(n)} \in H^\infty$, then*
$$\big\|(T_{\bar z^k} f)^{(n)}\big\|_\infty \le \operatorname{const} 2^n \log(1+k)\big\|f^{(k)}\big\|_\infty, \quad k > 0;$$
(ii) *if $f^{(n)} \in BMOA$, then*
$$\big\|(T_{\bar z^k} f)^{(n)}\big\|_{BMO} \le \operatorname{const} 2^n \big\|f^{(k)}\big\|_{BMO}, \quad k > 0;$$
(iii) *if $f^{(n)} \in H^1$, then*
$$\big\|(T_{\bar z^k} f)^{(n)}\big\|_1 \le \operatorname{const} 2^n \log(1+k)\big\|f^{(k)}\big\|_1, \quad k > 0;$$
(iv) *if $f^{(n)} \in \mathbb{P}_+ L^1$, then*
$$\big\|(T_{\bar z^k} f)^{(n)}\big\|_{\mathbb{P}_+ L^1} \le \operatorname{const} 2^n \big\|f^{(k)}\big\|_{\mathbb{P}_+ L^1}, \quad k > 0.$$

Proof. For a positive integer k consider the sequence $\big\{\gamma_j^{(k)}\big\}_{j\ge 0}$ defined by
$$\gamma_j^{(k)} = 1 - \frac{j}{j+k}, \quad j \ge 0.$$
Obviously, the sequence $\big\{\gamma_j^{(k)}\big\}_{j\ge 0}$ is convex and $\gamma_0^{(k)} = 1$. By Polya's theorem (see Zygmund, Ch. V, Th. 1.5), there exists a probability measure μ_k on $\mathbb{T}$ such that $\hat\mu_k(j) = \gamma_{|j|}^{(k)}$, $j \in \mathbb{Z}$. Let δ be the unit mass at $1 \in \mathbb{T}$. Consider the n-fold convolution of the measure $\delta - \mu_k$ and denote it by ν_k^n. We have
$$\hat\nu_k^n(j) = \left(\frac{|j|}{|j|+k}\right)^n, \quad j \in \mathbb{Z},$$
and $\|\nu_k^n\|_{\mathcal{M}(\mathbb{T})} \le 2^n$, where $\mathcal{M}(\mathbb{T})$ is the space of Borel measures on $\mathbb{T}$.

Let $f^{(n)} \in H^\infty$. Then
$$\begin{aligned}
\|(T_{\bar z^k} f)^{(n)}\|_\infty &= \left\|\sum_{j\ge 0} j^n \hat f(j+k) z^j\right\|_\infty \\
&= \left\|\sum_{j\ge 0}\left(\frac{j}{j+k}\right)^n (j+k)^n \hat f(j+k) z^j\right\|_\infty \\
&\le \|\nu_k^n\|_{\mathcal{M}(\mathbb{T})}\left\|\sum_{j\ge 0}(j+k)^n \hat f(j+k) z^{j+k}\right\|_\infty \\
&\le \operatorname{const} 2^n \log(1+k)\|f^{(n)}\|_\infty.
\end{aligned}$$
The last inequality is a consequence of the well-known estimate for the Dirichlet kernel:
$$\left\|\sum_{j=-k}^{k} z^j\right\|_{L^1} \le \operatorname{const}\log(1+k)$$

(see Zygmund [1], Ch. II, §12).

To prove (ii) we use the same scheme and the following obvious inequality:

$$\|T_{\bar{z}^k} f\|_{BMOA} \le \text{const}\,\|f\|_{BMOA}, \quad f \in BMOA.$$

The proof of (iii) is the same as the proof of (i) while the proof of (iv) is the same as the proof of (ii). ■

We need one more lemma.

Lemma 4.7. $\big(L^1 + \widetilde{L^1}\big)^{(1)} \subset VMO$.

Proof. Let us first prove that $\big(L^1 + \widetilde{L^1}\big)^{(1)} \subset BMO$. Clearly, it is sufficient to show that if f is an analytic function in $\mathbb{D}$ such that $f' \in \mathbb{P}_+ L^1$, then $f \in BMOA$. Then there exists a function ψ such that $\psi' \in L^1$ and $\mathbb{P}_+\psi = f$. Since such a function ψ must be in L^∞, it follows that $f \in BMOA$.

To complete the proof, it remains to observe that the set of trigonometric polynomials is dense in $\big(L^1 + \widetilde{L^1}\big)^{(1)}$. ■

Proof of Theorem 4.5. (A1) is obvious. To verify that $Z^{(n)}$ is an algebra, it is sufficient to show that

$$f,\, g \in Z_+^{(n)} \quad \Longrightarrow \quad fg \in Z^{(n)} \quad \text{and} \quad \mathbb{P}_+ \bar{f} g \in Z^{(n)}.$$

Suppose that $f,\, g \in Z_+^{(n)}$. Let us first prove that $fg \in Z^{(n)}$. We use the formula

$$(fg)^{(n)} = f^{(n)} g + \binom{n}{1} f^{(n-1)} g' + \cdots + \binom{n}{n-1} f' g^{(n-1)} + f g^{(n)}. \tag{4.4}$$

Let us show that if $\varphi' \in Z_+$ and $\psi' \in Z_+$, then $\varphi\psi \in Z_+$. If Z_+ is C_A, H^∞, $VMOA$, $BMOA$, or H^1, this is obvious, since in this case φ and ψ must be in C_A, and so $\varphi\psi \in C_A \subset Z_+$.

Suppose now that $Z_+ = \mathbb{P}_+ L^1$. By Lemma 4.7, $\varphi,\, \psi \in BMOA \subset H^2$. Hence, $\varphi\psi \in H^1 \subset \mathbb{P}_+ L^1$.

It follows that the terms $f^{(k)} g^{(n-k)}$, $1 \le k \le n-1$, on the right-hand side of (4.4) belong to Z.

Let us show that $f^{(n)} g$ and $f g^{(n)}$ are in Z. If $Z = C_A + \overline{C_A}$, $Z = H^\infty + \overline{H^\infty}$, or $Z = H^1 + \overline{H^1}$, this is obvious, since $f,\, g \in C_A$.

Let $Z = VMO$. It is sufficient to show that if $\varphi \in VMOA$ and $\psi' \in VMOA$, then $\varphi\psi \in VMOA$. It follows by duality from Lemma 4.6, (iii) that

$$\|z^k \varphi\|_{VMOA} \le \text{const} \cdot \log(2+k) \|\varphi\|_{VMOA}.$$

It is also obvious that $\psi \in \Lambda_\alpha$ for any $\alpha < 1$. It is well known that $\Lambda_\beta \subset \mathcal{F}(\ell^1)$ for $\beta > 1/2$ (Bernshtein's theorem, see Kahane [1], Ch. II, §6), which implies that

$$\sum_{k \ge 0} \log(2+k) |\hat{\psi}(k)| < \infty.$$

We have

$$\varphi\psi = \sum_{k\geq 0} \hat{\psi}(k) z^k \varphi$$

and

$$\left\| \sum_{k\geq 0} \hat{\psi}(k) z^k \varphi \right\|_{VMOA} \leq \sum_{k\geq 0} \|z^k\varphi\|_{VMOA} \leq \text{const} \sum_{k\geq 0} \log(2+k)|\hat{\psi}(k)|.$$

For $Z = BMO$ the proof is similar.

Finally, consider the case $Z = L^1 + \widetilde{L^1}$, $n \geq 2$. It is sufficient to show that if $\varphi \in \mathbb{P}_+ L^1$ and $\psi'' \in \mathbb{P}_+ L^1$, then $\varphi\psi \in \mathbb{P}_+ L^1$. By Lemma 4.7, $\psi' \in VMOA$, and so $\psi \in \Lambda_a$ for any $\alpha < 1$. The rest of the proof is exactly the same as in the case $Z = VMO$.

Let us now prove that $\mathbb{P}_+ \bar{f} g \in Z_+^{(n)}$. Assume first that Z is VMO, BMO, $C_A + \overline{C_A}$ or $H^\infty + \overline{H^\infty}$ and $n \geq 1$, or Z is $H^1 + \overline{H^1}$ or $L^1 + \widetilde{L^1}$ and $n \geq 2$. As we have already seen above, this implies that $f \in \Lambda_\alpha$ for any $\alpha < 1$, and so

$$\sum_{k\geq 0} \log(2+k)|\hat{f}(k)| < \infty,$$

$$\mathbb{P}_+ \bar{f} g = \sum_{k\geq 0} \overline{\hat{f}(k)} T_{\bar{z}^k} g, \tag{4.5}$$

and by Lemma 4.6, the series on the right-hand side of (4.5) converges in the norm of Z_+.

It remains to consider the case $Z = H^1 + \overline{H^1}$ and $n = 1$. In this case $f \in H^\infty$. Since the trigonometric polynomials are dense in Z it is sufficient to show that

$$\|\mathbb{P}_+ \bar{f} g\|_{Z_+^{(1)}} \leq \text{const}\, \|f\|_{H^\infty} \|g\|_{Z_+^{(1)}}, \quad g \in \boldsymbol{\mathcal{P}}_A.$$

The dual space to $Z_+^{(1)}$ can naturally be identified with the space $\{\psi' : \ \psi \in BMOA\}$ with respect to the pairing (4.2). This space admits the following description:

$$\psi \in \left(Z_+^{(1)}\right)^* \iff |\psi(\zeta)|^2(1-|\zeta|^2)d\boldsymbol{m}_2(\zeta) \text{ is a Carleson measure} \tag{4.6}$$

(this follows from the characterization of BMO in terms of Carleson measures; see Appendix 2.5). We have

$$\begin{aligned} |(\mathbb{P}_+ \bar{f} g, \psi)| &= |(\bar{f} g, \psi)| = |(g, f\psi)| \\ &\leq \text{const}\, \|g\|_{Z_+^{(1)}} \|f\psi\|_{\left(Z_+^{(1)}\right)^*}, \quad \psi \in \left(Z_+^{(1)}\right)^*. \end{aligned}$$

The result follows now from the fact that multiplication by f is bounded on $\left(Z_+^{(1)}\right)^*$, which is an immediate consequence of (4.6).

Let us verify (A4). If $Z = VMO$, $Z = C_A + \overline{C_A}$, $Z = H^1 + \overline{H^1}$, or $Z = L^1 + \widetilde{L^1}$, the trigonometric polynomials are dense in the Banach algebra $Z^{(n)}$, and the result is almost obvious. Indeed, if ω is a complex homomorphism, let $\lambda = \omega(z)$. Then $\omega(\varphi) = \varphi(\lambda)$ for any trigonometric polynomial φ. Therefore the linear functional $\varphi \mapsto \varphi(\lambda)$ extends by continuity to $Z^{(n)}$. Obviously, this implies that $\lambda \in \mathbb{T}$ and $\omega(\varphi) = \varphi(\lambda)$ for any $\varphi \in Z^{(n)}$. Hence, all complex homomorphisms on $Z^{(n)}$ are point evaluations at points of $\mathbb{T}$. This implies (A4).

Suppose now that $Z = BMO$ or $Z = H^\infty + \overline{H^\infty}$. Let $f^{(n)} \in Z$ and $\inf_{\tau\in\mathbb{T}} |f(\tau)| > 0$. Then $f \in \Lambda_a$ for $\alpha < 1$ and $1/f \in \Lambda_a$ for $\alpha < 1$. It is easy to see now that

$$\left(\frac{1}{f}\right)^{(n)} = -\frac{f^{(n)}}{f^2} + g,$$

where g is a function that belongs to Λ_α for all $\alpha < 1$.

It is sufficient to prove the following fact. Let $\varphi \in Z$ and $\psi \in \Lambda_\alpha$ for all $\alpha < 1$. Then $\varphi\psi \in Z$. Put $\varphi_+ \stackrel{\text{def}}{=} \mathbb{P}_+\varphi$, $\varphi_- \stackrel{\text{def}}{=} \mathbb{P}_-\varphi$, $\psi_+ \stackrel{\text{def}}{=} \mathbb{P}_+\psi$, and $\psi_- \stackrel{\text{def}}{=} \mathbb{P}_-\psi$. We have

$$\varphi\psi = \varphi_+\psi_+ + \varphi_-\psi_- + \varphi_+\psi_- + \varphi_-\psi_+.$$

Let us show that $\varphi_+\psi_+ \in Z$ and $\varphi_-\psi_+ \in Z$. The proof that $\varphi_-\psi_- \in Z$ and $\varphi_+\psi_- \in Z$ are the same.

Consider first the case $Z = H^\infty + \overline{H^\infty}$. Obviously, $\varphi_+\psi_+ \in H^\infty$. It is also clear that $\varphi_-\psi_+ \in L^\infty$, and so it is sufficient to show that $\mathbb{P}_+\varphi_-\psi_+ \in H^\infty$. However, this follows from Lemma 4.6, (i).

Consider now the case $Z = BMO$. When verifying the axiom (A2) we have shown that the product of a $BMOA$ function and a function that belongs to $(\Lambda_\alpha)_+$ for some $\alpha > 1/2$ must be in $BMOA$. This implies that $\varphi_+\psi_+ \in BMOA$. Let us prove now that $\varphi_-\psi_+ \in BMO$. There exists a function $\xi \in BMOA$ such that $\varphi_- + \xi \in L^\infty$. We have

$$\varphi_-\psi_+ = (\varphi_- + \xi)\psi_+ + \xi\psi_+.$$

Obviously, $(\varphi_- + \xi)\psi_+ \in L^\infty$. We have already proved that the product of a $BMOA$ function and a function in Λ_α for some $\alpha > 1/2$ is in $BMOA$, which implies that $\xi\psi_+ \in BMOA$.

It remains to verify (A3). For $Z = VMO$, $Z + C_A + \overline{C_A}$, $Z = H^1 + \overline{H^1}$, or $Z + L^1 + \widetilde{L^1}$ the trigonometric polynomials are dense in $Z^{(n)}$, and so (A3) follows immediately from Theorem 2.1. To verify (A3) for $Z = BMO$ or $Z = H^\infty + \overline{H^\infty}$ we make use of Theorem 2.2.

Let $Z = BMO$. Put

$$Y = \left(\mathcal{F}\ell^1\right)_+ = \left\{ f \in C_A : \sum_{k\ge 0} |\hat{f}(k)| < \infty \right\}.$$

If $f \in Y$, it follows immediately Lemma 4.6 (ii) that $T_{\bar{f}} Z_+^{(n)} \subset Z_+^{(n)}$. By Theorem 2.2, it is sufficient to prove that the identical inclusion of Y in $Z_+^{(n)}$ is compact. Let $1/2 < \beta < \alpha < n$. Then $BMO^{(n)} \subset \Lambda_\alpha$ and $\Lambda_\beta \subset \mathcal{F}\ell^1$. The result follows now from the fact that the inclusion $\Lambda_\alpha \to \Lambda_\beta$ is compact (see the proof of Theorem 4.3).

Now let $Z = H^\infty + \overline{H^\infty}$. Put

$$Y = \left\{ f \in C_A : \sum_{k \ge 0} \log(2 + |k|)|\hat{f}(k)| < \infty \right\}.$$

It follows immediately from Lemma 4.7 (i) that $T_{\bar{f}} Z_+^{(n)} \subset Z_+^{(n)}$ for $f \in Y$. It remains to show that the identical inclusion of Y in $Z_+^{(n)}$ is compact. Again, let $1/2 < \beta < \alpha < n$. Then $(H^\infty + \overline{H^\infty})^{(n)} \subset \Lambda_\alpha$ and $\Lambda_\beta \subset Y$. As in the previous case the result follows from the fact that the inclusion $\Lambda_\alpha \to \Lambda_\beta$ is compact. ■

Theorem 4.5 implies the following interesting corollary.

Corollary 4.8. *Let n be a positive integer. Then the following assertions hold:*

(i) *if $f^{(n)} \in C_A$, then $(\mathcal{A}\bar{f})^{(n)} \in C_A$;*

(ii) *if $f^{(n)} \in H^\infty$, then $(\mathcal{A}\bar{f})^{(n)} \in H^\infty$;*

(iii) *if $f^{(n)} \in H^1$, then $(\mathcal{A}\bar{f})^{(n)} \in H^1$.*

The Spaces $\mathcal{F}\ell^p_w$

Let $w = \{w_n\}_{n \ge 0}$ be a sequence of positive numbers and let $1 \le p \le \infty$. Consider the space $\mathcal{F}\ell^p_w$ of formal trigonometric series

$$f = \sum_{n \in \mathbb{Z}} \hat{f}(n) z^n$$

such that

$$\|f\|_{\mathcal{F}\ell^p_w} \stackrel{\text{def}}{=} \begin{cases} \left(\sum_{n \in \mathbb{Z}} |\hat{f}(n) w_{|n|}|^p \right)^{1/p}, & p < \infty, \\ \sup_n |\hat{f}(n) w_{|n|}|, & p = \infty. \end{cases}$$

We also consider the space $\mathcal{F}c_{0,w}$ that can be defined as the closure of the set of polynomials in $\mathcal{F}\ell^\infty_w$.

It is easy to verify (see Nikol'skii [1]) that under the condition

$$\sup_m \left(\sum_{k+j=m} \left(\frac{w_{|m|}}{w_{|k|} w_{|j|}} \right)^{p'} \right)^{1/p'} < \infty, \quad p > 1, \tag{4.7}$$

$$w_{|k+j|} \le \text{const} \cdot w_{|k|} w_{|j|}, \quad j, k \in \mathbb{Z}, \quad p = 1, \tag{4.8}$$

the space $\mathcal{F}\ell^p_w$ is an algebra (i.e., $\mathcal{F}\ell^p_w \subset L^\infty$ and $\mathcal{F}\ell^p_w$ is an algebra under pointwise multiplication).

If $\mathcal{F}\ell^p_w$ is an algebra, then $\inf_{n\geq 0} w_n > 0$. Indeed,

$$w_0 = \|z^{-n} \cdot z^n\| \leq \text{const}\, \|z^{-n}\| \cdot \|z^n\| = w_n^2.$$

Clearly, if

$$\limsup_{n\to\infty} w_n^{1/n} = 1, \tag{4.9}$$

then elements of $\mathcal{F}\ell^p_w \not\subset AC_\delta$ for any $\delta \in (0,1)$, where AC_δ is the space of functions that extend analytically to the annulus $\{\zeta \in \mathbb{C} :\ \delta < |\zeta| < 1/\delta\}$.

It is now easy to see that the following theorem holds.

Theorem 4.9. *Let $w = \{w_n\}_{n\geq 0}$ be a sequence of positive numbers satisfying* (4.9)*. The following assertions hold:*

(i) *if $1 < p < \infty$ and w satisfies* (4.7)*, then $\mathcal{F}\ell^p_w$ is a decent space;*

(ii) *if $p = 1$ and w satisfies* (4.8)*, then $\mathcal{F}\ell^1_w$ is a decent space;*

(iii) *if w satisfies* (4.7) *with $p' = 1$, then $\mathcal{F}c_{0,w}$ is a decent space.*

Consider now the nonseparable space $\mathcal{F}\ell^\infty_w$. It is not obvious for this space that conditions (4.7) and (4.9) imply that the space of maximal ideals of $\mathcal{F}\ell^\infty_w$ can be identified with $\mathbb{T}$. To prove this, we need one more lemma.

If X is a Banach space of functions on $\mathbb{T}$, we say that X is *homogeneous* if

$$f \in X, \quad \tau \in \mathbb{T} \quad \Longrightarrow \quad f_\tau \in X \quad \text{and} \quad \|f_\tau\|_X = \|f\|_X,$$

where $f_\tau(\zeta) \stackrel{\text{def}}{=} (\bar{\tau}\zeta)$, $\zeta \in \mathbb{T}$.

Lemma 4.10. *Let X be a homogeneous Banach algebra of functions on $\mathbb{T}$ that is invariant under complex conjugation and such that $C^\infty(\mathbb{T}) \subset X \subset C(\mathbb{T})$. Suppose that there exists a homogeneous Banach space L of functions on $\mathbb{T}$ such that*

(i) *the trigonometric polynomials are dense in L;*

(ii) $X \subset L \subset C(\mathbb{T})$*;*

(iii) $\|f^2\|_X \leq \text{const}\, \|f\|_X \|f\|_L$ *for every $f \in X$.*

Then each multiplicative linear functional on X is a point evaluation at some point of $\mathbb{T}$.

Proof. Denote by $\boldsymbol{r}(f)$ the spectral radius of an element f in X. Since $\boldsymbol{r}(f)$ is the sup-norm of the Gelfand transform of f, it follows that

$$\boldsymbol{r}(f_1 + f_2) \leq \boldsymbol{r}(f_1) + \boldsymbol{r}(f_2), \quad f_1,\, f_2 \in X.$$

Let X_0 be the closure of the set of trigonometric polynomials in X. It is clear that the space of maximal ideals of X_0 can be identified naturally with $\mathbb{T}$. Let us show that for every $f \in X$

$$\boldsymbol{r}(f) = \max_{\tau\in\mathbb{T}} |f(\tau)|. \tag{4.10}$$

Indeed, let K_n be the Fejér kernel and $f_n = f * K_n$, $n \in \mathbb{Z}_+$. Since X and L are homogeneous, it is easy to see that convolution with K_n on X and L has norm one, and so

$$\lim_{n\to\infty} \|f - f_n\|_L = 0 \quad \text{and} \quad \|f_n\|_X \leq \|f\|_X.$$

We have

$$\begin{aligned}\boldsymbol{r}(f) &\le \boldsymbol{r}(f_n) + \boldsymbol{r}(f - f_n) \le \boldsymbol{r}(f_n) + \|(f - f_n)^2\|_X^{1/2} \\ &\le \boldsymbol{r}(f_n) + \text{const}\, \|f - f_n\|_X^{1/2} \|f - f_n\|_L^{1/2} \\ &\le \boldsymbol{r}(f_n) + \text{const}\, \|f\|_X^{1/2} \|f - f_n\|_L^{1/2}.\end{aligned}$$

Since $f_n \in X_0$, clearly, $\boldsymbol{r}(f_n) = \|f_n\|_\infty$. Hence, $\lim_{n\to\infty} \boldsymbol{r}(f_n) = \|f\|_\infty$, and since $\lim_{n\to\infty} \|f - f_n\|_L = 0$, it follows that $\boldsymbol{r}(f) \le \|f\|_\infty$. The opposite inequality $\boldsymbol{r}(f) \ge \|f\|_\infty$ is obvious.

Suppose now that φ is a complex homomorphism on X that is not a point evaluation at any point of $\mathbb{T}$. Since $\mathbb{T}$ is compact, there is a finite collection $\{f_1, \cdots, f_n\}$ of elements of X such that

$$\inf_{\tau\in\mathbb{T}} \sum_{k=1}^{n} |f_k(t)|^2 = \delta > 0, \quad \varphi(f_k) = 0, \quad k = 1, \cdots, n.$$

Put $f = \sum\limits_{k=1}^{n} |f_k|^2 \in X$. Clearly,

$$\varphi(f) = \sum_{k=1}^{n} \varphi(\overline{f_k})\varphi(f_k) = 0.$$

Hence, f is not invertible in X. On the other hand, since $f \in C(\mathbb{T})$ and $1/f \in C(\mathbb{T})$, there exists a trigonometric polynomial q such that $\|fq - \mathbf{1}\|_{L^\infty} < 1$. It follows now from (4.10) that $\boldsymbol{r}(fq) < 1$, and so fq is invertible in X. Clearly, this implies that f is invertible in X. So we have got a contradiction with the assumption of the existence of a complex homomorphism φ that is not a point evaluation at any point of $\mathbb{T}$. ■

Now we are in a position to obtain the following result.

Theorem 4.11. *Let $w = \{w_n\}_{n\ge 0}$ be a nondecreasing sequence of positive numbers which satisfies* (4.7) *with $p' = 1$ and such that*

$$\sup_n \frac{w_{2n}}{w_n} < \infty.$$

Then $\mathcal{F}\ell_w^\infty$ is a decent space.

Proof. (A1) is obvious. We have already observed that (A2) is satisfied. Let us verify (A3). Let $Y = (\mathcal{F}\ell^1)_+$. To show that $\mathcal{F}\ell_w^\infty \subset \mathcal{F}\ell^1$, it is sufficient to prove that $\sum\limits_{n\ge 0} w_n^{-1} < \infty$. We have from (4.7)

$$\sum_{k=0}^{m} \frac{1}{w_k} \le \sum_{k=0}^{m} \frac{w_m}{w_{n-k} w_k} \le \text{const}\,.$$

In fact, the identical inclusion of $\mathcal{F}\ell^1$ in $\mathcal{F}\ell_w^\infty$ is compact since

$$\left\| \sum_{|k|\ge n} \hat{f}(k) z^k \right\|_{\mathcal{F}\ell^1} \le \|f\|_{\mathcal{F}\ell_w^\infty} \sum_{|k|\ge n} \frac{1}{w_k} \to 0 \quad \text{as } n \to \infty.$$

Since the sequence w is nondecreasing, it follows that the norms of the Toeplitz operators $T_{\bar{z}^n}$ on $\mathcal{F}\ell^\infty_w$ do not exceed 1, and so $T_{\bar{f}}\mathcal{F}\ell^\infty_w \subset \mathcal{F}\ell^\infty_w$ for every $f \in Y$. (A3) follows now from Theorem 2.2.

It remains to verify (A4). Let us show that $X = \mathcal{F}\ell^\infty_w$ and $L = \mathcal{F}\ell^1$ satisfy the hypotheses of Lemma 4.10. We have already proved (ii); (i) is obvious. Let us establish (iii). Let $f \in \mathcal{F}\ell^\infty_w$. Then

$$\|f^2\|_{\mathcal{F}\ell^\infty_w} = \sup_{n\in\mathbb{Z}} w_{|n|}|\widehat{f^2}(n)|.$$

Let us first estimate $\sup\limits_{n\ge 0} w_{|n|}|\widehat{f^2}(n)|$. We have

$$\begin{aligned}
\sup_{n\ge 0} w_{|n|}|\widehat{f^2}(n)| &\le \sup_{n\ge 0} w_n \sum_{k\in\mathbb{Z}} |\hat{f}(k)|\cdot|\hat{f}(n-k)| \\
&\le 2\sup_{n\ge 0} w_n \sum_{k\ge \frac{n}{2}} |\hat{f}(k)|\cdot|\hat{f}(n-k)| \\
&= 2\sup_{n\ge 0} \sum_{k\ge \frac{n}{2}} w_k|\hat{f}(k)|\cdot|\hat{f}(n-k)|\cdot\frac{w_n}{w_k} \\
&\le 2\sup_{n\ge 0} \frac{w_{2n}}{w_n}\|f\|_{\mathcal{F}\ell^\infty_w}\|f\|_{\mathcal{F}\ell^1}.
\end{aligned}$$

The estimate of $\sup\limits_{n\le 0} w_{|n|}|\widehat{f^2}(n)|$ is similar. ■

Carleman Classes

Let $\mathcal{E}$ be a a Banach space of functions on $\mathbb{T}$ and $\{M_n\}_{n\ge 0}$ an increasing and logarithmically convex sequence with $M_0 = 1$. The Carleman class $C(\mathcal{E};\{M_n\})$ consists of infinitely differentiable functions f on $\mathbb{T}$ such that $f^{(n)} \in \mathcal{E}$, $n \in \mathbb{Z}_+$, and

$$\left\|f^{(n)}\right\|_{\mathcal{E}} \le C_f Q_f^n \cdot n! \cdot M_n, \quad n \in \mathbb{Z}_+,$$

for some positive constants C_f and Q_f.

Carleman classes are not Banach spaces. One can introduce in a natural way a locally convex topology of inductive limit on a Carleman class. However, we do not need this.

Lemma 4.12. *Suppose that $\mathcal{E}$ is a Banach algebra of functions on $\mathbb{T}$ whose space of maximal ideals can naturally be identified with $\mathbb{T}$. Let $f \in C(\mathcal{E};\{M_n\})$ and let φ be a function analytic in a neighborhood of $f(\mathbb{T})$. Then $\varphi\circ f \in C(\mathcal{E};\{M_n\})$.*

Proof. We use Faa di Bruno's formula (see Bourbaki [1], Ch. 1, §3, Ex. 7a)

$$(\varphi\circ f)^{(n)} =$$

$$\sum_{m_1+2m_2\cdots+nm_n=n} \frac{n!}{m_1!\cdots m_n!}(\varphi^{(m_1+\cdots+m_n)}\circ f)\left(\frac{f^{(1)}}{1!}\right)^{m_1}\cdots\left(\frac{f^{(n)}}{n!}\right)^{m_n}.$$

Since $\mathcal{E}$ is a Banach algebra and φ is analytic in a neighborhood of $f(\mathbb{T})$, it follows easily from the Riesz–Dunford integral formula (see Gamelin [1], Ch. 1, §5) that

$$\|\varphi^{(m)} \circ f\|_{\mathcal{E}} \le m! C_\varphi Q_\varphi^m.$$

For brevity we use the notation $m = m_1 + \cdots + m_n$. Then

$$\begin{aligned} &\left\| (\varphi^{(m)} \circ f) \left(\frac{f^{(1)}}{1!}\right)^{m_1} \cdots \left(\frac{f^{(n)}}{n!}\right)^{m_n} \right\|_{\mathcal{E}} \\ \le\ & \|\varphi^{(m)} \circ f\|_{\mathcal{E}} \left\|\frac{f^{(1)}}{1!}\right\|_{\mathcal{E}}^{m_1} \cdots \left\|\frac{f^{(n)}}{n!}\right\|_{\mathcal{E}}^{m_n} \\ \le\ & m! C_\varphi Q_\varphi^m C_f^m Q_f^m M_1^{m_1} \cdots M_n^{m_n}. \end{aligned}$$

Since $\{M_n\}_{n\ge 0}$ is logarithmically convex, the sequence $\{M_n^{1/n}\}_{n\ge 1}$ is nondecreasing. Hence,

$$\begin{aligned} M_1^{m_1} \cdots M_n^{m_n} &= M_1^{m_1} (M_2^{1/2})^{2m_2} \cdots (M_n^{1/n})^{nm_n} \\ &\le (M_n^{1/n})^{m_1 + 2m_2 + \cdots + nm_n} = M_n. \end{aligned}$$

Combining all the estimates, we obtain

$$\|(\varphi \circ f)^{(n)}\|_{\mathcal{E}} \le C_\varphi Q_f^n n! M_n \sum_{m_1 + 2m_2 + \cdots + nm_n = n} \frac{m!}{m_1! \cdots m_2!} (C_f Q_\varphi)^m.$$

It remains to observe that

$$\sum_{m_1 + 2m_2 + \cdots + nm_n = n;\ m_1 + m_2 + \cdots + m_n = m} \frac{m!}{m_1! \cdots m_2!} = \begin{pmatrix} n-1 \\ m-1 \end{pmatrix} \tag{4.11}$$

and

$$\sum_{m=1}^{n} \begin{pmatrix} n-1 \\ m-1 \end{pmatrix} = 2^{n-1}.$$

To prove (4.11), we set in Faa di Bruno's formula $\varphi(t) = t^m$, $f(t) = t(1-t)^{-1}$ and compute the left-hand and right-hand sides at $t = 0$. ■

Note that under the hypotheses of the lemma the space $C(\mathcal{E}; \{M_n\})$ is an algebra. This follows from the identity $4fg = (f+g)^2 - (f-g)^2$.

Corollary 4.13. *Let $1 < p < \infty$ and $\mathcal{E} = L^p$. Then the space $C(\mathcal{E}; \{M_n\})$ satisfies the conclusion of Lemma 4.12.*

Proof. Put

$$M_n' \stackrel{\text{def}}{=} \frac{M_{n+1}}{M_1}, \quad n \in \mathbb{Z}_+.$$

Consider the Sobolev space $W_p^1 = \{\varphi \in L^p : \varphi' \in L^p\}$. It is easy to see that W_p^1 is an algebra whose space of maximal ideals can naturally be identified with $\mathbb{T}$. It is also clear that

$$C(L^p; \{M_n\}) = C(W_p^1; \{M_n'\}).$$

The result now follows from Lemma 4.12. ■

The following result shows that if the M_n do not grow terribly rapidly, then $C(C(\mathbb{T});\{M_n\})$ is invariant under the Riesz projection.

Theorem 4.14. *Suppose $\{M_n\}_{n\ge 0}$ is an increasing logarithmically convex sequence such that*

$$\frac{M_{n+1}}{M_n} \le cK^{K^n} \quad \text{for some } c,\ K > 0. \tag{4.12}$$

Then $\mathbb{P}_+ C(C(\mathbb{T});\{M_n\}) \subset C(C(\mathbb{T});\{M_n\})$.

Proof. Let Υ_N be the de la Vallée Poussin kernel defined by (4.3). Let g be a function in the Zygmund class Λ_1. It is easy to see from the definition of Λ_1 in terms of convolutions with the trigonometric polynomials W_n (see Appendix 2.6) that

$$\|\mathbb{P}_+ g - \Upsilon_N * \mathbb{P}_+ g\|_{L^\infty} \le \text{const}\cdot\frac{1}{N}\|\mathbb{P}_+ g\|_{\Lambda_1} \le \text{const}\cdot\frac{1}{N}\|g\|_{\Lambda_1}.$$

Indeed, to prove this inequality, it is sufficient to consider the case $N = 2^n$ and use the elementary identity

$$\mathbb{P}_+ g - \Upsilon_{2^n} * \mathbb{P}_+ g = \sum_{j\ge n+1} W_j * \mathbb{P}_+ g.$$

Since $\Upsilon_N * g$ is a trigonometric polynomial of degree at most $2N$,

$$\left\|\sum_{j=0}^m z^j\right\|_{L^1} \le \text{const}\log(1+m),$$

and the L^1-norms of the Υ_N are uniformly bounded, it follows that

$$\|\mathbb{P}_+\Upsilon_N * g\|_{L^\infty} \le \text{const}\cdot\log(1+N)\|\Upsilon_N * g\|_{L^\infty} \le \text{const}\cdot\log(1+N)\|g\|_{L^\infty}.$$

Hence, for $g \in C^1(\mathbb{T})$ we have

$$\begin{aligned}
\|\mathbb{P}_+ g\|_{L^\infty} &\le \|\mathbb{P}_+\Upsilon_N * g\|_{L^\infty} + \|\mathbb{P}_+ g - \Upsilon_N\mathbb{P}_+ g\|_{L^\infty}\\
&\le \text{const}\cdot\log(1+N)\|g\|_{L^\infty} + \text{const}\cdot\frac{1}{N}\|g'\|_{L^\infty}.
\end{aligned}$$

Suppose now that $f \in C(C(\mathbb{T});\{M_n\})$ and

$$\|f^{(n)}\|_{L^\infty} \le C_f Q_f^n \cdot n! \cdot M_n.$$

Without loss of generality we may assume that the number K in (4.12) is an integer. Now let $g = f^{(n)}$, $N = K^{K^n}$. Then

$$\begin{aligned}
\|\mathbb{P}_+ f^{(n)}\|_{L^\infty} &\le \text{const}\cdot K^n\|f^{(n)}\|_{L^\infty} + \text{const}\cdot K^{-K^n}\|f^{(n+1)}\|_{L^\infty}\\
&\le \text{const}(K+Q_f)^n\cdot n!\cdot M_n\\
&+ \text{const}\,Q_f^{n+1}\cdot(n+1)!\cdot M_{n+1}K^{-K^n}\\
&\le \text{const}(K+Q_f)^n\cdot n!\cdot M_n + \text{const}\,Q_f^{n+1}\cdot n!(n+1)M_n\\
&\le \text{const}(K+Q_f)^n\cdot n!\cdot M_n,
\end{aligned}$$

which proves that $\mathbb{P}_+ f \in C(C(\mathbb{T});\{M_n\})$. ■

We are now able to prove that under the hypotheses of Theorem 4.14 the Carleman class $C(C(\mathbb{T});\{M_n\})$ has the heredity property.

Theorem 4.15. *Suppose that $\{M_n\}_{n\geq 0}$ is an increasing and logarithmically convex sequence satisfying* (4.12). *Then the Carleman class $C(C(\mathbb{T});\{M_n\})$ satisfies the axioms* (B1)–(B3).

Proof. By Lemma 4.12 and Theorem 4.14, (B1) and (B3) are satisfied. Let us verify (B2). Put

$$Y = \{f : \sum_{n\in\mathbb{Z}} |\hat{f}(n)| \log(2+|n|) < \infty\}.$$

Suppose that $\varphi \in C(C(\mathbb{T});\{M_n\})$ and

$$\|\varphi^{(n)}\|_{L^\infty} \leq C_\varphi Q_\varphi^n \cdot n! \cdot M_n.$$

It follows from Lemma 4.6 (i) that for $f \in Y_+$

$$\|\varphi^{(n)}\|_{L^\infty} \leq \text{const}\, \|f\|_Y (2Q_\varphi)^n \cdot n! \cdot M_n. \quad \blacksquare$$

Finally, we prove that for $1 < p < \infty$ the Carleman class $C(L^p;\{M_n\})$ also has the hereditary property even without any assumption on the sequence $\{M_n\}_{n\geq 0}$.

Theorem 4.16. *Let $1 < p < \infty$ and let $\{M_n\}_{n\geq 0}$ be an increasing and logarithmically convex sequence. Then the Carleman class $C(L^p;\{M_n\})$ satisfies the axioms* (B1)–(B3).

Proof. (B1) and (B3) follows immediately from Corollary 4.13. It is also obvious that (B2) holds with $Y = \mathcal{F}\ell^1$. $\blacksquare$

Consider the important example of Carleman classes. Let $0 < \alpha < \infty$. The *Gevrey class* G_α is the Carleman class $C(C(\mathbb{T});\{M_n\})$, where $M_n = n^{n/\alpha}$. Clearly, this sequence satisfies (7.4.12), and so by Theorem 4.15, $\mathcal{A}_m G_\alpha \subset G_\alpha$, $m \in \mathbb{Z}_+$, $\alpha > 0$.

It can easily be shown that a function $f \in C(\mathbb{T})$ belongs to G_α, $\alpha > 0$, if and only if there exist K, $\delta > 0$ such that

$$|\hat{f}(n)| \leq K \exp(-\delta|n|^{\alpha/(1+\alpha)}), \quad n \in \mathbb{Z}.$$

The Space AC_δ

For $\delta \in [0,1)$ the space AC_δ consists of functions on $\mathbb{T}$ that extend analytically to the annulus $\{\zeta \in \mathbb{C} : \ \delta < |\zeta| < \delta^{-1}\}$. It is easy to see that AC_δ satisfies (B1) and (B2) (in (B2) we can take $Y = \mathcal{F}\ell^1$). Therefore Lemma 3.1 holds for $X = AC_\delta$. In fact, we can slightly improve Lemma 3.1 for $X = AC_\delta$.

Theorem 4.17. *Let h be an outer function such that $h, h^{-1} \in H^2$. Then $\mathbb{P}_-\bar{h}/h \in AC_\delta$ if and only if $|h|^2 \in AC_\delta$.*

Proof. The theorem follows immediately from Lemma 3.1 and the obvious observation that the conditions $|h|^2 \in AC_\delta$ and $h^{-1} \in H^2$ imply that $\inf\limits_{\tau\in\mathbb{T}} |h(\tau)| > 0$. $\blacksquare$

The space AC_δ does not satisfy (B3). Indeed, if $f \in AC_\delta$, f has no zeros on $\mathbb{T}$ but the analytic extension of f in $\{\zeta \in \mathbb{C} : \ \delta < |\zeta| < \delta^{-1}\}$ has a zero, then $\inf_{\tau \in \mathbb{T}} |f(\tau)| > 0$ but f is not invertible in AC_δ.

Moreover, $\mathcal{A}(AC_\delta) \not\subset AC_\delta$. Indeed, let $h = z - 2/(1+\delta)$ and $u = \bar{z}\bar{h}/h$. Then

$$\mathbb{P}_- z^2 u = \mathbb{P}_- z(\bar{z} - 2/(1+\delta)) \cdot (z - 2/(1+\delta))^{-1} = 0.$$

Hence, $\mathbb{P}_- u \in AC_\delta$. On the other hand,

$$\mathbb{P}_+ z^2 u = \left(1 - \frac{2z}{1+\delta}\right)\left(z - \frac{2}{1+\delta}\right)^{-1} \notin AC_\delta,$$

since $1 < 2/(1+\delta) < 1/\delta$. Consequently, $\mathcal{A}\mathbb{P}_- u = -\mathbb{P}_+ u \notin AC_\delta$.

The Space $C(\mathbb{T})$

The operator $\mathcal{A}$ of best approximation does not act on $C(\mathbb{T})$. This has been proved implicitly in §1.5 (see the remark after Corollary 1.5.7). Indeed, let α be a real continuous function on $\mathbb{T}$ such that its harmonic conjugate $\tilde{\alpha}$ is discontinuous. Let $u = \bar{z}e^{\mathrm{i}\tilde{\alpha}}$. Then $u \in H^\infty + C$ and $\operatorname{dist}_{L^\infty}(u, H^\infty) = 1$. Let $u = f + g$, where $f \in C(\mathbb{T})$ and $g \in H^\infty$. Clearly, $\operatorname{dist}_{L^\infty}(f, H^\infty) = 1$ and $\mathcal{A}f = -g \notin C(\mathbb{T})$.

It is also easy to show that $\mathcal{A}_m C(\mathbb{T}) \not\subset C(\mathbb{T})$ for any $m \in \mathbb{Z}_+$. To show this we can consider the function $u = \bar{z}^{m+1}e^{\mathrm{i}\tilde{\alpha}}$, observe that $\mathcal{A}_m u = \mathbb{O}$, and apply the same reasoning as in the case $m = 0$.

The Spaces $C^n(\mathbb{T})$, $n \geq 1$

We are going to prove here that the space $C^n(\mathbb{T})$ of n times continuously differentiable functions on $\mathbb{T}$ does not have the hereditary property. Consider the space

$$H^\infty + C^n(\mathbb{T}) \stackrel{\text{def}}{=} \{\varphi = f + g : \ f \in H^\infty, \ g \in C^n(\mathbb{T})\}.$$

We need the following fact.

Theorem 4.18. *The space $H^\infty + C^n(\mathbb{T})$ is an algebra with respect to pointwise multiplication on $\mathbb{T}$.*

Proof. Clearly, it is sufficient to show that the product of an H^∞ function and a function in $C^n(\mathbb{T})$ belongs to $H^\infty + C^n(\mathbb{T})$. This is equivalent to the following assertion. Let $f \in H^\infty$, $g \in C^n(\mathbb{T})$, then $\bar{f}g \in \overline{H^\infty} + C^n(\mathbb{T})$. Put

$$VMOA^{(n)} = \{\varphi \in VMOA : \ \varphi^{(n)} \in VMOA\},$$
$$BMOA^{(n)} = \{\varphi \in BMOA : \ \varphi^{(n)} \in BMOA\}$$

Obviously, $VMOA^{(n)} = \mathbb{P}_+ C^n(\mathbb{T})$. It is easy to see that $\bar{f}g \in \overline{H^\infty} + C^n(\mathbb{T})$ if and only if

$$\mathbb{P}_+ \bar{f}g = \mathbb{P}_+(\bar{f}\mathbb{P}_+ g) \in VMOA^{(n)}.$$

Since $\mathbb{P}_+\bar{f}q$ is a polynomial for any polynomial q and the polynomials are dense in $VMOA^{(n)}$, it is sufficient to show that the Toeplitz operator $T_{\bar{f}}$ is a bounded operator from $VMOA^{(n)}$ to $BMOA^{(n)}$. Consider the space

$$I_{-n}H^1 = \{\psi^{(n)} : \psi \in H^1\}.$$

Clearly, the dual space $(I_{-n}H^1)^*$ can be identified with $BMOA^{(n)}$ with respect to the pairing (4.2). The space $I_{-n}H^1$ admits the following description:

$$\varphi \in I_{-n}H^1 \quad \Longleftrightarrow \quad \int_{\mathbb{T}} \left(\int_0^1 |\varphi(r\zeta)|^2 (1-r)^{2n-1} dr \right)^{1/2} d\boldsymbol{m}(\zeta) < \infty \tag{4.13}$$

(see Appendix 2.6). It follows from (4.13) that multiplication by $f \in H^\infty$ is a bounded operator on $I_{-n}H^1$.

Let ψ be an analytic polynomial in $I_{-n}H^1$. We have

$$\begin{aligned} |(\psi, \mathbb{P}_+(\bar{f}\mathbb{P}_+ g))| &= |(\psi, \bar{f}\mathbb{P}_+ g)| = |(f\psi, \mathbb{P}_+ g)| \\ &\le \operatorname{const} \|f\psi\|_{I_{-n}H^1} \|\mathbb{P}_+ g\|_{VMOA^{(n)}} \\ &\le \operatorname{const} \|f\|_{H^\infty} \|\psi\|_{I_{-n}H^1} \|g\|_{C^n(\mathbb{T})}, \end{aligned}$$

which proves that $\mathbb{P}_+(\bar{f}\mathbb{P}_+ g)$ determines a continuous linear functional on $I_{-n}H^1$, and so $\mathbb{P}_+(\bar{f}\mathbb{P}_+ g) \in BMOA^{(n)}$. ■

Theorem 4.19. *Let n be a positive integer. Then $\mathcal{A}C^n(\mathbb{T}) \not\subset C^n(\mathbb{T})$.*

Proof. Let α be a real function in $C^n(\mathbb{T})$ such that its harmonic conjugate $\tilde{\alpha}$ is not in $C^n(\mathbb{T})$. Put $u = \bar{z}e^{\mathrm{i}\tilde{\alpha}}$. We have

$$u = \bar{z}e^{\alpha + \mathrm{i}\tilde{\alpha}} e^{-\alpha}.$$

Obviously, $e^{\alpha+\mathrm{i}\tilde{\alpha}} \in H^\infty$, $e^{-\alpha} \in C^n(\mathbb{T})$, and so by Theorem 4.18, $u \in H^\infty + C^n(\mathbb{T})$ but $u \notin C^n(\mathbb{T})$. Clearly, $\operatorname{dist}_{L^\infty}(u, H^\infty) = 1$. Let $u = f + g$, where $f \in C^n(\mathbb{T})$ and $g \in H^\infty$. Then $\operatorname{dist}_{L^\infty}(f, H^\infty) = 1$ and $\mathcal{A}f = -g$. However, $g \notin C^n(\mathbb{T})$, since $u \notin C^n(\mathbb{T})$. ■

As in the case of the space $C(\mathbb{T})$ one can also show that $\mathcal{A}_m C^n(\mathbb{T}) \not\subset C^n(\mathbb{T})$ for any $m \in \mathbb{Z}_+$.

The Space $C_A + \overline{C_A}$

The space $C_A + \overline{C_A}$ does not have the hereditary property, i.e., $\mathcal{A}(C_A + \overline{C_A}) \not\subset C_A + \overline{C_A}$. Clearly, to show this, it is sufficient to prove the following fact.

Theorem 4.20. *There exists a function φ in C_A such that the function $\mathcal{A}\bar{\varphi}$ is discontinuous.*

Proof. Let $\frac{1}{3} < \gamma_1 \le \frac{1}{2}$ and $\gamma_2 > \frac{1}{2}$. Consider a function ψ on $\mathbb{T}$ that satisfies

$$\psi(e^{it}) = \begin{cases} -\gamma_1 \log|\log t|, & 0 < t \le \frac{1}{2}, \\ -\gamma_2 \log|\log|t||, & -\frac{1}{2} \le t < 0, \end{cases}$$

and has continuous derivative in $\mathbb{T} \setminus \{1\}$.

Define the function $\varphi \in H^\infty$ by

$$\varphi = c e^{\psi + \mathrm{i}\tilde{\psi}},$$

where $c > 0$. It is easy to see that the function $\tilde{\psi}$ is continuous on $\mathbb{T} \setminus \{1\}$. We also have

$$\lim_{\zeta \in \mathbb{T},\ \zeta \to 1} |\varphi(\zeta)| = \lim_{\zeta \in \mathbb{T},\ \zeta \to 1} e^{\psi(\zeta)} = 0.$$

Therefore φ is a continuous function on $\mathbb{T}$, and so $\varphi \in C_A$.

It is easy to see that

$$\int_0^{1/2} \frac{|\varphi(e^{\mathrm{i}t})|^2}{t} = \infty \tag{4.14}$$

while

$$\int_{-1/2}^{0} \frac{|\varphi(e^{\mathrm{i}t})|^2}{|t|} < \infty. \tag{4.15}$$

Denote by ξ the harmonic extension of $|\varphi|^2$ to the unit disk and by $\tilde{\xi}$ its harmonic conjugate, which is a harmonic function in $\mathbb{D}$. It follows from (4.14) and (4.15) that

$$\lim_{r \to 1} \tilde{\xi}(r) = \infty \tag{4.16}$$

(see Zygmund [1], Ch. III, Theorem 7.20).

Let $f = \mathcal{A}\bar{\varphi}$. We claim that $f \notin C(\mathbb{T})$. Suppose that $f \in C(\mathbb{T})$. Choosing the right c in the definition of φ, we may assume that $\operatorname{dist}_{L^\infty}(\bar{f}, H^\infty) = 1$. Hence, $\bar{\varphi} - f$ is a continuous unimodular function on $\mathbb{T}$.

Since f is continuous, there exists $\delta > 0$ such that

$$|f(\zeta)| \ge \frac{1}{2}, \quad \zeta \in \Omega_\delta \stackrel{\text{def}}{=} \{\zeta \in \mathbb{D} : \ |1 - \zeta| < \delta\}. \tag{4.17}$$

Let $f = \vartheta h$, where $h = \exp(\log|f| + \mathrm{i}\widetilde{\log|f|})$ is an outer function and ϑ is an inner function. It follows from (4.17) that ϑ extends analytically across the arc $I_\delta \stackrel{\text{def}}{=} \{\zeta \in \mathbb{T} : \ |1 - \zeta| < \delta\}$ and has an analytic logarithm in Ω_δ. Then f has an analytic logarithm in Ω_δ given by

$$\log f(\zeta) = \int_{\mathbb{T}} \log|f(\tau)| \frac{\tau + \zeta}{\tau - \zeta} d\boldsymbol{m}(\tau) + \log\vartheta(\zeta). \tag{4.18}$$

Clearly, $\log\vartheta$ is bounded in Ω_δ.

Since $|\bar{\varphi} - f| = 1$ on $\mathbb{T}$, it follows that

$$|f|^2 = 1 + 2\operatorname{Re}(\varphi \cdot f) - |\varphi|^2. \tag{4.19}$$

We have for $\zeta \in I_\delta$

$$\begin{aligned}
\log|f(\zeta)| &= \frac{1}{2}\log|f(\zeta)|^2 \\
&= \frac{1}{2}\left(2\operatorname{Re}(\varphi(\zeta)f(\zeta)) - |\varphi(\zeta)|^2 - \frac{(2\operatorname{Re}(\varphi(\zeta)f(\zeta)) - |\varphi(\zeta)|^2)^2}{2}\right) \\
&+ \chi_1(\zeta) \\
&= \operatorname{Re}(\varphi(\zeta)f(\zeta)) - \frac{1}{2}|\varphi(\zeta)|^2 - (\operatorname{Re}(\varphi(\zeta)f(\zeta)))^2 + \chi_2(\zeta) \\
&= \operatorname{Re}(\varphi(\zeta)f(\zeta)) - \frac{1}{2}|\varphi(\zeta)|^2 - \frac{1}{2}|\varphi(\zeta)f(\zeta)|^2 - \frac{1}{2}\operatorname{Re}(\varphi(\zeta)f(\zeta))^2 \\
&+ \chi_2(\zeta),
\end{aligned}$$

where χ_1 and χ_2 are functions on I_δ such that

$$|\chi_j(\zeta)| \le \operatorname{const}|\varphi(\zeta)|^3, \quad j = 1, 2, \quad \zeta \in I_\delta.$$

In view of (4.19)

$$\begin{aligned}
\frac{1}{2}|\varphi(\zeta)f(\zeta)|^2 &= \frac{1}{2}|\varphi(\zeta)|^2\left(1 + 2\operatorname{Re}(\varphi(\zeta)\cdot f(\zeta)) - |\varphi(\zeta)|^2\right) \\
&= \frac{1}{2}|\varphi(\zeta)|^2 + \chi_3(\zeta),
\end{aligned}$$

and so

$$\log|f(\zeta)| = \operatorname{Re}(\varphi(\zeta)f(\zeta)) - \frac{1}{2}\operatorname{Re}(\varphi(\zeta)f(\zeta))^2 - |\varphi(\zeta)|^2 + \chi_4(\zeta), \tag{4.20}$$

where χ_3 and χ_4 are functions on I_δ such that

$$|\chi_j(\zeta)| \le \operatorname{const}|\varphi(\zeta)|^3, \quad j = 3, 4, \quad \zeta \in I_\delta.$$

We can extend the above equalities from I_δ to the entire circle $\mathbb{T}$ and consider them as the definitions of the functions χ_j, $1 \le j \le 4$, on $\mathbb{T} \setminus I_\delta$. It is easy to see that the χ_j are integrable on $\mathbb{T}$ and continuous on I_δ.

Since $\gamma_1 > \frac{1}{3}$ and $\gamma_2 > \frac{1}{3}$, it follows that

$$\int_{\mathbb{T}} \frac{|\chi_4(\tau)|}{|1-\tau|} d\boldsymbol{m}(\tau) < \infty,$$

which implies

$$\left|\int_{\mathbb{T}} \chi_4(\tau)\frac{\tau + r}{\tau - r} d\boldsymbol{m}(\tau)\right| \le \operatorname{const}, \quad 0 < r < 1, \tag{4.21}$$

(see Zygmund [1], Ch. III, Theorem 7.20).

Since both φ and f belong to C_A, we have for $\zeta \in \mathbb{D}$

$$\int_{\mathbb{T}} \operatorname{Re}(\varphi(\tau)f(\tau))\frac{\tau + \zeta}{\tau - \zeta} d\boldsymbol{m}(\tau) = \varphi(\zeta)f(\zeta) - \operatorname{Im}(\varphi(0)f(0)), \tag{4.22}$$

$$\int_{\mathbb{T}} \operatorname{Re}(\varphi(\tau)f(\tau))^2\frac{\tau + \zeta}{\tau - \zeta} d\boldsymbol{m}(\tau) = (\varphi(\zeta)f(\zeta))^2 - \operatorname{Im}(\varphi(0)f(0))^2. \tag{4.23}$$

Recall that

$$\int_{\mathbb{T}} |\varphi(\tau)|^2 \frac{\tau+\zeta}{\tau-\zeta} \boldsymbol{dm}(\tau) = \xi(\zeta) + \mathrm{i}\tilde{\xi}(\zeta), \quad \zeta \in \mathbb{D}. \tag{4.24}$$

It follows now from (4.18) and (4.20) that for $\zeta \in \Omega_\delta$

$$\begin{aligned} \log f(\zeta) &= \int_{\mathbb{T}} \operatorname{Re}(\varphi(\tau)f(\tau)) \frac{\tau+\zeta}{\tau-\zeta} \boldsymbol{dm}(\tau) \\ &- \frac{1}{2}\int_{\mathbb{T}} \operatorname{Re}(\varphi(\tau)f(\tau))^2 \frac{\tau+\zeta}{\tau-\zeta} \boldsymbol{dm}(\tau) \\ &- \int_{\mathbb{T}} |\varphi(\tau)|^2 \frac{\tau+\zeta}{\tau-\zeta} \boldsymbol{dm}(\tau) \\ &+ \int_{\mathbb{T}} \chi_4(\tau) \frac{\tau+\zeta}{\tau-\zeta} \boldsymbol{dm}(\tau) + \log\vartheta(\zeta). \end{aligned}$$

It is now easy to see from (4.22), (4.23), and (4.24) that for $\zeta \in \Omega_\delta$ we have

$$\begin{aligned} \log f(\zeta) &= \varphi(\zeta)f(\zeta) - \frac{1}{2}(\varphi(\zeta)f(\zeta))^2 + \xi(\zeta) + i\tilde{\xi}(\zeta) \\ &+ \int_{\mathbb{T}} \chi_4(\tau) \frac{\tau+\zeta}{\tau-\zeta} \boldsymbol{dm}(\tau) + \log\vartheta(\zeta) + \mathrm{const}. \end{aligned}$$

It follows now from (4.16) and (4.21) that

$$\lim_{r\to 1} \operatorname{Im} \log f(r) = \infty,$$

which contradicts our assumption that $f \in C_A$. ■

Note, however, that for $n \geq 1$ the condition $\varphi^{(n)} \in C_A$ implies $(\mathcal{A}\bar{\varphi})^{(n)} \in C_A$ (see Corollary 4.8).

5. Badly Approximable Functions

In this section we find a necessary and sufficient condition for a function $f \in BMOA$ to be the best approximant to a function $\varphi \in VMO$. Clearly, f is the best approximant if and only if the error function $u = \varphi - f$ satisfies the condition $\operatorname{dist}_{L^\infty}(u, H^\infty) = \|u\|_{L^\infty}$. Such functions u are called *badly approximable*. We characterize here the badly approximable functions in VMO. We also obtain similar results for best approximation by meromorphic functions.

Recall that by Theorem 1.6, the function $f = \mathcal{A}\varphi$ belongs to VMO, and so $u \in VMO$. By Corollary 1.1.6, u has constant modulus and unless $\varphi \in H^\infty$, u must be invertible in $H^\infty + C$. In §3.3 we have defined the winding number $\operatorname{wind} u$ for such functions u and proved that $\operatorname{wind} u = -\operatorname{ind} T_u$.

Theorem 5.1. *Let u be a nonzero function such that $\mathbb{P}_-u \in VMO$. Then u is badly approximable if and only if u has constant modulus, $u \in QC$, and* $\operatorname{wind} u < 0$.

Proof. Suppose that u is badly approximable. Then $\mathcal{A}\mathbb{P}_-u = -\mathbb{P}_+u$ and by Theorem 1.6, $\mathbb{P}_+u \in VMO$, and so $u \in QC$. By Corollary 1.1.6, u has the form

$$u = c\bar{z}\bar{\vartheta}\bar{h}/h,$$

where $c \in \mathbb{C}$, ϑ is an inner function, and h is an outer function in H^2. Thus u has constant modulus. Clearly, $h \in \operatorname{Ker} T_u$. By Theorem 3.1.4, $\operatorname{Ker} T_u^* = \{\mathbb{O}\}$, and so $\operatorname{ind} T_u > 0$, which means that $\operatorname{wind} u < 0$.

Suppose now that u has constant modulus and $\operatorname{wind} u < 0$. Without loss of generality we may assume that $|u(\zeta)| = 1$ for almost all $\zeta \in \mathbb{T}$. Since u is in VMO, the Toeplitz operator T_u is Fredholm and

$$\operatorname{ind} T_u = \dim \operatorname{Ker} T_u = -\operatorname{wind} u > 0.$$

So T_u is not left invertible, and by Theorem 3.1.11, $\operatorname{dist}_{L^\infty}(u, H^\infty) = 1$, which precisely means that u is badly approximable. ■

Theorem 5.1 can be reformulated as follows.

Theorem 5.2. *Let φ be a function in VMO such that $\mathbb{P}_-\varphi \neq \mathbb{O}$ and let f be a function in $VMOA$. Then $f = \mathcal{A}\varphi$ if and only if the function $\varphi - f$ has constant modulus on $\mathbb{T}$ and* $\operatorname{wind}(\varphi - f) < 0$.

We are going to obtain an analog of Theorem 5.2 for best approximation by functions in $BMOA_{(m)}$. Suppose that $\varphi \in VMO$ and the singular value $s_m(H_\varphi)$ of H_φ has multiplicity μ. Then there exists $k \in \mathbb{Z}_+$ such that

$$s_k(H_\varphi) = \cdots = s_{k+\mu-1}(H_\varphi) > s_{k+\mu}(H_\varphi), \quad k \le m \le k + \mu - 1. \tag{5.1}$$

Consider the error function $u = \varphi - A_m\varphi$. It belongs to VMO and $-\operatorname{wind} u = 2k + \mu$ (see Theorem 4.1.7). Let us prove the converse. We show that if f is a function in

$$VMOA_{(m)} \stackrel{\text{def}}{=} BMOA_{(m)} \cap VMO$$

and $-\operatorname{wind}(\varphi - f)$ is sufficiently large, then $f = \mathcal{A}_m\varphi$.

Theorem 5.3. *Let $m \in \mathbb{Z}_+$, $\varphi \in VMO$, and $f \in VMOA_{(m)}$. Suppose that $\varphi - f$ has nonzero constant modulus on $\mathbb{T}$ and*

$$N \stackrel{\text{def}}{=} -\operatorname{wind}(\varphi - f) > 2m.$$

Then $f = \mathcal{A}_m\varphi$ and the singular value $s_m(H_\varphi)$ has multiplicity at least $N - 2m$.

Proof. Suppose that $g \in BMOA_{(m)}$ and

$$\|\varphi - g\|_\infty < \|\varphi - f\|_\infty. \tag{5.2}$$

Put $u = \varphi - f$. Without loss of generality we may assume that u is a unimodular function. We have

$$\varphi - g = u + f - g.$$

By Theorem 5.1, u is a badly approximable function. Consequently, $\mathbb{P}_-(f-g) \neq \mathbb{O}$. Clearly, $f - g \in BMOA_{(2m)}$. Since $\dim \operatorname{Ker} T_u = N > 2m$, it follows from Lemma 4.1.8 that $\|H_{u+f-g}\| > 1$. Hence, $\|u+f-g\|_\infty > 1$, which contradicts (5.2).

Suppose now that the singular value $s_m(H_\varphi)$ of H_φ has multiplicity μ and (5.1) holds. By Theorem 4.1.7, $N = 2k+\mu$. Therefore $\mu = N-2k \geq N-2m$, which completes the proof. ■

Theorem 5.3 implies the following criterion for a function f to be the best approximation of φ by functions in $BMOA_{(k)}$.

Theorem 5.4. *Let $k \in \mathbb{Z}_+$ and $\varphi \in VMO \setminus BMOA_{(k)}$. Suppose that $f \in VMOA_{(k)}$. Then $f = \mathcal{A}_k\varphi$ if and only if*

$$|\varphi - f| = \text{const} \ \ \textit{on } \mathbb{T} \quad \textit{and} \quad N \stackrel{\text{def}}{=} -\operatorname{wind}(\varphi - f) > 2k. \tag{5.3}$$

If (5.3) *holds, then $s_k(H_\varphi)$ is a singular value of H_φ of multiplicity $N-2k$.*

Theorem 5.4 follows immediately from Theorems 5.3 and 4.1.7.

Now we are going obtain analogs of the above results for a considerably bigger class of functions.

Theorem 5.5. *Let u be a function in L^∞ such that $\|H_u\|_{\mathrm{e}} < \|H_u\|$. Then u is badly approximable if and only if φ has constant modulus, the Toeplitz operator T_u is Fredholm, and* $\operatorname{ind} T_u > 0$.

Proof. Suppose that φ is badly approximable. Since $\|H_u\|_{\mathrm{e}} < \|H_u\|$, the Hankel operator H_u attains its norm on the unit ball of H^2 and by Theorem 1.1.6, u has constant modulus and has the form

$$u = c\bar{z}\vartheta\frac{\bar{h}}{h},$$

where $c \in \mathbb{C}$, ϑ is an inner function, and h is an outer function in H^2. Clearly, we may assume that $|c| = 1$. Hence, u is a unimodular function. By Theorem 4.4.10, T_u has dense range in H^2. Consider the subspace

$$E = \{f \in H^2 : \ \|H_u f\|_2 = \|f\|_2\} = \operatorname{Ker} T_u.$$

Since $\|H_u\|_{\mathrm{e}} < \|H_u\| = 1$, it follows that $\dim E < \infty$. By Corollary 4.4.7, $\|H_u\|_{\mathrm{e}} = \|H_{\bar{u}}\|_{\mathrm{e}}$, and so by Corollary 3.1.16, T_u is Fredholm. Clearly, $h \in \operatorname{Ker} T_u$, which implies that $\operatorname{ind} T_u > 0$.

Suppose now that u has constant modulus, T_u is Fredholm, and $\operatorname{ind} T_u > 0$. Without loss of generality we may assume that u is unimodular. Since $\operatorname{ind} T_u > 0$, it follows that $\operatorname{Ker} T_u \neq \{\mathbb{O}\}$. Let $f \in \operatorname{Ker} T_u$. It is easy to see that $\|H_u f\|_2 = \|f\|_2$, and so $\|H_u\| = 1$. Hence, u is badly approximable. ■

Similarly, one can state analogs of Theorem 5.3 for functions φ satisfying $\|H_\varphi\|_{\mathrm{e}} < s_m(H_\varphi)$. We state an analog of Theorem 5.4.

Theorem 5.6. *Let $k \in \mathbb{Z}_+$ and φ is a function in BMO such that $\|H_\varphi\|_{\mathrm{e}} < s_k(H_\varphi)$. Suppose that $f \in BMOA_{(k)}$. Then f is the best approximation of φ by functions in $BMOA_{(k)}$ if and only if*

$$|\varphi - f| = \text{const}, \quad T_{\varphi-f} \quad \textit{is Fredholm, and} \quad N \stackrel{\text{def}}{=} \operatorname{ind} T_{\varphi-f} > 2k. \tag{5.4}$$

If (5.3) holds, then $s_k(H_\varphi)$ is a singular value of H_φ of multiplicity $N-2k$.

Proof. By Theorem 4.1.3, φ has a unique best approximation by functions in $BMOA_{(k)}$. Suppose that $\deg \mathbb{P}_- f = k$ and f is the unique best approximation of φ. Let μ be the multiplicity of the singular value $s_k(H_\varphi)$ of the Hankel operator H_φ. We have

$$s_k(H_\varphi) = \cdots = s_{k+\mu-1}(H_\varphi) > s_{k+\mu}(H_\varphi). \tag{5.5}$$

By Theorem 4.1.7, $|\varphi - f| = s_k(H_\varphi)$ almost everywhere on $\mathbb{T}$ and $\dim \operatorname{Ker} T_{\varphi-f} = 2k + \mu$. Let us show that $T_{\varphi-f}$ is Fredholm. Let $u = s_k^{-1}(H_\varphi)(\varphi - f)$. Then u is unimodular, $\operatorname{Ker} T_u \neq \{\mathbb{O}\}$, and so $\operatorname{Ker} T_u$ has dense range. Since the space $E = \operatorname{Ker} T_u$ is finite-dimensional, it follows from Corollary 4.4.7 that $\|H_u\|_{\mathrm{e}} = \|H_{\bar{u}}\|_{\mathrm{e}}$, and by Corollary 3.1.16, T_u is Fredholm, and so $T_{\varphi-f}$ is Fredholm.

Suppose now that (5.4) holds. Put $u = \varphi - f$. Without loss of generality we may assume that u is unimodular. Suppose that $g \in BMOA_{(k)}$ and $\|\varphi - g\|_\infty < 1$. We have $\varphi - g = u + (f - g)$ and $f - g \in BMO_{(2k)}$. This contradicts Lemma 4.1.8.

Let us prove now that $s_k(H_\varphi)$ has multiplicity $N - 2k$. Let μ be the multiplicity of $s_k(H_\varphi)$. Clearly, (5.5) holds. It follows from Theorem 4.1.7 that $\operatorname{Ker} T_{\varphi-f} = 2k + \mu$, which proves that $\mu = N - 2k$. ■

6. Perturbations of Multiple Singular Values of Hankel Operators

In this section we use the results of §5 to study the behavior of multiple singular values of Hankel operators under perturbation of their symbols by rational functions of degree 1 and under multiplication of their symbols by z and $\bar{z}$.

Let $s > 0$ and let φ be a function in VMO such that s is a singular value of the Hankel operator H_φ of multiplicity μ and suppose that

$$s = s_k(H_\varphi) = s_{k+1}(H_\varphi) = \cdots = s_{k+\mu-1}(H_\varphi) > s_{k+\mu}(H_\varphi). \tag{6.1}$$

Recall that for a function φ satisfying (6.1), $\mathcal{A}_m\varphi = \mathcal{A}_k\varphi$ for $k \le m \le k+\mu-1$ and $r \stackrel{\text{def}}{=} \mathbb{P}_-\mathcal{A}_k\varphi \in \mathcal{R}_k$, i.e., r is a rational function of degree k (see §4.1). Moreover, r is the only function in $\mathcal{R}_{k+\mu-1}$ for which $\|H_{\varphi-r}\| = s$.

Consider first the generic case of perturbation of φ by rational fractions with one pole.

Theorem 6.1. *Let $\varphi \in VMO$. Suppose that s is a singular value of H_φ of multiplicity $\mu \ge 2$ and* (6.1) *holds. Let $\lambda \in \mathbb{D}$, $\zeta \in \mathbb{C}$ be such that $\zeta \ne 0$ and λ is not a pole of $\mathbb{P}_-\mathcal{A}_k\varphi$. If $\psi = \varphi + \zeta/(z-\lambda)$, then*

$$s_k(H_\psi) > s > s_{k+\mu-1}(H_\psi).$$

Moreover, s is a singular value of H_ψ if and only $\mu \ge 3$, in which case its multiplicity is $\mu - 2$ and

$$s = s_{k+1}(H_\psi) = \cdots = s_{k+\mu-2}(H_\psi). \tag{6.2}$$

Proof. Let $r = \mathbb{P}_-\mathcal{A}_k\varphi$. Let us show that $s_k(H_\psi) > s$. Since H_ψ is a rank one perturbation of H_φ, it follows that $s_k(H_\psi) \ge s_{k+1}(H_\varphi) = s$. Suppose that $s_k(H_\psi) = s$. Let r_1 be the function in $\mathcal{R}_k$ such that $\|H_{\psi-r_1}\| = s$. Then $s_k(H_\varphi) = \|H_{\varphi-r_2}\|$, where $r_2 = r_1 + \zeta/(z-\lambda) \in \mathcal{R}_{k+1} \subset \mathcal{R}_{k+\mu-1}$. As we have noticed before the statement of Theorem 6.1, this implies that $r_2 = r$.

Let us show that this is impossible. By the assumption, λ is not a pole of ρ. Therefore if $r_2 = r$, then λ is not a pole of r_2. Hence, $k > 0$ and $r_1 = r_3 - \zeta/(z-\lambda)$, where $r_3 \in \mathcal{R}_{k-1}$, and so $s_k(H_\varphi) = \|H_{\varphi-r_3}\|$, which contradicts the fact that $s_{k-1}(H_\varphi) > s_k(H_\varphi)$.

Let us show that $s_{k+\mu-1}(H_\psi) < s$. Since H_ψ is a rank one perturbation of H_φ, it follows that $s_{k+\mu-1}(H_\psi) \le s_{k+\mu-2}(H_\varphi) = s$. Suppose that $s_{k+\mu-1}(H_\varphi) = s$. Then

$$s_k(H_\psi) > s = s_{k+1}(H_\psi) = \cdots = s_{k+\mu-1}(H_\psi).$$

Let ν be the multiplicity of the singular value s of H_ψ. Then $\nu \ge \mu - 1$. By the Adamyan–Arov–Krein theorem there exists a unique rational function ρ in $\mathcal{R}_{k+\mu-1}$ such that $\|H_{\psi-\rho}\| = s$. Let

$$\rho_1 \stackrel{\text{def}}{=} r + \zeta/(z-\lambda) \in \mathcal{R}_{k+1} \subset \mathcal{R}_{k+\mu-1}.$$

Clearly, $\|H_{\psi-\rho_1}\| = s$. Hence, $\rho = \rho_1$ and $\psi - \mathcal{A}_{k+1}\psi = \varphi - \mathcal{A}_k\varphi$. By Corollary 1.7, $\varphi - \mathcal{A}_k\varphi \in QC$. It follows from Theorem 4.1.7 that $\varphi - \mathcal{A}_k\varphi$ has constant modulus and $\operatorname{wind}(\varphi - \mathcal{A}_k\varphi) = -(2k+\mu)$. On the other hand,

$$\operatorname{wind}(\psi - \mathcal{A}_{k+1}\psi) = -(2(k+1)+\nu).$$

However, $2(k+1)+\nu > 2k+\mu$, which proves that the assumption $s_{k+\mu-1}(H_\varphi) = s$ is false.

If $\mu = 2$, it is obvious that s is not a singular value of H_ψ. If $\mu \ge 3$, then (6.2) is an immediate consequence of the fact that H_ψ is a rank one perturbation of H_φ, which completes the proof. ■

Corollary 6.2. *Suppose that s is a singular value of H_φ of multiplicity μ and* (6.1) *holds. Let $l \ge (\mu-1)/2$ and let Λ be a set of l points in $\mathbb{D}$ none of which is a pole of $\mathbb{P}_-\mathcal{A}_k\varphi$. Then there exist arbitrarily small numbers ζ_λ, $\lambda \in \Lambda$, such that the Hankel operator H_ψ,*

$$\psi = \varphi + \sum_{\lambda\in\Lambda} \frac{\zeta_\lambda}{z-\lambda},$$

satisfies

$$s_k(H_\psi) > s_{k+1}(H_\psi) > \cdots > s_{k+\mu-1}(H_\psi) > s_{k+\mu}(H_\psi).$$

The result follows easily from Theorem 6.1.

Consider now the case when λ is a pole of $\mathbb{P}_-\mathcal{A}_k\varphi$.

Theorem 6.3. *Let $\varphi \in VMO$, let s be a singular value of H_φ of multiplicity $\mu \ge 2$, and suppose that* (6.1) *holds. If λ is a pole of $\mathbb{P}_-\mathcal{A}_k\varphi$, $\zeta \in \mathbb{C}$, $\psi = \varphi + \zeta/(z-\zeta)$, then either s is a singular value of H_ψ of multiplicity μ and*

$$s = s_k(H_\psi) = s_{k+1}(H_\psi) = \cdots = s_{k+\mu-1}(H_\psi) > s_{k+\mu}(H_\psi) \tag{6.3}$$

or s is a singular value of H_ψ of multiplicity $\mu+2$ and

$$s = s_{k-1}(H_\psi) = s_k(H_\psi) = \cdots = s_{k+\mu}(H_\psi) > s_{k+\mu+1}(H_\psi). \tag{6.4}$$

Proof. Let $r = \mathbb{P}_-\mathcal{A}_k\varphi \in \mathcal{R}_k$. It is clear that under the hypotheses of the theorem $r + \zeta/(z-\lambda) \in \mathcal{R}_k$. Since H_ψ is a rank one perturbation of H_φ, it follows that $s_k(H_\psi) \ge s$. On the other hand, $\|H_{\psi-(r+\zeta/(z-\lambda))}\| = s$, and so $s_k(H_\psi) = s$. Hence, $\mathcal{A}_k\psi = \mathcal{A}_k\varphi + \zeta/(z-\lambda)$. Therefore $\psi - \mathcal{A}_k\psi = \varphi - \mathcal{A}_k\varphi$, $\psi - \mathcal{A}_k\psi$ has constant modulus $\psi - \mathcal{A}_k\psi \in QC$, and $\operatorname{wind}\psi - \mathcal{A}_k\psi = -(2k+\mu)$.

Consider first the case when $r + \zeta/(z-\lambda) \in \mathcal{R}_{k-1}$. Then the same reasoning as in the proof of Theorem 6.1 shows that (6.4) holds.

Suppose now that $r + \zeta/(z-\lambda) \notin \mathcal{R}_{k-1}$. Then again the same reasoning shows that (6.3) holds. ■

We proceed now to the study of the behavior of multiple singular values of a Hankel operator under multiplication of its symbol by z or $\bar{z}$. We consider a more general problem when z is replaced by a Blaschke factor b_λ, $\lambda \in \mathbb{D}$,

$$b_\lambda(z) = \frac{z-\lambda}{1-\bar{\lambda}z}.$$

Theorem 6.4. *Let $\varphi \in VMO$. Suppose that s is a singular value of H_φ of multiplicity $\mu \ge 2$ and* (6.1) *holds. Let $\lambda \in \mathbb{D}$ and $\psi = \bar{b}_\lambda\varphi$. Then the following assertions hold:*

(1) *if* $(\mathbb{P}_-\mathcal{A}_k\varphi)(\lambda) = 0$*, then* s *is a singular value of* H_ψ *of multiplicity* $\mu + 1$ *and*

$$s = s_k(H_\psi) = s_{k+1}(H_\psi) = \cdots = s_{k+\mu}(H_\psi) > s_{k+\mu+1}(H_\psi); \tag{6.5}$$

(2) *if* $(\mathbb{P}_-\mathcal{A}_k\varphi)(\lambda) \neq 0$*, then* s *is a singular value of* H_ψ *of multiplicity* $\mu - 1$ *and*

$$s = s_{k+1}(H_\psi) = s_{k+2}(H_\psi) = \cdots = s_{k+\mu-1}(H_\psi) > s_{k+\mu}(H_\psi). \tag{6.6}$$

Proof. Consider the function $\bar{b}_\lambda\mathcal{A}_k\varphi$. The function $\psi - \bar{b}_\lambda\mathcal{A}_k\varphi$ has constant modulus on $\mathbb{T}$, belongs to QC, and

$$\operatorname{wind}\bar{b}_\lambda(\varphi - \mathcal{A}_k\varphi) = -(2k + \mu + 1).$$

If $(\mathbb{P}_-\mathcal{A}_k\varphi)(\lambda) = 0$, then $\mathbb{P}_-\bar{b}_\lambda\mathcal{A}_k\varphi$ is a rational function of degree k. It follows from Theorem 5.4 that $\mathcal{A}_k\psi = \bar{b}_\lambda\mathcal{A}_k\varphi$ and (6.5) holds. If $(\mathbb{P}_-\mathcal{A}_k\varphi)(\lambda) \neq 0$, then $\mathbb{P}_-\bar{b}_\lambda\mathcal{A}_k\varphi$ is a rational function of degree $k + 1$. Again, it follows from Theorem 5.4 that $\mathcal{A}_{k+1}\psi = \bar{b}_\lambda\mathcal{A}_k\varphi$ and (6.6) holds. ■

Theorem 6.5. *Let* $\varphi \in VMO$*. Suppose that* s *is a singular value of* H_φ *of multiplicity* $\mu \geq 2$ *and* (6.1) *holds. Let* $\lambda \in \mathbb{D}$ *and* $\psi = b_\lambda\varphi$*. Then the following assertions hold:*

(1) *if* λ *is a pole of* $\mathbb{P}_-\mathcal{A}_k\varphi$*, then* s *is a singular value of* H_ψ *of multiplicity* $\mu + 1$ *and*

$$s = s_{k-1}(H_\psi) = s_k(H_\psi) = \cdots = s_{k+\mu-1}(H_\psi) > s_{k+\mu}(H_\psi); \tag{6.7}$$

(2) *if* λ *is not a pole of* $\mathbb{P}_-\mathcal{A}_k\varphi$*, then* s *is a singular value of* H_ψ *of multiplicity* $\mu - 1$ *and*

$$s = s_k(H_\psi) = s_{k+2}(H_\psi) = \cdots = s_{k+\mu-2}(H_\psi) > s_{k+\mu-1}(H_\psi). \tag{6.8}$$

Proof. Consider the function $b_\lambda\mathcal{A}_k\varphi$. The function $\psi - b_\lambda\mathcal{A}_k\varphi$ has constant modulus on $\mathbb{T}$, belongs to QC, and $\operatorname{wind} b_\lambda(\varphi - \mathcal{A}_k\varphi) = -(2k + \mu - 1)$. If λ is a pole of $\mathbb{P}_-\mathcal{A}_k\varphi$, then $\mathbb{P}_-b_\lambda\mathcal{A}_k\varphi$ is a rational function of degree $k - 1$. It follows from Theorem 5.4 that $\mathcal{A}_{k-1}\psi = b_\lambda\mathcal{A}_k\varphi$ and (6.7) holds. If λ is not a pole of $\mathbb{P}_-\mathcal{A}_k\varphi$, then $\mathbb{P}_-b_\lambda\mathcal{A}_k\varphi$ is a rational function of degree k. Again, it follows from Theorem 5.4 that $\mathcal{A}_k\psi = b_\lambda\mathcal{A}_k\varphi$ and (6.8) holds. ■

7. The Boundedness Problem

We have seen in §1 that the operator $\mathcal{A}$ of best approximation by analytic functions (as well as the operators $\mathcal{A}_m$) is bounded on quasi-Banach $\mathcal{R}$-spaces. In particular, $\mathcal{A}$ is bounded on VMO and the Besov spaces $B_p^{1/p}$,

$0 < p < \infty$. In this section we see that those spaces are rather exceptional. We show that the operator $\mathcal{A}$ is unbounded on the Hölder–Zygmund classes Λ_s, $0 < s < \infty$, Besov spaces B_p^s, $0 < p < \infty$, $s > 1/p$, and the space $\mathcal{F}\ell^1$ of functions with absolutely continuous Fourier series. We also obtain similar results for the operators $\mathcal{A}_m$ of best approximation by meromorphic functions.

Hölder and Besov Spaces

Theorem 7.1. *Let $0 < p \le \infty$ and $s > 1/p$. Then there exists a sequence of functions $\{\varphi_n\}$ in B_p^s such that*

$$\|\varphi_n\|_{B_p^s} \le \text{const},$$

but

$$\lim_{n\to\infty} \|\mathcal{A}\varphi_n\|_{B_p^s} = \infty.$$

Recall that $B_\infty^s = \Lambda_s$.

For $\rho \in (0,1)$ consider the conformal mapping ω_ρ from the unit disk $\mathbb{D}$ onto the disk $\{\zeta : \ |1-\zeta| < 1\}$ defined by

$$\omega_\rho(\zeta) = 1 + \frac{\zeta - \rho}{1 - \rho\zeta}.$$

Define the functions ξ_ρ and η_ρ on $\mathbb{T}$ by

$$\xi_\rho(\zeta) = \frac{\omega_\rho(\zeta) - \bar{\zeta}^2}{|\omega_\rho(\zeta) - \bar{\zeta}^2|}(|\omega_\rho(\zeta) - \bar{\zeta}^2| - 1);$$

$$\eta_\rho(\zeta) = \bar{\zeta}^2 + \xi_\rho(\zeta).$$

Note that for ρ sufficiently close to 1 the function $\omega_\rho(z) - \bar{z}^2$ is separated away from 0 since $\omega_\rho(\zeta)$ is close to 0 when ζ lies outside a small neighborhood of 1.

Lemma 7.2. *For ρ sufficiently close to* 1 *the following equality holds:*

$$\mathcal{A}\eta_\rho = \omega_\rho.$$

Proof. We assume that ρ is sufficiently close to 1, and so the function $\omega_\rho(z) - \bar{z}^2$ is separated away from 0 on $\mathbb{T}$. By Theorem 5.1, it is sufficient to show that $\eta_\rho - \omega_\rho$ has constant modulus on $\mathbb{T}$ and negative winding number. We have

$$\begin{aligned} \eta_\rho(\zeta) - \omega_\rho(\zeta) &= \bar{\zeta}^2 + \frac{\omega_\rho(\zeta) - \bar{\zeta}^2}{|\omega_\rho(\zeta) - \bar{\zeta}^2|}(|\omega_\rho(\zeta) - \bar{\zeta}^2| - 1) - \omega_\rho(\zeta) \\ &= -\frac{\omega_\rho(\zeta) - \bar{\zeta}^2}{|\omega_\rho(\zeta) - \bar{\zeta}^2|}. \end{aligned}$$

So $\eta_\rho - \omega_\rho$ is unimodular. Let us show that its winding number is -1. Clearly,

$$\begin{aligned}\operatorname{wind}(\eta_\rho - \omega_\rho) &= \operatorname{wind}(\omega_\rho(z) - \bar{z}^2) = \operatorname{wind}(\bar{z}^2(z^2\omega_\rho(z) - 1)) \\ &= -2 + \operatorname{wind}(z^2\omega_\rho(z) - 1) = -2 + \operatorname{wind}_1(z^2\omega_\rho(z)),\end{aligned}$$

where wind_1 means winding number with respect to 1.

Let us now show that $\operatorname{wind}_1(z^2\omega_\rho(z)) = 1$. Suppose that ρ is sufficiently close to 1. Let θ be a positive number such that $\cos\theta = \rho$ and $\tau = e^{i\theta}$. It is easy to check that $\omega_\rho(\tau) = 1 - \bar{\tau}$. When ζ is moving along $\mathbb{T}$ counterclockwise from 1 to τ, $\omega_\rho(\zeta)$ is moving counterclockwise along the circle $\{\zeta : |1-\zeta| = 1\}$ from 2 to $1-\bar{\tau}$ while ζ^2 lies in a small neighborhood of 1. Therefore $\zeta^2\omega_\rho(\zeta)$ is moving along a curve close to the arc $[2, 1-\bar{\tau}]$ of the circle $\{\zeta : |1-\zeta| = 1\}$. Next, when ζ is moving from τ to $\bar{\tau}$ along $\mathbb{T}$, $\omega_\rho(\zeta)$ is moving within a small neighborhood of 0 from $1-\bar{\tau}$ to $1-\tau$ while ζ^2 has modulus 1. Therefore $\zeta^2\omega_\rho(\zeta)$ is moving within a small neighborhood of 0. Finally, when ζ is moving along $\mathbb{T}$ from $\bar{\tau}$ to 1, $\omega_\rho(\zeta)$ is moving along the circle $\{\zeta : |1-\zeta| = 1\}$ from $1-\tau$ to 2 while ζ^2 lies in a small neighborhood of 1. Therefore $\zeta^2\omega_\rho(\zeta)$ is moving along a curve close to the arc $[1-\tau, 2]$ of the circle $\{\zeta : |1-\zeta| = 1\}$.

This shows that $\operatorname{wind}_1(z^2\omega_\rho(z)) = 1$. ■

To prove Theorem 7.1, it is sufficient to show that $\|\omega_\rho\|_{B_p^s} \to \infty$ as $\rho \to 1$ and

$$\lim_{\rho\to 1} \frac{\|\omega_\rho\|_{B_p^s}}{\|\xi_\rho\|_{B_p^s}} = \infty.$$

Lemma 7.3. *There exist positive constants c_1 and c_2 such that*

$$c_1(1-\rho)^{1/p-s} \le \|\omega_\rho\|_{B_p^s} \le c_2(1-\rho)^{1/p-s}.$$

Proof. We use the following (quasi)norm on the space $(B_p^s)_+$ of functions analytic in $\mathbb{D}$:

$$\|f\|_{B_p^s} = \|f\|_{L^\infty} + \left(\int_{\mathbb{D}} (1-|\zeta|)^{(n-s)p-1}|f^{(n)}(\zeta)|^p d\boldsymbol{m}_2(\zeta)\right)^{1/p}, \quad p < \infty,$$

and

$$\|f\|_{B_\infty^s} = \|f\|_{L^\infty} + \sup_{\zeta\in\mathbb{D}}(1-|\zeta|)^{n-s}|f^{(n)}(\zeta)|,$$

where n is an integer greater than s (see Appendix 2.6).

It is easy to see that

$$\omega_\rho^{(n)}(\zeta) = n!\frac{\rho^{n-1}(1-\rho^2)}{(1-\rho\zeta)^{n+1}}, \quad n > 0. \tag{7.1}$$

Suppose that ρ is sufficiently close to 1 so that $\rho^{n-1} \ge \frac{1}{2}$.

Consider first the case $p = \infty$. It is easy to see that

$$\sup_{\zeta\in\mathbb{D}}(1-|\zeta|)^{n-s}|\omega_\rho^{(n)}(\zeta)| = \sup_{r\in(0,1)}(1-r)^{n-s}|\omega_\rho^{(n)}(r)|.$$

Put $r = \rho$. We have

$$\sup_{\zeta\in\mathbb{D}}(1-|\zeta|)^{n-s}|\omega_\rho^{(n)}(\zeta)| \ge \text{const}\,\frac{(1-\rho^2)(1-\rho)^{n-s}}{(1-\rho^2)^{n+1}} \ge \text{const}(1-\rho)^{-s}.$$

To obtain an upper estimate, we assume that $\rho^{m+1} \le r \le \rho^m$, $m \in \mathbb{Z}_+$. We have

$$\begin{aligned}\frac{(1-r)^{n-s}(1-\rho^2)}{(1-\rho r)^{n+1}} &\le \frac{(1-\rho^{m+1})^{n-s}(1-\rho^2)}{(1-\rho^{m+1})^{n+1}}\\ &= \frac{1-\rho^2}{(1-\rho^{m+1})^{1+s}} \le \text{const}(1-\rho)^{-s}.\end{aligned}$$

Suppose now that $0 < p < \infty$. We have

$$\|\omega_\rho\|_{B_p^s}^p \ge \int_\Omega (1-|\zeta|)^{(n-s)p-1}|\omega_\rho^{(n)}(\zeta)|^p d\boldsymbol{m}_2(\zeta),$$

where $\Omega = \{\zeta = re^{\mathrm{i}\theta} \in \mathbb{D} :\ \rho < r < 1,\ 0 \le \theta \le 1-\rho\}$. We have

$$\begin{aligned}\|\omega_\rho\|_{B_p^s}^p &\ge \text{const}\int_\Omega (1-|\zeta|)^{(n-s)p-1}\left(\frac{1-\rho}{|1-\rho\zeta|^{n+1}}\right)^p d\boldsymbol{m}_2(\zeta)\\ &\ge \text{const}(1-\rho)^{-np}\int_\Omega (1-|\zeta|)^{(n-s)p-1}d\boldsymbol{m}_2(\zeta)\\ &\ge \text{const}(1-\rho)^{-np}(1-\rho)\int_0^{1-\rho} r^{(n-s)p-1}dr = \text{const}(1-\rho)^{1-sp}.\end{aligned}$$

Let us now obtain an upper estimate for $\|\omega_\rho\|_{B_p^s}$. It is sufficient to estimate

$$N_\rho \stackrel{\text{def}}{=} \left(\int_{\mathbb{D}} (1-|\zeta|)^{(n-s)p-1}|\omega_\rho^{(n)}(\zeta)|^p d\boldsymbol{m}_2(\zeta)\right)^{1/p}.$$

We have

$$\begin{aligned}N_\rho^p &\le \text{const}(1-\rho)^p\int_{\mathbb{D}} (1-|\zeta|)^{(n-s)p-1}\left(\frac{1}{|1-\rho\zeta|^{n+1}}\right)^p d\boldsymbol{m}_2(\zeta)\\ &\le \text{const}(1-\rho)^p\int_0^1 (1-r)^{(n-s)p-1}\left(\int_{\mathbb{T}}\frac{1}{|1-\rho r\zeta|^{(n+1)p}}d\boldsymbol{m}(\zeta)\right)dr.\end{aligned}$$

Let us estimate

$$\int_{\mathbb{T}}\frac{1}{|1-a\zeta|^q}d\boldsymbol{m}(\zeta)$$

for $0 < a < 1$ and $q > 1$. Consider the substitution $w = (a-\zeta)(1-a\zeta)^{-1}$. Then $\zeta = (a-w)(1-aw)^{-1}$. It is easy to see that

$$\int_{\mathbb{T}}\frac{1}{|1-a\zeta|^q}d\boldsymbol{m}(\zeta) = \frac{1}{(1-a^2)^{q-1}}\int_{\mathbb{T}}|1-aw|^{q-2}d\boldsymbol{m}(\zeta) \le \frac{\text{const}}{(1-a)^{q-1}}.$$

Therefore

$$\begin{aligned} N_\rho^p &\le \text{const}\cdot(1-\rho)^p\int_0^1 \frac{(1-r)^{(n-s)p-1}}{(1-\rho r)^{(n+1)p-1}}dr \\ &\le \text{const}\cdot(1-\rho)^p\int_t^1 \frac{(1-r)^{(n-s)p-1}}{(1-\rho r)^{(n+1)p-1}}dr \\ &+ \text{const}\cdot(1-\rho)^p\int_0^t \frac{(1-r)^{(n-s)p-1}}{(1-\rho r)^{(n+1)p-1}}dr. \end{aligned}$$

We have

$$\begin{aligned} &(1-\rho)^p\int_t^1 \frac{(1-r)^{(n-s)p-1}}{(1-\rho r)^{(n+1)p-1}}dr \\ &\qquad\le \text{const}\,\frac{(1-\rho)^p}{(1-\rho)^{(n+1)p-1}}\int_t^1 (1-r)^{(n-s)p-1}dr \\ &\qquad\le \text{const}\,\frac{(1-\rho)^p}{(1-\rho)^{(n+1)p-1}}(1-\rho)^{(n-s)p} \\ &\qquad= \text{const}(1-\rho)^{1-sp}. \end{aligned}$$

On the other hand,

$$\begin{aligned} (1-\rho)^p\int_0^t \frac{(1-r)^{(n-s)p-1}}{(1-\rho r)^{(n+1)p-1}}dr &\le \text{const}\cdot(1-\rho)^p\int_0^t (1-r)^{-(1+s)p}dr \\ &\le \text{const}\cdot(1-\rho)^p(1-\rho)^{-(1+s)p+1} \\ &= \text{const}\cdot(1-\rho)^{1-sp}. \end{aligned}$$

This yields the desired upper estimate for $\|\omega_\rho\|_{B_p^s}$. ∎

Lemma 7.4. *There exist positive constants* c_1 *and* c_2 *such that*

$$c_1(1-\rho)^{-n} \le \|\omega_\rho^{(n)}\|_{L^\infty} \le c_2(1-\rho)^{-n}, \quad \rho\in(0,1), \quad n\in\mathbb{Z}_+.$$

The result follows immediately from (7.1).

Lemma 7.5. *For* ρ *sufficiently close to* 1 *there exists a positive constant* c *such that*

$$\|\xi_\rho\|_{L^\infty} \le c(1-\rho)^{1/2}.$$

Proof. As in the proof of Lemma 7.2, $\tau\in\mathbb{T}$, $\operatorname{Im}\tau>0$, and $\operatorname{Re}\tau=\rho$. Suppose that ζ belongs to the arc $[\bar\tau,\tau]$ of $\mathbb{T}$. Then it is easy to see that

$$\big||\omega_\rho(\zeta)-\bar\zeta^2|-1\big| \le |\tau^2-1| \le \text{const}(1-\rho)^{1/2}.$$

Suppose now that ζ does not belong to the arc $[\bar\tau,\tau]$ of $\mathbb{T}$. Then $\omega_\rho(\zeta)$ lies on the arc $[1-\bar\tau,1-\tau]$ of the circle $\{\zeta:\ |1-\zeta|=1\}$. It is easy to see that

$$\big||\omega_\rho(\zeta)-\bar\zeta^2|-1\big| \le |\tau-1| \le \text{const}(1-\rho)^{1/2}. \quad ∎$$

Clearly, Theorem 7.1 is an immediate consequence of the following result.

Lemma 7.6. *There exists a number $\gamma > 1/p - s$ such that for ρ sufficiently close to* 1

$$\|\xi_\rho\|_{B_p^s} \le \operatorname{const} \cdot (1-\rho)^\gamma.$$

Proof. We represent ξ_ρ as $\xi_\rho = f_\rho(g_\rho - 1)$, where

$$f_\rho(\zeta) \stackrel{\text{def}}{=} \frac{\omega_\rho(\zeta) - \bar\zeta^2}{|\omega_\rho(\zeta) - \bar\zeta^2|}, \quad g_\rho(\zeta) = |\omega_\rho(\zeta) - \bar\zeta^2|.$$

Since for ρ sufficiently close to 1 the function $\omega_\rho(z) - \bar z^2$ is separated away from zero, it follows that

$$\|f_\rho\|_{B_p^s} \le \operatorname{const} \cdot \|\omega_\rho\|_{B_p^s},$$

where the constant does not depend on ρ. This follows from the description of B_p^s in terms of pseudoanalytic continuation (see Appendix 2.6).

We are going to use the following formula, which can easily be verified by induction:

$$(\Delta_\tau^n \varphi\psi)(\zeta) = \sum_{k=0}^n \binom{n}{k} (\Delta_\tau^{n-k}\varphi)(\tau^k\zeta)(\Delta_\tau^k\psi)(\zeta). \tag{7.2}$$

Here φ and ψ are functions on $\mathbb{T}$, and Δ_τ, $\tau \in \mathbb{R}$, is the difference operator,

$$(\Delta_\tau\varphi)(\zeta) = \varphi(\tau\zeta) - \varphi(\zeta), \quad \zeta \in \mathbb{T},$$

(see Appendix 2.6).

Suppose first that $p = \infty$. Consider the following seminorm on B_∞^s:

$$\|\varphi\|_{s,n} = \sup_{\tau \ne 1} \frac{\|(\Delta_\tau^n\varphi)\|_{L^\infty}}{|1-\tau|^s},$$

where n is an integer greater than s. Since, obviously, $\|\xi_\rho\|_{L^\infty} \le \operatorname{const}$, it is sufficient to estimate $\|\xi_\rho\|_{s,n}$.

Let us first estimate $\|g_\rho\|_{s,n}$. Let δ be a positive number that will be specified later. Suppose that $|1-\tau| \ge (1-\rho)^\delta$. We have

$$\frac{|(\Delta_\tau^n g_\rho)(\zeta)|}{|1-\tau|^s} \le \operatorname{const} \cdot \|\xi_\rho\|_{L^\infty}(1-\rho)^{-\delta s} \le \operatorname{const} \cdot (1-\rho)^{1/2-\delta s}$$

by Lemma 7.5. Suppose now that $|1-\tau| < (1-\rho)^\delta$. Then

$$\begin{aligned} \frac{|(\Delta_\tau^n g_\rho)(\zeta)|}{|1-\tau|^s} &\le \operatorname{const} \cdot \|g_\rho^{(n)}\|_{L^\infty}|1-\tau|^{n-s} \le \operatorname{const} \cdot \|\omega_\rho^{(n)}\|_{L^\infty}|1-\tau|^{n-s} \\ &\le \operatorname{const} \cdot (1-\rho)^{-n}(1-\rho)^{(n-s)\delta} = \operatorname{const} \cdot (1-\rho)^{\delta n - \delta s - n} \end{aligned}$$

by Lemma 7.4.

Choose now δ such that $\frac{1}{2} - \delta s = \delta n - \delta d - n$. Thus $\delta = 1 + \frac{1}{2n}$. Then

$$\|g_\rho\|_{s,n} \le \operatorname{const} \cdot (1-\rho)^{-s+1/2-s/2n}. \tag{7.3}$$

Note that $\frac{1}{2} - \frac{s}{2n} > 0$.

Let us now estimate $\|\xi_\rho\|_{s,n}$. We apply (7.2) with $\varphi = f_\rho$ and $\psi = g_\rho - 1$. Let $0 < k < n$ and let $s = s_1 + s_2$, where $0 < s_1 < n-k$, $0 < s_2 < k$. We have

$$\begin{aligned}\frac{|(\Delta_\tau^{n-k} f_\rho)(\tau^k\zeta)(\Delta_\tau^k(g_\rho - 1))(\zeta)|}{|1-\tau|^s} &\le \operatorname{const}\cdot\|f_\rho\|_{s_1,n-k}\|g_\rho\|_{s_2,k}\\ &\le \operatorname{const}\cdot(1-\rho)^{-s_1}(1-\rho)^{-s_2+1/2-s_2/2k}\\ &\le \operatorname{const}\cdot(1-\rho)^{-s+1/2-s_2/2k}\end{aligned}$$

by (7.3) and Lemma 7.3. Next, if $k = 0$, then

$$\frac{|(\Delta_\tau^n f_\rho)(\zeta)(g_\rho - 1)(\zeta)|}{|1-\tau|^s} \le \operatorname{const}\cdot\|\omega_\rho\|_{\Lambda_s}\|\xi_\rho\|_{L^\infty} \le \operatorname{const}\cdot(1-\rho)^{-s+1/2}$$

by Lemmas 7.5 and 7.3. Finally, if $k = n$, then

$$\frac{|f_\rho(\tau^n\zeta)(\Delta_t^n(g_\rho - 1))(\zeta)|}{|1-\tau|^s} \le \operatorname{const}\cdot(1-\rho)^{-s+1/2-s/2n}$$

by (7.3), which completes the proof of the lemma for $p = \infty$.

Suppose now that $0 < p < \infty$. Consider the following seminorm (quasi-seminorm if $p < 1$) on B_p^s:

$$\|\varphi\|_{p,s,n} = \left(\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|(\Delta_\tau^n\varphi)(\zeta)|^p}{|1-\tau|^{1+sp}} d\boldsymbol{m}(\zeta) d\boldsymbol{m}(\tau)\right)^{1/p}.$$

Here n is an integer greater than s. As in the case $p = \infty$ it is sufficient to estimate $\|\xi_\rho\|_{p,s,n}$.

Consider first the case $2 < p < \infty$. Let us estimate $\|g_\rho\|_{p,s,n}$. Let δ be a positive number whose choice will be specified later. Let

$$J_1 = \{e^{\mathrm{i}t} : \ t \in [-\pi,\pi], \ |t| \ge (1-\rho)^\delta\}, \quad J_2 = \{e^{\mathrm{i}t} : \ |t| < (1-\rho)^\delta\}.$$

We have

$$\begin{aligned}\int_{J_1}\int_{\mathbb{T}} \frac{|(\Delta_\tau^n\gamma_\rho)(\zeta)|^p}{|1-\tau|^{1+sp}} d\boldsymbol{m}(\zeta) d\boldsymbol{m}(\tau) &\le \operatorname{const}\cdot(1-\rho)^{p/2}\int_{(1-\rho)^\delta}^{\pi} t^{-1-sp}dt\\ &\le \operatorname{const}\cdot(1-\rho)^{p(1/2-\delta s)}\end{aligned}$$

by Lemma 7.5. Next,

$$\begin{aligned}\int_{J_2}\int_{\mathbb{T}} \frac{|(\Delta_\tau^n\gamma_\rho)(\zeta)|^p}{|1-\tau|^{1+sp}} d\boldsymbol{m}(\zeta) d\boldsymbol{m}(\tau) &\le \operatorname{const}\cdot\|\omega_\rho^{(n)}\|_{L^\infty}\int_0^{(1-\rho)^\delta} t^{np-sp-1}dt\\ &\le \operatorname{const}\cdot(1-\rho)^{-np}(1-\rho)^{\delta(np-sp)}\\ &\le \operatorname{const}\cdot(1-\rho)^{p(\delta n-\delta s-n)}.\end{aligned}$$

Let us now choose δ such that $p(1/2-\delta s) = p(\delta n - \delta s - n)$, i.e., $\delta = 1 + \frac{1}{2n}$. Then

$$\|g_\rho\|_{p,s,n} \le \operatorname{const}\cdot(1-\rho)^{1/2-s-s/2n}. \tag{7.4}$$

We can pick now $n > \frac{s}{1-2/p}$. Then $\frac{1}{2} - s - \frac{s}{2n} > \frac{1}{p} - s$.

Again, we use formula (7.2). Let $0 < k < n$ and let $s = s_1 + s_2$, where $0 < s_1 < (n-k)(1-2/p)$, $0 < s_2 < k(1-2/p)$. We have

$$\iint\limits_{\mathbb{T}\times\mathbb{T}} \frac{|(\Delta_\tau^{n-k} f_\rho)(\tau^k\zeta)(\Delta_\tau^k(g_\rho - 1))(\zeta)|^p}{|t|^{1+sp}} d\boldsymbol{m}(\zeta) d\boldsymbol{m}(\tau) \le$$

$$\left(\iint\limits_{\mathbb{T}\times\mathbb{T}} \frac{|(\Delta_\tau^{n-k} f_\rho)(\tau^k\zeta)|^{2p} d\boldsymbol{m}(\zeta) d\boldsymbol{m}(\tau)}{|1-\tau|^{1+2s_1p}} \iint\limits_{\mathbb{T}\times\mathbb{T}} \frac{|(\Delta_\tau^k g_\rho)(\zeta)|^{2p} d\boldsymbol{m}(\zeta) d\boldsymbol{m}(\tau)}{|1-\tau|^{1+2s_2p}}\right)^{1/2}$$

$$\le \|f_\rho\|_{2p,s_1 n-k}^p \|g_\rho\|_{2p,s_2,k}^p \le \operatorname{const}(1-\rho)^{1/2-ps_1}(1-\rho)^{\mu p} \le \operatorname{const}(1-\rho)^{\gamma p},$$

by Lemma 7.3 and (7.4), where $\mu > \frac{1}{2p} - s_2$ and $\gamma = \frac{1}{2p} - s_1 + \mu > 1/p - s$.

Let now $k = 0$. Then

$$\left(\iint\limits_{\mathbb{T}\times\mathbb{T}} \frac{|(\Delta_\tau^n f_\rho)(\zeta)((g_\rho - 1))(\zeta)|^p}{|1-\tau|^{1+sp}} d\boldsymbol{m}(\zeta) d\boldsymbol{m}(\tau)\right)^{1/p} \le \|f_\rho\|_{p,s,n} \|\xi_\rho\|_{L^\infty}$$

$$\le \operatorname{const} \cdot (1-\rho)^{\frac{1}{p}-s}(1-\rho)^{\frac{1}{2}} = \operatorname{const} \cdot (1-\rho)^{1/2+1/p-s},$$

by Lemmas 7.5 and 7.3.

Finally, let $k = n$. We have

$$\left(\iint\limits_{\mathbb{T}\times\mathbb{T}} \frac{|(f_\rho)(\tau^n\zeta)(\Delta_\tau^n(g_\rho - 1))(\zeta)|^p}{|1-\tau|^{1+sp}} d\boldsymbol{m}(\zeta) d\boldsymbol{m}(\tau)\right)^{1/p}$$

$$\le \operatorname{const} \cdot (1-\rho)^{1/2-s-s/2n}$$

by (7.4), which completes the proof in the case $p > 2$.

Consider now the case $p \le 2$. First of all, let us observe that if $n > s$ and ρ is sufficiently close to 1, then

$$\|g_\rho\|_{p,s,n} \le \operatorname{const} \cdot (1-\rho)^{1/p-s}. \tag{7.5}$$

Indeed, by Lemma 7.3, $\|\omega_\rho - \bar{z}^2\|_{B_p^s} \le \operatorname{const} \cdot (1-\rho)^{1/p-s}$. Since B_p^s is an algebra (see Appendix 2.6), it follows that $\|g_\rho^2\|_{B_p^s} \le \operatorname{const} \cdot (1-\rho)^{1/p-s}$. To obtain (7.5), it suffices to observe that for ρ sufficiently close to 1, g_ρ is uniformly separated away from 0 and apply to g_ρ^2 an analytic branch of the square root (see Appendix 2.6).

Let now q be a positive number such that $qp > 4$ and let $q' = q/(q-1)$. We have by Hölder's inequality

$$\|g_\rho\|^p_{p,s,n} = \iint\limits_{\mathbb{T}\times\mathbb{T}} \frac{|(\Delta^n_\tau g_\rho)(\zeta)|^{p/2}}{|1-\tau|^{sp/2}} \cdot \frac{|(\Delta^n_\tau g_\rho)(\zeta)|^{p/2}}{|1-\tau|^{sp/2}} \frac{d\boldsymbol{m}(\zeta)d\boldsymbol{m}(\tau)}{|1-\tau|}$$

$$\le \left(\iint\limits_{\mathbb{T}\times\mathbb{T}} \frac{|(\Delta^n_\tau g_\rho)(\zeta)|^{pq/2}}{|1-\tau|^{spq/2}} \frac{d\boldsymbol{m}(\zeta)d\boldsymbol{m}(\tau)}{|1-\tau|}\right)^{1/q}$$

$$\times \left(\iint\limits_{\mathbb{T}\times\mathbb{T}} \frac{|(\Delta^n_\tau g_\rho)(\zeta)|^{pq'/2}}{|1-\tau|^{spq'/2}} \frac{d\boldsymbol{m}(\zeta)d\boldsymbol{m}(\tau)}{|1-\tau|}\right)^{1/q'}$$

$$= \|g_\rho\|^{p/2}_{pq/2,s,n} \|g_\rho\|^{p/2}_{pq'/2,s,n} \le \mathrm{const}\cdot(1-\rho)^{\gamma p/2}(1-\rho)^{(2/pq'-s)p/2}$$

by (7.5), where $\gamma > \frac{2}{pq} - s$ (it is possible to find such a γ since $pq > 2$ and the lemma has already been proved for $p > 2$). It follows that

$$\|g_\rho\|_{p,s,n} \le \mathrm{const}\cdot(1-\rho)^{(\gamma+2/pq'-s)/2}.$$

Clearly,

$$\frac{1}{2}\left(\gamma + \frac{2}{pq'} - s\right) > \frac{1}{2}\left(\frac{2}{pq} - s + \frac{2}{pq'} - s\right) = 1/p - s.$$

Finally, to estimate $\|\xi_\rho\|_{p,s,n}$, we apply formula (7.2) and argue as in the case $p > 2$. ■

We can now obtain a similar result for the operators $\mathcal{A}_m$ of best approximation by meromorphic functions.

Theorem 7.7. *Let $0 < p \le \infty$ and let $s > 1/p$. For each $m \in \mathbb{Z}_+$ there exists a sequence $\{\varphi_n\}$ of functions in B^s_p such that*

$$\|\varphi_n\|_{B^s_p} \le \mathrm{const},$$

but

$$\lim_{n\to\infty} \|\mathcal{A}_m\varphi_n\|_{B^s_p} = \infty.$$

The proof of Theorem 7.7 is almost the same as that of Theorem 7.1. The only difference is that in the definition of ξ_ρ and η_ρ we have to replace $\bar\zeta^2$ with $\bar\zeta^{m+2}$. It can be shown in the same way as in Lemma 7.2 that

$$\mathrm{wind}(\eta_\rho - \omega_\rho) = -(m+1).$$

Then $s_j(H_{\eta_\rho}) = 1$, $0 \le j \le m$. Hence, $\mathcal{A}_m\eta_\rho = \mathcal{A}\eta_\rho = \omega_\rho$. The rest of the proof is exactly the same as for the case $m = 0$.

The Space $\mathcal{F}\ell^1$

Let us now proceed to the space $\mathcal{F}\ell^1$ of functions with absolutely converging Fourier series.

Theorem 7.8. *The operator $\mathcal{A}$ of best approximation by analytic functions is unbounded on the space $\mathcal{F}\ell^1$.*

Proof. Let K be a positive integer. We construct a function $\varphi \in \mathcal{F}\ell^1$ such that $\|\mathcal{A}\varphi\|_{\mathcal{F}\ell^1} > K\|\varphi\|_{\mathcal{F}\ell^1}$.

Let $r > 1$. Consider the function

$$u_r = \frac{\bar{z} - r}{z - r} = -\frac{\bar{z}}{r} + \left(1 - \frac{1}{r^2}\right)\sum_{j \ge 0} \frac{z^j}{r^j}.$$

Let $\varepsilon > 0$. For $d = 1, 2, \cdots, K$ we inductively find numbers $r_d > 1$, positive integers N_d and $M_d > N_d$, such that

$$|1 - 1/r_d| < \frac{\varepsilon}{4}, \quad \sum_{j=0}^{N_d} |\hat{u}_{r_d}(j)| < \frac{\varepsilon}{4}, \quad \text{and} \quad \|\mathbb{P}_+ u_{r_d}\|_{\mathcal{F}\ell^1} > 2 - \frac{\varepsilon}{4}, \tag{7.6}$$

and trigonometric polynomials q_d of the form

$$q_d(z) = -\bar{z} + \sum_{j=N_d+1}^{M_d} \hat{q}_d(j) z^j \tag{7.7}$$

such that

$$\hat{q}_d(j) \ge 0 \quad \text{for} \quad j \ge 0; \quad q_d(1) = 1, \quad \text{and} \quad \|u_{r_d} - q_d\|_{\mathcal{F}\ell^1} < \varepsilon, \tag{7.8}$$

as follows.

Put $N_1 = K + 1$. It is easy to see that we can choose $r_1 > 1$ so that (7.6) holds. Clearly, there exist a positive integer M_1 and a trigonometric polynomial q_1 of the form (7.7) that satisfies (7.8).

Put $g_1 = q_1$ and $N_2 = M_1$. We can now find $r_2 > 1$, a positive integer M_2, and a trigonometric polynomial q_2 of the form (7.7) that satisfies (7.8). Consider $g_2 \stackrel{\text{def}}{=} g_1 q_2 = q_1 q_2$. We have

$$g_2 = (-\bar{z} + \mathbb{P}_+ q_1)(-\bar{z} + \mathbb{P}_+ q_2) = \bar{z}^2 - \bar{z}\mathbb{P}_+ q_1 + q_1 \mathbb{P}_+ q_2.$$

Clearly, $\mathbb{P}_- g_2 = \bar{z}^2$ and $\hat{g}_2(j) = 0$ for $0 \le j \le K$. Let us show that $\|\mathbb{P}_+ g_2\|_{\mathcal{F}\ell^1} \ge 4$. Since the sets $\{j \in \mathbb{Z} : (\bar{z}\mathbb{P}_+ q_1)\hat{\ }(j) \ne 0\}$ and $\{j \in \mathbb{Z} : (q_1 \mathbb{P}_+ q_2)\hat{\ }(j) \ne 0\}$ are obviously disjoint, it follows that

$$\begin{aligned} \|\mathbb{P}_+ g_2\|_{\mathcal{F}\ell^1} &= \|\bar{z}\mathbb{P}_+ q_1\|_{\mathcal{F}\ell^1} + \|q_1 \mathbb{P}_+ q_2\|_{\mathcal{F}\ell^1} \\ &\ge \|\mathbb{P}_+ q_1\|_{\mathcal{F}\ell^1} + |q_1(1)(\mathbb{P}_+ q_2)(1)| = 4 \end{aligned}$$

(the last equality follows easily from (7.7) and (7.8)).

Suppose now that we have already found $r_1, \cdots, r_l$, $N_1, \cdots, N_l$, $M_1, \cdots, M_l$, trigonometric polynomials $q_1, \cdots, q_l$, $l < K$, such that (7.6)–(7.8) hold and the function $g_l \stackrel{\text{def}}{=} q_1 q_2 \cdots q_l$ has the following properties:

$$g_l(1) = 1, \quad \hat{g}_l(j) = 0 \quad \text{for} \quad 0 \le j \le K - l + 1 \quad \text{and} \quad \|\mathbb{P}_+ g_l\|_{\mathcal{F}\ell^1} \ge 2l.$$

Let us find q_{l+1}. Put

$$N_{l+1} = \max\{j \in \mathbb{Z} : \ \hat{g}_l(j) \ne 0\} + K.$$

Clearly, we can find $r_{l+1} > 1$, a positive integer M_{l+1}, and a trigonometric polynomial q_{l+1} such that (7.6)–(7.8) hold. Put $g_{l+1} = g_l q_{l+1}$. It is easy to see that $\mathbb{P}_- g_{l+1} = (-1)^{l+1}\bar{z}^{l+1}$, $g_{l+1}(1) = 1$, and $\hat{g}_{l+1}(j) = 0$ for $0 \le j \le K - l$. Then the same reasoning as above shows that

$$\begin{aligned} \|\mathbb{P}_+ g_{l+1}\|_{\mathcal{F}\ell^1} &= \|\bar{z}\mathbb{P}_+ g_l\|_{\mathcal{F}\ell^1} + \|g_l \mathbb{P}_+ q_{l+1}\|_{\mathcal{F}\ell^1} \\ &\ge \|\mathbb{P}_+ g_l\|_{\mathcal{F}\ell^1} + |g_l(1)(\mathbb{P}_+ q_{l+1})(1)| \ge 2l + 2 = 2(l+1). \end{aligned}$$

Proceeding in this way we can construct $q_1, \cdots, q_K$.

Consider the function $u = \bar{z} u_{r_1} \cdots u_{r_K}$. Clearly, u is a unimodular function and $\|H_u f\|_2 = \|f\|_2$, where

$$f(z) = \prod_{d=1}^{K} \frac{1}{z - r_d}.$$

Therefore $\|H_u\| = 1$, and so $\mathcal{A}\mathbb{P}_- u = -\mathbb{P}_+ u$. Now put $\varphi \stackrel{\text{def}}{=} \mathbb{P}_- u$.

Put now $v = \bar{z} q_1 \cdots q_K$. We have $\|q_d - u_{r_d}\|_{\mathcal{F}\ell^1} < \varepsilon$, $\|\mathbb{P}_- v\|_{\mathcal{F}\ell^1} = 1$, and $\|\mathbb{P}_+ v\|_{\mathcal{F}\ell^1} \ge 2K$. It is easy to see that if ε is sufficiently small, then

$$\|\mathcal{A}\varphi\|_{\mathcal{F}\ell^1} = \|\mathbb{P}_+ u\|_{\mathcal{F}\ell^1} > K\|\mathbb{P}_- u\|_{\mathcal{F}\ell^1} = K\|\varphi\|_{\mathcal{F}\ell^1}. \quad \blacksquare$$

A similar result holds for the operators $\mathcal{A}_m$.

Theorem 7.9. *Let $m \in \mathbb{Z}_+$. Then the operator $\mathcal{A}_m$ is unbounded on $\mathcal{F}\ell^1$.*

The proof is almost the same. The only difference is that we should define $u = \bar{z}^{m+1} u_{r_1} \cdots u_{r_K}$. Then $\mathcal{A}_m \mathbb{P}_u = -\mathbb{P}_+ u$ and it is easy to see that

$$\lim_{K \to \infty} \frac{\|\mathbb{P}_+ u\|_{\mathcal{F}\ell^1}}{\|\mathbb{P}_- u\|_{\mathcal{F}\ell^1}} = \infty.$$

Finally, we obtain an estimate for $\|\mathcal{A}\varphi\|_{\mathcal{F}\ell^1}$ for functions φ in $BMOA_{(m)}$.

Theorem 7.10. *Let $\varphi \in BMOA_{(m)}$. Then*

$$\|\mathcal{A}\varphi\|_{\mathcal{F}\ell^1} \le \text{const} \cdot m \|\varphi\|_{\mathcal{F}\ell^1}.$$

Proof. Clearly, $\mathcal{A}\varphi = \mathcal{A}\mathbb{P}_-\varphi + \mathbb{P}_+\varphi$. Hence, we may assume that $\mathbb{P}_+\varphi = \mathbb{O}$. Since $B_1^1 \subset \mathcal{F}\ell^1$ and the operator $\mathcal{A}$ is bounded on B_1^1, we have

$$\|\mathcal{A}\varphi\|_{\mathcal{F}\ell^1} \le \text{const} \cdot \|\mathcal{A}\varphi\|_{B_1^1} \le \text{const} \cdot \|\varphi\|_{B_1^1}.$$

Next, it follows from Corollary 6.1.2 that

$$\begin{aligned}\|\varphi\|_{B_1^1} &\le \operatorname{const}\cdot\|H_\varphi\|_{\boldsymbol{S}_1} \le \operatorname{const}\cdot\operatorname{rank}H_\varphi\cdot\|H_\varphi\| \\ &\le \operatorname{const}\cdot m\|\varphi\|_{L^\infty} \le \operatorname{const}\cdot m\|\varphi\|_{\mathcal{F}\ell^1},\end{aligned}$$

since $\varphi \in BMOA_{(m)}$. ■

8. Arguments of Unimodular Functions

We have frequently used Sarason's theorem (Theorem 3.3.10), which asserts that a unimodular function u is in VMO if and only if u admits a representation $u = z^n e^{\mathrm{i}\varphi}$, where $n \in \mathbb{Z}$ and φ is a real function in VMO. We will need analogs of this result for other function spaces. It is not difficult to obtain similar results for spaces of "smooth" functions (e.g., Λ_α). However, for spaces that contain unbounded functions (such as $B_p^{1/p}$, $1 < p < \infty$) such results are not trivial. To obtain analogs of Sarason's theorem for such spaces, we are going to use the results on best approximation obtained earlier in this chapter.

We consider a linear space X, $\boldsymbol{\mathcal{P}} \subset X \subset VMO$, of functions on $\mathbb{T}$ which satisfies the following axioms:

(C1) $f \in X \implies \bar{f} \in X$ *and* $\mathbb{P}_+ f \in X$*;*

(C2) $\mathcal{A}X \subset X$*;*

(C3) *if* $f \in X \bigcap L^\infty$ *and* φ *is analytic in a neighborhood of* $f(\mathbb{T})$*, then* $\varphi \circ f \in X$*.*

Remark. Note that (C3) implies that X is an algebra with respect to pointwise multiplication, since $4fg = (f+g)^2 - (f-g)^2$.

Theorem 8.1. *Suppose that* X *is a space of functions on* $\mathbb{T}$ *that satisfies the axioms* (C1)–(C3) *and let* u *be a unimodular function in* X*. Then* u *admits a representation* $u = z^n e^{\mathrm{i}\varphi}$*, where* $n \in \mathbb{Z}$ *and* φ *is a real function in* X*.*

Proof. Since $X \subset VMO$, the Toeplitz operator T_u is Fredholm. Multiplying u by z^n, we can reduce the situation to the case $\operatorname{ind} T_u = 0$, that is, T_u is invertible. In this case $\operatorname{dist}_{L^\infty}(u, H^\infty) < 1$ and $h = \mathcal{A}u$ is an outer function (see Theorem 3.1.13). By (C2), $h \in X$. Since $X \bigcap L^\infty$ is an algebra, it follows that $\bar{u}h \in X$. We have

$$\|\mathbf{1} - \bar{u}h\|_{L^\infty} = \|u - h\|_{L^\infty} < 1,$$

and so the essential range of the function $\bar{u}h$ is contained in the disk $\{\zeta \in \mathbb{C} : |1-\zeta| < \delta\}$ for some $\delta < 1$. Therefore we can represent the function $\bar{u}h$ in the form $\bar{u}h = |h|\exp(\mathrm{i}\psi)$, where ψ is a real-valued function such that $\|\psi\|_{L^\infty} < \pi/2$.

Clearly, $\inf\limits_{\tau\in\mathbb{T}} |h(\tau)| > 0$, and so by (C3), $\log|h| = \frac{1}{2}\log(h\bar{h}) \in X$. It follows from (C1) that $\widetilde{\log|h|} \in X$. Applying an analytic branch of logarithm to

$\bar{u}h$, we obtain

$$\log \bar{u}h = \log|h| + \mathrm{i}\psi \in X$$

(see (C3)). Therefore $\psi \in X$.

By Theorem 3.3.10, u can be represented as $u = \exp(\mathrm{i}\beta)$, where β is a real function in VMO. We have

$$\mathbf{1} = u\frac{|h|}{h}\exp(\mathrm{i}\psi) = \exp\left(\mathrm{i}(\beta - \widetilde{\log|h|} + \psi + c)\right)$$

for some $c \in \mathbb{R}$. It follows from Corollary 3.2.7 that

$$\beta - \widetilde{\log|h|} + \psi = \text{const},$$

and so $\beta \in X$. ■

It is easy to see that all spaces considered in §4 that are hereditary for $\mathcal{A}$ satisfy (C1)–(C3).

Remark. If X is a space of functions satisfying (C1)–(C3) and $X \subset L^\infty$, then it is easy to see that the converse to Theorem 8.1 holds: if $n \in \mathbb{Z}$ and $\varphi \in X$, then $z^n e^{\mathrm{i}\varphi} \in X$. However, this is also true for the Besov spaces $B_p^{1/p}$, $1 < p < \infty$, which contain unbounded functions.

Corollary 8.2. *Let u be a unimodular function and let $0 < p < \infty$. Then $u \in B_p^{1/p}$ if and only if u admits a representation $u = z^n e^{\mathrm{i}\varphi}$, where $n \in \mathbb{Z}$ and φ is a real-valued function in $B_p^{1/p}$.*

Proof. If $p \le 1$, then $B_p^{1/p} \subset L^\infty$, and the result follows from the Remark preceding the lemma. If $p > 1$, then it follows immediately from the definition of $B_p^{1/p}$ in terms of the difference operator Δ_τ (see Appendix 2.6) that $f \circ \varphi \in B_p^{1/p}$ whenever $\varphi \in B_p^{1/p}$ and f satisfies a Lipschitz condition

$$|f(\zeta_1) - f(\zeta_2)| \le \text{const}\,|\zeta_1 - \zeta_2|. \quad ■$$

9. Schmidt Functions of Hankel Operators

In this section we study properties of Schmidt functions of Hankel operators whose symbols belong to certain function spaces. In particular we prove that if X is a function space satisfying the axioms (A1)–(A4) or the axioms (B1)–(B3) and $\varphi \in X$, then all Schmidt functions of the Hankel operator H_φ belong to X.

Recall that for $\varphi \in VMO$ the Schmidt functions of the compact Hankel operator H_φ are eigenfunctions of the operator $H_\varphi^* H_\varphi$ that correspond to nonzero eigenvalues (see §4.1).

Consider first the case when X is a space of functions on $\mathbb{T}$, $\boldsymbol{\mathcal{P}} \subset X \subset VMO$, that satisfies the following axioms.

(C1) $\quad f \in X \quad \Longrightarrow \quad \bar{f} \in X \quad$ *and* $\quad \mathbb{P}_+ f \in X$;

(C2) $\quad \mathcal{A}X \subset X$;

(C3′) *if* $f \in X$ *and* φ *is analytic in a neighborhood of* $f(\mathbb{T})$*, then* $\varphi \circ f \in X$*.*

Recall that axioms (C1) and (C2) have been introduced in §7. The axiom C3′ is slightly stronger than (C3). In particular, it implies that X is an algebra with respect to pointwise multiplication. It is easy to see that if X satisfies the axioms (A1)–(A4) or the axioms (B1)–(B3), then X also satisfies (C1), (C2), and (C3′).

Lemma 9.1. *Suppose that* X *satisfies* (C1), (C2), *and* (C3′). *Then* $\varphi - \mathcal{A}\varphi$ *has the form* $cz^n\bar{h}/h$*, where* $c \in \mathbb{C}$*,* n *is a negative integer, and* h *is an outer function invertible in* X*.*

Proof. Suppose that $\mathcal{A}\varphi \neq \varphi$. Then $\varphi - \mathcal{A}\varphi = cu$, where c is a nonzero complex number and u is a unimodular function (see Theorem 1.1.4). By (C2), $u \in X$. It follows from Theorem 8.1 that u admits a representation $u = z^n e^{\mathrm{i}\xi}$, where ξ is a real function in X. Since $\mathcal{A}\varphi$ is the best approximation to φ, it follows that

$$\operatorname{dist}_{L^\infty}(u, H^\infty) = \|H_u\| = 1,$$

and so the Toeplitz operator T_u is not left invertible, which implies that $n < 0$ (see Theorem 3.1.11 and Corollary 3.2.6).

Put now $h \stackrel{\text{def}}{=} \exp\left(\frac{1}{2}(\tilde{\xi} - \mathrm{i}\xi)\right)$. Then h is an outer function. It follows from (C1) and (C3′) that $h \in X$, and so $h^{-1} \in X$. ■

Lemma 9.2. *Suppose that* X *satisfies* (C1), (C2), *and* (C3′). *Let* φ *be a function in* X *such that* $\mathbb{P}_-\varphi \neq \mathbb{O}$*, and let* f *be a maximizing vector for the Hankel operator* H_φ *on* H^2*, i.e.,* $\|H_\varphi f\|_2 = \|H_\varphi\| \cdot \|\varphi\|_2$*. Then* $f \in X$*.*

Proof. Put $v = \varphi - \mathcal{A}\varphi$. Then $H_v = H_\varphi$ and $\|v\|_\infty = \|H_\varphi\|$. By Lemma 9.1, $v = c\bar{z}^n\bar{h}/h$, where n is a positive integer, $c \in \mathbb{C}$, and h is an outer function invertible in X. Clearly,

$$\{g \in H^2 : \ \|H_v g\|_2 = \|H_v\| \cdot \|g\|_2\} = \operatorname{Ker} T_v.$$

If q is an analytic polynomial of degree at most $n-1$, then obviously $qh \in KerT_v$. On the other hand, $T_v = cT_{\bar{z}^n}T_{\bar{h}}T_{1/h}$, and so $\dim \operatorname{Ker} T_v = \dim \operatorname{Ker} T_{\bar{z}^n} = n$. It follows that

$$\operatorname{Ker} T_v = \{qh : \ q \in \boldsymbol{\mathcal{P}}_+, \ \deg q \leq n-1\} \subset X,$$

which proves the result. ■

Theorem 9.3. *Suppose that* X *satisfies* (C1), (C2), *and* (C3′). *Let* φ *be a function in* X *such that* $\mathbb{P}_-\varphi \neq \mathbb{O}$*. Then all Schmidt functions of* H_φ *belong to* X*.*

Proof. Suppose that $H_\varphi^* H_\varphi f = s^2 f$, where $f \in H^2$ and s is a nonzero singular value of H_φ. If $s = s_0(H_\varphi)$, it follows from Lemma 9.2 that $f \in X$. Assume that $s < s_0(H_\varphi)$.

Consider the space

$$E = \{\xi \in H^2 : \ H_\varphi^* H_\varphi \xi = s^2 \xi\}.$$

Let $d = \dim E$. We can now choose $m > 0$ so that $s = s_m(H_\varphi)$ and

$$s_{m-1}(H_\varphi) > s_m(H_\varphi) = \cdots = s_{m+d-1}(H_\varphi) > s_{m+d}(H_\varphi).$$

Put $g = \frac{1}{s}H_\varphi f$ and $\varphi_s = g/f$. Then φ_s is a unimodular function (see Corollary 4.1.5). Clearly, $H_{\varphi_s}f = g$, and so $\|H_{\varphi_s}f\|_2 = \|H_{\varphi_s}f\| \cdot \|f\|_2$, which means that f is a maximizing vector of $H_{\varphi_s}f$.

It has been proved in the proof of Theorem 4.1.1 that the Hankel operators H_φ and $H_{s\varphi_s}$ coincide on a subspace of H^2 of codimension m. Hence, by Kronecker's theorem $\mathbb{P}_-(\varphi-\varphi_s)$ is a rational function of degree m. Since the rational functions belong to X, it follows that $\mathbb{P}_-\varphi_s \in X$, and so by Lemma 9.2, $f \in X$. ■

The same can be proved about Schmidt functions of H_φ^*. Indeed, if g is an eigenfunction of $H_\varphi H_\varphi^*$ that corresponds to a nonzero eigenvalue s^2, then there exists a function $f \in H^2$ such that $H_\varphi^* H_\varphi f = s^2 f$ and $H_\varphi f = sg$. It follows from Theorem 9.3 that $f \in X$. This implies that $g \in X$.

To complete this section we describe Schmidt functions of compact Hankel operators as well as Schmidt functions of Hankel operators of class $\boldsymbol{S}_p$.

Theorem 9.4. *Let φ be a function in L^2 such the Hankel operator H_φ is compact (or $H_\varphi \in \boldsymbol{S}_p$, $0 < p < \infty$). Suppose that f is a Schmidt function of H_φ. Then f has the form $f = qe^\psi$, where q is an analytic polynomial and $\psi \in VMOA$ (or $\psi \in \left(B_p^{1/p}\right)_+$).*

Proof. Let $X = VMO$ if H_φ is compact or $X = B_p^{1/p}$ if $H_\varphi \in \boldsymbol{S}_p$. As in Theorem 9.3 it is sufficient to consider the case when f is a maximizing vector of H_φ.

Consider the function $v = \varphi - \mathcal{A}\varphi$. By Theorem 8.1, v has the form $v = c\bar{z}^n e^{\mathrm{i}\eta}$, where η is a real function in X. As in Theorem 9.3, $n > 0$. Clearly, $e^{\mathrm{i}\eta} = \bar{h}/h$, where $h = \exp(\frac{1}{2}(\tilde{\eta} - \mathrm{i}\eta))$. Then h is an outer function and $\psi \stackrel{\text{def}}{=} \frac{1}{2}(\tilde{\eta} - \mathrm{i}\eta) \in X$. We have shown in the proof of Theorem 9.3 that f has the form qh, where q is a polynomial of degree at most $n-1$. ■

Remark 1. The same can be proved about Schmidt functions of H_φ^*. This follows from the fact that the operator J defined in the proof of Lemma 4.1.6 by $Jg = \bar{z}\bar{g}$ maps the eigenspace of $H_\varphi^* H_\varphi$, which corresponds to a positive number s^2 onto the eigenspace of $H_\varphi H_\varphi^*$, which corresponds to the same eigenvalue.

Remark 2. Any function f of the form $f = qe^\psi$, where q is an analytic polynomial and $\psi \in VMOA$ (or $\psi \in \left(B_p^{1/p}\right)_+$),q is a Schmidt function of a compact Hankel operator (a Hankel operator of class $\boldsymbol{S}_p$). Indeed, let $n = \deg q + 1$, $h = e^\psi$ and $\varphi \stackrel{\text{def}}{=} \bar{z}^n \bar{h}/h = \bar{z}^n \exp(-2\mathrm{i}\,\mathrm{Im}\,\psi) \in X$. Then H_φ is compact (or $H_\varphi \in B_p^{1/p}$) and f is a maximizing vector of H_φ, since $H_\varphi f = \bar{z}^n \bar{h}$.

Corollary 9.5. *If H_φ is compact and f is a Schmidt function of H_φ. Then $f \in \bigcap_{p<\infty} H^p$.*

Proof. The result follows immediately from Theorem 9.3 and Corollary 3.2.6. ■

Note that it is not true that for compact Hankel operators (or Hankel operators of class $\boldsymbol{S}_p$ with $p > 1$) their Schmidt functions always belong to VMO (or to $B_p^{1/p}$). However, for $p \le 1$ this is true.

Corollary 9.6. *Suppose that H_φ is a Hankel operator of class $\boldsymbol{S}_p$, $0 < p \le 1$. Then the Schmidt functions of H_φ belong to $B_p^{1/p}$.*

The result follows from the fact that if $\psi \in B_p^{1/p}$, $0 < p \le 1$, then $e^\psi \in B_p^{1/p}$ (see Appendix 2.6).

10. Continuity in the sup-Norm

In this section we are going to study continuity properties of the operator $\mathcal{A}$ of best approximation by analytic functions in the sup-norm. To be more precise, we consider here the operator $\mathcal{A}$ as a nonlinear operator from $C(\mathbb{T})$ to H^∞.

The continuity problem is very important in applications. Suppose we are given a function $\varphi \in C(\mathbb{T})$ and we want to evaluate $\mathcal{A}\varphi$. We can approximate φ by functions φ_n for which it is easier to evaluate $\mathcal{A}\varphi_n$ (e.g., φ_n can be a polynomial, a rational function, etc.). The problem of whether the functions $\mathcal{A}_m\varphi_n$ give a good approximation for $\mathcal{A}_m\varphi$ reduces to the problem of whether φ is a continuity point of the operator $\mathcal{A}$.

It is obvious that the zero function is a continuity point of $\mathcal{A}$ since $\|\varphi - \mathcal{A}\varphi\|_\infty \le \|\varphi\|_\infty$, and so $\|\mathcal{A}\varphi\|_\infty \le 2\|\varphi\|_\infty$. In other words, the operator $\mathcal{A} : C(\mathbb{T}) \to H^\infty$ *is bounded.* It is also obvious that any function in C_A is a continuity point of $\mathcal{A}$ since $\mathcal{A}\varphi = \varphi$ for $\varphi \in C_A$. The main result of this section shows that there are no other continuity points of $\mathcal{A}$ in the sup-norm. We also obtain similar results for the operators $\mathcal{A}_m$ of best approximation by meromorphic functions with at most m poles.

Such results look disappointing for practical purposes. However, we shall see in §11 that if X is a decent space and we consider the continuity problem for the operator $\mathcal{A} : X \to X_+$ in the norm of X, the situation is quite different.

Theorem 10.1. *Let φ be a continuous function on $\mathbb{T}$ such that $\varphi \notin C_A$. Then there exists a sequence $\varphi_n \in C(\mathbb{T})$ such that*

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_\infty \to 0 \quad \text{but} \quad \liminf_{n\to\infty} \|\mathcal{A}\varphi_n - \mathcal{A}\varphi\|_\infty > 0.$$

Note that it follows from Theorem 10.1 that the operator $\mathcal{A}$ defined on $H^\infty + C$ is discontinuous in the L^∞-norm at any point $f \in (H^\infty + C) \setminus H^\infty$.

Indeed, if $f = \varphi + \psi$, where $\varphi \in C(\mathbb{T})$ and $\psi \in H^\infty$, then $\mathcal{A}f = \mathcal{A}\varphi + \psi$, and since $\mathcal{A}$ is discontinuous at φ by Theorem 10.1, it follows that it is discontinuous at f.

To prove Theorem 10.1, we need the following construction. Let $0 < \delta < 1/4$. Put

$$I \stackrel{\text{def}}{=} \left\{1 + e^{\mathrm{i}\theta} : \ \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right\}$$

and

$$\Omega_\delta \stackrel{\text{def}}{=} \{\zeta \in \mathbb{C} : \ \operatorname{dist}(\zeta, I) < \delta\}.$$

It is well known (see Goluzin [1], Ch. X, §1) that if ι is a conformal map of $\mathbb{D}$ onto Ω_δ, then ι extends to a continuous map of $\operatorname{clos}\mathbb{D}$ onto $\operatorname{clos}\Omega_\delta$. We can always choose such a conformal map ι that $\iota(1) = -\delta$.

Remark. In fact, we do not need the result that ι extends by continuity. For our purpose we can replace Ω_δ with $\iota(\{\zeta \in \mathbb{C} : \ |\zeta| < r\}$, where r is sufficiently close to 1 and replace ι with $\iota_r \stackrel{\text{def}}{=} \iota(rz)$.

Let now $\rho \in (0,1)$. Consider the conformal map $\gamma_\rho : \mathbb{D} \to \mathbb{D}$ defined by

$$\gamma_\rho(\zeta) = \frac{\rho - \zeta}{1 - \rho\zeta}, \quad \zeta \in \mathbb{D}.$$

It is easy to see that

$$\operatorname{Re}\gamma_\rho(\zeta) \le \rho \quad \Longleftrightarrow \quad \operatorname{Re}\zeta \ge \rho. \tag{10.1}$$

Proof of Theorem 10.1. Consider separately two cases. Suppose first that $\mathcal{A}\varphi \notin C(\mathbb{T})$. Let $\{\varphi_n\}$ be a sequence of trigonometric polynomials that converges to φ in the sup-norm. It has been shown in §4 that $\mathcal{A}\varphi_n \in C^\infty$. If the sequence $\{\mathcal{A}\varphi_n\}$ converged in the sup-norm to $\mathcal{A}\varphi$, it would follow that $\mathcal{A}\varphi \in C(\mathbb{T})$, which contradicts our assumption.

Suppose now that $\mathcal{A}\varphi \in C(\mathbb{T})$. Put $u = \varphi - \mathcal{A}\varphi$. Then $\mathcal{A}u = \mathbb{O}$. Clearly, it is sufficient to show that $\mathcal{A}$ is discontinuous at u. Without loss of generality we may assume that the function u is unimodular and $u(1) = 1$. By Theorem 5.1, u has negative winding number.

Consider the function $\psi_\rho \stackrel{\text{def}}{=} \iota \circ \gamma_\rho$. It is a conformal map of $\mathbb{D}$ onto Ω_δ. Define the function ξ_ρ by

$$\xi_\rho = \frac{\psi_\rho - u}{|\psi_\rho - u|}\left(|\psi_\rho - u| - 1\right).$$

Let ρ be so close to 1 that $|u(\zeta) - 1| \le \delta$ and $|\iota(\zeta) + \delta| \le \delta$ whenever $\operatorname{Re}\zeta \ge \rho$. Let us show that $\|\xi_\rho\|_\infty \le 2\delta$. Suppose first that $\operatorname{Re}\zeta \ge \rho$. Then $|u(\zeta) - 1| \le \delta$ and $\big||\psi_\rho(\zeta) - 1| - 1\big| \le \delta$, which implies that

$$\big||\psi_\rho(\zeta) - u(\zeta)| - 1\big| \le 2\delta.$$

Suppose now that $\operatorname{Re}\zeta \le \rho$. It follows from (10.1) that $\gamma_\rho(\zeta) \ge \rho$, and so $|\psi_\rho(\zeta) + \delta| \le \delta$. Since $|u(\zeta)| = 1$, it follows that $\big||\psi_\rho(\zeta) - u(\zeta)| - 1\big| \le 2\delta$. This completes the proof of the fact that $\|\xi_\rho\|_\infty \le 2\delta$.

In particular, the above reasoning shows that for ρ sufficiently close to 1 the function $|\psi_\rho - u|$ is separated away from zero, and so $\xi_\rho \in C(\mathbb{T})$.

To complete the proof of the theorem it is sufficient to show that $\mathcal{A}(u+\xi_\rho) = \psi_\rho$. Indeed, we can make δ as small as possible. On the other hand, $\mathcal{A}u = \mathbb{O}$, while $\|\psi_\rho\|_\infty \geq 1$.

To prove that $\mathcal{A}(u+\xi_\rho) = \psi_\rho$, we use Theorem 5.1. We show that the function $(u+\xi_\rho)-\psi_\rho$ has constant modulus and negative winding number. It is easy to see that

$$(u+\xi_\rho) - \psi_\rho = \frac{u-\psi_\rho}{|u-\psi_\rho|}.$$

We have

$$\operatorname{wind}((u+\xi_\rho)-\psi_\rho) = \operatorname{wind}\left(\frac{u-\psi_\rho}{|u-\psi_\rho|}\right) = \operatorname{wind}(u-\psi_\rho),$$

since $|u-\psi_\rho|$ is separated away from 0. Hence,

$$\operatorname{wind}((u+\xi_\rho)-\psi_\rho) = \operatorname{wind} u + \operatorname{wind}(1-\bar{u}\psi_\rho).$$

Since $\operatorname{wind} u < 0$, it is sufficient to show that $\operatorname{wind}(1-\bar{u}\psi_\rho) = 0$. This follows from the fact that the function $\bar{u}\psi_\rho - 1$ on $\mathbb{T}$ does not take values in $\mathbb{R}_+ = \{\zeta \in \mathbb{C} : \operatorname{Im}\zeta = 0,\ \operatorname{Re}\zeta \geq 0\}$. Indeed, if $\operatorname{Re}\zeta \geq \rho$, then $\bar{u}(\zeta)$ takes values in a small neighbourhood of 1 while ψ_ρ takes values in $\partial\Omega_\rho$, and so $\bar{u}(\zeta)\psi_\rho(\zeta) - 1 \notin \mathbb{R}_+$. On the other hand, if $\operatorname{Re}\zeta \leq \rho$, then $\bar{u}(\zeta)\psi_\rho(\zeta)$ is within a small neighborhood of the origin, and so $\bar{u}(\zeta)\psi_\rho(\zeta) - 1 \notin \mathbb{R}_+$. This completes the proof of the theorem. ■

We can now prove a similar result for the operators $\mathcal{A}_m$ of best approximation by meromorphic functions of degree at most m. Again, we consider the operator $\mathcal{A}_m$ here as an operator from $C(\mathbb{T})$ to L^∞.

Theorem 10.2. *Let φ be a continuous function on $\mathbb{T}$ such that $\varphi \notin H^\infty_{(m)}$. Then the operator $\mathcal{A}_m : C(\mathbb{T}) \to L^\infty$ is discontinuous at φ.*

Proof. The proof is almost the same as the proof of Theorem 10.1. If $\mathcal{A}_m\varphi \notin C(\mathbb{T})$, we can approximate φ by a sequence of trigonometric polynomials $\{\varphi_n\}$ and as in the case $m = 0$ it is obvious that $\{\mathcal{A}_m\varphi_n\}$ does not converge to $\mathcal{A}_m\varphi$.

If $\mathcal{A}_m\varphi \in C(\mathbb{T})$, we consider the function $u = \varphi - \mathcal{A}_m\varphi$. Again, without loss of generality we may assume that u is unimodular and $u(1) = 1$. Suppose now that the singular value $s_m(H_\varphi)$ has multiplicity μ, $k \leq m \leq k+\mu-1$, and

$$s_k(H_\varphi) = \cdots = s_{k+\mu-1}(H_\mu).$$

Then $\operatorname{wind} u = -(2k+\mu)$ (see Theorem 4.1.7). We can define the functions ξ_ρ and ψ_ρ in the same way as in the proof of Theorem 10.1. We claim that $\mathcal{A}_m(\varphi+\xi_\rho) = \mathcal{A}_m\varphi + \psi_\rho$. Indeed, we have already proved that the function

$$\varphi + \xi_\rho - (\mathcal{A}_m\varphi + \psi_\rho) = u + \xi_\rho - \psi_\rho$$

is unimodular and its winding number is equal to $\operatorname{wind} u = -(2k + \mu)$. Clearly, $\deg \mathbb{P}_-(\mathcal{A}_m\varphi + \psi_\rho) = k$. It follows from Theorem 5.3 that $\mathcal{A}_k(\varphi + \xi_\rho) = \mathcal{A}_k\varphi + \psi_\rho$. Consider the singular value $s_k(H_{\varphi+\xi_\rho})$. Clearly, if $k > 0$, then $s_k(H_{\varphi+\xi_\rho}) < s_{k-1}(H_{\varphi+\xi_\rho})$. Let ν be the multiplicity of the singular value $s_k(H_{\varphi+\xi_\rho})$. Then by Theorem 4.1.7,

$$\operatorname{wind}(\varphi + \xi_\rho - (\mathcal{A}_k\varphi + \psi_\rho)) = 2k + \nu,$$

and so $\mu = \nu$. Hence, $\mathcal{A}_k(\varphi+\xi_\rho) = \mathcal{A}_m(\varphi+\xi_\rho)$, which completes the proof. ■

11. Continuity in Decent Banach Spaces

In this section we study continuity properties of the operator $\mathcal{A}$ of best approximation by analytic functions in the norm of a decent space X. As we have already mentioned in the previous section, such continuity problems are very important in applications.

We show that if $\varphi \in X$ and $\mathbb{P}_-\varphi \neq \mathbb{O}$, then φ is a continuity point of $\mathcal{A}$ if and only if $s_0(H_\varphi)$ is a singular value of H_φ of multiplicity 1. We also consider the problem of whether a function φ in X is a continuity point of the operator $\mathcal{A}_m$ of best approximations by functions in $BMOA_{(m)}$. We shall see that the answer to this question depends on the multiplicity of the singular value $s_m(H_\varphi)$ of the Hankel operator H_φ.

These results do not seem to be very helpful for practical purposes in case of multiple singular values. To find a practical tool in this case, we study continuity properties of the operators $\mathcal{A}_m$ as operators from X to L^q, $q < \infty$, and we show that such operators are continuous everywhere.

Let X be a decent space of functions on $\mathbb{T}$ (see §2). Then for $\varphi \in X$ the Hankel operator H_φ maps X_+ to X_-. Recall that the operator H_φ^* is an operator from H^2_- into H^2 and $H_\varphi^* g = \mathbb{P}_+\bar{\varphi}g$. It follows from (A1)–(A4) that $H_\varphi^* H_\varphi$ maps X_+ into itself and it is a compact operator on X_+.

Lemma 11.1. *Let X be a decent space of functions on $\mathbb{T}$ and let $\varphi \in X$. Then the operator $H_\varphi^* H_\varphi$ has the same spectrum on H^2 and X_+. If λ is a nonzero point of the spectrum, then for every $n \geq 1$*

$$\begin{aligned}\{f \in X_+ : (H_\varphi^* H_\varphi - \lambda I)^n \varphi = \mathbb{O}\} &= \{f \in H^2 : (H_\varphi^* H_\varphi - \lambda I)^n \varphi = \mathbb{O}\} \\ &= \{f \in H^2 : (H_\varphi^* H_\varphi - \lambda I) \varphi = \mathbb{O}\}.\end{aligned}$$

Proof. Since the operator $H_\varphi^* H_\varphi$ is compact on both H^2 and X_+, it follows that the spectrum of $H_\varphi^* H_\varphi$ on both H^2 and X_+ consists of 0 and eigenvalues. Since $X_+ \subset H^2$, it follows that each eigenvector of the operator $H_\varphi^* H_\varphi$ on X_+ is also an eigenvector of the operator $H_\varphi^* H_\varphi$ on H^2. Theorem 9.3 says that each eigenvector of the operator $H_\varphi^* H_\varphi$ on H^2 belongs to X_+. Hence, the operator $H_\varphi^* H_\varphi$ has the same spectrum on X_+

and H^2. To complete the proof, we note that since the operator $H_\varphi^* H_\varphi$ on H^2 is self-adjoint, we have

$$\{f \in H^2 : (H_\varphi^* H_\varphi - \lambda I)^n \varphi = \mathbb{O}\} \quad = \quad \{f \in H^2 : (H_\varphi^* H_\varphi - \lambda I)\varphi = \mathbb{O}\}, \quad n \geq 1. \quad \blacksquare$$

Let $\varphi \in X$, $m \in \mathbb{Z}_+$. We denote by $\mu_m(H_\varphi)$ the multiplicity of the singular value of the Hankel operator $H_\varphi : H^2 \to H_-^2$:

$$\mu_m(\varphi) = \dim\{f \in H^2 : H_\varphi^* H_\varphi f = s_m^2(H_\varphi)f\}.$$

Theorem 11.2. *Let $m \in \mathbb{Z}_+$ and let X be a decent space. Suppose that φ is a function in X such that $\mu_m(H_\varphi) = 1$. Then φ is a continuity point of the operator $\mathcal{A}_m$ in the norm of X.*

We need the following lemma.

Lemma 11.3. *Let X, m, and φ satisfy the hypotheses of Theorem 11.2. Let f be an eigenfunction of $H_\varphi^* H_\varphi$ with eigenvalue $s_m^2(H_\varphi)$. Then f has no zeros on $\mathbb{T}$.*

Proof. Let E be the S-invariant subspace of H^2 spanned by f. It has been shown in the proof of Theorem 4.1.1 (see §4.1) that E has the form bH^2, where b is a finite Blaschke product of degree m. Then f can be represented as $f = bh$, where h is an outer function in X_+. It has also been shown in the proof of Theorem 4.1.1 that $H_\varphi f = c\bar{z}\bar{b}\bar{h}$, where $c \in \mathbb{C}$.

Let $v \stackrel{\text{def}}{=} \varphi - \mathcal{A}_m\varphi$. Then $v = (H_\varphi f)/f$ (see (4.1.12)). We have

$$v = \frac{H_\varphi f}{f} = c\bar{z}\frac{\bar{b}}{b}\frac{\bar{h}}{h}.$$

By Theorem 4.1.7, $\operatorname{wind} v = -(2m+1)$. Clearly,

$$\operatorname{wind} v = -1 + \operatorname{wind}\frac{\bar{b}}{b} + \operatorname{wind}\frac{\bar{h}}{h} = -(1+2m) + \operatorname{wind}\frac{\bar{h}}{h},$$

and so

$$\operatorname{wind}\frac{\bar{h}}{h} = 0.$$

Let $u \stackrel{\text{def}}{=} \bar{z}\bar{h}/h$. Then $\operatorname{wind} u = -1$, and so the singular value $1 = s_0(H_u)$ of the Hankel operator H_u on H^2 has multiplicity 1 and h is a maximizing vector of H_u. If $h(\tau) = 0$ for some point $\tau \in \mathbb{T}$, then it has been shown in §2 that the function $h/(1-\bar{\tau}z)$ belongs to X_+ and is also a maximizing vector of H_u. This contradicts the fact that the multiplicity of $s_0(H_u)$ is 1. Therefore h has no zeros on $\mathbb{T}$, which completes the proof of the lemma. $\blacksquare$

Proof of Theorem 11.2. Let f be an eigenfunction of $H_\varphi^* H_\varphi$ with eigenvalue $s_m^2(H_\varphi)^2$. As we have already mentioned in the proof of Lemma 11.3,

$$\mathcal{A}_m\varphi = \varphi - \frac{H_\varphi f}{f}.$$

By Lemma 11.1, $f \in X$. Let $\mathcal{J}$ be a positively oriented Jordan curve that surrounds $s_m^2(H_\varphi)$ and such that there are no points of the spectrum of $H_\varphi^* H_\varphi$ on $\mathcal{J}$ and the only point of the spectrum of $H_\varphi^* H_\varphi$ inside $\mathcal{J}$ is $s_m^2(H_\varphi)$. By Lemma 11.1 the same can be said about the spectrum of the operator $H_\varphi^* H_\varphi$ on X_+.

Obviously, the operators $H_{\varphi_n}^* H_{\varphi_n}$ converge to $H_\varphi^* H_\varphi$ in the operator norm on H^2 and on X_+. It follows that if n is sufficiently large, then the spectrum of $H_{\varphi_n}^* H_{\varphi_n}$ (whether on H^2 or X_+) has no points on $\mathcal{J}$ and has one point inside $\mathcal{J}$.

Consider now the spectral projections $\mathcal{P}$ and $\mathcal{P}_n$ on X_+ defined by

$$\mathcal{P} = \frac{1}{2\pi i} \int_{\mathcal{J}} (\zeta I - H_\varphi^* H_\varphi)^{-1} d\zeta, \quad \mathcal{P}_n = \frac{1}{2\pi i} \int_{\mathcal{J}} (\zeta I - H_{\varphi_n}^* H_{\varphi_n})^{-1} d\zeta. \tag{11.1}$$

It is easy to see from the above integral formulas that $\mathcal{P}_n \to \mathcal{P}$ in the operator norm. Clearly, for sufficiently large n the operator $\mathcal{P}_n$ is a projection onto a one-dimensional subspace. Put $f_n = \mathcal{P}_n f$. Then for sufficiently large n, $f_n \neq \mathbb{O}$ and

$$H_{\varphi_n}^* H_{\varphi_n} f_n = s_m^2(H_{\varphi_n}) f_n.$$

It follows from the convergence of $\mathcal{P}_n$ to $\mathcal{P}$ that $f_n \to f$ in X_+, and so $f_n \to f$ in L^∞. It follows now from Lemma 11.3 that for large n the function f_n does not vanish on $\mathbb{T}$. Therefore by the axiom (A4)

$$\lim_{n\to\infty} \|1/f_n - 1/f\|_X = 0.$$

Since

$$\mathcal{A}_m \varphi_n = \varphi_n - \frac{H_{\varphi_n} f_n}{f_n},$$

it follows that $\lim_{n\to\infty} \|\mathcal{A}_m \varphi_n - \mathcal{A}_m \varphi\|_X = 0$. ■

Let us proceed now to the case of a multiple singular value $s_m(H_\varphi)$. Suppose that $\mu = \mu_m(H_\varphi) \geq 2$ and

$$s_k(H_\varphi) = \cdots = s_{k+\mu-1}(H_\varphi) > s_{k+\mu}(H_\varphi), \quad k \leq m \leq k + \mu - 1. \tag{11.2}$$

Consider first the case $m \neq k + (\mu - 1)/2$.

Theorem 11.4. *Let $m \in \mathbb{Z}_+$, $\varphi \in VMO$, $\mu = \mu_m(H_\varphi) \geq 2$, and suppose that* (11.2) *holds. Assume that $m \neq k + (\mu-1)/2$ and Λ is a finite set in $\mathbb{D}$ of at least $(\mu - 1)/2$ points that are not poles of $\mathbb{P}_- \mathcal{A}_k \varphi$. Then there exist sequences $\{\zeta_\lambda^{(n)}\}_{n\geq 0}$, $\lambda \in \Lambda$, of nonzero complex numbers such that $\lim_{n\to\infty} \zeta_\lambda^{(n)} = 0$, $\lambda \in \Lambda$, and*

$$\|\mathcal{A}_m \varphi_n - \mathcal{A}_m \varphi\|_{BMO} \not\to 0,$$

where

$$\varphi_n = \varphi + \sum_{\lambda \in \Lambda} \frac{\zeta_\lambda^{(n)}}{z - \lambda}.$$

Remark. Note that all functions φ_n belong to a finite-dimensional subspace, and so $\varphi_n - \varphi \to \mathbb{O}$ in any norm. On the other hand, if X satisfies (A1)–(A4), then the norm in X is certainly stronger than the norm in BMO. So Theorem 11.4 says that the convergence of φ_n to φ in any norm does not guarantee the convergence of $\mathcal{A}_m\varphi_n$ to $\mathcal{A}_m\varphi$ even in the BMO norm.

We need the following lemma.

Lemma 11.5. *Let v, v_n be unimodular functions such that the Toeplitz operators T_v and T_{v_n} are Fredholm. If $\operatorname{ind} T_{v_n} \neq \operatorname{ind} T_v$, then*

$$\|v_n - v\|_{BMO} \not\to 0.$$

Proof. Multiplying v and v_n by $z^{\operatorname{ind} T_v}$, passing, if necessary, to complex conjugation and to a subsequence, we can assume that $\operatorname{ind} T_v = 0$ and $\operatorname{ind} T_{v_n} > 0$. Then T_v is invertible while T_{v_n} is not left invertible. It follows that $\|H_v\| < 1$ while $\|H_{v_n}\| = 1$ (see Theorem 3.1.11). It remains to observe that the convergence of v_n to v in BMO implies that $\lim\limits_{n\to\infty} \|H_{v_n} - H_v\| = 0$ (see Theorem 1.1.3), which leads to a contradiction. ■

Proof of Theorem 11.4. By Corollary 6.2, there exist sequences $\{\zeta_\lambda^{(n)}\}_{n\ge 0}$ tending to 0 such that the multiplicity of the singular value $s_m(H_{\varphi_n})$ equals 1. Then the function $\varphi_n - \mathcal{A}_m\varphi_n$ has constant modulus and by Theorem 4.1.7,

$$\operatorname{ind} T_{\varphi_n - \mathcal{A}_m\varphi_n} = 2m+1,$$

while

$$\operatorname{ind} T_{\varphi - \mathcal{A}_m\varphi} = 2k + \mu \neq 2m+1,$$

since $m \neq k + (\mu-1)/2$.

Put

$$v = \frac{1}{s_m(H_\varphi)}(\varphi - \mathcal{A}_m\varphi), \quad v_n = \frac{1}{s_m(H_{\varphi_n})}(\varphi_n - \mathcal{A}_m\varphi_n).$$

Clearly, $\lim\limits_{n\to\infty} s_m(H_{\varphi_n}) = s_m(H_\varphi)$, the functions v and v_n are unimodular. The result follows now from Lemma 11.4. ■

Now we are in a position to describe the continuity points of the operator of best approximation $\mathcal{A}$ by analytic functions.

Theorem 11.6. *Let X be a decent function space and let φ be a function in X such that $\mathbb{P}_-\varphi \neq \mathbb{O}$. Then φ is a continuity point of the operator $\mathcal{A}$ in the norm of X if and only if the multiplicity of the singular value $s_0(H_\varphi)$ of the Hankel operator $H_\varphi : H^2 \to H^2_-$ is 1.*

The result follows immediately from Theorems 11.2 and 11.4.

Let us now proceed to the case $m = k+(\mu-1)/2$. This case is considerably more complicated. We prove that if $\mu \ge 2$ and $X = B_1^1$, $X = \mathcal{F}\ell^1$, or $X = \lambda_\alpha$, $\alpha > 0$, $\alpha \notin \mathbb{Z}$, then φ is not a continuity point of $\mathcal{A}_m$ in the norm of X. Let us first consider the case when $X = B_1^1$ or $X = \mathcal{F}\ell^1$.

Theorem 11.7. *Suppose that $X = B_1^1$ or $X = \mathcal{F}\ell^1$. Let $m \in \mathbb{Z}_+$, $\varphi \in X$, $\mu = \mu_m(H_\varphi) \ge 2$, and suppose that (11.2) holds with $m = k + (\mu - 1)/2$. Then there exists a sequence $\{\varphi_n\}$ of functions in X converging to φ such that*

$$\|\mathcal{A}_m\varphi_n - \mathcal{A}_m\varphi\|_{BMO} \not\to 0 \quad as \quad n \to \infty.$$

To prove the theorem, we need a lemma.

For $\rho \in (0,1)$ consider the conformal map $\gamma_\rho : \mathbb{D} \to \mathbb{D}$ defined by

$$\gamma_\rho(\zeta) = \frac{\rho - \zeta}{1 - \rho\zeta}.$$

Lemma 11.8. *Let φ be a function in X_+ such that $\varphi(1) = 0$. Then*

$$\lim_{\rho\to 1} \|\varphi\gamma_\rho - \varphi\|_X = 0, \tag{11.3}$$

where $\rho \in (0,1)$.

Proof. Let us first prove the lemma for $X = B_1^1$. We have $\varphi\gamma_\rho - \varphi = \varphi(\gamma_\rho - 1)$. Since X is a Banach algebra and $\sup\limits_\rho \|\gamma_\rho\|_X < \infty$ (see Lemma 1.2), it is sufficient to prove (11.3) for analytic polynomials φ such that $\varphi(1) = 0$. Let us show that

$$\lim_{\rho\to 1} \int_{\mathbb{D}} |(\varphi(\gamma_\rho - 1))''| d\boldsymbol{m}_2 = 0.$$

It is easy to see that

$$\lim_{\rho\to 1} \gamma_\rho(\zeta) - 1 = 0, \quad \lim_{\rho\to 1} \gamma\,'_\rho(\zeta) = 0, \quad \lim_{\rho\to 1} \gamma\,''_\rho(\zeta) = 0, \quad \zeta \in \mathbb{D},$$

and the convergence is uniform on each set of the form $\{\zeta \in \mathbb{D} : |1-\zeta| \ge \delta\}$, $\delta \ge 0$. Clearly, it is sufficient to prove that

$$\lim_{\rho\to 1} \int_{\mathbb{D}} |\gamma_\rho - 1| d\boldsymbol{m}_2 = 0, \quad \lim_{\rho\to 1} \int_{\mathbb{D}} |\gamma\,'_\rho| d\boldsymbol{m}_2 = 0, \quad \lim_{\rho\to 1} \int_{\mathbb{D}} |\varphi\gamma\,''_\rho| d\boldsymbol{m}_2 = 0.$$

It is very easy to see that the first two limits are zero. Let us show that the third one is zero. Since $\sup\limits_\rho \|\gamma_\rho\|_X < \infty$, it follows that

$$\sup_\rho \int_{\mathbb{D}} |\gamma\,''_\rho| d\boldsymbol{m}_2 < \infty.$$

Let $\varepsilon > 0$. Since $\varphi(1) = 0$, we can choose a positive δ such that

$$\sup_\rho \int_{\{\zeta\in\mathbb{D}:|1-\zeta|<\delta\}} |\varphi\gamma\,''_\rho| d\boldsymbol{m}_2 < \frac{\varepsilon}{2}.$$

If we now choose ρ sufficiently close to 1, then

$$\int_{\{\zeta\in\mathbb{D}:|1-\zeta|\ge\delta\}} |\varphi\gamma\,''_\rho| d\boldsymbol{m}_2 < \frac{\varepsilon}{2},$$

since $\lim\limits_{\rho\to 1} \gamma\,''_\rho(\zeta) = 0$ uniformly on $\{\zeta \in \mathbb{D} : |1-\zeta| \ge \delta\}$. This completes the proof in the case $X = B_1^1$.

Suppose now that $X = \mathcal{F}\ell^1$. It is very easy to verify that

$$\sup_\rho \|\gamma_\rho\|_{\mathcal{F}\ell^1} < \infty.$$

Therefore it is sufficient to prove that if φ is an analytic polynomial and $\varphi(1) = 0$, then

$$\lim_{\rho\to 1} \|\varphi(\gamma_\rho - 1)\|_{\mathcal{F}\ell^1} = 0.$$

We have already proved the same convergence in B_1^1. The result follows now from the fact that the norm of B_1^1 is stronger than the norm of $\mathcal{F}\ell^1$. ■

Proof of Theorem 11.7. Without loss of generality we may assume that $\varphi \in X_-$. It follows from Lemma 11.8 that

$$\lim_{\rho\to 1} \|(\varphi - \varphi(0))\bar{\gamma}_\rho - (\varphi - \varphi(0))\|_X = 0. \tag{11.4}$$

We have

$$\mathcal{A}_m\varphi = \mathcal{A}_m(\varphi - \varphi(0)) + \varphi(0).$$

Similarly,

$$\mathcal{A}_m((\varphi - \varphi(0))\bar{\gamma}_\rho + \varphi(0)) = \mathcal{A}_m((\varphi - \varphi(0))\bar{\gamma}_\rho) + \varphi(0).$$

Thus it is sufficient to show that

$$\|\mathcal{A}_m((\varphi - \varphi(0))\bar{\gamma}_\rho) - \mathcal{A}_m(\varphi - \varphi(0))\|_{BMO} \not\to 0 \quad \text{as} \quad \rho \to 1. \tag{11.5}$$

Let $\xi = \varphi - \varphi(0)$ and $s = s_m(H_\varphi)$. Then $H_\varphi = H_\xi$. By Theorem 6.4, s is a singular value of $H_{\bar{\gamma}_\rho\xi}$ and

$$s = s_{k+1}(H_{\bar{\gamma}_\rho\xi}) = \cdots = s_{k+\mu-1}(H_{\bar{\gamma}_\rho\xi})$$

and $\mathcal{A}_{k+1}(\bar{\gamma}_\rho\xi) = \bar{\gamma}_\rho\mathcal{A}_k\xi$. Clearly, $\mathcal{A}_m(\bar{\gamma}_\rho\xi) = \mathcal{A}_{k+1}(\bar{\gamma}_\rho\xi)$ and $\mathcal{A}_m\xi = \mathcal{A}_{k+1}\xi$. It follows from (11.4) that to prove (11.5) we have to show that

$$\|\bar{\gamma}_\rho(\xi - \mathcal{A}_k\xi) - (\xi - \mathcal{A}_k\xi)\|_{BMO} \not\to 0 \quad \text{as} \quad \rho \to 1.$$

Clearly, the functions $\xi - \mathcal{A}_k\xi$ and $\bar{\gamma}_\rho(\xi - \mathcal{A}_k\xi)$ have the same constant modulus, belong to QC, and

$$\operatorname{ind} T_{\xi - \mathcal{A}_k\xi} = 2k + \mu, \quad \operatorname{ind} T_{\bar{\gamma}_\rho(\xi - \mathcal{A}_k\xi)} = 2k + \mu + 1.$$

The result follows now from Lemma 11.5. ■

We proceed now to the case $X = \lambda_\alpha$, $\alpha > 0$, $\alpha \notin \mathbb{Z}$.

Theorem 11.9. *Let $m \in \mathbb{Z}_+$, φ be a function in λ_α, $\alpha > 0$, $\alpha \notin \mathbb{Z}$, $\mu = \mu_m(H_\varphi) \geq 2$, and suppose that* (11.2) *holds with $m = k + (\mu - 1)/2$. Then there exists a sequence $\{\varphi_n\}$ of functions in λ_α converging to φ such that*

$$\|\mathcal{A}_m\varphi_n - \mathcal{A}_m\varphi\|_{BMO} \not\to 0 \quad \textit{as} \quad n \to \infty.$$

To prove Theorem 11.9, we obtain an analog of Lemma 11.8.

Lemma 11.10. *Let $\alpha > 0$, $\alpha \notin \mathbb{Z}$, and let d be the integer satisfying $d-1 < \alpha < d$. If f is a function in $(\lambda_\alpha)_+$ such that $f^{(s)}(1) = 0$, $0 \le s \le d-1$, then*

$$\lim_{\rho\to 1} \|f\gamma_\rho - f\|_{\Lambda_\alpha} = 0, \tag{11.6}$$

where $\rho \in (0,1)$.

To prove Lemma 11.10, we need one more lemma.

Lemma 11.11. *Let f be a function in Λ_α, $\alpha > 0$, such that $f^{(j)}(1) = 0$, $0 \le j \le d-1$, where $d-1 < \alpha < d$. Then*

$$\sup_{\rho\in(0,1)} \|f(\gamma_\rho - 1)\|_{\Lambda_\alpha} < \infty.$$

Proof of Lemma 11.11. Clearly, it is sufficient to prove that

$$\sup_{\rho\in(0,1)} \sup_{\zeta\in\mathbb{D}} |(f(\gamma_\rho - 1))^{(d)}(\zeta)|(1-|\zeta|)^{d-\alpha} < \infty, \quad d-1 < \alpha < d,$$

(see Appendix 2.6). This reduces to the following inequalities:

$$\sup_{\rho\in(0,1)} \sup_{\zeta\in\mathbb{D}} \frac{|f^{(d-j)}(\zeta)|}{|1-\rho\zeta|^{j+1}}(1-\rho)|(1-|\zeta|)^{d-\alpha} < \infty, \quad 1 \le j \le d, \tag{11.7}$$

and

$$\sup_{\rho\in(0,1)} \sup_{\zeta\in\mathbb{D}} |f^{(d)}(\zeta)| \cdot |\gamma_\rho(\zeta) - 1|(1-|\zeta|)^{d-\alpha} = 0. \tag{11.8}$$

Since $f^{(s)}(1) = 0$, $0 \le s \le d-1$, it follows that

$$|f^{(d-j)}(\zeta)| \le \text{const} \cdot |1-\zeta|^{\alpha-d+j},$$

and so

$$\frac{|f^{(d-j)}(\zeta)|}{|1-\rho\zeta|^{j+1}}(1-\rho)|(1-|\zeta|)^{d-\alpha} \le \text{const}\left(\frac{|1-\zeta|}{|1-\rho\zeta|}\right)^j \frac{1-\rho}{|1-\rho\zeta|} \cdot \frac{(1-|\zeta|)^{d-\alpha}}{|1-\zeta|^{d-\alpha}},$$

which easily implies (11.7).

Inequality (11.8) follows easily from the fact that

$$|f^{(d)}(\zeta)| \le \text{const}(1-|\zeta|)^{\alpha-d}. \quad \blacksquare$$

Proof of Lemma 11.10. By Lemma 11.11, it is sufficient to show that (11.6) holds for f in a dense subset of the space

$$Y \stackrel{\text{def}}{=} \{g \in (\lambda_\alpha)_+ : g^{(j)}(1) = 0,\ 0 \le j \le d-1\}.$$

It is easy to see that if $f \in Y$, then we can approximate f by polynomials g in Y. If g is a polynomial in Y, we can consider the polynomial $g_n = g - c_n z^n$, where c_n is chosen so that $g^{(d)}(1) = 0$, $n \ge 1$. It is easy to see that $\|z^n\|_{\Lambda_\alpha} \le \text{const} \cdot n^\alpha$ and $|c_n| \le \text{const} \cdot n^{-d}$, and so $\lim_{n\to\infty} \|c_n z^n\|_{\lambda_\alpha} = 0$.

Therefore it is sufficient to prove (11.6) for functions f in $C^d(\mathbb{T})$ such that $f^{(j)}(1) = 0$, $0 \le j \le d$. For such functions f it is easy to verify that

$$\lim_{\rho \to 1} \|f(\gamma_\rho - 1)\|_{C^d(\mathbb{T})} = 0,$$

which immediately implies (11.6). ■

Proof of Theorem 11.9. Clearly, we can assume that $\varphi \in (\lambda_\alpha)_-$. Let q be an analytic polynomial of degree at most $d-1$ such that $(\varphi - q)^{(j)}(1) = 0$, $0 \le j \le d-1$. If we apply Lemma 11.10 to the function $z^{d-1}(\overline{\varphi - q})$ in $(\lambda_\alpha)_+$, we find that

$$\lim_{\rho \to 1} \|(\varphi - q)(\bar{\gamma}_\rho - 1)\|_{\Lambda_\alpha} = 0.$$

Clearly, $\mathcal{A}_m(\varphi) = q + \mathcal{A}_m(\varphi - q)$ and $\mathcal{A}_m((\varphi - q)\bar{\gamma}_\rho + q) = q + \mathcal{A}_m((\varphi - q)\bar{\gamma}_\rho)$. Now put $\xi = \varphi - q$. The rest of the proof is exactly the same as the proof of Theorem 11.7. ■

As we have already seen, in the case when $s_m(H_\varphi)$ is a multiple singular value of H_φ and $\lim\limits_{n \to \infty} \|\varphi_n - \varphi\|_X = 0$ we should not expect the convergence of $\mathcal{A}_m \varphi_n$ to $\mathcal{A}_m \varphi$ even in the norm of BMO. Let us now see whether we can relax the norm of BMO so that $\mathcal{A}_m \varphi_n$ converges to $\mathcal{A}_m \varphi$.

Theorem 11.12. *Suppose that $m \in \mathbb{Z}_+$, X is a decent function space and let $\varphi \in X$. If $\{\varphi_n\}$ is a sequence of functions in X which converges to φ in the norm of X, then for all $q < \infty$*

$$\lim_{n \to \infty} \|\mathcal{A}_m \varphi_n - \mathcal{A}_m \varphi\|_{L^q} = 0.$$

Proof. The case $s_m(H_\varphi) = 0$ is obvious. Assume that $s_m(H_\varphi) > 0$. Let $s = s_m(H_\varphi)$, $\mu = \mu_m(\varphi)$, and

$$s = s_k(H_\varphi) = \cdots = s_{k+\mu-1}(H_\varphi).$$

As in the proof of Theorem 11.2 consider a positively oriented Jordan curve $\mathcal{J}$ that surrounds $s_m^2(H_\varphi)$ and such that there are no points of the spectrum of $H_\varphi^* H_\varphi$ on $\mathcal{J}$ and the only point of the spectrum of $H_\varphi^* H_\varphi$ inside $\mathcal{J}$ is $s_m^2(H_\varphi)$. Then for large values of n the points of $\mathcal{J}$ are in the resolvent set of $H_{\varphi_n}^* H_{\varphi_n}$ and there are μ points of the spectrum of $H_{\varphi_n}^* H_{\varphi_n}$ (counted with multiplicities) inside $\mathcal{J}$. Recall that by Lemma 11.1, the operators $H_\varphi^* H_\varphi$ and $H_{\varphi_n}^* H_{\varphi_n}$ have the same spectra on H^2 and X_+.

Consider the spectral projections $\mathcal{P}$ and $\mathcal{P}_n$ on X_+ defined by (11.1). Then $\lim_{n \to \infty} \mathcal{P}_n = \mathcal{P}$ in the operator norm of X_+. Let f_n be an eigenfunction of $H_{\varphi_n}^* H_{\varphi_n}$ with eigenvalue $s_m^2(H_{\varphi_n})$. We normalize it by the condition $\|f_n\|_X = 1$. Clearly, for large values of n, $\mathcal{P} f_n \ne \mathbb{O}$, and so $\mathcal{P} f_n$ is an eigenfunction of $H_\varphi^* H_\varphi$ with eigenvalue $s_m^2(H_\varphi)$.

Let now f be an arbitrary eigenfunction of $H_\varphi^* H_\varphi$ with eigenvalue $s_m^2(H_\varphi)$. Put

$$v = \frac{1}{s_m(H_\varphi)} \cdot \frac{H_\varphi f}{f} = \frac{1}{s_m(H_\varphi)} (\varphi - \mathcal{A}_m \varphi).$$

Then v is a unimodular function, $\|H_v f\|_{L^2} = \|f\|_{L^2}$, and the dimension of the subspace

$$E = \{g \in H^2 : \|H_v g\|_{L^2} = \|g\|_{L^2}\}$$

is equal to $2k+\mu$ (see §4.1). By the remark after Theorem 2.4, $v \in X$ and by Theorem 8.1, v admits a representation

$$v = \bar{z}^{2k+\mu}\frac{\bar{h}}{h},$$

where h is an outer function in X_+ that does not vanish on $\mathbb{T}$. It is easy to see that

$$E = \{ph : p \in \boldsymbol{\mathcal{P}}_+,\ \deg p < 2k+\mu\}.$$

Therefore for each n there exists a polynomial p_n in $\boldsymbol{\mathcal{P}}_+$ such that $\mathcal{P} f_n = p_n h$.

Let us show that

$$\lim_{n\to\infty}(\varphi_n - \mathcal{A}_m\varphi_n) = \varphi - \mathcal{A}_m\varphi \quad \text{in} \quad L^q.$$

We have

$$\varphi_n - \mathcal{A}_m\varphi_n = \frac{H_{\varphi_n} f_n}{f_n}, \quad \varphi - \mathcal{A}_m\varphi = \frac{H_\varphi p_n h}{p_n h}.$$

It follows from the convergence of $\mathcal{P}_n$ to $\mathcal{P}$ that

$$\|f_n - p_n h\|_X = \|\mathcal{P}_n f_n - \mathcal{P} f_n\|_X \to 0 \quad \text{as} \quad n \to \infty.$$

Next, since $\lim\limits_{n\to\infty}\|\varphi_n - \varphi\|_X = 0$, we have

$$\lim_{n\to\infty}\|H_{\varphi_n} f_n - H_\varphi p_n h\|_X = 0, \quad \lim_{n\to\infty}\|(p_n h)(H_{\varphi_n} f_n) - f_n H_\varphi p_n h\|_X = 0.$$

Since $C(\mathbb{T}) \subset X$, it follows that there exists a sequence $\{\delta_n\}$ of positive numbers with zero limit such that

$$\|f_n - p_n h\|_{L^\infty} \le \delta_n; \quad \|(p_n h)(H_{\varphi_n} f_n) - f_n H_\varphi p_n h\|_{L^\infty} \le \delta_n.$$

Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that

$$\lim_{n\to\infty}\varepsilon_n = 0, \quad \lim_{n\to\infty}\frac{\delta_n}{\varepsilon_n^2} = 0. \tag{11.9}$$

Put

$$\Omega_n = \{\zeta \in \mathbb{T} : |p_n(\zeta)h(\zeta)| \le \varepsilon_n\}.$$

We have to show that

$$\lim_{n\to\infty}\int_{\mathbb{T}}\left|\frac{(H_{\varphi_n} f_n)(\zeta)}{f_n(\zeta)} - \frac{(H_\varphi p_n h)(\zeta)}{(p_n h)(\zeta)}\right|^q d\boldsymbol{m}(\zeta) = 0.$$

It follows easily from (11.9) that

$$\lim_{n\to\infty}\int_{\mathbb{T}\setminus\Omega_n}\left|\frac{(H_{\varphi_n} f_n)(\zeta)}{f_n(\zeta)} - \frac{(H_\varphi p_n h)(\zeta)}{(p_n h)(\zeta)}\right|^q d\boldsymbol{m}(\zeta) = 0.$$

Since almost everywhere

$$\left|\frac{H_\varphi p_n h}{p_n h}\right| = s_m(H_\varphi), \quad \left|\frac{H_{\varphi_n} f_n}{f_n}\right| = s_m(H_{\varphi_n}) \to s_m(H_\varphi) \quad \text{as} \quad n \to \infty,$$

it follows that the proof will be completed as far as we show that

$$\lim_{n\to\infty} \boldsymbol{m}(\Omega_n) = 0.$$

Put $M = \min\{|h(\zeta)| : \zeta \in \mathbb{T}\}$. Obviously,

$$|p_n(\zeta)| \le \frac{\varepsilon_n}{M}, \quad \zeta \in \Omega_n. \tag{11.10}$$

Next, since $\lim\limits_{n\to\infty} \|p_n h - f_n\|_X = 0$ and $\|f_n\|_X = 1$, it follows that for large values of n,

$$\|p_n\|_X \ge \frac{\text{const}}{\|h\|_X}.$$

Finally, since p_n ranges over the finite-dimensional subspace of polynomials of degree at most $2k + \mu - 1$, we have

$$\|p_n\|_{L^\infty} \ge \text{const} > 0 \tag{11.11}$$

for large values of n.

The result follows now from (11.10) and (11.11) and the following well-known fact:

Let Ω be a closed subset of $\mathbb{T}$ of measure ε and let p be a polynomial of degree N. Then

$$\max_{\zeta\in\mathbb{T}} |p(\zeta)| \le C(N, \varepsilon) \max_{\zeta\in\Omega} |p(\zeta)|,$$

where $C(N, \varepsilon)$ depends only on N and ε (see Gaier [1]; see also Nazarov [1], which contains much stronger results). ■

12. The Recovery Problem in Spaces of Measures and Interpolation by Analytic Functions

In this section we consider applications of best approximation results obtained in §1. First we obtain so-called recovery theorems for measures. As an application of such recovery theorems we obtain an interpolation theorem for functions in $B_p^{1/p} \cap C_A$.

Let X be a space of distributions on $\mathbb{T}$. The recovery problem for measures with respect to the space X is the problem of whether of whether the following implication holds:

$$\mu \in \mathcal{M}(\mathbb{T}), \quad \mathbb{P}_-\mu \in X \quad \Longrightarrow \quad \mu \in X.$$

Here $\mathcal{M}(\mathbb{T})$ denotes the space of complex regular Borel measures on $\mathbb{T}$.

Let us mention two well-known recovery theorems for measures. The famous F. and M. Riesz theorem can be stated in the following way:

$$\mu \in \mathcal{M}(\mathbb{T}), \quad \mathbb{P}_-\mu \in L^1(\mathbb{T}) \quad \Longrightarrow \quad \mu \in L^1(\mathbb{T}).$$

The other classical example is due to Rajchman [1]:

$$\mu \in \mathcal{M}(\mathbb{T}), \quad \mathbb{P}_-\mu \in \mathcal{F}c_0 \quad \Longrightarrow \quad \mu \in \mathcal{F}c_0,$$

where $\mathcal{F}c_0$ is the space of pseudofunctions, i.e., the space of distributions ξ on $\mathbb{T}$ satisfying

$$\lim_{|n|\to\infty} |\hat{\xi}(n)| = 0.$$

In this section we show that the results of §1 on best approximation in Besov spaces $B_p^{1/p}$ lead to other recovery theorems for measures.

Recall that $B_p^{-1/p'}$, $1 < p < \infty$, is the space of distributions ξ on $\mathbb{T}$ for which

$$\sum_{n\in\mathbb{Z}} 2^{-n(p-1)} \|\xi * W_n\|_{L^p}^p < \infty,$$

and an analytic function f in $\mathbb{D}$ belongs to $B_p^{-1/p'}$ if and only if

$$\int_{\mathbb{D}} |f(\zeta)|^p (1-|\zeta|)^{p-2} d\boldsymbol{m}_2(\zeta) < \infty$$

(see Appendix 2.6). The space $B_p^{-1/p'}$ can naturally be identified with $\left(B_{p'}^{1/p'}\right)^*$. Note that the space $B_2^{-1/2}$ coincides with the space of distributions ξ with finite energy

$$\mathcal{E}(\xi) = \sum_{n\in\mathbb{Z},\, n\neq 0} \frac{|\hat{\xi}(n)|^2}{|n|} < \infty.$$

The following result is one more recovery theorem for measures.

Theorem 12.1. *Let $1 < p < \infty$ and $\mu \in \mathcal{M}(\mathbb{T})$. If $\mathbb{P}_-\mu \in B_p^{-1/p'}$, then $\mu \in B_p^{-1/p'}$.*

Proof. Let $0 < r < 1$ and $\mu_r = P_r * \mu$, where P_r is the Poisson kernel. Since $\left(B_{p'}^{1/p'}\right)^* = B_p^{-1/p'}$, there exists a function $\varphi \in (B_{p'}^{1/p'})_-$ such that

$$\|\varphi\|_{B_{p'}^{1/p'}} \le 1 \quad \text{and} \quad \|\mathbb{P}_+\mu_r\|_{B_p^{-1/p'}} \le \operatorname{const}\left|\int_{\mathbb{T}} z\varphi d\mu_r\right|.$$

Let $f = \mathcal{A}\varphi$. Then

$$\|\varphi - f\|_{L^\infty} = \operatorname{dist}_{L^\infty}(\varphi, H^\infty) \le \operatorname{const}\|\varphi\|_{B_{p'}^{1/p'}},$$

since $B_{p'}^{1/p'} \subset VMO$. By Theorem 1.6,

$$f \in B_{p'}^{1/p'} \quad \text{and} \quad \|f\|_{B_{p'}^{1/p'}} \le \operatorname{const}\|\varphi\|_{B_{p'}^{1/p'}}.$$

We have

$$\begin{aligned}
\|\mathbb{P}_+\mu_r\|_{B_p^{-1/p'}} &\le \text{const}\left|\int_{\mathbb{T}} z\varphi d\mu_r\right| \\
&\le \text{const}\left|\int_{\mathbb{T}} z(\varphi-f)d\mu_r\right| + \text{const}\left|\int_{\mathbb{T}} zfd\mu_r\right| \\
&\le \text{const}\,\|\varphi-f\|_{L^\infty}\|\mu\|_{\mathcal{M}(\mathbb{T})} \\
&+ \text{const}\,\|zf\|_{B_{p'}^{1/p'}}\|\mathbb{P}_-\mu_r\|_{B_p^{-1/p'}} \\
&\le \text{const}\,\|\varphi\|_{B_{p'}^{1/p'}}\left(\|\mu\|_{\mathcal{M}(\mathbb{T})}+\|\mathbb{P}_-\mu_r\|_{B_p^{-1/p'}}\right).
\end{aligned}$$

Since $\|\varphi\|_{B_{p'}^{1/p'}} \le 1$, it follows that the norms $\|\mathbb{P}_+\mu_r\|_{B_p^{-1/p'}}$ are uniformly bounded, and so $\mathbb{P}_+\mu \in B_p^{-1/p'}$. ■

Let us now state the special case of Theorem 12.1 when $p = 2$.

Corollary 12.2. *If $\mu \in \mathcal{M}(\mathbb{T})$, then*

$$\sum_{n\ge1}\frac{|\hat{\mu}(-n)|^2}{n} < \infty \quad \Longrightarrow \quad \sum_{n\ge1}\frac{|\hat{\mu}(n)|^2}{n} < \infty.$$

Corollary 12.3. *If $\mu \in \mathcal{M}(\mathbb{T})$ and s is a positive integer, then*

$$\sum_{n\ge1}\frac{|\hat{\mu}(-n)|^{2s}}{n} < \infty \quad \Longrightarrow \quad \sum_{n\ge1}\frac{|\hat{\mu}(n)|^{2s}}{n} < \infty. \tag{12.1}$$

Proof of Corollary 12.3. Let $\nu \stackrel{\text{def}}{=} \underbrace{\mu * \mu * \cdots * \mu}_{s}$. Then

$$\sum_{n\ge1}\frac{|\hat{\nu}(-n)|^2}{n} < \infty.$$

By Lemma 12.2,

$$\sum_{n\ge1}\frac{|\hat{\nu}(n)|^2}{n} < \infty,$$

which means that

$$\sum_{n\ge1}\frac{|\hat{\mu}(n)|^{2s}}{n} < \infty. \quad ■$$

The very interesting problem of whether (12.1) holds for other values of $s \ge 1/2$ is still open.

The rest of this section is devoted to an application of Theorem 12.1 to an interpolation problem for a space of analytic functions. Let $1 < p < \infty$. Consider the space $C_A \cap B_p^{1/p}$ of functions analytic in $\mathbb{D}$. A closed subset E of $\mathbb{T}$ is called an *interpolation set* for $C_A \cap B_p^{1/p}$ if for each continuous function f on E there exists a function $\varphi \in C_A \cap B_p^{1/p}$ such that $\varphi\big|E = f$.

We consider the following norm on $B_p^{1/p}$:

$$\|f\|_{B_p^{1/p}} = \left(\|f\|_{L^p}^p + \int_{\mathbb{T}} \frac{1}{|1-\tau|^2} \int_{\mathbb{T}} |f(\tau\zeta) - f(\zeta)|^p d\boldsymbol{m}(\zeta) d\boldsymbol{m}(\tau)\right)^{1/p}.$$

Definition. Let E be a closed subset of $\mathbb{T}$. We define its *capacity* $\operatorname{cap}_p^{1/p}(E)$ by

$$\operatorname{cap}_p^{1/p}(E) \stackrel{\text{def}}{=} \inf\left\{\|\varphi\|_{B_p^{1/p}} : \varphi \in C(\mathbb{T}) \cap B_p^{1/p},\ |\varphi|\,\big|E \ge 1\right\}.$$

Let $\omega(\zeta) \stackrel{\text{def}}{=} \min\{|\zeta|, 1\}$. Clearly, $|\omega(\zeta_1) - \omega(\zeta_2)| \le |\zeta_1 - \zeta_2|$, and so

$$\|\omega \circ \varphi\|_{B_p^{1/p}} \le \|\varphi\|_{B_p^{1/p}}.$$

It follows that

$$\operatorname{cap}_p^{1/p}(E) \stackrel{\text{def}}{=} \inf\left\{\|\varphi\|_{B_p^{1/p}} : \varphi \in C(\mathbb{T}) \cap B_p^{1/p},\ 0 \le \varphi \le 1,\ \varphi\,\big|E \ge 1\right\}.$$

Note that for $p = 2$ the capacity $\operatorname{cap}_p^{1/p}$ is equivalent to the logarithmic capacity (see Landkof [1]). By analogy with the case $p = 2$, distributions in the space $B_{p'}^{-1/p}$ are called *distributions of finite p-energy.*

Theorem 12.4. *Let $1 < p < \infty$ and let E be a closed subset of $\mathbb{T}$. If $\operatorname{cap}_p^{1/p}(E) = 0$, then E is an interpolation set for $C_A \cap B_p^{1/p}$.*

Note that it follows from the results of Sjödin [1] that if E is an interpolation set for $C(\mathbb{T}) \cap B_p^{1/p}$ (i.e., the restrictions of functions in $C(\mathbb{T}) \cap B_p^{1/p}$ coincide with $C(E)$), then $\operatorname{cap}_p^{1/p}(E) = 0$. This implies that the converse to Theorem 12.4 also holds, i.e., if E is an interpolation set for $C_A \cap B_p^{1/p}$, then $\operatorname{cap}_p^{1/p}(E) = 0$.

We need the following lemma.

Lemma 12.5. *Let μ be a complex Borel measure on $\mathbb{T}$ with finite p-energy and let E be a closed subset of $\mathbb{T}$ such that $\operatorname{cap}_p^{1/p}(E) = 0$. Then $|\mu|(E) = 0$.*

Proof. Let us show that $\int_E f\,d\mu = 0$ for any smooth function f on $\mathbb{T}$. Let $\{\varphi_n\}$ be a sequence of functions in $C(\mathbb{T}) \cap B_p^{1/p}$ such that

$$0 \le \varphi_n \le 1, \quad \varphi_n|E \equiv 1, \quad \lim_{n\to\infty} \|\varphi_n\|_{B_p^{1/p}} = 0.$$

It is easy to see that $\|\varphi_n f\|_{B_p^{1/p}} \le \text{const}\,\|f\|_{C^1(\mathbb{T})}\|\varphi_n\|_{B_p^{1/p}}$. We have

$$\left|\int_{\mathbb{T}} \varphi_n f\,d\mu\right| \le \text{const}\,\|\mu\|_{B_{p'}^{-1/p}}\|\varphi_n f\|_{B_p^{1/p}} \le \text{const}\,\|\mu\|_{B_{p'}^{-1/p}}\|f\|_{C^1(\mathbb{T})}\|\varphi_n\|_{B_p^{1/p}}.$$

Hence,

$$\lim_{n\to\infty} \int_{\mathbb{T}} \varphi_n f\,d\mu = 0.$$

Since $\varphi_n|E \equiv 1$, it follows that

$$\left|\int_E f\,d\mu\right| \le \left|\int_{\mathbb{T}} \varphi_n\,d\mu\right| + \int_{\mathbb{T}\setminus E} |f|d|\mu|,$$

and so

$$\left|\int_E f\,d\mu\right| \le \int_{\mathbb{T}\setminus E} |f|d|\mu|. \tag{12.2}$$

It is easy to see that there exists a sequence $\{f_j\}$ of smooth functions on $\mathbb{T}$ such that $f_j\big|E = f\big|E$, $\lim\limits_{j\to\infty} |f_j(\tau)| = 0$ for $\tau \in \mathbb{T}\setminus E$, and $\|f_j\|_{C(\mathbb{T})} \le$ const. By Lebesgue's theorem it follows from (12.2) that $\int\limits_E f\,d\mu = 0$. ■

Proof of Theorem 12.4. Clearly, $C_A \cap B_p^{1/p} = C(\mathbb{T}) \cap \left(B_p^{1/p}\right)_+$. Consider the subspace D (diagonal) of the direct product $C(\mathbb{T}) \times \left(B_p^{1/p}\right)_+$ defined by

$$D = \left\{\{f,f\}:\ f \in C(\mathbb{T}) \cap \left(B_p^{1/p}\right)_+\right\} \subset C(\mathbb{T}) \times \left(B_p^{1/p}\right)_+$$

and define the linear operator $J : C(\mathbb{T}) \cap \left(B_p^{1/p}\right)_+ \to D$ by $Jf = \{f,f\}$. The dual space $\left(C(\mathbb{T}) \times \left(B_p^{1/p}\right)_+\right)^*$ can be identified naturally with the space $\mathcal{M}(\mathbb{T}) \times \left(B_{p'}^{-1/p}\right)_+$.

Since $\{z^n, z^n\} \in D$ for $n \in \mathbb{Z}_+$, it follows that the annihilator $D^\perp$ consists of the pairs $\{\mu, f\}$ in $\mathcal{M}(\mathbb{T}) \times \left(B_{p'}^{-1/p}\right)_+$ for which

$$\hat{\mu}(n) + \hat{f}(n) = 0, \quad n \in \mathbb{Z}_+.$$

Hence, $\mathbb{P}_+\mu \in B_{p'}^{-1/p}$ for $\{\mu, f\} \in D^\perp$. It follows from Theorem 12.1 that $\mu \in B_{p'}^{-1/p}$. By Theorem 12.5, $|\mu|(E) = 0$. Consider now the restriction operator

$$R : C(\mathbb{T}) \cap (B_p^{1/p})_+ \to C(E).$$

By the Banach theorem R is onto if and only if the operator $(J^*)^{-1}R^*$ is an isomorphism onto its range. We can identify $\big(C(E)\big)^*$ with the space $\mathcal{M}(E)$ of complex Borel measures on E. It is easy to see that if $\nu \in \mathcal{M}(E)$, then $(J^*)^{-1}R^*\nu = \{\nu, \mathbb{O}\} + D^\perp$. We have

$$\|(J^*)^{-1}R^*\nu\| = \inf\left\{\|\nu+\mu\|_{\mathcal{M}(\mathbb{T})} + \|\xi\|_{B_{p'}^{-1/p}}:\ \{\mu,\xi\} \in D^\perp\right\} \ge \|\nu\|_{\mathcal{M}(E)},$$

since $|\mu|(E) = 0$. Thus $(J^*)^{-1}R^*$ is an isomorphism onto its range, and so R is onto. ■

13. The Fefferman–Stein Decomposition in $B_p^{1/p}$

In this section we obtain an analog of the Fefferman–Stein decomposition for functions in BMO. Recall that a function $f \in L^1(\mathbb{T})$ belongs to BMO, (i.e.,

$$\sup_I \frac{1}{\boldsymbol{m}(I)} \int_I |f - f_I| d\boldsymbol{m} < \infty, \quad f_I \stackrel{\text{def}}{=} \frac{1}{\boldsymbol{m}(I)} \int_I f \, d\boldsymbol{m}, \tag{13.1}$$

the supremum in (13.1) being taken over all arcs I of $\mathbb{T}$) if and only if f can be represented as $f = \xi + \tilde{\eta}$, where ξ, $\eta \in L^\infty$. Such a representation of f is called the *Fefferman–Stein* decomposition of f.

The following theorem is an analog of the Fefferman–Stein decomposition for functions in $B_p^{1/p}$.

Theorem 13.1. *Let $1 < p < \infty$. If $f \in B_p^{1/p}$, then f can be represented as*

$$f = \xi + \tilde{\eta},$$

where ξ, $\eta \in B_p^{1/p} \cap L^\infty$.

Proof. We have $\mathbb{P}_- f \in B_p^{1/p}$. Let $g = \mathcal{A}(\mathbb{P}_- \varphi)$. Then by Theorem 1.6, $g \in B_p^{1/p}$ and $\varphi = \mathbb{P}_- f - g \in B_p^{1/p} \cap L^\infty$. Since $\mathbb{P}_- \varphi = (\varphi - \mathrm{i}\tilde{\varphi} - \hat{\varphi}(0))/2$, it follows that

$$\mathbb{P}_- f = \mathbb{P}_- \varphi = \frac{1}{2}(\varphi - \hat{\varphi}(0) - \mathrm{i}\tilde{\varphi}).$$

The function $\mathbb{P}_+ f$ admits a similar representation. ∎

Concluding Remarks

The heredity problem for function spaces was studied first by Shapiro [1], who proved that the space of functions which extend analytically to a neighborhood of the unit circle has the hereditary property. In Carleson and Jacobs [1] it was shown that the classes Λ_α, $\alpha \notin \mathbb{Z}$, have the hereditary property. The heredity problem and the recovery problem for unimodular functions were studied systematically in Peller and Khrushchëv [1]. We also mention here the survey article Kahane [2].

The notion of an $\mathcal{R}$-space was introduced in Peller and Khrushchëv [1]. Theorems 1.3, 1.4, 1.5, and 1.6 were obtained in Peller and Khrushchëv [1]. Theorems 1.10 and 1.11 are published here for the first time.

The presentation of §2 follows Peller and Khrushchëv [1]. However, in Peller and Khrushchëv [1] the axiom (A3) is different: *the set of trigonometric polynomials is dense in X*. The fact that Theorems 2.2 and 2.3 hold under less restrictive Axiom (A3) stated in §2 was proved in Alexeev and Peller [2]. Note that the proof of Theorem 2.3 uses an idea of Lax [1]. In that paper the author considered a self-adjoint operator A on a Hilbert space $\mathcal{H}$ and a Banach space X continuously imbedded in $\mathcal{H}$ such that $AX \subset X$. Under certain conditions it was shown in Lax [1] that if x is an eigenvector of A, then $x \in X$. Note also that the proof of Theorem 2.3 uses an idea of the paper Adamyan, Arov, and Krein [1] in which the authors considered the recovery problem for the space $\mathcal{F}\ell^1$. Theorem 2.1 was established in Peller [24]. Note that the nonseparable function spaces Λ_α, $B^s_{p\infty}$ as well as many other nonseparable Banach spaces (see §4) were treated in Peller and Khrushchëv [1] as spaces satisfying the axioms (B1)–(B3) stated in §3.

The results of §3 are due to Peller and Khrushchëv [1].

Theorem 4.1 was obtained in Peller [10]. Theorem 4.3 is due to Peller [24]. The fact that for $\varphi \in \Lambda_\alpha$ the Hankel operator H_φ is a compact operator from $(\Lambda_\alpha)_+$ to $(\Lambda_\alpha)_-$ was obtained for $\alpha \notin \mathbb{Z}$ in Gohberg and Budyanu [2] by a different method. The proof of this fact given in §4 was found in Peller [24]. It works in a much more general situation. Note that the fact that $T_{\bar{f}}$ is a bounded operator on $(\Lambda_\alpha)_+$ for any $f \in H^\infty$ was observed first in Khavin [1] and Shamoyan [1]. Theorem 4.4 is taken from Peller and Khrushchëv [1]. Theorem 4.5 for $Z = VMO, C_A + \overline{C_A}$, $\boldsymbol{H^1} + \overline{\boldsymbol{H^1}}$, $L^1 + \widetilde{L^1}$ was established in Peller and Khrushchëv [1]. The nonseparable spaces $Z^{(n)}$ with $Z = H^\infty + \overline{H^\infty}$ and $Z = BMO$ were treated in Peller and Khrushchëv as spaces satisfying the axioms (B1)–(B3). The fact that these nonseparable spaces satisfy (A3) was established in Peller [24]. Theorem 4.9 was obtained in Peller and Khrushchëv [1]. Again, Theorem 4.11 is contained in Peller [24] while in Peller and Khrushchëv [1] the axioms (B1)–(B3) were used to show that $\mathcal{F}\ell^\infty_w$ is hereditary. Theorems 4.14, 4.15, 4.16 and 4.17 are taken from Peller and Khrushchëv [1]. Theorem 4.18 is due to A.B. Aleksandrov (unpublished). Theorem 4.19 is new. The fact that $C_A + \overline{C_A}$ is not hereditary was discovered in Papadimitrakis [1]. We also mention the papers Krepkogorskii [1–2], in which the author studies interpolation spaces between $B_p^{1/p}$ and BMO which are $\mathcal{R}$-spaces, and the recent paper Gheorghe [1], in which the author studies generalized Besov spaces B_E associated with a monotone Riesz–Fischer space E (such spaces are also $\mathcal{R}$-spaces).

Theorem 5.1 for continuous functions u was obtained in Adamyan, Arov, and Krein [1] and Poreda [1]. Theorems 5.3 and 5.4 are contained in Adamyan, Arov, and Krein [3] and Hayashi, Trefethen, and Gutknekht [1]. Theorems 5.5 can be found in Alexeev and Peller [3], while Theorem 5.6 is published here for the first time.

The results of §6 were obtained in Peller [19].

Theorems 7.1 and 7.10 we found in Peller [20], while Theorem 7.8 was established in Papadimitrakis [3].

The results of §8 are due to Peller and Khrushchëv [1], while the results of §9 are taken from Peller [15].

Theorem 10.1 was obtained independently by different methods in Merino [1] and Papadimitrakis [2]. In §2 we use the method of Merino. Note that the continuity problem in the L^∞ norm for the operator $\mathcal{A}$ was posed in the preprint Helton and Schwartz [1], though the results of that preprint contain an error.

The results of §11 are taken from Peller [19]; the results for the space $\mathcal{F}\ell^1$ were established in Hayashi, Trefethen, and Gutknekht [1].

The results of §12 were obtained in Peller and Khrushchëv [1]. Note that Koosis [2] found another proof of Corollary 12.2.

The results of §13 are taken from Peller and Khrushchëv [1]. Note that in Adams and Frazier [1] Fefferman–Stein decomposition was obtained for other spaces of functions.

Let us also mention the papers Volberg and Tolokonnikov [1] and Tolokonnikov [2] in which the authors use other methods to find more hereditary function spaces.

Finally, we mention here the following results of Chui and Li [1]. They considered the nonlinear operator $\mathcal{G}_n$ defined as follows. Let $\varphi \in C(\mathbb{T})$. By the Adamyan–Arov–Krein theorem there exists a unique Hankel operator Γ_n of rank at most n such that $\|H_\varphi - \Gamma_n\| = s_n(H_\varphi)$. The function $\mathcal{G}_n\varphi$ is by definition the antianalytic symbol of Γ_n. By Kronecker's theorem $\mathcal{G}_n\varphi$ is a rational function of degree at most n. It was shown in Chui and Li [1] that if $s_{n-1}(H_\varphi) > s_n(H_\varphi)$, then φ is a continuity point of $\mathcal{G}_n$ in the L^∞ norm. On the other hand, the authors gave an example that shows that the restriction $s_{n-1}(H_\varphi) > s_n(H_\varphi)$ cannot be dropped.

8

An Introduction to Gaussian Spaces

In this chapter we give a brief introduction in the theory of Gaussian processes by means of the technique of Fock spaces. This approach, which appeared in quantum field theory, simplifies both Wiener's approach based on Hermite polynomials in several variables and Itô's method of stochastic integrals.

In §1 we define Gaussian variables and Gaussian subspaces. Then we define stationary Gaussian processes and the spectral measure associated to a stationary Gaussian process.

In §2 we construct the (bozon) Fock space of a Hilbert space. Then we define the so-called Vick transform that orthogonalizes the spaces of homogeneous polynomials of Gaussian functions of degree k. This technique allows us to pass from the subspace of Gaussian functions in $L^2(\Omega, \mathcal{A}, P)$ to the whole space $L^2(\Omega, \mathcal{A}, P)$ over a probability space $(\Omega, \mathcal{A}, P)$ and to introduce a probability preserving automorphism τ of $(\Omega, \mathcal{A}, P)$ associated with the stationary Gaussian process.

In §3 we consider various regularity conditions for stationary processes (i.e., conditions of weak dependence of the future on the past) and their relationships with mixing properties of the measure preserving automorphism τ defined in §2. Later in Chapter 9 we characterize various regularity conditions in terms of the spectral measure of the process.

Next, we study in §4 interpolation problems for stationary Gaussian processes. In other words, we consider so-called minimal stationary processes, i.e., processes $\{X_n\}_{n\in\mathbb{Z}}$ such that X_0 cannot be predicted by the X_n with $n \neq 0$. With each such process we associate the so-called interpolation error process. Then we study related properties of basisness and unconditional basisness of stationary processes.

Finally, in §5 we relate the theory of stationary processes with Hankel operators. Namely, we show that the product of the orthogonal projections onto the past and the future can be expressed in terms of a Hankel operator whose symbol is the so-called phase function of the process. Then we consider the following geometric problem in the theory of stationary processes. Given subspaces $\mathcal{K}$ and $\mathcal{L}$ of the Gaussian space, the problem is to find out whether there exists a stationary Gaussian process with past $\mathcal{K}$ and future $\mathcal{L}$. We show that this problem essentially reduces to the problem of the description of the operators on Hilbert spaces that are unitarily equivalent to the moduli of Hankel operators. The latter problem will be solved in Chapter 12.

1. Gaussian Spaces

Let (Ω, Σ, P) be a probability space, i.e., Ω is a space of elementary events, Σ is a σ-algebra of subsets of Ω (σ-algebra of events), and P is a probability measure on Σ (i.e., P is a positive measure such that $P(\Omega) = 1$). Events A, $B \in \Sigma$ are called *independent* if

$$P(A \cap B) = P(A)P(B).$$

By *random variables* we understand measurable (real or complex) functions on Ω. Random variables f and g are called *independent* if the events $\{\omega \in \Omega : f(\omega) \in F\}$ and $\{\omega \in \Omega : g(\omega) \in G\}$ are independent for any Borel sets F and G. For a random variable $f \in L^1(\Omega, \Sigma, P)$ its *mathematical expectation* (or its *mean*) $\mathbb{E}f$ is defined by

$$\mathbb{E}f = \int_\Omega f\, dP.$$

A measurable function $f : \Omega \to \mathbb{R}$ is called a *Gaussian random variable* (or simply Gaussian variable) if its distribution function is Gaussian, i.e.,

$$P\{\omega : f(\omega) < s\} = \frac{1}{\sqrt{2\pi}\sigma_f} \int_{-\infty}^{s} \exp\left(-\frac{t^2}{2\sigma_f^2}\right) dt,$$

where

$$\sigma_f^2 = \mathbb{E}|f|^2 = \int_\Omega |f|^2 dP$$

is the *variance* of f. Note that we consider only Gaussian variables with zero mean. For convenience we assume that the zero function on Ω is also a Gaussian variable.

If f is a Gaussian variable, it is easy to see that

$$\mathbb{E}e^{\mathrm{i}tf} = e^{-t^2\sigma_f^2/2}, \quad t \in \mathbb{R}. \tag{1.1}$$

It follows easily from the Fourier inversion formula that (1.1) implies that f is a Gaussian variable with variance σ_f^2.

A family $\{f_\alpha\}_{\alpha\in A}$ is called *Gaussian* if its linear span consists of Gaussian variables. If $f_1,\cdots,f_n$ is a Gaussian family and $\gamma=(\gamma_1\ \cdots\ \gamma_n)^{\mathrm{t}}\in\mathbb{R}^n$, then the Gaussian variable $f=\sum\limits_{k=1}^{n}\gamma_k f_k$ has variance

$$\mathbb{E}f^2=\sum_{j,k=1}^{n}\gamma_j\gamma_k\mathbb{E}(f_jf_k)=(Q\gamma,\gamma),$$

where the matrix Q is defined by $Q=\{\mathbb{E}(f_jf_k)\}_{1\le j,k\le n}$. Clearly, Q is symmetric and positive definite. It is easy to see that Q is invertible if and only if $f_1,\cdots,f_n$ are linearly independent. In this case the joint distribution $\mu_{f_1,\cdots,f_n}$ of the family $f_1,\cdots,f_n$ (i.e., the image of P under the mapping $\omega\mapsto(f_1(\omega),\cdots,f_n(\omega))$) is absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^n$ and it is easy to verify that the density $P_{f_1,\cdots,f_n}$ of $\mu_{f_1,\cdots,f_n}$ is given by

$$P_{f_1,\cdots,f_n}(x)=\frac{1}{(2\pi)^{n/2}}(\det Q)^{1/2}\exp\left(-\frac{1}{2}(Q^{-1}x,x)\right). \tag{1.2}$$

It follows immediately from (1.2) that if g_1 and g_2 are random variables that belong to the same Gaussian family, then they are independent if and only if they are orthogonal.

A closed subspace $\boldsymbol{G}$ of the real space $L^2_{\mathbb{R}}(\Omega,\Sigma,P)$ is called *Gaussian* if it consists of Gaussian variables. The following lemma shows that the closure of the linear span of a Gaussian family is a Gaussian subspace.

Lemma 1.1. *If $\{f_n\}_{n\ge0}$ is a sequence of Gaussian variables and $f=\lim\limits_{n\to\infty}f_n$ in $L^2(\Omega,\Sigma,P)$, then f is a Gaussian variable.*

Proof. Passing to a subsequence, we can assume that $f=\lim\limits_{n\to\infty}f_n$ almost everywhere (*almost surely*) on Ω. By Lebesgue's theorem

$$\mathbb{E}e^{\mathrm{i}tf}=\lim_{n\to\infty}\mathbb{E}e^{\mathrm{i}tf_n}=\exp\left(-\frac{t^2}{2}\lim_{n\to\infty}\sigma_{f_n}^2\right),$$

which implies that f is a Gaussian variable with variance $\lim\limits_{n\to\infty}\sigma_{f_n}^2$. ■

It is easy to see that there exist infinite-dimensional Gaussian spaces. Indeed, let (Ω_j,Σ_j,P_j), $j\in\mathbb{Z}$, be a copy of the probability space $(\mathbb{R},\mathcal{B},\mu)$, where $\mathcal{B}$ the set of Borel subsets of $\mathbb{R}$ and μ is the measure on $\mathbb{R}$ with density

$$\frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

Consider the infinite product $(\Omega,\Sigma,P)=\prod\limits_{j\in\mathbb{Z}}(\Omega_j,\Sigma_j,P_j)$. Let g_j, $j\in\mathbb{Z}$, be the function on (Ω,Σ,P) defined by

$$g_j(\cdots,\omega_{-2},\omega_{-1},\omega_0,\omega_1\omega_2,\cdots)=\omega_j.$$

It is easy to see that $\{g_j\}_{j\in\mathbb{Z}}$ is a sequence of independent Gaussian variables and the closed linear span of $\{g_j\}_{j\in\mathbb{Z}}$ is an infinite-dimensional Gaussian subspace space of $L^2_{\mathbb{R}}(\Omega,\Sigma,P)$.

We fix a Gaussian subspace $\boldsymbol{G}$ in $L^2_{\mathbb{R}}(\Omega,\Sigma,P)$ and we consider only Gaussian variables in $\boldsymbol{G}$. It is natural to assume that Σ is the smallest σ-algebra with respect to which all functions in $\boldsymbol{G}$ are measurable. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two closed subspaces of $\boldsymbol{G}$, and let $\Sigma_{\boldsymbol{A}}$ and $\Sigma_{\boldsymbol{B}}$ be the smallest σ-algebras with respect to which the functions in $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively, are measurable. We say that the subspaces $\boldsymbol{A}$ and $\boldsymbol{B}$ are *independent* if every function in $\boldsymbol{A}$ is independent with any function in $\boldsymbol{B}$. It follows immediately from (1.2) that $\boldsymbol{A}$ and $\boldsymbol{B}$ are independent if and only if they are orthogonal.

We are going to use the easily verifiable identity

$$n!\prod_{j=1}^{n} z_j=\sum_{r=1}^{n}(-1)^{n-r}\sum_{j_1<\cdots<j_r}(z_{j_1}+\cdots+z_{j_r})^n, \tag{1.3}$$

which holds in any commutative ring with a unit element.

Theorem 1.2. $L^2_{\mathbb{R}}(\Omega,\Sigma,P)=\operatorname{span}\{g^k:\ g\in\boldsymbol{G},\ k\in\mathbb{Z}_+\}$.

Proof. Suppose that f is orthogonal to all functions g^k, $g\in\boldsymbol{G}$, $k\in\mathbb{Z}_+$. It follows from (1.3) that f is orthogonal to any function of the form $q(g_1,\cdots,g_n)$, where q is a polynomial in n variables. Since the set of polynomials in n variables is dense in the space of L^2 functions on $\mathbb{R}^n$ with weight $\exp(-\|x\|^2/2)$, it follows that $\int f\,dP=0$ for every subset E of Ω that is measurable with respect to a finite collection of elements in $\boldsymbol{G}$. Such subsets generate Σ. Hence, $f=\mathbb{O}$. ■

With the Gaussian space $\boldsymbol{G}$ we associate its *complexification* $\boldsymbol{G}_{\mathrm{c}}=\boldsymbol{G}+\mathrm{i}\boldsymbol{G}$. Elements of this space are called complex Gaussian variables.

A Gaussian family $\{X_n\}_{n\in\mathbb{Z}}$ is called a *Gaussian process with discrete time.* We denote by $\boldsymbol{G}$ the closed Gaussian subspace spanned by X_n, $n\in\mathbb{Z}$. Here we consider only Gaussian process with discrete time and we call them simply *Gaussian processes.*

A Gaussian process $\{X_n\}_{n\in\mathbb{Z}}$ is called *stationary* if $\mathbb{E}X_jX_k$ depends only on $j-k$, i.e., there exists a sequence $\{Q_n\}_{n\in\mathbb{Z}}$ such that

$$\mathbb{E}X_jX_k=Q_{j-k}.$$

The sequence $\{Q_n\}_{n\in\mathbb{Z}}$ is called the *covariance sequence of the process.*

It is easy to see that $\{Q_n\}_{n\in\mathbb{Z}}$ is positive definite, since

$$\sum_{j,k=1}^{m}Q_{j-k}\zeta_j\bar{\zeta}_k=\mathbb{E}\left|\sum_{j=1}^{m}\zeta_jX_j\right|^2\geq 0$$

for any $m\in\mathbb{Z}_+$ and $\zeta_1,\cdots,\zeta_m\in\mathbb{C}$. By the Riesz–Herglotz theorem (see Riesz and Sz.-Nagy [1], §5.3) there exists a unique positive regular Borel measure μ on $\mathbb{T}$ such that

$$Q_n = \hat{\mu}(n), \quad n \in \mathbb{Z}. \tag{1.4}$$

The measure μ is called the *spectral measure of the process* $\{X_n\}_{n\in\mathbb{Z}}$.

Clearly, the sequence $\{Q_n\}_{n\in\mathbb{Z}}$ is real. This implies that μ is invariant under the transformation $\zeta \mapsto \bar{\zeta}$ of $\mathbb{T}$. Conversely, if μ is invariant under this transformation, then the sequence $\{Q_n\}_{n\in\mathbb{Z}}$ defined by (1.4) is real.

If μ is the spectral measure of a stationary Gaussian process $\{X_n\}_{n\in\mathbb{Z}}$, we can consider the mapping Φ defined on the linear combinations of the X_n by

$$\Phi\left(\sum_{j=1}^{m} c_j X_j\right) = \sum_{j=1}^{m} c_j z^{t_j}. \tag{1.5}$$

It is easy to see that Φ extends to an isometry of the complexification $\boldsymbol{G}_{\mathrm{c}}$ of $\boldsymbol{G}$ onto $L^2(\mathbb{T}, \mu)$. Φ is called the *spectral representation of the process.*

Theorem 1.3. *Let μ be a positive Borel measure on $\mathbb{T}$ that is invariant under the transformation $\zeta \mapsto \bar{\zeta}$ of $\mathbb{T}$. There exists a stationary Gaussian process $\{X_n\}_{n\in\mathbb{Z}}$ whose spectral measure coincides with μ.*

Proof. Let

$$\mathcal{H} \stackrel{\text{def}}{=} \{f \in L^2(\mu) : \; f(\bar{\zeta}) = \overline{f(\zeta)}, \; \zeta \in \mathbb{T}\}.$$

Then $\mathcal{H}$ is a real subspace of the complex Hilbert space $L^2(\mu)$.

Let $\boldsymbol{G}$ be an arbitrary separable infinite-dimensional Gaussian space. Let U be a unitary operator of $\boldsymbol{G}$ onto $\mathcal{H}$. Put

$$X_n = U^{-1} z^n, \quad n \in \mathbb{Z}.$$

It is easy to see that $\mathbb{E} X_j X_k = \hat{\mu}(j - k)$, which proves the theorem. ■

In what follows we are going to work with the complex Hilbert space $L^2(\Omega, \Sigma, P)$ and with the complex Gaussian space, which is the complexification of the real Gaussian space. For convenience we change the notation and we denote the complex Gaussian space by $\boldsymbol{G}$.

We are going to introduce the notion of canonical correlations for two subspaces of the Gaussian space. If f and g are nonzero functions in $\boldsymbol{G}$, their correlation is defined by

$$\frac{(f, g)}{\|f\| \cdot \|g\|}.$$

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are subspaces of $\boldsymbol{G}$, their *maximal canonical correlation* $\operatorname{cor}_0(\boldsymbol{A}, \boldsymbol{B})$ is defined by

$$\operatorname{cor}_0(\boldsymbol{A}, \boldsymbol{B}) = \sup\{|(f, g)| : \; f \in \boldsymbol{A}, \; \|f\| = 1, \; g \in \boldsymbol{B}, \; \|g\| = 1\}. \tag{1.6}$$

Note that $\operatorname{cor}_0(\boldsymbol{A}, \boldsymbol{B})$ is the cosine of the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$. Suppose now that there exist functions $f_0 \in \boldsymbol{A}$ and $g_0 \in \boldsymbol{B}$ at which the supremum in (1.6) is attained. We consider the subspaces

$$\boldsymbol{A}_1 \stackrel{\text{def}}{=} \{f \in \boldsymbol{A} : f \perp f_0\}, \quad \boldsymbol{B}_1 \stackrel{\text{def}}{=} \{g \in \boldsymbol{B} : g \perp g_0\}.$$

The *canonical correlation* $\operatorname{cor}_1(\boldsymbol{A}, \boldsymbol{B})$ is defined as $\operatorname{cor}_0(\boldsymbol{A_1}, \boldsymbol{B_1})$. If there exist functions $f_1 \in \boldsymbol{A_1}$ and $g_1 \in \boldsymbol{B_1}$ of norm 1 such that

$$|(f_1, g_1)| = \operatorname{cor}_0(\boldsymbol{A_1}, \boldsymbol{B_1}),$$

we can continue this process and consider the subspaces

$$\boldsymbol{A}_2 \stackrel{\text{def}}{=} \{f \in \boldsymbol{A}_1 : f \perp f_1\}, \quad \boldsymbol{B}_2 \stackrel{\text{def}}{=} \{g \in \boldsymbol{B}_1 : g \perp g_1\}.$$

This allows us to define $\operatorname{cor}_2(\boldsymbol{A}, \boldsymbol{B})$ as $\operatorname{cor}_0(\boldsymbol{A}_2, \boldsymbol{B}_2)$.

In this way, we obtain the sequence (finite or infinite) $\{\operatorname{cor}_n(\boldsymbol{A}, \boldsymbol{B})\}$ of *canonical correlations* of the pair $\{\boldsymbol{A}, \boldsymbol{B}\}$ of Gaussian subspaces. It is easy to see that the nth canonical correlation $\operatorname{cor}_n(\boldsymbol{A}, \boldsymbol{B})$ is the nth singular value of the operator $P_{\boldsymbol{A}} P_{\boldsymbol{B}}$, where $P_{\boldsymbol{A}}$ and $P_{\boldsymbol{B}}$ are the orthogonal projections onto $\boldsymbol{A}$ and $\boldsymbol{B}$.

2. The Fock Space

Let $\mathcal{H}$ be a Hilbert space. For every positive integer n we consider the *Hilbert tensor product*

$$\underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n} \stackrel{\text{def}}{=} \bigotimes^{n} \mathcal{H}.$$

The space $\bigotimes^{n} \mathcal{H}$ is a completion of the linear combinations of tensors $f_1 \otimes \cdots \otimes f_n$ with respect to the following inner product:

$$\begin{aligned}&(f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_n)_{\otimes} = (f_1, g_1) \cdots (f_n, g_n),\\ &f_1, \cdots, f_n, g_1, \cdots, g_n \in \mathcal{H}.\end{aligned}$$

On the space $\bigotimes^{n} \mathcal{H}$ we consider the projection Sym onto the subspace of symmetric tensors which is defined by

$$\operatorname{Sym}(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} f_{\sigma_1} \otimes \cdots \otimes f_{\sigma_n},$$

where Σ_n is the group of permutations of $\{1, \cdots, n\}$. We denote by

$$\underbrace{\mathcal{H} \circledS \cdots \circledS \mathcal{H}}_{n} \stackrel{\text{def}}{=} \mathcal{H}^n_{\circledS}$$

the range of Sym. It is easy to see that

$$\big(\operatorname{Sym}(f_1 \otimes \cdots \otimes f_n), \operatorname{Sym}(g_1 \otimes \cdots \otimes g_n)\big)_{\otimes} = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (f_{\sigma_1}, g_1) \cdots (f_{\sigma_n}, g_n). \tag{2.1}$$

For technical reasons it is convenient to endow $\mathcal{H}^n_{\circledS}$ with the new inner product

$$(f,g)_{\circledS} \stackrel{\text{def}}{=} n!(f,g)_{\otimes}, \quad f,\, g \in \mathcal{H}^n_{\circledS}.$$

We put $\mathcal{H}^0_{\circledS} \stackrel{\text{def}}{=} \mathbb{C}$.

It is easy to see that if $\mathcal{H} = L^2(\mathbb{T},\mu)$, then $\mathcal{H}_n$ can be identified naturally with the subspace of $L^2(\mathbb{T}^n, \overset{n}{\otimes}\, \mu)$ that consists of the functions that are symmetric with respect to permutations of the coordinates, and

$$(f,g)_{\circledS} = n! \underbrace{\int \cdots \int}_{n} f(\zeta_1, \cdots, \zeta_n)\overline{g(\zeta_1, \cdots, \zeta_n)}\, d\mu(\zeta_1)\cdots d\mu(\zeta_n).$$

Definition. The *Fock space* $\boldsymbol{\mathcal{F}}(\mathcal{H})$ *over* $\mathcal{H}$ is the infinite orthogonal sum

$$\bigoplus_{n\in\mathbb{Z}_+} \mathcal{H}^n_{\circledS}.$$

$\boldsymbol{\mathcal{F}}(\mathcal{H})$ is also called the *boson Fock space.*

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $T : \mathcal{H} \to \mathcal{K}$ be a bounded linear operator. Consider the operator

$$\bigotimes^n T \stackrel{\text{def}}{=} \underbrace{T \otimes \cdots \otimes T}_{n} : \bigotimes^n \mathcal{H} \to \bigotimes^n \mathcal{K}$$

defined by

$$\bigotimes^n T(f_1 \otimes \cdots \otimes f_n) = Tf_1 \otimes \cdots \otimes Tf_n.$$

It is easy to see that if T is an isometric operator, then $\overset{n}{\otimes}\, T$ is isometric, and if T is an orthogonal projection, then so is $\overset{n}{\otimes}\, T$. It is also easy to see that $\left(\overset{n}{\otimes}\, T\right)^* = \overset{n}{\otimes}\, T^*$ and $\overset{n}{\otimes}\, (TR) = \left(\overset{n}{\otimes}\, T\right)\left(\overset{n}{\otimes}\, R\right)$. To prove that for any contractive operator T (i.e., $\|T\| \le 1$) the operator $\overset{n}{\otimes}\, T$ is also contractive, we prove the following elementary lemma.

Lemma 2.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $T : \mathcal{H} \to \mathcal{K}$ be a contraction. Then there exists a unitary operator U on $\mathcal{H} \oplus \mathcal{K}$ such that $T = \mathcal{P}UJ$, where J is the canonical imbedding of $\mathcal{H}$ in $\mathcal{H} \oplus \mathcal{K}$ and $\mathcal{P}$ is the orthogonal projection of $\mathcal{H} \oplus \mathcal{K}$ onto $\mathcal{K}$.*

Proof. We define U by the block matrix:

$$\begin{pmatrix} (I - T^*T)^{1/2} & T^* \\ T & -(I - TT^*)^{1/2} \end{pmatrix}.$$

It follows from Lemma 2.1.7 that the conclusion of the lemma holds. ■

Corollary 2.2. *If $T : \mathcal{H} \to \mathcal{K}$ is a contraction, then $\overset{n}{\otimes}\, T$ is also a contraction.*

Proof. The result follows from Lemma 2.1 and preceding remarks. ■

It is easy to see that if T is a bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, then $\bigotimes^n T$ maps symmetric tensors to symmetric tensors. Thus we can consider its restriction to the space of symmetric tensors:

$$\underbrace{T\circledS\cdots\circledS T}_{n} : \mathcal{H}^n_{\circledS} = \mathcal{H}\circledS\cdots\circledS\mathcal{H} \to \mathcal{K}^n_{\circledS} = \mathcal{K}\circledS\cdots\circledS\mathcal{K}.$$

We say that a vector h in $\boldsymbol{\mathcal{F}}(\mathcal{H})$ is a finite particle vector if at most finitely many projections of h onto $\mathcal{H}_n$ can be nonzero. If T is a contraction from $\mathcal{H}$ to $\mathcal{K}$ we can define its *Fock power* $\boldsymbol{\mathcal{F}}(T) : \boldsymbol{\mathcal{F}}(\mathcal{H}) \to \boldsymbol{\mathcal{F}}(K)$ on the finite particle vectors by

$$\boldsymbol{\mathcal{F}}(T)|\mathcal{H}_n = \underbrace{T\circledS\cdots\circledS T}_{n}, \quad n \ge 1, \quad \boldsymbol{\mathcal{F}}(T)|\mathcal{H}_0 = I.$$

It follows from the above remarks that for any contraction T the operator $\boldsymbol{\mathcal{F}}(T)$ extends to a contractive operator from $\boldsymbol{\mathcal{F}}(\mathcal{H})$ to $\boldsymbol{\mathcal{F}}(\mathcal{K})$. It is also clear that $(\boldsymbol{\mathcal{F}}(T))^* = \boldsymbol{\mathcal{F}}(T^*)$, $\boldsymbol{\mathcal{F}}(T)$ is isometric if $\boldsymbol{\mathcal{F}}(T)$ is, and $\boldsymbol{\mathcal{F}}(TR) = \boldsymbol{\mathcal{F}}(T)\boldsymbol{\mathcal{F}}(R)$ for any contractive operators T and R.

We are going to introduce on the space $L^2(\Omega, \Sigma, P)$ the structure of a Fock space. For a positive integer n we denote by G^n the space of homogeneous polynomials of degree n of elements of the Gaussian space $\boldsymbol{G} \subset L^2(\Omega, \Sigma, P)$. Put $G^0 = \mathbb{C}$.

Lemma 2.3. *The spaces G^n, $n \in \mathbb{Z}_+$, are linearly independent.*

Proof. Let $g_1, \cdots, g_m$ be Gaussian variables in $\boldsymbol{G}$ and let q_j, $0 \le j \le n$, be a homogeneous polynomial in m variables of degree j such that

$$\sum_{j=0}^{n} q_j(g_1, \cdots, g_m) = 0 \quad \text{almost surely on } \Omega.$$

Without loss of generality we may assume that the Gaussian variables $g_1, \cdots, g_m$ are real. Consider the image in $\mathbb{R}^m$ of probability measure under the mapping

$$\omega \mapsto (g_1(\omega), \cdots, g_m(\omega)).$$

It is easy to see from (1.2) that this measure is concentrated on a subspace L of $\mathbb{R}^m$ and it is mutually absolutely continuous with Lebesgue measure on L. Hence,

$$\sum_{j=0}^{n} q_j(x_1, \cdots, x_m) = 0, \quad (x_1, \cdots, x_m) \in L.$$

But the q_j have different degrees of homogeneity, and so $q_j \big| L = \mathbb{O}$, $0 \le j \le n$. Consequently, $q_j(g_1, \cdots, g_m) = \mathbb{O}$. ■

Let us now introduce the *Vick transform*, which orthogonalizes the subspaces G^n. For $f \in G^n$ the Vick transform $: f :$ is defined by

$$: f := \begin{cases} f - \mathcal{P}_n f, & n > 0, \\ f, & n = 0, \end{cases}$$

where $\mathcal{P}_n$ is the orthogonal projection onto $\operatorname{span}\{G^k : 0 \le k \le n-1\}$. The Vick transform is defined on the set of all polynomials of elements of $\boldsymbol{G}$ which is dense in $L^2(\Omega, \Sigma, P)$ by Lemma 2.3. We introduce the notation

$$\Gamma(\boldsymbol{G})_n \stackrel{\text{def}}{=} : G^n :, \quad n \in \mathbb{Z}_+.$$

It follows from Lemma 2.3 that

$$L^2(\Omega, \Sigma, P) = \bigoplus_{n \ge 0} \Gamma(\boldsymbol{G})_n. \tag{2.2}$$

Lemma 2.4. *Let f be a real function in $\boldsymbol{G}$. There exist real numbers $a_0, a_1, \cdots, a_{n-1}$ such that*

$$: f^n := f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0.$$

Proof. We choose $a_0, a_1, \cdots, a_{n-1}$ so that

$$f_n \stackrel{\text{def}}{=} f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 \ \perp \ f^j, \quad 0 \le j \le n-1.$$

Let $\boldsymbol{G}(f) \stackrel{\text{def}}{=} \boldsymbol{G} \ominus \{\lambda f : \lambda \in \mathbb{C}\}$. Since orthogonality in $\boldsymbol{G}$ is equivalent to independence, we have

$$\mathbb{E}(p(f)q(h)) = \mathbb{E}p(f)\mathbb{E}q(h), \quad h \in \boldsymbol{G}(f),$$

for any polynomials p and q.

Let us show that f_n is orthogonal to G^k, $0 \le k \le n-1$. By polarization formula (1.3), it is sufficient to prove that $\mathbb{E}(f_n(\lambda f + g)^k) = 0$ for $g \in \boldsymbol{G}(f)$, $\lambda \in \mathbb{C}$, and $0 \le k \le n-1$. We have

$$\begin{aligned} \mathbb{E}(f_n(\lambda f + g)^k) &= \sum_{j=0}^{k} \lambda^j \binom{k}{j} \mathbb{E}(f_n f^j g^{k-j}) \\ &= \sum_{j=0}^{k} \lambda^j \binom{k}{j} \mathbb{E}(f_n f^j) \mathbb{E} g^{k-j} = 0. \quad \blacksquare \end{aligned}$$

Remark. Let $\{H_n\}_{n \ge 0}$,

$$H_n(x) = x^n + a_{n-1}^{(n)} x^{n-1} + \cdots + a_1^{(n)} x + a_0^{(n)}, \quad n \in \mathbb{Z}_+,$$

be the sequence of *Hermite polynomials*, i.e., the sequence of orthogonal polynomials in the weighted space $L^2(\mathbb{R}, \exp(-x^2/2))$. If f is a real function in $\boldsymbol{G}$ and $\mathbb{E}f^2 = 1$, then it is easy to see that $: f^n := H_n(f)$.

Theorem 2.5. *Let n be a positive integer. Then the mapping*

$$: f_1 \cdots f_n : \to \operatorname{Sym}(f_1 \otimes \cdots \otimes f_n), \quad f_1, \cdots, f_n \in \boldsymbol{G},$$

extends uniquely to an isometry of $\Gamma(\boldsymbol{F})_n$ onto $\boldsymbol{G}^n_{\circledS}$.

Proof. By (2.1) we have to show that

$$\mathbb{E}(: g_1 \cdots g_n :: f_1 \cdots f_n :) = \sum_{\sigma \in \Sigma_n} \mathbb{E}(g_{\sigma_1} f_1) \cdots \mathbb{E}(g_{\sigma_n} f_n) \tag{2.3}$$

for $f_1, \cdots, f_n, g_1, \cdots, g_n \in \boldsymbol{G}$. Without loss of generality we may assume that the Gaussian variables are real. By (1.3), it suffices to prove (2.3) for $g_j = g \in \boldsymbol{G}$, $1 \le j \le n$, and $f_j = f \in \boldsymbol{G}$, $1 \le j \le n$. Thus we have to verify the formula

$$\mathbb{E}(: g^n :\, : f^n :) = n!(\mathbb{E}(fg))^n$$

for real functions f and g of norm 1. Clearly, $g = (\mathbb{E}(gf))f + h$, where $h \perp f$, and so h and f are independent Gaussian variables.

By the definition of the Vick transform, $\mathbb{E}(f^k : f^n :) = 0$ for $0 \le k < n$, and so

$$\mathbb{E}(: h^{n-k} f^k :\, : f^n :) = \mathbb{E}(h^{n-k} f^k \ : f^n :) = (\mathbb{E}h^{n-k})\mathbb{E}(f^k : f^n :) = 0. \tag{2.4}$$

Since $g = (\mathbb{E}(gf))f + h$, it follows from (2.4) that

$$\mathbb{E}(: g^n :\, : f^n :) = (\mathbb{E}(gf))^n \mathbb{E}(: f^n :\, : f^n :).$$

It follows easily from the remark after Lemma 2.4 that

$$\mathbb{E}(: f^n :\, : f^n :) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} H_n^2(x) e^{-x^2/2} dx. \tag{2.5}$$

The result follows from the well-known fact that the right-hand side of (2.5) is equal to $n!$ (see Jackson [1], Ch. IX). ■

Corollary 2.6. *Let $\{e_n\}_{n \ge 0}$ be an orthonormal basis in $\boldsymbol{G}$. Then the functions*

$$\frac{: e_{j_1}^{n_1} \cdots e_{j_k}^{n_k} :}{\sqrt{n_1! \cdots n_k!}}, \quad j_1 < j_2 < \cdots < j_k, \quad n_1 + \cdots + n_k = n,$$

form an orthonormal basis in $\Gamma(\boldsymbol{G})_n$.

The result follows immediately from (2.3).

Theorem 2.5 together with (2.2) allows us to identify $L^2(\Omega, \Sigma, P)$ with the Fock space $\boldsymbol{\mathcal{F}}(\boldsymbol{G})$. If $\boldsymbol{A}$ is a subspace of $\boldsymbol{G}$ and $\Sigma_{\boldsymbol{A}}$ is the σ-algebra that consists of the sets measurable with respect to $\boldsymbol{A}$, then it follows from Theorem 2.5 that $\boldsymbol{\mathcal{F}}(\boldsymbol{A})$ can be identified in a natural way with the subspace $L^2(\Omega, \Sigma_{\boldsymbol{A}}, P)$ of $\Sigma_{\boldsymbol{A}}$-measurable functions.

Let $P_{\boldsymbol{A}}$ be the orthogonal projection onto $\boldsymbol{A}$. Then $\boldsymbol{\mathcal{F}}(P_{\boldsymbol{A}})$ is the orthogonal projection onto $L^2(\Omega, \Sigma_{\boldsymbol{A}}, P)$. This orthogonal projection is denoted by $\mathbb{E}_{\Sigma_{\boldsymbol{A}}}$ and is called *conditional expectation with respect to* $\Sigma_{\boldsymbol{A}}$. $\mathbb{E}_{\Sigma_{\boldsymbol{A}}}$ is uniquely defined by the following properties:

$$\int_A f\, dP = \int_A \mathbb{E}_{\Sigma_{\boldsymbol{A}}} f\, dP, \quad \mathbb{E}_{\Sigma_{\boldsymbol{A}}} f \in L^2(\Omega, \Sigma_{\boldsymbol{A}}, P)$$

for every $f \in L^2(\Omega, \Sigma, P)$ and $A \in \Sigma_{\boldsymbol{A}}$. It is easy to see that $\mathbb{E}_{\Sigma_{\boldsymbol{A}}} f$ is the Radon–Nikodym derivative of the measure

$$A \mapsto \int_A f\, dP, \quad A \in \Sigma_{\boldsymbol{A}}.$$

We return now to stationary Gaussian processes. Let $\{X_n\}_{n\in\mathbb{Z}}$ be a stationary Gaussian process. We are going to find a realization of the probability space as a sequence space. Let Ω be the set of sequences

$$\{\omega_n\}_{n\in\mathbb{Z}}, \quad \omega_n \in \mathbb{R}.$$

Let Σ be the σ-algebra spanned by sets of the form

$$\Xi_{m,A} = \big\{\{\omega_n\}_{n\in\mathbb{Z}} : \ (\omega_{-m}, \omega_{-m+1}, \cdots, \omega_m) \in A\big\}, \tag{2.6}$$

where $m \in \mathbb{Z}_+$ and A is a Borel subset of $\mathbb{R}^{2m+1}$. We define $P(\Xi_{m,A})$ as the probability of the event that $(X_{-m}, \cdots, X_m) \in A$. It is well known that P extends uniquely to a probability measure on Σ (Kolmogorov's theorem, see Gikhman and Skorokhod [1]). It is easy to see that the coordinate functions g_n, $n \in \mathbb{Z}$,

$$g_n(\{\omega_j\}_{j\in\mathbb{Z}}) = \omega_n$$

form a Gaussian family and the Gaussian processes $\{X_n\}_{n\in\mathbb{Z}}$ and $\{\omega_n\}_{n\in\mathbb{Z}}$ have the same covariance sequences. Thus we can assume that the initial process $\{X_n\}_{n\in\mathbb{Z}}$ is realized on our sequence space Ω.

Consider the operator V on $\boldsymbol{G}$ defined by $VX_n = X_{n+1}$. Since the process $\{X_n\}_{n\in\mathbb{Z}}$ is stationary, it follows that V extends uniquely to a unitary operator on $\boldsymbol{G}$. Let us show that there exists a measure preserving automorphism $\tau : \Omega \to \Omega$ such that $(Vg)(\omega) = g(\tau\omega)$, $g \in \boldsymbol{G}$. (Recall that a measure preserving automorphism $\tau : \Omega \to \Omega$ is a bijection of Ω onto itself modulo a set of measure 0 such that $\tau^{-1}A \in \Sigma$ if an only if $A \in \Sigma$, and $P(\tau^{-1}A) = P(A)$ for $A \in \Sigma$.) Indeed, we can define τ on the sequence space Ω by

$$(\tau\omega)_n = \omega_{n+1}, \quad n \in \mathbb{Z}, \quad \omega = \{\omega_j\}_{j\in\mathbb{Z}}.$$

It is easy to see that τ is a measure preserving automorphism, since for sets $\Xi_{m,A}$ of the form (2.6) the equality

$$P(\tau^{-1}\Xi_{m,A}) = P(\Xi_{m,A})$$

follows immediately from the fact that the process $\{X_n\}_{n\in\mathbb{Z}}$ is stationary. The equality $(Vg)(\omega) = g(\tau\omega)$, $g \in \boldsymbol{G}$ is obvious. Using this formula, we can extend the operator V to a unitary operator U on $L^2(\Omega, \Sigma, P)$ by $Uf = f(\tau\omega)$, $f \in L^2(\Omega, \Sigma, P)$. It is easy to see that $U\Gamma(\boldsymbol{G}_m) \subset \Gamma(\boldsymbol{G}_m)$, $m \in \mathbb{Z}_+$, and $U = \boldsymbol{\mathcal{F}}(V)$ if we identify $L^2(\Omega, \Sigma, P)$ with $\boldsymbol{\mathcal{F}}(\boldsymbol{G})$.

Let μ be the spectral measure of our process and let Φ be its spectral representation defined by (1.5). Denote by M_z multiplication by z on $L^2(\mu)$. Clearly, $V = \Phi^* M_z \Phi$. Then

$$U = \boldsymbol{\mathcal{F}}(\Phi^*)\boldsymbol{\mathcal{F}}(M_z)\boldsymbol{\mathcal{F}}(\Phi). \tag{2.7}$$

3. Mixing Properties and Regularity Conditions

In this section we study the extrapolation problem for stationary Gaussian processes, i.e., the problem of when the future can be predicted by the past. We consider various regularity conditions for stationary processes (i.e., conditions of weak dependence of the future on the past) and their relationship with mixing properties of the measure preserving automorphism τ introduced in the previous section.

We introduce the following notation. Let $\{X_n\}_{n\in\mathbb{Z}}$ be a stationary Gaussian process. Put

$$\boldsymbol{G}_m = \operatorname{span}\{X_j : j < m\}, \quad \boldsymbol{G}^m = \operatorname{span}\{X_j : j \ge m\}.$$

The subspace $\boldsymbol{G}_m$ is called the *past of the process before time* m, the subspace $\boldsymbol{G}^m$ is called the *future of the process from time* m. The subspace $\boldsymbol{G}_0$ is called simply the *past* of the process while the subspace $\boldsymbol{G}^0$ is called the *future* of the process.

We also denote by Σ_m and Σ^m the σ-algebras of events that are measurable with respect to functions in $\boldsymbol{G}_m$ and $\boldsymbol{G}^m$, respectively.

Let μ be the spectral measure of the process. Consider the images of the past and the future under the spectral representation Φ defined by (1.5).

$$\Phi(\boldsymbol{G}_m) = z^m H^2_-(\mu) = \operatorname{span}_{L^2(\mu)}\{z^j : j < m\}$$

and

$$\Phi(\boldsymbol{G}^m) = z^m H^2(\mu) = \operatorname{span}_{L^2(\mu)}\{z^j : j \ge m\}.$$

We are going to study various conditions on a stationary process that describe the dependence of the future on the past.

Ergodicity

Recall that a measure preserving automorphism τ is called *ergodic* if the condition $\tau^{-1}A = A$, $A \in \Sigma$, implies that $P(A) = 0$ or $P(A) = 1$.

Let $\{X_n\}_{n\in\mathbb{Z}}$ be a stationary Gaussian process. Consider the corresponding measure preserving automorphism τ of (Ω, Σ, P). Denote by μ the spectral measure of the process. Recall that μ is said to be a continuous measure if $\mu(\{\zeta\}) = 0$ for any $\zeta \in \mathbb{T}$.

Theorem 3.1. *The transformation τ is ergodic if and only if the measure μ is continuous.*

Proof. Suppose that $\zeta_0 \in \mathbb{T}$ and $\mu(\{\zeta_0\}) > 0$. Then $g = \Phi^{-1}(\chi_{\{\zeta_0\}})$ is a nonzero function in $\boldsymbol{G}$. Let us apply the operator U, $(Uf)(\omega) = f(\tau\omega)$, $\omega \in \Omega$, $f \in L^2(\Omega, \Sigma, P)$. We have $Ug = \zeta_0 g$, and so $U|g| = |g|$. Obviously, the set $A = \{\omega \in \Omega : |g(\omega)| > 1\}$ is invariant under τ and $0 < P(A) < 1$.

Suppose now that μ is a continuous measure and f is a nonconstant function such that $Uf = f$. By (2.7), there exists a positive integer m and a nonzero $h \in \underbrace{L^2(\mathbb{T},\mu) ⓢ \cdots ⓢ L^2(\mathbb{T},\mu)}_{m}$ such that $(\underbrace{M_z \otimes \cdots \otimes M_z}_{m})h = h$. Hence,

$$((\underbrace{M_z \otimes \cdots \otimes M_z}_{m})^j h, h) = (h, h), \quad j \in \mathbb{Z}.$$

In other words,

$$\int_{\mathbb{T}^m} (\zeta_1 \cdots \zeta_m)^j |h(\zeta)|^2 d(\otimes^n \mu)(\zeta) = \int_{\mathbb{T}^m} |h(\zeta)|^2 d(\otimes^n \mu)(\zeta), \quad j \in \mathbb{Z}, \tag{3.1}$$

(here $\zeta = (\zeta_1 \cdots, \zeta_m)$).

Consider the map $D_m : \mathbb{T}^m \to \mathbb{T}$, $D_m(\zeta_1, \cdots, \zeta_m) = \zeta_1 \cdots \zeta_m$. Clearly, D_m transforms the measure $\bigotimes^m \mu$ to the mth convolution power $\mu^{*m} \stackrel{\text{def}}{=} \underbrace{\mu * \cdots * \mu}_{m}$ of μ on $\mathbb{T}$, i.e., $\mu^{*m}(E) = \bigotimes^m \mu(D_m^{-1}(E))$, $E \subset \mathbb{T}$. Consider now the image of the measure $|h|^2 d(\otimes^m \mu)$ under the map D_n. It is easy to see from (3.1) that it is the measure $\|h\|_\otimes^2 \delta_1$, where δ_1 is the unit point mass at 1. Since the measure $|h|^2 d(\otimes^m \mu)$ is absolutely continuous with respect to $\otimes^m \mu$, it follows that $\|h\|_\otimes^2 \delta_1$ is absolutely continuous with respect to μ^{*m}. Since obviously, the convolution of continuous measures is continuous, we get a contradiction. ■

Mixing

Consider a stronger property than ergodicity. We say that τ has the *mixing property* (or simply, is mixing) if

$$\lim_{n\to\infty} P(A \cap \tau^{-n} B) = P(A)P(B), \quad A, B \in \Sigma. \tag{3.2}$$

It is easy to see that τ is mixing if and only if

$$\lim_{n\to\infty} (U^n f, g) = \int_\Omega f\, dP \overline{\int_\Omega g\, dP}, \quad f, g \in L^2(\Omega, \Sigma, P). \tag{3.3}$$

Indeed, (3.2) means that (3.3) holds for all characteristic functions f and g. It remains to observe that the linear combinations of characteristic functions are dense in $L^2(\Omega, \Sigma, P)$.

Theorem 3.2. *The automorphism τ has the mixing property if and only if*

$$\lim_{n\to\infty} \hat{\mu}(n) = 0. \tag{3.4}$$

Proof. Suppose that τ is mixing. Then (3.3) holds for all f and g in $\boldsymbol{G}$. Let us apply the spectral representation Φ. We obtain

$$\lim_{n\to\infty} \int_{\mathbb{T}} \zeta^n \varphi(\zeta) \overline{\psi(\zeta)} d\mu(\zeta) = 0, \quad \varphi, \psi \in L^2(\mathbb{T}, \mu). \tag{3.5}$$

If we take $\varphi = \psi = \mathbf{1}$, we obtain (3.4).

Suppose now that (3.4) holds. Then (3.5) holds for trigonometric polynomials φ and ψ. Since the trigonometric polynomials are dense in $L^2(\mathbb{T},\mu)$, (3.5) holds for arbitrary φ and ψ in $L^2(\mathbb{T},\mu)$. Applying Φ^{-1}, we find that (3.3) holds for all Gaussian functions f and g, i.e.,

$$\lim_{n\to\infty}(V^n f, g) = 0, \quad f,\, g \in \boldsymbol{G}. \tag{3.6}$$

Since (3.3), obviously, holds for constants f and g, it remains to show that

$$\lim_{n\to\infty}\big((V^n f_1 \otimes \cdots \otimes V^n f_m), (g_1 \otimes \cdots \otimes g_m)\big)_{\otimes} = 0,$$

for any positive integer m and Gaussian variables $f_1, \cdots, f_m, g_1, \cdots, g_m$. However, this is an immediate consequence of (3.6). ■

Nondeterministic Processes

A Gaussian process $\{X_n\}_{n\in\mathbb{Z}}$ is called *deterministic* if $\Sigma_0 = \Sigma$, and it is called *nondeterministic* otherwise. Obviously, the process is deterministic if and only if $\boldsymbol{G}_0 = \boldsymbol{G}$. It is also very easy to see that if the process is deterministic, then $\boldsymbol{G}_m = \boldsymbol{G}$ and $\Sigma_m = \Sigma$ for any $m \in \mathbb{Z}$.

Consider the Lebesgue decomposition $\mu = \mu_{\rm s} + \mu_{\rm a}$ of the spectral measure μ, where $\mu_{\rm s}$ is singular and $\mu_{\rm a}$ is absolutely continuous with respect to Lebesgue measure. Let w be the Radon–Nikodym derivative of $\mu_{\rm a}$, i.e., $d\mu_{\rm a} = w\, d\boldsymbol{m}$.

Theorem 3.3. *The process $\{X_n\}_{n\ge 0}$ is nondeterministic if and only if*

$$\int_{\mathbb{T}} \log w\, d\boldsymbol{m} > -\infty. \tag{3.7}$$

The theorem follows immediately from the Szegö–Kolmogorov alternative (see Appendix 2.4).

Uniform Mixing with Respect to the Past

We say that τ is a *uniform mixing with respect to the past* if

$$\lim_{n\to\infty}\sup_{A\in\Sigma_0}|P(A\cap\tau^{-n}B) - P(A)P(B)| = 0 \quad \text{for any } B\in\Sigma.$$

Clearly, this condition is equivalent to the following:

$$\lim_{n\to-\infty}\sup_{A\in\Sigma_n}|P(A\cap B) - P(A)P(B)| = 0 \quad \text{for any } B\in\Sigma. \tag{3.8}$$

We introduce the following notation:

$$\xi_A \stackrel{\text{def}}{=} \chi_A - P(A), \quad A\in\Sigma.$$

We denote by $\mathcal{P}_n$ and $\mathcal{P}^n$ the orthogonal projections from $\boldsymbol{G}$ onto $\boldsymbol{G}_n$ and $\boldsymbol{G}^n$.

Theorem 3.4. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be a stationary Gaussian process. The following are equivalent:*

(i) (3.8) *holds;*

(ii) $\bigcap\limits_{n\in\mathbb{Z}} \Sigma_n = \{\varnothing, \Omega\}$*;*

(iii) $\bigcap\limits_{n\in\mathbb{Z}} \boldsymbol{G}_n = \{\mathbb{O}\}$*;*

(iv) $\lim\limits_{n\to\infty} \mathcal{P}_{-n} = \mathbb{O}$ *in the strong operator topology;*

(v) μ *is absolutely continuous with respect to* $\boldsymbol{m}$ *and its Radon–Nikodym derivative* w *has the form* $w = |h|^2$, *where* h *is an outer function in* H^2.

Proof. (i)⇒(ii). Suppose that $B \in \bigcap\limits_{n\in\mathbb{Z}} \Sigma_n$. Then setting $A = B$ in (3.8), we find that $(P(B))^2 = P(B)$.

(ii)⇒(iii). Suppose that $g \in \bigcap\limits_{n\in\mathbb{Z}} \boldsymbol{G}_n$ and $g \ne \mathbb{O}$. It follows that $\{\omega \in \Omega : |g(\omega)| < 1\} \in \bigcap\limits_{n\in\mathbb{Z}} \Sigma_n$.

(iii)⇒(iv). It is easy to see that $\mathcal{P} \stackrel{\text{def}}{=} \lim\limits_{n\to\infty} \mathcal{P}_{-n}$ is an orthogonal projection. If $g \in \operatorname{Range} \mathcal{P}$, then $g \in \bigcap\limits_{n\in\mathbb{Z}} \boldsymbol{G}_n$, and so $g = \mathbb{O}$.

(iv)⇒(i). We have $\boldsymbol{\mathcal{F}}(\mathcal{P}_n) = \mathbb{E}_{\Sigma_n}$. It is easy to see that

$$\lim_{n\to\infty} (\underbrace{\mathcal{P}_n \otimes \cdots \otimes \mathcal{P}_n}_{m})(g_1 \otimes \cdots \otimes g_m) = 0, \quad m > 0.$$

Hence,

$$\lim_{n\to-\infty} \|\mathbb{E}_{\Sigma_n} f\|_{L^2(\Omega,\Sigma,P)} = 0$$

for any $f \in L^2(\Omega, \Sigma, P)$ that is orthogonal to the constants. Let $B \in \Sigma$ and $A \in \Sigma_n$. We have

$$|\mathbb{E}(\xi_A \xi_B)| = |\mathbb{E}(\xi_A \mathbb{E}_{\Sigma_n} \xi_B)| \le \|\xi_A\|^{1/2}_{L^2(\Omega,\Sigma,P)} \|\mathbb{E}_{\Sigma_n} \xi_B\|^{1/2}_{L^2(\Omega,\Sigma,P)}.$$

The result follows now from the obvious identity

$$\mathbb{E}(\xi_A \xi_B) = P(A \cap B) - P(A)P(B).$$

(iii)⇔(v). This is an immediate consequence of the Szegö–Kolmogorov alternative (see Appendix 2.4). ■

Clearly, w admit a representation $w = |h|^2$ for an outer function h in H^2 if and only if w satisfies (3.7).

Note that if $\mu = \mu_{\mathrm{s}} + \mu_{\mathrm{a}}$, $d\mu_{\mathrm{a}} = w\, d\boldsymbol{m}$, and w satisfies (3.7), then it follows from the Szegö–Kolmogorov alternative that

$$\bigcap_{n\in\mathbb{Z}} \boldsymbol{G}_n = \Phi^{-1} L^2(\mathbb{T}, \mu_{\mathrm{s}}).$$

Stationary processes satisfying condition (iii) are called *regular*. The function w is called the *spectral density* of the regular process $\{X_n\}_{n\in\mathbb{Z}}$.

It is easy to see that the process is regular if and only if $\bigcap\limits_{n\in\mathbb{Z}} \boldsymbol{G}^n = \{\mathbb{O}\}$, which in turn is equivalent to the condition $\bigcap\limits_{n\in\mathbb{Z}} \Sigma^n = \{\varnothing, \Omega\}$.

Uniform Mixing with Respect to the Past and the Future

We say that τ is a *uniform mixing with respect to the past and the future* if

$$\lim_{n\to\infty} \sup_{A\in\Sigma_0, B\in\Sigma^0} |P(A\cap\tau^{-n}B) - P(A)P(B)| = 0.$$

Clearly, this condition is equivalent to the condition

$$\lim_{n\to\infty} \sup_{A\in\Sigma_0, B\in\Sigma^n} |P(A\cap B) - P(A)P(B)| = 0. \tag{3.9}$$

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are subspaces of $\boldsymbol{G}$, we denote the cosine of the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$ by $\cos(\widehat{\boldsymbol{A},\boldsymbol{B}})$,

$$\begin{aligned}\cos(\widehat{\boldsymbol{A},\boldsymbol{B}}) &= \sup\{|(a,b)|: \ \|a\| = \|b\| = 1, \ a\in\boldsymbol{A}, \ b\in\boldsymbol{B}\}\\ &= \operatorname{cor}_0(\boldsymbol{A},\boldsymbol{B}) = \|P_{\boldsymbol{A}}P_{\boldsymbol{B}}\|.\end{aligned}$$

We say that the subspaces $\boldsymbol{A}$ and $\boldsymbol{B}$ are at nonzero angle if $\cos(\widehat{\boldsymbol{A},\boldsymbol{B}}) < 1$.

If Ξ_1 and Ξ_2 are σ-subalgebras of Σ, put

$$k(\Xi_1,\Xi_2) \stackrel{\text{def}}{=} \sup_{A\in\Xi_1, B\in\Xi_2} |P(A\cap B) - P(A)P(B)|.$$

Theorem 3.5. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be subspaces of $\boldsymbol{G}$, and let $\Sigma_{\boldsymbol{A}}$ and $\Sigma_{\boldsymbol{B}}$ be the σ-algebras of events measurable with respect to $\boldsymbol{A}$ and $\boldsymbol{B}$. Then*

$$\begin{aligned}4k(\Sigma_{\boldsymbol{A}},\Sigma_{\boldsymbol{B}}) &\le \cos(\widehat{\boldsymbol{A},\boldsymbol{B}})\\ &= \cos(L^2(\Omega,\Sigma_{\boldsymbol{A}},P)\ominus\widehat{\mathbb{C}, L^2}(\Omega,\Sigma_{\boldsymbol{B}},P)\ominus\mathbb{C})\\ &\le \sin(2\pi k(\Sigma_{\boldsymbol{A}},\Sigma_{\boldsymbol{B}})).\end{aligned}$$

Recall that we identify the constant functions on Ω with $\mathbb{C}$.

Proof. We identify $L^2(\Omega,\Sigma_{\boldsymbol{A}},P)\ominus\mathbb{C}$ with $\boldsymbol{\mathcal{F}}_1(\boldsymbol{A}) \stackrel{\text{def}}{=} \boldsymbol{\mathcal{F}}(\boldsymbol{A})\ominus\mathbb{C}$ and $L^2(\Omega,\Sigma_{\boldsymbol{B}},P)\ominus\mathbb{C}$ with $\boldsymbol{\mathcal{F}}_1(\boldsymbol{B}) \stackrel{\text{def}}{=} \boldsymbol{\mathcal{F}}(\boldsymbol{B})\ominus\mathbb{C}$. Then

$$\cos(L^2(\Omega,\Sigma_{\boldsymbol{A}},P)\ominus\widehat{\mathbb{C}, L^2}(\Omega,\Sigma_{\boldsymbol{B}},P)\ominus\mathbb{C}) = \|P_{\boldsymbol{\mathcal{F}}_1(\boldsymbol{A})}P_{\boldsymbol{\mathcal{F}}_1(\boldsymbol{B})}\|.$$

Clearly,

$$P_{\boldsymbol{\mathcal{F}}_1(\boldsymbol{A})}P_{\boldsymbol{\mathcal{F}}_1(\boldsymbol{B})}f = \boldsymbol{\mathcal{F}}(P_{\boldsymbol{A}})\boldsymbol{\mathcal{F}}(P_{\boldsymbol{B}})f = \boldsymbol{\mathcal{F}}(P_{\boldsymbol{A}}P_{\boldsymbol{B}})f, \quad f\in\boldsymbol{\mathcal{F}}_1(\boldsymbol{G}).$$

It follows that

$$\|P_{\boldsymbol{\mathcal{F}}_1(\boldsymbol{A})}P_{\boldsymbol{\mathcal{F}}_1(\boldsymbol{B})}\| \le \|P_{\boldsymbol{A}}P_{\boldsymbol{B}}\|.$$

The opposite inequality is obvious.

Let us estimate $k(\Sigma_{\boldsymbol{A}}, \Sigma_{\boldsymbol{B}})$. We have

$$\begin{aligned} k(\Sigma_{\boldsymbol{A}}, \Sigma_{\boldsymbol{B}}) &\le \sup_{A\in\Sigma_{\boldsymbol{A}}, B\in\Sigma_{\boldsymbol{B}}} |\mathbb{E}(\xi_A\xi_B)| \\ &\le \cos(\widehat{\boldsymbol{A}, \boldsymbol{B}}) \sup_{A\in\Sigma_{\boldsymbol{A}}, B\in\Sigma_{\boldsymbol{B}}} (\mathbb{E}\xi_A^2)^{1/2}(\mathbb{E}\xi_B^2)^{1/2} = \frac{1}{4}\cos(\widehat{\boldsymbol{A}, \boldsymbol{B}}), \end{aligned}$$

since it is very easy to see that $\sup\limits_A \mathbb{E}\xi_A^2 = 1/4$.

Finally, let $f \in \boldsymbol{A}$ and $g \in \boldsymbol{B}$ be Gaussian variables such that $\mathbb{E}|f|^2 = 1$, $\mathbb{E}|g|^2 = 1$, and

$$|\mathbb{E}fg| \ge \cos(\widehat{\boldsymbol{A}, \boldsymbol{B}}) - \varepsilon,$$

where $\varepsilon > 0$. Put

$$A = \{\omega \in \Omega : f(\omega) > 0\}, \quad B = \{\omega \in \Omega : g(\omega) > 0\}.$$

It is easy to see that $P(A) = P(B) = 1/2$. If can be shown by direct computations from (1.2) (with $f_1 = f$ and $f_2 = g$) that

$$P(A \cap B) = \frac{1}{4} + \frac{1}{2\pi}\arcsin|\mathbb{E}fg|.$$

Hence,

$$P(A \cap B) - P(A)P(B) = \frac{1}{2\pi}\arcsin|\mathbb{E}fg|,$$

and so

$$\cos(\widehat{\boldsymbol{A}, \boldsymbol{B}}) \le \sin(2\pi k(\Sigma_{\boldsymbol{A}}, \Sigma_{\boldsymbol{B}})). \quad \blacksquare$$

Corollary 3.6. $k(\Sigma_{\boldsymbol{A}}, \Sigma_{\boldsymbol{B}}) > 0$ *if and only if* $\cos(\widehat{\boldsymbol{A}, \boldsymbol{B}}) > 0$. $k(\Sigma_{\boldsymbol{A}}, \Sigma_{\boldsymbol{B}}) < 1/4$ *if and only if* $\cos(\widehat{\boldsymbol{A}, \boldsymbol{B}}) < 1$.

Let us return to a stationary Gaussian process $\{X_n\}_{n\in\mathbb{Z}}$. We introduce the sequence

$$\rho_n = \cos(\widehat{\boldsymbol{G}_0, \boldsymbol{G}^n}) = \operatorname{cor}_0(\boldsymbol{G}_0, \boldsymbol{G}^n), \quad n \in \mathbb{Z}_+. \tag{3.10}$$

The process $\{X_n\}_{n\in\mathbb{Z}}$ is called *completely regular* if

$$\lim_{n\to\infty} \rho_n = 0.$$

Theorem 3.7. *Let* $\{X_n\}_{n\in\mathbb{Z}}$ *be a stationary Gaussian process. The following are equivalent:*

(i) $\{X_n\}_{n\in\mathbb{Z}}$ *is completely regular;*

(ii) τ *is a uniform mixing with respect to the past and the future;*

(iii) $\lim\limits_{n\to\infty} k(\Sigma_0, \Sigma^n) = 0$.

This is an immediate consequence of Theorem 3.5.

In Chapter 9 we characterize the completely regular processes in terms of their spectral measures.

Fast Mixing

Here we consider the processes for which the sequence (3.10) decreases rapidly. The process is called *completely regular of order* α, $0 < \alpha < \infty$, if

$$\rho_n \le \mathrm{const}(1+n)^{-\alpha}.$$

By Theorem 3.5, this is equivalent to the fact that

$$k(\Sigma_0, \Sigma^n) \le \mathrm{const}(1+n)^{-\alpha}.$$

Completely regular processes of order α are characterized in Chapter 9 in terms of their spectral measures.

We also characterize in Chapter 9 the processes with exponential decay of the ρ_n. As the limit case we consider there the processes for which Σ^n is independent of Σ_0 for some $n \in \mathbb{Z}_+$.

Canonical Correlations of the Past and the Future

Let $\{X_n\}_{n\in\mathbb{Z}}$ be a *regular* stationary Gaussian process. Consider the sequence $\{\mathrm{cor}_m(\boldsymbol{G}_0, \boldsymbol{G}^0)\}_{m\ge 0}$ of canonical correlations of the past $\boldsymbol{G}_0$ and the future $\boldsymbol{G}^0$. We can consider the class of stationary Gaussian processes for which

$$\lim_{m\to\infty} \mathrm{cor}_m(\boldsymbol{G}_0, \boldsymbol{G}^0) = 0.$$

However, it turns out that these are precisely the completely regular processes. We shall prove this in Chapter 9.

We impose now a stronger condition on the sequence of canonical correlations. We say that a regular stationary Gaussian process is *p-regular*, $0 < p < \infty$, if

$$\{\mathrm{cor}_m(\boldsymbol{G}_0, \boldsymbol{G}^0)\}_{m\ge 0} \in \ell^p. \tag{3.11}$$

Clearly, (3.11) is equivalent to the condition $\mathcal{P}_0\mathcal{P}^0 \in \boldsymbol{S}_p$.

Note that the operators $\mathcal{P}_0\mathcal{P}^0$ and $\mathcal{P}_0\mathcal{P}^n$ differ from each other by a finite rank operator. Hence, if the process is p-regular, then $\mathcal{P}_0\mathcal{P}^n \in \boldsymbol{S}_p$ for every $n \in \mathbb{Z}_+$.

Theorem 3.8. *A stationary Gaussian process $\{X_n\}_{n\in\mathbb{Z}}$ is p-regular, $0 < p < \infty$, if and only if there exists $N \in \mathbb{Z}_+$ such that $\mathbb{E}_{\Sigma_0}\mathbb{E}_{\Sigma^n} \in \boldsymbol{S}_p$ for $n \ge N$.*

We need the following lemma.

Lemma 3.9. *Let $T \in \boldsymbol{S}_p$, $0 < p < \infty$. Then for any positive integer n*

$$\left\| \bigotimes^n T \right\|_{\boldsymbol{S}_p} = \|T\|_{\boldsymbol{S}_p}^n.$$

Proof. Let $T = \sum_{j\ge 0} s_j(\cdot,\xi_j)\eta_j$ be the Schmidt expansion of T. It is easy to see that

$$\bigotimes^n T = \sum_{j_1,\cdots,j_n\ge 0} s_{j_1}\cdots s_{j_n}(\cdot,\xi_{j_1}\otimes\cdots\otimes\xi_{j_n})\eta_{j_1}\otimes\cdots\otimes\eta_{j_n}$$

is the Schmidt expansion of $\bigotimes^n T$. Hence,

$$\left\|\bigotimes^n T\right\|_{\boldsymbol{S}_p}^p = \sum_{j_1,\cdots,j_n\ge 0}(s_{j_1}\cdots s_{j_n})^p = \left(\sum_{j\ge 0} s_j^p\right)^n = \|T\|_{\boldsymbol{S}_p}^{pn}. \quad \blacksquare$$

Corollary 3.10. *If* $T\in\boldsymbol{S}_p$, $0<p<\infty$, *and* $\|T\|_{\boldsymbol{S}_p}<1$, *then* $\boldsymbol{\mathcal{F}}(T)\in\boldsymbol{S}_p$ *and* $\|\boldsymbol{\mathcal{F}}(T)\|_{\boldsymbol{S}_p}\le(1-\|T\|_{\boldsymbol{S}_p}^p)^{-1/p}$.

This follows immediately from Lemma 3.9.

Proof of Theorem 3.8. Suppose that the process is p-regular. Consider the operators $\mathcal{P}_0\mathcal{P}^n$, $n\in\mathbb{Z}_+$. Since the process is regular, the sequence $\{\mathcal{P}^n\}_{n\ge 0}$ tends to $\mathbb{O}$ in the strong operator topology. Therefore

$$\lim_{n\to\infty}\|\mathcal{P}_0\mathcal{P}^n\|_{\boldsymbol{S}_p} = \|(\mathcal{P}_0\mathcal{P}^0)\mathcal{P}^n\|_{\boldsymbol{S}_p} = 0.$$

Let m be a positive integer for which $\|\mathcal{P}_0\mathcal{P}^m\|_{\boldsymbol{S}_p}<1$. By Corollary 3.10, $\boldsymbol{\mathcal{F}}(\mathcal{P}_0\mathcal{P}^m) = \mathbb{E}_{\Sigma_0}\mathbb{E}_{\Sigma^m}\in\boldsymbol{S}_p$.

Suppose now that $\mathbb{E}_{\Sigma_0}\mathbb{E}_{\Sigma^m} = \boldsymbol{\mathcal{F}}(\mathcal{P}_0\mathcal{P}^m)\in\boldsymbol{S}_p$. Clearly, this implies that $\mathcal{P}_0\mathcal{P}^m\in\boldsymbol{S}_p$. It remains to show that the process is regular. Assume the contrary. By the Szegö–Kolmogorov alternative, there exists a nonzero subspace $\boldsymbol{L}$ of $\boldsymbol{G}$ such that $\boldsymbol{L}\subset\boldsymbol{G}_0\cap\boldsymbol{G}^n$ for every $n\in\mathbb{Z}_+$. It follows that $\|\mathcal{P}_0\mathcal{P}^n\|\ge 1$ for any $n\in\mathbb{Z}_+$. Therefore

$$\|\underbrace{\mathcal{P}_0\mathcal{P}^n ⓢ\cdots ⓢ\mathcal{P}_0\mathcal{P}^n}_{j}\|\ge 1$$

for any $j\in\mathbb{Z}_+$. Hence, $\mathbb{E}_{\Sigma_0}\mathbb{E}_{\Sigma^n}\notin\boldsymbol{S}_p$ for any $n\in\mathbb{Z}_+$. $\blacksquare$

In Chapter 9 we characterize the p-regular processes in terms of their spectral measures.

Consider now the case of 2-regular stationary Gaussian processes. Such processes are also called *absolutely regular*. They admit the following characterization.

Theorem 3.11. *A stationary Gaussian process is absolutely regular if and only if*

$$\lim_{n\to\infty}\mathbb{E}\sup_{A\in\Sigma^n}|\mathbb{E}_{\Sigma_0}\xi_A|^2 = 0. \tag{3.12}$$

We can understand the supremum in (3.12) as the supremum in the space of measurable functions.

It is easy to see that (3.12) is equivalent to the condition

$$\lim_{n\to\infty}\mathbb{E}\sup_{A\in\Sigma^n}|\mathbb{E}_{\Sigma_0}\xi_A|^p = 0,\quad p<\infty.$$

We need the following lemma.

Lemma 3.12. *Let T be an operator on $L^2(\Omega, \Sigma, P)$. Then $T \in \boldsymbol{S}_2$ if and only if*

$$\int_\Omega \sup_{\|f\|_2 \le 1} |(Tf)(\omega)|^2 dP(\omega) < \infty. \tag{3.13}$$

Moreover, the integral in (3.13) *is equal to $\|T\|^2_{\boldsymbol{S}_2}$.*

Proof. If $T \in \boldsymbol{S}_2$, then

$$Tf = \sum_j s_j (f, \xi_j)\eta_j,$$

where $\{\xi_j\}$ and $\{\eta\}_j$ are orthonormal systems. We have

$$\begin{aligned} \sup_{\|f\|_2 \le 1} |(Tf)(\omega)|2 &\le \sup_{\|f\|_2 \le 1} \left(\sum_j |(f, \xi_j)|^2 \right) \left(\sum_j s_j^2 |\eta_j(\omega)|^2 \right) \\ &= \left(\sum_j s_j^2 |\eta_j(\omega)|^2 \right). \end{aligned}$$

Hence,

$$\int_\Omega \sup_{\|f\|_2 \le 1} |Tf|^2 dP = \sum_j s_j^2 = \|T\|^2_{\boldsymbol{S}_2} < \infty.$$

Suppose now that (3.13) holds. Let $\mathcal{P}$ be a finite rank orthogonal projection. Then $T\mathcal{P} \in \boldsymbol{S}_2$. We have

$$\sup_{\|f\|_2 \le 1} |(T\mathcal{P}f)(\omega)|^2 \le \sup_{\|f\|_2 \le 1} |(Tf)(\omega)|^2.$$

Therefore

$$\|T\mathcal{P}\|_{\boldsymbol{S}_2} \le \int_\Omega \sup_{\|f\|_2 \le 1} |Tf|^2 dP.$$

It follows that $T \in \boldsymbol{S}_2$. ■

Proof of Theorem 3.11. Suppose that $\mathbb{E}_{\Sigma_0}\mathbb{E}_{\Sigma^m} \in \boldsymbol{S}_2$ for some $m \in \mathbb{Z}_+$. Then

$$\lim_{j \to \infty} \|\mathcal{P}_{\boldsymbol{G}_0} \mathcal{P}_{\boldsymbol{G}^j}\|_{\boldsymbol{S}_2} = 0$$

(see the proof of Theorem 3.8). Put

$$T_j = \mathbb{E}_{\Sigma_0}\mathbb{E}_{\Sigma^j} P_{L^2(\Omega) \ominus \mathbb{C}}.$$

By Lemma 3.10, $\lim_{j \to \infty} \|T_j\|_{\boldsymbol{S}_2} = 0$. Applying Lemma 3.12, we find that

$$\mathbb{E} \sup_{A \in \Sigma^j} |\mathbb{E}_{\Sigma_0} \xi_A|^2 \le \|T_j\|^2_{\boldsymbol{S}_2} \|\xi_A\|_{L^2} \le \frac{1}{4} \|T_j\|^2_{\boldsymbol{S}_2} \to 0 \quad \text{as} \quad j \to \infty.$$

Suppose now that (3.12) holds. Let $\boldsymbol{L}^m$ be a finite-dimensional subspace of $\boldsymbol{G}^m$. Consider the Schmidt expansion of the finite rank operator $P_{\boldsymbol{G}_0}P_{\boldsymbol{L}^m}$:

$$P_{\boldsymbol{G}_0}P_{\boldsymbol{L}^m}h = \sum_{k=0}^{d} s_k(h, f_k)g_k, \quad h \in \boldsymbol{L}^m.$$

Clearly, we can assume that the Gaussian variables f_k and g_k are real.

Let us show that

$$(f_k, g_j) = \begin{cases} s_k, & k = j, \\ 0, & k \neq j. \end{cases}$$

Indeed,

$$(f_k, f_j) = s_j^{-1}(f_k, P_{\boldsymbol{G}_0}g_j) = s_j^{-1}(P_{\boldsymbol{G}_0}f_k, g_j) = s_j^{-1}(f_k, g_j) = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}$$

Let p_{fg}, p_f, and p_g be the densities of the joint distribution of the random variables $f_0, \cdots, f_d, g_0, \cdots, g_d$, $f_0, \cdots, f_d$, and $g_0, \cdots, g_d$, respectively. We denote by $\Sigma(f)$ and $\Sigma(g)$ the σ-subalgebras of Σ generated by $f_0, \cdots, f_d$ and $g_0, \cdots, g_d$. We have

$$\begin{aligned}
\mathbb{E} \sup_{A \in \Sigma^m} |\mathbb{E}_{\Sigma_0}\xi_A| &\geq \mathbb{E} \sup_{A \in \Sigma(f)} |\mathbb{E}_{\Sigma_0}\xi_A| \\
&\geq \int_{\mathbb{R}^{d+1}} p_g(y) \left(\sup_{Q \subset \mathbb{R}^{d+1}} \int_Q \left(\frac{p_{fg}(x,y)}{p_g(y)} - p_f(x) \right) dx \right) dy \\
&= \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} \left(p_{fg}(x,y) - p_f(x)p_g(y) \right)^+ dx\, dy \\
&= \frac{1}{2} \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} |(p_{fg}(x,y) - p_f(x)p_g(y)| dx\, dy \stackrel{\text{def}}{=} K,
\end{aligned}$$

where as usual $\varphi^+(t) \stackrel{\text{def}}{=} \max\{\varphi(t), 0\}$.

Put

$$\varphi = \log \frac{p_{fg}}{p_f p_g}.$$

It can be shown by elementary computations that the integral

$$M \stackrel{\text{def}}{=} \int\int e^{\mathrm{i}\varphi} p_a p_b \, dx\, dy$$

satisfies the identity

$$|M|^2 = \prod_{k=0}^{d} \left| 1 + (2\mathrm{i} + 1) \frac{s_k^2}{1 - s_k^2} \right|.$$

We need the following elementary lemma.

Lemma 3.13. *Let $s \in \mathbb{R}$. Then $|e^{\mathrm{i}s} - 1| \leq 3|e^s - 1|$.*

Let us first complete the proof of Theorem 3.11.

Since

$$2K = \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} |e^{\varphi} - 1| p_f p_g \, dx \, dy,$$

we have by Lemma 3.13,

$$\begin{aligned} |M| &\geq 1 - \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} |e^{\mathrm{i}\varphi} - 1| p_f p_g \, dx \, dy \\ &\geq 1 - 3 \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} |e^{\varphi} - 1| p_f p_g \, dx \, dy = 1 - 6K. \end{aligned}$$

Hence,

$$(1 - 6K)^{-2} \geq |M|^{-2} \geq \prod_{k=0}^{d} (1 + s_k^2) \geq 1 + \sum_{k=0}^{d} s_k^2.$$

Suppose now that m is such that $K < 1/12$. Then

$$\|P_{\boldsymbol{G}_0} P_{\boldsymbol{L}^m}\|_{\boldsymbol{S}_2} = \sum_{k=0}^{d} s_k^2 \leq \text{const}$$

for an arbitrary subspace $\boldsymbol{L}^m$ of $\boldsymbol{G}^m$. It follows that $P_{\boldsymbol{G}_0} P_{\boldsymbol{G}^m} \in \boldsymbol{S}_2$.

It remains to show that the process is regular. Assume the contrary. Suppose that $A \in \Sigma_0 \cap \Sigma^n$, $n \geq 0$, and $0 < P(A) < 1$. Then $\mathbb{E}|\mathbb{E}_{\Sigma_0} \xi_A| = \mathbb{E}|\xi_A| > 0$, which contradicts (3.12). ■

Proof of Lemma 3.13. Let $s \geq -2\log 3$. We have

$$\left| \frac{e^{is} - 1}{e^s - 1} \right| = e^{-s/2} \left| \frac{\sin s/2}{\sinh s/2} \right| \leq e^{-s/2} \leq 3.$$

If $s < -2\log 3$, then

$$|e^{\mathrm{i}s} - 1| \leq 2 < 3\left(1 - \frac{1}{9}\right) < 3(1 - e^s). \quad ■$$

We can consider a condition stronger than (3.12):

$$\alpha_n \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{\omega \in \Omega} \sup_{A \in \Sigma^n} |(\mathbb{E}_{\Sigma_0} \xi_A)(\omega)| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.14}$$

It turns out, however, that this condition does not distinguish a nontrivial class of processes.

Theorem 3.14. *If a stationary Gaussian process satisfies* (3.14)*, then there exists $n \in \mathbb{Z}_+$ such that σ-algebras Σ_0 and Σ^n are independent.*

Such processes will be described in terms of their spectral measure in Chapter 9.

We need the following lemma.

Lemma 3.15.

$$\alpha_n = \sup_{A \in \Sigma^n,\, B \in \Sigma_0,\, P(B) > 0} \frac{|P(A \cap B) - P(A)P(B)|}{P(B)}.$$

Proof. If $A \in \Sigma^n$, $B \in \Sigma_0$, then

$$|P(A \cap B) - P(A)P(B)| = |\mathbb{E}(\chi_B \mathbb{E}_{\Sigma_0} \xi_A)| \le \alpha_n \|\chi_B\|_{L^1} = \alpha_n P(B).$$

Let us prove the opposite inequality. It is easy to see that

$$\alpha_n = \operatorname*{ess\,sup}_{\omega \in \Omega} \sup_{A \in \Sigma^n} (\mathbb{E}_{\Sigma_0} \xi_A)(\omega).$$

Pick $A \in \Sigma^n$ such that

$$\operatorname*{ess\,sup}_{\omega \in \Omega} (\mathbb{E}_{\Sigma_0} \xi_A)(\omega) > \alpha_n - \varepsilon$$

and put

$$B = \{\omega \in \Omega : \ (\mathbb{E}_{\Sigma_0} \xi_A)(\omega) > \alpha_n - \varepsilon\}.$$

We have

$$P(A \cap B) = \mathbb{E}(\chi_A \chi_B) = \mathbb{E}((\mathbb{E}_{\Sigma_0} \chi_A)\chi_B) = \mathbb{E}((\mathbb{E}_{\Sigma_0} \xi_A)\chi_B) + P(A)P(B).$$

Hence,

$$P(A \cap B) - P(A)P(B) = \mathbb{E}((\mathbb{E}_{\Sigma_0} \xi_A)\chi_B) \ge (\alpha_n - \varepsilon)P(B). \quad \blacksquare$$

Proof of Theorem 3.14. Assume the contrary. Choose $n \in \mathbb{Z}_+$ such that $\alpha_n < (2\pi e)^{-1/2}$. Let $f \in \boldsymbol{G}^n$ and $g \in \boldsymbol{G}_0$ be real Gaussian variables such that $\mathbb{E}f^2 = \mathbb{E}g^2 = 1$, and $\mathbb{E}(fg) = \rho > 0$. Let p_f, p_g, and p_{fg} be the distributions of f, g, and (f, g). Put

$$B_k = \{\omega \in \Omega : \ g(\omega) \ge k/\rho\} \quad \text{and} \quad A = \{\omega \in \Omega : \ f(\omega) \in [0,1]\},$$

where $k > 0$. Then

$$\begin{aligned}
&P(A \cap B_k) - P(A)P(B_k) \\
&= \int_{k/\rho}^{\infty} \int_0^1 (p_{fg}(x,y) - p_f(x)p_g(y))dx\,dy \\
&= \frac{1}{2\pi} \int_{k/\rho}^{\infty} e^{-y^2/2} \int_0^1 \left(\frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2} \frac{(x-\rho y)^2}{1-\rho^2} \right) - e^{-x^2/2} \right) dx\,dy.
\end{aligned}$$

For $y \ge k/\rho$ and $x \in [0,1]$ we have

$$\begin{aligned}
&\frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2} \frac{(x-\rho y)^2}{1-\rho^2} \right) - e^{-x^2/2} \\
&\ge \frac{1}{\sqrt{e}} \left| 1 - \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2} \frac{(k-1)^2}{1-\rho^2} + \frac{1}{2} \right) \right|.
\end{aligned}$$

Keeping in mind that

$$P(B_k) = \frac{1}{\sqrt{2\pi}} \int_{k/\rho} e^{-y^2/2} dy,$$

we obtain

$$\begin{aligned}
\alpha_n &\geq \sup_k \frac{|P(A \cap B_k) - P(A)P(B_k)|}{P(B_k)} \\
&\geq \frac{1}{\sqrt{2\pi e}} \sup_k \left| 1 - \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\frac{(k-1)^2}{1-\rho^2} + \frac{1}{2} \right) \right| = \frac{1}{\sqrt{2\pi e}}.
\end{aligned}$$

We have obtained a contradiction. ■

A stronger condition than p-regularity is p-regularity of order α. A stationary Gaussian process $\{X_n\}_{n\in\mathbb{Z}}$ is called *p-regular of order α*, $0 < \alpha < \infty$, if

$$\|\mathcal{P}_0 \mathcal{P}^n\|_{\boldsymbol{S}_p} \leq \mathrm{const}(1+n)^{-\alpha}, \quad n \in \mathbb{Z}_+.$$

In Chapter 9 we describe such processes in spectral terms. We also consider in Chapter 9 processes for which $\mathcal{P}_0\mathcal{P}^0$ has finite rank, i.e., only finitely many canonical correlations of the future and the past are nonzero.

4. Minimality and Basisness

In this section we study the interpolation property for stationary processes. In other words, for a stationary Gaussian process $\{X_n\}_{n\in\mathbb{Z}}$ we study the problem of when X_0 can be predicted by the past $\boldsymbol{G}_0$ and the future $\boldsymbol{G}^1$.

We say that a regular stationary Gaussian process is *minimal* if

$$X_0 \notin \mathrm{span}\{X_j : \; j \in \mathbb{Z}, \; j \neq 0\}.$$

We also consider stronger conditions, namely basisness and unconditional basisness.

Suppose that $\{X_n\}_{n\in\mathbb{Z}}$ is a minimal stationary process. For $n \in \mathbb{Z}$ consider the best prediction $\hat{X}_n$ of X_n from the Gaussian variables X_j, $j \neq n$, i.e., $\hat{X}_n$ is the projection of X_n onto

$$\mathrm{span}\{X_j : \; j \neq n\}.$$

Let $Y_n \stackrel{\mathrm{def}}{=} X_n - \hat{X}_n$ be the error in this prediction. The sequence $\{Y_n\}_{n\in\mathbb{Z}}$ is called the *interpolation error process.*

Theorem 4.1. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be a minimal stationary process. Then $\{Y_n\}_{n\in\mathbb{Z}}$ is a stationary process.*

Proof. Consider the unitary operator V on $\boldsymbol{G}$ introduced in §2 which is defined by $VX_j = X_{j+1}$, $j \in \mathbb{Z}$. Then it is easy to see that $V\hat{X}_j = \hat{X}_{j+1}$ and $VY_j = Y_{j+1}$, $j \in \mathbb{Z}$. It follows that

$$(Y_{n+1}, Y_{k+1}) = (VY_n, VY_k) = (Y_n, Y_k), \quad n, \, k \in \mathbb{Z},$$

and so the process $\{Y_n\}_{n\in\mathbb{Z}}$ is stationary. ■

In Chapter 9 we describe the minimal processes in spectral terms and we find the spectral density of the interpolation error process in terms of the spectral density of the initial process.

We proceed to a condition stronger than minimality. We consider the class of stationary Gaussian processes $\{X_n\}_{n\in\mathbb{Z}}$ for which the sequence $\{X_n\}_{n\in\mathbb{Z}}$ forms a *basis* in the Hilbert space $\boldsymbol{G}$. This means that for each $f \in \boldsymbol{G}$ there exists a unique sequence $\{\beta_j\}_{j\in\mathbb{Z}}$ of complex numbers such that

$$f = \lim_{k\to\infty,\, m\to-\infty} \sum_{j=m}^{k} \beta_j X_j.$$

Define the linear functionals λ_j on $\boldsymbol{G}$ by

$$\lambda_j(f) = \beta_j. \tag{4.1}$$

Basisness is equivalent to the important property that the past and the future are at nonzero angle.

Theorem 4.2. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be a stationary Gaussian process. The following are equivalent:*

(i) *the sequence $\{X_n\}_{n\in\mathbb{Z}}$ forms a basis;*

(ii) *the subspaces $\boldsymbol{G}_0$ and $\boldsymbol{G}^0$ are at nonzero angle;*

(iii)

$$\sup_{A\in\Sigma_0,\, B\in\Sigma^0} |P(A\cap B) - P(A)P(B)| < 1/4.$$

Proof. (i)$\Rightarrow$(ii). Suppose that the sequence $\{X_n\}_{n\in\mathbb{Z}}$ forms a basis in $\boldsymbol{G}$. It follows easily from the closed graph theorem that the linear functionals λ_j defined by (4.1) are continuous and it follows from the Banach–Steinhaus theorem that the norms of the linear operators

$$f \mapsto \sum_{j=m}^{k} \lambda_j(f) X_j$$

are uniformly bounded. Hence, the operator

$$f \mapsto \lim_{k\to\infty} \sum_{j=0}^{k} \lambda_j(f) X_j$$

defined on the linear combinations of the X_j extends by continuity to a bounded projection $\mathcal{Q}^{\mathrm{f}}$ onto the future $\boldsymbol{G}^0$. It is easy to see that $\operatorname{Ker} \mathcal{Q}^{\mathrm{f}} = \boldsymbol{G}_0$. It is also evident that $\mathcal{Q}_{\mathrm{p}} \stackrel{\mathrm{def}}{=} I - \mathcal{Q}^{\mathrm{f}}$ is a projection onto the past $\boldsymbol{G}_0$. Therefore the angle between $\boldsymbol{G}_0$ and $\boldsymbol{G}^0$ is nonzero.

(ii)$\Rightarrow$(i). Suppose now that the angle between $\boldsymbol{G}_0$ and $\boldsymbol{G}^0$ is nonzero. Then $\boldsymbol{G}_0 + \boldsymbol{G}^0 = \boldsymbol{G}$ and there exists a projection $\mathcal{Q}^{\mathrm{f}}$ onto $\boldsymbol{G}^0$ such that $\operatorname{Ker} \mathcal{Q}^{\mathrm{f}} = \boldsymbol{G}_0$. Let V be the "translation" operator on $\boldsymbol{G}$ defined by $VX_n = X_{n+1}$. Then $V^k \mathcal{Q}^{\mathrm{f}} V^{-k}$ is a projection onto $\boldsymbol{G}^k$, $k \in \mathbb{Z}$. Define the linear functionals λ_j, $j \in \mathbb{Z}$, on $\boldsymbol{G}$

$$\lambda_j f = V^j \mathcal{Q}^{\mathrm{f}} V^{-j} f - V^{j+1} \mathcal{Q}^{\mathrm{f}} V^{-j-1} f.$$

It is easy to see that

$$\sum_{j=m}^{k} \lambda_j(f)X_j = V^m \mathcal{Q}^{\mathrm{f}} V^{-m} f - V^{k+1}\mathcal{Q}^{\mathrm{f}} V^{-k-1} f.$$

Hence, the linear operators

$$f \mapsto \sum_{j=m}^{k} \lambda_j(f)X_j$$

are uniformly bounded, and so $\{X_j\}_{j\in\mathbb{Z}}$ is a basis sequence.

(ii)$\Leftrightarrow$(iii). This is an immediate consequence of Corollary 3.6. ∎

Note that if $\{X_n\}_{n\in\mathbb{Z}}$ is a basis, then the process is regular. In Chapter 9 we characterize the stationary processes with basisness property in spectral terms.

Consider now the class of stationary processes $\{X_n\}_{n\in\mathbb{Z}}$ for which the sequence $\{X_n\}_{n\in\mathbb{Z}}$ forms an unconditional basis in the Hilbert space $\boldsymbol{G}$. This means that

$$\lim_{E} \sum_{j\in E} \lambda_j(f)X_j = f, \quad \text{for any} \quad f \in \boldsymbol{G},$$

where the limit is taken over all finite subsets E of $\mathbb{Z}$; the finite subsets are ordered by inclusion. The following theorem characterizes the processes with unconditional basisness property.

Theorem 4.3. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be a stationary process with spectral measure μ. The sequence $\{X_n\}_{n\in\mathbb{Z}}$ forms an unconditional basis if and only if μ is absolutely continuous with respect to Lebesgue measure and its Radon–Nikodym derivative w satisfies the conditions*

$$w \in L^\infty, \quad w^{-1} \in L^\infty. \tag{4.2}$$

Proof. Suppose that μ is absolutely continuous and (4.2) holds. Then as a set $L^2(w)$ coincides with $L^2(\mathbb{T}, \boldsymbol{m})$. It follows that the sequence $\{z^n\}_{n\in\mathbb{Z}}$ forms an unconditional basis in $L^2(w)$. Consider now the spectral representation of the process $\Phi : \boldsymbol{G} \to L^2(w)$, $\Phi X_n = z^n$, $n \in \mathbb{Z}$. Since Φ is a unitary operator, it follows that the sequence $\{X_n\}_{n\in\mathbb{Z}}$ forms an unconditional basis in $\boldsymbol{G}$.

If $\{X_n\}_{n\in\mathbb{Z}}$ is an unconditional basis in $\boldsymbol{G}$, then $\{z^n\}_{n\in\mathbb{Z}}$ is an unconditional basis in the Hilbert space $L^2(\mu)$. It is well known that this basis must be equivalent to an orthonormal basis, i.e., for sequences $\{c_j\}_{j\in\mathbb{Z}}$ with finitely many nonzero terms the following inequalities hold:

$$k\left(\sum_{j\in\mathbb{Z}} |c_j|^2\right)^{1/2} \le \left\|\sum_{j\in\mathbb{Z}} c_j z^j\right\|_{L^2(\mu)} \le K\left(\sum_{j\in\mathbb{Z}} |c_j|^2\right)^{1/2}$$

for some k, $K \in (0, \infty)$ (see N.K. Nikol'skii [2], Lect. VI, Sect. 3). It follows that as a set $L^2(\mu)$ coincides with $L^2(\mathbb{T}, \boldsymbol{m})$. It is evident that μ is absolutely continuous and w satisfies (4.2). ■

5. Scattering Systems and Hankel Operators

We consider here regular stationary Gaussian processes $\{X_n\}_{n\in\mathbb{Z}}$ and we express the products $P_{\boldsymbol{G}_0} P_{\boldsymbol{G}^m}$, $m \in \mathbb{Z}_+$, in terms of Hankel operators. The symbol of this Hankel operator is a so-called phase function of the process. Then we consider a more general situation. We introduce the notion of a scattering system and define a Hankel operator that corresponds to the scattering system. Its symbol is a so-called scattering function of the scattering system. We associate with a regular stationary Gaussian process a scattering system and show that the corresponding scattering function is a phase function of the process.

Theorem 5.1. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be a regular stationary Gaussian process with spectral density $|h|^2$, where h is an outer function in H^2. Then*

$$P_{\boldsymbol{G}_0} P_{\boldsymbol{G}^m} = \mathcal{V}_1^* H_{z^m \bar{h}/h} \mathbb{P}_+ \mathcal{V}_2, \tag{5.1}$$

where $\mathcal{V}_1$ and $\mathcal{V}_2$ are the unitary operators from $\boldsymbol{G}$ onto $L^2(\mathbb{T}, \boldsymbol{m})$ defined by $\mathcal{V}_1 f = \bar{h}\Phi f$ and $\mathcal{V}_2 f = \bar{z}^m h \Phi f$.

Proof. Multiplication by $1/h$ is a unitary operator of L^2 onto $L^2(|h|^2)$. It is easy to see that it maps H^2 onto $H^2(|h|^2)$. Indeed, $z^n h/h = z^n$, $n \in \mathbb{Z}_+$, and since h is outer, the linear combinations of $z^n h$, $n \geq 0$, are dense in H^2. Similarly, one can prove that multiplication by $1/\bar{h}$ maps H^2_- unitarily onto $H^2_-(|h|^2)$. Hence,

$$\psi \mapsto (1/\bar{h})\mathbb{P}_- \bar{h}\psi$$

is the orthogonal projection from $L^2(|h|^2)$ onto $H^2_-(|h|^2)$ while

$$\psi \mapsto (z^m/h)\mathbb{P}_+ \bar{z}^m h\psi$$

is the orthogonal projection from $L^2(|h|^2)$ onto $z^m H^2(|h|^2)$. Therefore

$$P_{\boldsymbol{G}_0} P_{\boldsymbol{G}^m} f = \Phi^* \big(1/\bar{h}\mathbb{P}_- \big((z^m \bar{h}/h)\mathbb{P}_+ (\bar{z}^m h \Phi f)\big)\big),$$

which proves (5.1). ■

In particular,

$$P_{\boldsymbol{G}_0} P_{\boldsymbol{G}^0} = \mathcal{V}_1^* H_{\bar{h}/h} \mathbb{P}_+ \mathcal{V}_2.$$

Definition. Let w be the spectral density of a stationary process. Let h be an outer function in H^2 such that $|h|^2 = w$. The unimodular function $\bar{h}/h$ is called a *phase function* of the process. It is determined by w modulo a unimodular constant factor.

We can also consider slightly more general objects, *stationary sequences* in $\boldsymbol{G}$, i.e., sequences $\{X_n\}_{n\in\mathbb{Z}}$ such that $\mathbb{E}X_j \overline{X}_k$ depends only on $k-j$. The difference between stationary sequences and stationary Gaussian processes is that the Gaussian variables X_n in the case of a stationary Gaussian process are *real*.

Let $\{X_n\}_{n\in\mathbb{Z}}$ be a stationary sequence in $\boldsymbol{G}$. As in the case of stationary Gaussian processes the sequence $\{Q_n\}_{n\in\mathbb{Z}}$ defined by $Q_n = \mathbb{E}X_n\overline{X}_0$ is positive definite and by the Riesz-Herglotz theorem there exists a positive regular Borel measure μ on $\mathbb{T}$ such that $\hat{\mu}(n) = Q_n$, $n \in \mathbb{Z}$. Among all positive measures μ the ones that correspond to stationary Gaussian processes are invariant under the transformation $\zeta \mapsto \bar{\zeta}$ of $\mathbb{T}$ onto itself. We can define in a similar way the past $\boldsymbol{G}_0$ and the future $\boldsymbol{G}^m$ related to a stationary sequence $\{X_n\}_{n\in\mathbb{Z}}$. If we can restrict ourselves to the case of *regular* stationary sequences, i.e., stationary sequences for which $\bigcap\limits_{m\ge 0} \boldsymbol{G}^m = \{\mathbb{O}\}$, then Theorem 5.1 also holds for such stationary sequences.

If $\{X_n\}_{n\in\mathbb{Z}}$ is a regular stationary Gaussian sequence, V is the unitary operator on $\boldsymbol{G}$, defined by $VX_n = X_{n+1}$, then the quadruple $(\boldsymbol{G}, V, \boldsymbol{G}^0, \boldsymbol{G}_0)$ forms a scattering system. Let us define this notion.

Definition. Let $\mathcal{H}$ be a Hilbert space and let U be a unitary operator on $\mathcal{H}$. A subspace $\mathcal{N}_+$ of $\mathcal{H}$ is called *outgoing* for U if

$$U\mathcal{N}_+ \subset \mathcal{N}_+, \qquad \bigcap_{n=-\infty}^{\infty} U^n\mathcal{N}_+ = \{\mathbb{O}\}, \quad \text{and} \quad \operatorname{span}\{U^n\mathcal{N}_+ :\ n \in \mathbb{Z}\} = \mathcal{H}.$$

A subspace $\mathcal{N}_-$ of $\mathcal{H}$ is called *incoming* for U if

$$U^*\mathcal{N}_- \subset \mathcal{N}_-, \qquad \bigcap_{n=-\infty}^{\infty} U^n\mathcal{N}_- = \{\mathbb{O}\}, \quad \text{and} \quad \operatorname{span}\{U^n\mathcal{N}_- :\ n \in \mathbb{Z}\} = \mathcal{H}.$$

If $\mathcal{N}_+$ is an outgoing subspace for U and $\mathcal{N}_-$ is an incoming subspace for U, the quadruple $(\mathcal{H}, U, \mathcal{N}_+, \mathcal{N}_-)$ is called a *scattering system*.

Suppose that $(\mathcal{H}, U, \mathcal{N}_+, \mathcal{N}_-)$ is a scattering system. Let $\mathcal{K}_+ = \mathcal{N}_+ \ominus U\mathcal{N}_+$. Then there exists a unitary operator $\mathcal{V}_+$ from $\mathcal{H}$ onto $L^2(\mathcal{K}_+)$ such that $\mathcal{V}_+\mathcal{N}_+ = H^2(\mathcal{K}_+)$ and $\mathcal{V}_+U\mathcal{V}_+^* = \mathcal{S}_{\mathcal{K}_+}$ is multiplication by z on $L^2(\mathcal{K}_+)$. Indeed, it is easy to see that the spaces $U^j\mathcal{K}_+$ are pairwise orthogonal and

$$\mathcal{H} = \bigoplus_{j\in\mathbb{Z}} U^j\mathcal{K}_+,$$

and we can define $\mathcal{V}_+$ by

$$\mathcal{V}_+\sum_{j\in\mathbb{Z}} U^j x_j = \sum_{j\in\mathbb{Z}} z^j x_j, \quad x_j \in \mathcal{K}_+.$$

Similarly, there exists a unitary operator $\mathcal{V}_-$ of $\mathcal{H}$ onto $L^2(\mathcal{K}_-)$ (where $\mathcal{K}_- \stackrel{\text{def}}{=} \mathcal{N}_- \ominus U^*\mathcal{N}_-$) such that $\mathcal{V}_-(\mathcal{N}_-) = H^2_-(\mathcal{K}_-)$ and $\mathcal{V}_-U\mathcal{V}_-^* = \mathcal{S}_{\mathcal{K}_-}$.

Since the spectral multiplicity of $\mathcal{S}_{\mathcal{K}_+}$ is equal to the spectral multiplicity of U and the same is true for the spectral multiplicity of $\mathcal{S}_{\mathcal{K}_-}$, it follows that $\dim\mathcal{K}_+ = \dim\mathcal{K}_-$, and we can identify both $\mathcal{K}_+$ and $\mathcal{K}_-$ with a Hilbert space $\mathcal{K}$.

The operator $\mathfrak{S} \stackrel{\text{def}}{=} \mathcal{V}_-\mathcal{V}_+^*$ on $L^2(\mathcal{K})$ is called the *abstract scattering operator*. It is defined modulo constant unitary factors. Clearly,

$$\mathfrak{S}\mathcal{S}_{\mathcal{K}} = \mathcal{V}_-\mathcal{V}_+^*\mathcal{S}_{\mathcal{K}} = \mathcal{V}_-U\mathcal{V}_+^* = \mathcal{S}_{\mathcal{K}}\mathfrak{S},$$

i.e., $\mathfrak{S}$ commutes with $\mathcal{S}_{\mathcal{K}}$. Then there exists a unitary-valued function $\Xi \in L^\infty(\mathcal{B}(\mathcal{K}))$ such that

$$(\mathfrak{S}f)(\zeta) = \Xi(\zeta)f(\zeta), \quad f \in L^2(\mathcal{K}),\ \zeta \in \mathbb{T},$$

(see Appendix 1.4). The operator function Ξ is called a *scattering function of the scattering system*. Clearly, it is defined modulo constant unitary factors.

We have

$$P_{N_-}P_{N_+} = \mathcal{V}_-^*\mathbb{P}_-\mathcal{V}_-\mathcal{V}_+^*\mathbb{P}_+\mathcal{V}_+ = \mathcal{V}_-^*\mathbb{P}_-\mathfrak{S}\mathbb{P}_+\mathcal{V}_+ = \mathcal{V}_-^*H_\Xi\mathbb{P}_+\mathcal{V}_+$$

and

$$P_{\mathcal{N}_-^\perp}P_{N_+} = \mathcal{V}_-^*\mathbb{P}_+\mathcal{V}_-\mathcal{V}_+^*\mathbb{P}_+\mathcal{V}_+ = \mathcal{V}_-^*\mathbb{P}_+\mathfrak{S}\mathbb{P}_+\mathcal{V}_+ = \mathcal{V}_-^*T_\Xi\mathbb{P}_+\mathcal{V}_+.$$

Recall that $P_{\mathcal{L}}$ is the orthogonal projection onto $\mathcal{L}$. Thus the Hankel operator H_Ξ and the Toeplitz operator T_Ξ appear naturally when studying scattering systems.

Let us return now to stationary sequences. Consider a regular stationary sequence $\{X_n\}_{n\in\mathbb{Z}}$ in $\boldsymbol{G}$. Then it is easy to see that the future $\boldsymbol{G}^0$ is an outgoing subspace for the unitary operator V on G defined by $VX_n = X_{n+1}$ and the past $\boldsymbol{G}_0$ is an incoming subspace for V. Thus the quadruple $(\boldsymbol{G}, V, \boldsymbol{G}^0, \boldsymbol{G}_0)$ is a scattering system. We are going to compute now its scattering function.

Let $w = |h|^2$ be the spectral density of the process, where h is an outer function. We can identify $\boldsymbol{G}$ with $L^2(w)$ via the unitary map Φ introduced in §4. It is easy to see that the spaces $\mathcal{K}_+$ and $\mathcal{K}_-$ are one-dimensional and

$$\mathcal{K}_+ = \left\{\frac{\lambda}{h}:\ \lambda \in \mathbb{C}\right\} \quad \text{and} \quad \mathcal{K}_- = \left\{\bar{z}\frac{\lambda}{\bar{h}}:\ \lambda \in \mathbb{C}\right\}.$$

Indeed, to see this, it is sufficient to consider the unitary operators $\mathcal{V}_1$ and $\mathcal{V}_2$ from $\boldsymbol{G}$ onto $L^2(\mathbb{T}, \boldsymbol{m})$ introduced in Theorem 5.1 with $m = 0$ and observe that $\mathcal{V}_2\mathcal{K}_+$ is the space of constants and $\mathcal{V}_1\mathcal{K}_- = \{\lambda\bar{z}:\ \lambda \in \mathbb{C}\}$. Let us identify both $\mathcal{K}_+$ and $\mathcal{K}_-$ with $\mathbb{C}$. We can define now the unitary operators $\mathcal{V}_-$: and $\mathcal{V}_+$ from $L^2(w)$ onto L^2 by

$$\mathcal{V}_+\varphi = h\varphi \quad \text{and} \quad \mathcal{V}_-\varphi = \bar{h}\varphi.$$

Then the abstract scattering operator $\mathfrak{S}$ is given by

$$\mathfrak{S}f = \mathcal{V}_-\mathcal{V}_+^*f = \frac{h}{\bar{h}}f,$$

and so the phase function $\bar{h}/h$ is a scattering function of the scattering system $(\boldsymbol{G}, V, \boldsymbol{G}^0, \boldsymbol{G}_0)$, and

$$P_{\boldsymbol{G}_0}P_{\boldsymbol{G}^0} = \mathcal{V}_-^*H_{\bar{h}/h}\mathbb{P}_+\mathcal{V}_+.$$

If we consider the scattering system $(\boldsymbol{G}, V, \boldsymbol{G}^m, \boldsymbol{G}_0)$, then it is easy to show that its scattering function is $z^m\bar{h}/h$ and

$$P_{\boldsymbol{G}_0}P_{\boldsymbol{G}^m} = \mathcal{V}_-^* H_{z^m\bar{h}/h}\mathbb{P}_+\mathcal{V}_+,$$

which is just formula (5.1).

6. Geometry of Past and Future

We consider the following geometric problem in this section. Let $\boldsymbol{G}$ be a Gaussian subspace and let $\boldsymbol{A}$ and $\boldsymbol{B}$ be subspaces of $\boldsymbol{G}$ such that $\operatorname{clos}(\boldsymbol{A}+\boldsymbol{B}) = \boldsymbol{G}$. The problem is to determine whether there exists a stationary Gaussian process $\{X_n\}_{n\in\mathbb{Z}}$ such that $\boldsymbol{G}_0 = \boldsymbol{A}$ and $\boldsymbol{G}^0 = \boldsymbol{B}$. We show in this section that this geometric problem is closely related to the problem of describing operators unitarily equivalent to moduli of Hankel operators. The latter problem will be solved in Chapter 12, which will allow us to solve the above geometric problem in the case when $\boldsymbol{A}$ and $\boldsymbol{B}$ are at nonzero angle.

The spectral representation Φ of a stationary Gaussian process $\{X_n\}_{n\in\mathbb{Z}}$ transforms the Gaussian space $\boldsymbol{G}$ to the space $L^2(\mu)$. If the process is deterministic, then $\boldsymbol{G}_0 = \boldsymbol{G}^0 = \boldsymbol{G}$, and so the geometry of past and future is trivial. If the process is nondeterministic and $d\mu = d\mu_{\rm s} + w\,d\boldsymbol{m}$, then $\Phi\boldsymbol{G}^0 = H^2(w) + L^2(\mu_{\rm s})$ and $\Phi\boldsymbol{G}_0 = H^2_-(w) + L^2(\mu_{\rm s})$. Hence, we can pass to the orthogonal complement to $\Phi^*(L^2(\mu_{\rm s}))$ and consider the subspaces $\boldsymbol{G}_0\ominus\Phi^*(L^2(\mu_{\rm s}))$ and $\boldsymbol{G}^0\ominus\Phi^*(L^2(\mu_{\rm s}))$ of the Gaussian space $\boldsymbol{G}\ominus\Phi^*(L^2(\mu_{\rm s}))$. This allows us to reduce the initial problem to the case when there exists a *regular* stationary Gaussian process $\{X_n\}_{n\in\mathbb{Z}}$ such that $\boldsymbol{A}$ is its past and $\boldsymbol{B}$ is its future. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are at nonzero angle we reduce this problem to the problem of description of the operators unitarily equivalent to the moduli of Hankel operators.

Suppose now that $\boldsymbol{A}$ and $\boldsymbol{B}$ are subspaces of $\boldsymbol{G}$ such that $\operatorname{clos}(\boldsymbol{A}+\boldsymbol{B}) = \boldsymbol{G}$. We consider the problem of when there exists a stationary sequence $\{X_n\}_{n\in\mathbb{Z}}$ of vectors in $\boldsymbol{G}$ such that $\boldsymbol{G}_0 = \boldsymbol{A}$ and $\boldsymbol{G}^0 = \boldsymbol{B}$.

We consider the set $\mathcal{T}$ of triples $(\mathcal{K}, \mathcal{L}, \mathcal{H})$, where $\mathcal{K}$ and $\mathcal{L}$ are subspaces of a complex infinite-dimensional separable Hilbert space $\mathcal{H}$ such that $\operatorname{clos}(\mathcal{K}+\mathcal{L}) = \mathcal{H}$. Two triples $(\mathcal{K}_1, \mathcal{L}_1, \mathcal{H}_1)$ and $(\mathcal{K}_2, \mathcal{L}_2, \mathcal{H}_2)$ in $\mathcal{T}$ are said to be *equivalent* if there exists a unitary operator $\mathcal{W}$ of $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $\mathcal{W}\mathcal{K}_1 = \mathcal{K}_2$ and $\mathcal{W}\mathcal{L}_1 = \mathcal{L}_2$. We want to find out which triples $(\mathcal{K}, \mathcal{L}, \mathcal{H})$ are equivalent to a triple of the form $(\boldsymbol{G}_0, \boldsymbol{G}^0, \boldsymbol{G})$.

With each triple $t = (\mathcal{K}, \mathcal{L}, \mathcal{H}) \in \mathcal{T}$ we associate the operator $P_\mathcal{L}P_\mathcal{K}P_\mathcal{L}$ and the following numbers:

$$n_+(t) = \dim \mathcal{K}^\perp \cap \mathcal{L}, \quad n_-(t) = \dim \mathcal{K} \cap \mathcal{L}^\perp,$$

where as usual $P_\mathcal{M}$ is the orthogonal projection onto a subspace $\mathcal{M}$.

Theorem 6.1. *Triples $t_1 = (\mathcal{K}_1, \mathcal{L}_1, \mathcal{H}_1)$ and $t_2 = (\mathcal{K}_2, \mathcal{L}_2, \mathcal{H}_2)$ are equivalent if and only if the operators $P_{\mathcal{L}_1} P_{\mathcal{K}_1} P_{\mathcal{L}_1}$ and $P_{\mathcal{L}_2} P_{\mathcal{K}_2} P_{\mathcal{L}_2}$ are unitarily equivalent and $n_\pm(t_1) = n_\pm(t_2)$.*

Proof. It is evident that if t_1 and t_2 are equivalent, then $P_{\mathcal{L}_1} P_{\mathcal{K}_1} P_{\mathcal{L}_1}$ and $P_{\mathcal{L}_2} P_{\mathcal{K}_2} P_{\mathcal{L}_2}$ are unitarily equivalent and $n_\pm(t_1) = n_\pm(t_2)$.

Let us establish the converse. Clearly,

$$\mathcal{K}_j \cap \mathcal{L}_j = \{x \in \mathcal{H}_j : \ P_{\mathcal{L}_j} P_{\mathcal{K}_j} P_{\mathcal{L}_j} x = x\}, \quad j = 1, 2.$$

Hence, $\dim \mathcal{K}_1 \cap \mathcal{L}_1 = \mathcal{K}_2 \cap \mathcal{L}_2$ and we can replace t_j with

$$(\mathcal{K}_j \ominus (\mathcal{K}_j \cap \mathcal{L}_j), \mathcal{L}_j \ominus (\mathcal{K}_j \cap \mathcal{L}_j), \mathcal{H}_j \ominus (\mathcal{K}_j \cap \mathcal{L}_j)), \quad j = 1, 2.$$

Thus without loss of generality we may assume that $\mathcal{K}_j \cap \mathcal{L}_j = \{\mathbb{O}\}$, $j = 1, 2$.

Lemma 6.2. *If t_1 and t_2 are equivalent, then*

$$\dim \mathcal{H}_1 \ominus \mathcal{L}_1 = \dim \mathcal{H}_2 \ominus \mathcal{L}_2 \tag{6.1}$$

and

$$\dim \operatorname{Ker}(P_{\mathcal{L}_1} P_{\mathcal{K}_1} P_{\mathcal{L}_1} \big| \mathcal{L}_1) = \dim \operatorname{Ker}(P_{\mathcal{L}_2} P_{\mathcal{K}_2} P_{\mathcal{L}_2} \big| \mathcal{L}_2). \tag{6.2}$$

Proof. Let us first prove (6.1). Since $\mathcal{H}_j = \operatorname{clos}(\mathcal{K}_j + \mathcal{L}_j)$, it follows under the assumption $\mathcal{K}_j \cap \mathcal{L}_j = \{\mathbb{O}\}$ that $\dim(\mathcal{H}_j \ominus \mathcal{L}_j) = \dim \mathcal{K}_j$, $j = 1, 2$. It is easy to see that

$$\dim \mathcal{K}_j = \dim P_{\mathcal{L}_j} \mathcal{K}_j + \dim \mathcal{K}_j \cap \mathcal{L}_j^\perp = \dim P_{\mathcal{L}_j} \mathcal{K}_j + n_-(t_j) \quad j = 1, 2.$$

Equality (6.1) follows now from the fact that $\operatorname{clos} \operatorname{Range} P_{\mathcal{L}_j} P_{\mathcal{K}_j} P_{\mathcal{L}_j} = \operatorname{clos} P_{\mathcal{L}_j} \mathcal{K}_j$, $j = 1, 2$.

Let us now prove (6.2). It is easy to see that

$$\operatorname{Ker} P_{\mathcal{L}_j} P_{\mathcal{K}_j} P_{\mathcal{L}_j} = (P_{\mathcal{L}_j} \mathcal{K}_j)^\perp = \mathcal{L}_j^\perp \oplus (\mathcal{L}_j \ominus P_{\mathcal{L}_j} \mathcal{K}_j), \quad j = 1, 2.$$

Therefore $\operatorname{Ker}(P_{\mathcal{L}_j} P_{\mathcal{K}_j} P_{\mathcal{L}_j} \big| \mathcal{L}_j) = \mathcal{L}_j \ominus P_{\mathcal{L}_j} \mathcal{K}_j$, $j = 1, 2$. Clearly,

$$\mathcal{L}_j \ominus P_{\mathcal{L}_j} \mathcal{K}_j = \mathcal{L}_j \cap \mathcal{K}_j^\perp, \quad j = 1, 2,$$

and so

$$\dim \operatorname{Ker}(P_{\mathcal{L}_j} P_{\mathcal{K}_j} P_{\mathcal{L}_j} \big| \mathcal{L}_j) = \dim \mathcal{L}_j \cap \mathcal{K}_j^\perp = n_+(t_j) \quad j = 1, 2,$$

which proves (6.2). ∎

Let us continue the proof of Theorem 6.1. Let $\mathcal{W}$ be a unitary operator from $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $\mathcal{W} P_{\mathcal{L}_1} P_{\mathcal{K}_1} P_{\mathcal{L}_1} = P_{\mathcal{L}_2} P_{\mathcal{K}_2} P_{\mathcal{L}_2} \mathcal{W}$. Then $\mathcal{W} \operatorname{Ker} P_{\mathcal{L}_1} P_{\mathcal{K}_1} P_{\mathcal{L}_1} = \operatorname{Ker} P_{\mathcal{L}_2} P_{\mathcal{K}_2} P_{\mathcal{L}_2}$. We can now change the unitary operator $\mathcal{W}$ on $\operatorname{Ker} P_{\mathcal{L}_1} P_{\mathcal{K}_1} P_{\mathcal{L}_1}$. Let $\mathcal{W}_\#$ be a unitary operator from $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $\mathcal{W}_\# x = \mathcal{W} x$ for $x \perp \operatorname{Ker} P_{\mathcal{L}_1} P_{\mathcal{K}_1} P_{\mathcal{L}_1}$, $\mathcal{W}_\# \big| \mathcal{L}_1^\perp$ is an arbitrary unitary operator from $\mathcal{L}_1^\perp$ onto $\mathcal{L}_2^\perp$, and $\mathcal{W}_\# \big| (\mathcal{L}_1 \cap \operatorname{Ker} P_{\mathcal{L}_1} P_{\mathcal{K}_1} P_{\mathcal{L}_1}) = \mathcal{L}_2 \cap \operatorname{Ker} P_{\mathcal{L}_2} P_{\mathcal{K}_2} P_{\mathcal{L}_2}$. By Lemma 6.2, such a unitary operator $\mathcal{W}_\#$ exists.

The operator $\mathcal{W}_\#$ can identify $\mathcal{H}_1$ with $\mathcal{H}_2$ and $\mathcal{L}_1$ with $\mathcal{L}_2$. Without loss of generality we may assume now that $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$, and $P_{\mathcal{L}} P_{\mathcal{K}_1} P_{\mathcal{L}} = P_{\mathcal{L}} P_{\mathcal{K}_2} P_{\mathcal{L}}$.

Under the assumption that $\mathcal{L} \cap \mathcal{K}_j = \{\mathbb{O}\}$, $j = 1, 2$ we have

$$\mathcal{H} = \operatorname{clos}(\mathcal{L} + P_{\mathcal{K}_j}\mathcal{L}) \oplus \mathcal{K}_j \cap \mathcal{L}^\perp, \quad j = 1, 2,$$

and

$$\mathcal{L} \cap P_{\mathcal{K}_j}\mathcal{L} = \{\mathbb{O}\}, \quad j = 1, 2.$$

We define a unitary operator $\mathcal{U}$ on $\mathcal{H}$ as follows. We put $\mathcal{U}x = x$ for $x \in \mathcal{L}$. On $P_{\mathcal{K}_1}\mathcal{L}$ we define $\mathcal{U}$ by $\mathcal{U}P_{\mathcal{K}_1}x = P_{\mathcal{K}_2}x$, $x \in \mathcal{L}$. Finally, $\mathcal{U}\big|\mathcal{K}_1 \cap \mathcal{L}^\perp$ is an arbitrary unitary operator from $\mathcal{K}_1 \cap \mathcal{L}^\perp$ onto $\mathcal{K}_2 \cap \mathcal{L}^\perp$.

Note first of all that $\mathcal{U}$ is well-defined on $P_{\mathcal{K}_1}\mathcal{L}$. Indeed, if $P_{\mathcal{K}_1}x_1 = P_{\mathcal{K}_1}x_2$ for $x_1, x_2 \in \mathcal{L}$, then

$$\mathbb{O} = P_\mathcal{L}P_{\mathcal{K}_1}P_\mathcal{L}(x_1 - x_2) = P_\mathcal{L}P_{\mathcal{K}_2}P_\mathcal{L}(x_1 - x_2) = P_\mathcal{L}P_{\mathcal{K}_2}(x_1 - x_2).$$

Hence,

$$\begin{aligned} 0 &= (P_\mathcal{L}P_{\mathcal{K}_2}(x_1 - x_2), (x_1 - x_2)) = (P_{\mathcal{K}_2}(x_1 - x_2), (x_1 - x_2)) \\ &= (P_{\mathcal{K}_2}(x_1 - x_2), P_{\mathcal{K}_2}(x_1 - x_2)) = \|P_{\mathcal{K}_2}(x_1 - x_2)\|^2, \end{aligned}$$

and so $P_{\mathcal{K}_2}x_1 = P_{\mathcal{K}_2}x_2$. It is also clear that $\mathcal{U}$ maps $P_{\mathcal{K}_1}\mathcal{L}$ onto $P_{\mathcal{K}_2}\mathcal{L}$.

To prove that $\mathcal{U}$ is a unitary operator on $\mathcal{H}$ it suffices to verify that its restriction to $\operatorname{clos}(\mathcal{L} + P_{\mathcal{K}_1}\mathcal{L})$ is unitary. Let $y_1, y_2, z_1, z_2 \in \mathcal{L}$. Let $v_1 = y_1 + P_{\mathcal{K}_1}z_1$, $v_2 = y_2 + P_{\mathcal{K}_1}z_2$. We have

$$\begin{aligned} (v_1, v_2) &= (y_1, y_2) + (y_1, P_{\mathcal{K}_1}z_2) + (P_{\mathcal{K}_1}z_1, y_2) + (P_{\mathcal{K}_1}z_1, P_{\mathcal{K}_1}z_2) \\ &= (y_1, y_2) + (P_\mathcal{L}P_{\mathcal{K}_1}P_\mathcal{L}y_1, z_2) \\ &+ (P_\mathcal{L}P_{\mathcal{K}_1}P_\mathcal{L}z_1, y_2) + (P_\mathcal{L}P_{\mathcal{K}_1}P_\mathcal{L}z_1, z_2) \\ &= (y_1, y_2) + (P_\mathcal{L}P_{\mathcal{K}_2}P_\mathcal{L}y_1, z_2) \\ &+ (P_\mathcal{L}P_{\mathcal{K}_2}P_\mathcal{L}z_1, y_2) + (P_\mathcal{L}P_{\mathcal{K}_2}P_\mathcal{L}z_1, z_2) \\ &= (y_1, y_2) + (y_1, P_{\mathcal{K}_2}z_2) + (P_{\mathcal{K}_2}z_1, y_2) + (P_{\mathcal{K}_2}z_1, P_{\mathcal{K}_2}z_2) \\ &= (y_1 + P_{\mathcal{K}_2}z_1, y_2 + P_{\mathcal{K}_2}z_2) = (\mathcal{U}v_1, \mathcal{U}v_2). \end{aligned}$$

It is easy to see that $\mathcal{U}$ maps $\mathcal{H}$ onto itself. It is also clear that $\mathcal{U}$ maps $\mathcal{L}_1$ onto $\mathcal{L}_2$ and $\mathcal{K}_1$ onto $\mathcal{K}_2$. ■

Note that for each $k, m \in \mathbb{Z}_+ \cup \{\infty\}$ and for each self-adjoint operator A on a Hilbert space $\mathcal{N}$ such that $\mathbb{O} \le A \le I$, $\operatorname{Ker} A = \{\mathbb{O}\}$, and $\operatorname{Ker}(I - A) = \{\mathbb{O}\}$ there exists a triple $t = (\mathcal{K}, \mathcal{L}, \mathcal{H}) \in \mathcal{T}$ such that $n_+(t) = k$, $n_-(t) = m$ and $P_\mathcal{L}P_\mathcal{K}P_\mathcal{L}\big|\mathcal{L} \ominus \mathcal{K}^\perp$ is unitarily equivalent to $\mathcal{A}$. Indeed, it is easy to reduce the problem to the case $n_+(t) = n_-(t) = 0$. Consider the operator $\mathcal{P}$ on $\mathcal{N} \oplus \mathcal{N}$ given by the block matrix

$$\mathcal{P} = \begin{pmatrix} A & A^{1/2}(I - A)^{1/2} \\ A^{1/2}(I - A)^{1/2} & I - A \end{pmatrix}.$$

It is easy to see that $\mathcal{P}$ is a self-adjoint projection. Put $\mathcal{K} \stackrel{\text{def}}{=} \operatorname{Range}\mathcal{P}$, $\mathcal{L} \stackrel{\text{def}}{=} \mathcal{N} \oplus \{\mathbb{O}\}$ and $\mathcal{H} = \operatorname{clos}(\mathcal{K} + \mathcal{L})$. Clearly, $P_\mathcal{L}P_\mathcal{K}P_\mathcal{L}\big|\mathcal{L}$ is unitarily equivalent to A. It is easy to verify that $\mathcal{K} \cap \mathcal{L}^\perp = \mathcal{K}^\perp \cap \mathcal{L} = \{\mathbb{O}\}$.

We return now to the triple $(\boldsymbol{G}_0, \boldsymbol{G}^0, \boldsymbol{G})$ associated with a stationary sequence $\{X_n\}_{n\in\mathbb{Z}}$ in $\boldsymbol{G}$.

Theorem 6.3. *Let $\{X_n\}_{n\in\mathbb{Z}}$ in $\boldsymbol{G}$ be a stationary sequence in $\boldsymbol{G}$. Then*

$$n_-(\boldsymbol{G}_0, \boldsymbol{G}^0, \boldsymbol{G}) = n_+(\boldsymbol{G}_0, \boldsymbol{G}^0, \boldsymbol{G}). \tag{6.3}$$

Proof. Let J be the unitary operator on $\boldsymbol{G}$ defined by

$$JX_n = X_{-n-1}, \quad n \in \mathbb{Z},$$

on the linear span of the X_n. Clearly, J extends to a unitary operator on $\boldsymbol{G}$. It is easy to see that $J\boldsymbol{G}_0 = \boldsymbol{G}^0$ and $J\boldsymbol{G}^0 = \boldsymbol{G}_0$. This implies (6.3). ■

Theorem 6.4. *Let $\{X_n\}_{n\in\mathbb{Z}}$ in $\boldsymbol{G}$ be a regular stationary sequence in $\boldsymbol{G}$. Then the operator $P_{\boldsymbol{G}^0}P_{\boldsymbol{G}_0}P_{\boldsymbol{G}^0}\big|\boldsymbol{G}^0$ is unitarily equivalent to the square of the modulus of a Hankel operator on H^2.*

Proof. Let h be an outer function in H^2 such that $|h|^2$ is the spectral density of the stationary sequence $\{X_n\}_{n\in\mathbb{Z}}$. By Theorem 5.1,

$$P_{\boldsymbol{G}^0}P_{\boldsymbol{G}_0}P_{\boldsymbol{G}^0}\big|\boldsymbol{G}^0 = \mathcal{V}_2^* H^*_{\bar{h}/h} H_{\bar{h}/h} \mathcal{V}_2,$$

and so $P_{\boldsymbol{G}^0}P_{\boldsymbol{G}_0}P_{\boldsymbol{G}^0}\big|\boldsymbol{G}^0$ is unitarily equivalent to $H^*_{\bar{h}/h} H_{\bar{h}/h}$. ■

It is easy to see that if Γ is a Hankel operator from H^2 to H^2_-, then

(a) Γ *is noninvertible;*

and

(b) *either* $\operatorname{Ker}\Gamma = \{\mathbb{O}\}$ *or* $\dim \operatorname{Ker}\Gamma = \infty$.

Indeed, (a) follows from the obvious fact that $\lim_{n\to\infty} \|\Gamma z^n\| = 0$. To prove (b), we observe that $\operatorname{Ker}\Gamma$ is invariant under multiplication by z, and so by Beurling's theorem (see Appendix 2.2) either $\operatorname{Ker}\Gamma = \{\mathbb{O}\}$ or $\operatorname{Ker}\Gamma = \vartheta H^2$ for an inner function ϑ.

Corollary 6.5.

(i) *The operator $P_{\boldsymbol{G}^0}P_{\boldsymbol{G}_0}P_{\boldsymbol{G}^0}\big|\boldsymbol{G}^0$ is noninvertible;*

(ii) *Either $n_+(\boldsymbol{G}_0, \boldsymbol{G}^0, \boldsymbol{G}) = 0$ or $n_+(\boldsymbol{G}_0, \boldsymbol{G}^0, \boldsymbol{G}) = \infty$.*

Proof. It is easy to see that (i) follows from Theorem 6.4 and (a) while (ii) follows from Theorem 6.4 and (b). ■

We are going prove the converse of Theorem 6.4 in the case when the subspaces are at nonzero angle. If $\cos(\widehat{\boldsymbol{G}_0, \boldsymbol{G}^0}) < 1$, then the corresponding Hankel operator has norm less than one. We consider now a triple $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{G}) \in \mathcal{T}$ with $\cos(\widehat{\boldsymbol{A}, \boldsymbol{B}}) < 1$.

Theorem 6.6. *Let $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{G}) \in \mathcal{T}$ be a triple such that $\boldsymbol{A}$ and $\boldsymbol{B}$ are at nonzero angle and $n_-(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{G}) = n_+(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{G})$. If the operator $P_{\boldsymbol{B}}P_{\boldsymbol{A}}P_{\boldsymbol{B}}\big|\boldsymbol{B}$ is unitarily equivalent to the square of the modulus of a Hankel operator, then there exists a stationary sequence $\{X_n\}_{n\in\mathbb{Z}}$ with $\boldsymbol{G}_0 = \boldsymbol{A}$ and $\boldsymbol{G}^0 = \boldsymbol{B}$.*

Proof. Let φ be a function in L^∞ such that $\|\varphi\|_{L^\infty} < 1$ and $P_{\boldsymbol{B}}P_{\boldsymbol{A}}P_{\boldsymbol{B}}\big|\boldsymbol{B}$ is unitarily equivalent to $H_\varphi^* H_\varphi$. It follows that $\|H_\varphi\| < 1$. By Theorem 1.1.7, there exists a unimodular function u such that $H_u = H_\varphi$ and $\|H_{\bar{z}u}\| = 1$. By Theorem 3.1.14, the Toeplitz operator T_u is invertible. Then by Corollary 3.2.2, u can be represented in the form $u = \bar{h}/h$, where h is an outer function in H^2. Put $w = |h|^2$.

It follows that $P_{\boldsymbol{B}}P_{\boldsymbol{A}}P_{\boldsymbol{B}}\big|\boldsymbol{B}$ is unitarily equivalent to $H^*_{\bar{h}/h}H_{\bar{h}/h}$. Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are at nonzero angle and $n_-(\boldsymbol{A},\boldsymbol{B},\boldsymbol{G}) = n_+(\boldsymbol{A},\boldsymbol{B},\boldsymbol{G})$, we have $\dim B^\perp = \infty$. Thus $P_{\boldsymbol{B}}P_{\boldsymbol{A}}P_{\boldsymbol{B}}$ is unitarily equivalent to $P_{H^2(w)}P_{H^2_-(w)}P_{H^2(w)}$. It is also easy to see that

$$n_+(\boldsymbol{A},\boldsymbol{B},\boldsymbol{G}) = \dim \operatorname{Ker} H_{\bar{h}/h} = n_+\big(H^2_-(w), H^2(w), L^2(w)\big).$$

It follows now from Theorems 5.1 and 6.1 that $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{G})$ is equivalent to the triple $\big(H^2_-(w), H^2(w), L^2(w)\big)$. Clearly, $H^2_-(w)$ is the past and $H^2(w)$ is the future associated with the stationary sequence $\{z^n\}_{n\in\mathbb{Z}}$. ■

Theorem 6.7. *If under the hypotheses of Theorem 6.6 the operator $P_{\boldsymbol{B}}P_{\boldsymbol{A}}P_{\boldsymbol{B}}\big|\boldsymbol{B}$ is unitarily equivalent to the square of the modulus of a self-adjoint Hankel operator, and $\boldsymbol{A}$ and $\boldsymbol{B}$ are Gaussian subspaces of a Gaussian space $\boldsymbol{G}$, then there exists a stationary Gaussian process $\{X_n\}_{n\in\mathbb{Z}}$ with $\boldsymbol{G}_0 = \boldsymbol{A}$ and $\boldsymbol{G}^0 = \boldsymbol{B}$.*

Proof. The proof is the practically same as the proof of Theorem 6.6. By the Remark following Theorem 1.1.7 and Corollary 3.2.4, we can find an outer function h with real Fourier coefficients such that $H_\varphi = H_{\bar{h}/h}$. Thus for the stationary sequence $\{z^n\}_{n\in\mathbb{Z}}$ in $L^2(w)$ the numbers $(z^j, z^k)_{L^2(w)}$ are real, and so the triple $(H^2_-(w), H^2(w), L^2(w))$ is equivalent to a triple $(\boldsymbol{G}_0, \boldsymbol{G}^0, \boldsymbol{G})$ associated with a stationary Gaussian process. ■

In §12.8 we describe the operators that are unitarily equivalent to the moduli of Hankel operators. We shall prove there that if A is an operator on Hilbert space such that $A \ge \mathbb{O}$, A is noninvertible and $\operatorname{Ker} A$ is either trivial or infinite-dimensional, then A is unitarily equivalent to the modulus of a Hankel operator. In fact we show in §12.8 that if we can construct a Hankel operator whose modulus is unitarily equivalent to A, we can construct a self-adjoint Hankel operator with the same property. This will lead to a solution of the problem to describe the triples $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{G}) \in \mathcal{T}$ such that $\boldsymbol{A} = \boldsymbol{G}_0$, $\boldsymbol{B} = \boldsymbol{G}^0$ for a stationary process in $\boldsymbol{G}$ for which $\boldsymbol{G}_0$ and $\boldsymbol{G}^0$ are at nonzero angle.

Concluding Remarks

We refer the reader to Lamperti [1] for background in probability theory. Basic facts about stationary Gaussian processes can be found in Rozanov

[1], Ibragimov and Rozanov [1], Dym and McKean [1], Gikhman and Skorokhod [1], and Yaglom [1]. The important notion of canonical correlations was introduced in Yaglom [2].

There are several methods of reducing probability problems of Gaussian processes to geometric problems in Gaussian spaces. Wiener [1] elaborated the method of orthogonal expansion in Hermite polynomials of infinitely many variables (see also Masani [1]). Later Segal [1] observed that such expansions are closely related to Fock spaces introduced for the needs of quantum mechanics. The use of Fock spaces turned out to be very fruitful both for quantum field theory and probability theory. More information about Fock spaces can be found in Simon [1], Reed [1], and Nelson [1]. We also recommend the recent book Janson [2].

The presentation of §2 and §3 follows the survey article Peller and Khrushchëv [1]. We mention here the papers Vershik [1–2] in which the technique of Fock spaces was used to study spectral properties of the operator U defined by (2.7). Theorem 3.1 is due to Maruyama [1]. Theorem 3.2 was used in Girsanov [1] to construct a mixing measure preserving automorphism with simple continuous singular spectrum. In connection with nondeterministic processes we mention here *completely nondeterministic processes.* These are processes with trivial intersection of the past and the future, i.e., $\boldsymbol{G}_0 \cap \boldsymbol{G}^0 = \{\mathbb{O}\}$. Such processes were characterized in spectral terms in Bloomfield, Jewell, and Hayashi [1]. Theorem 3.4 is well-known, see e.g., Ibragimov and Rozanov [1]. Theorem 3.5 was obtained in Kolmogorov and Rozanov [1]. Theorem 3.8 was proved in Peller and Khrushchëv [1]. Theorem 3.11 was published in Ibragimov and Rozanov [1]. The proof given in the book is taken from Peller and Khrushchëv [1]; it was inspired by Dym and McKean [1]. Note that a stationary Gaussian process is absolutely regular if and only if it is *informationally regular*; the last notion was introduced in Gelfand and Yaglom [1]; see Ibragimov and Rozanov [1], and Dym and McKean [1]. Theorem 3.14 appeared in Ibragimov and Linnik [1]. The notion of a minimal stationary process was introduced in Kolmogorov [1]. Theorem 4.2 is a consequence of the Kolmogorov–Rozanov theorem and well-known facts about bases. Theorem 4.3 is well-known.

Formula (5.1) was used systematically in Peller and Khrushchëv [1]. Scattering theory was developed by Lax and Phillips [1]. Hankel and Toeplitz operators associated with scattering systems were considered in Mitter and Avniel [1].

Section 6 follows the paper Khrushchëv and Peller [1], in which the problem of the geometry of past and future was stated. The results on the classification of triples in $\mathcal{T}$ can be found in the paper Davis [1] in which there are many other results on pairs of subspaces of a Hilbert space.

9

Regularity Conditions for Stationary Processes

In this chapter we characterize different regularity conditions introduced in the previous chapter in spectral terms. In §1 we characterize the minimal stationary processes and find the spectral density of the interpolation error process in terms of the spectral density of the initial process. In §2 we consider the angles between the past and the future of a stationary process. We characterize the processes with nonzero angles between the past and the future. In the next section we consider various regularity conditions for stationary processes (such as complete regularity, complete regularity of order α, p-regularity, etc.) and we characterize such regularity conditions in spectral terms. Note that the original proofs of these results were quite different for different regularity conditions; some proofs were quite complicated. In Peller and Khrushchëv [1] a single approach to all regularity conditions was found. This approach is based on Hankel operators and the results on best approximation given in Chapter 7 and it simplifies the original proofs. Finally, in §4 we consider several stronger regularity conditions and we also characterize them in spectral terms.

1. Minimality in Spectral Terms

In this section we prove the Kolmogorov theorem, which characterizes the minimal stationary process in spectral terms. Given a minimal stationary process we also find the spectral density of the interpolation error process and the canonical correlations between its past and future.

Theorem 1.1. *Let w be the spectral density of a regular stationary process. Then the process is minimal if and only if $w^{-1} \in L^1$.*

Proof. Suppose that the process is minimal. Then

$$\mathbf{1} \notin \operatorname{span}_{L^2(w)}\{z^j : \ j \in \mathbb{Z}, \ j \neq 0\}.$$

Let g be a function in $L^2(w)$ such that

$$\int_{\mathbb{T}} g(\zeta)\bar{\zeta}^j w(\zeta) d\boldsymbol{m}(\zeta) = \begin{cases} 0, & j \neq 0, \\ 1, & j = 0. \end{cases} \tag{1.1}$$

Consider the function gw on $\mathbb{T}$. Clearly, it belongs to L^1. It follows from (1.1) that $\widehat{gw}(j) = 0$, $j \neq 0$, and $\widehat{gw}(0) = 1$. Hence, $g(\zeta)w(\zeta) = 1$, $\zeta \in \mathbb{T}$, and so $g = w^{-1}$. Since $g \in L^2(w)$, we have

$$\int_{\mathbb{T}} w^{-1} d\boldsymbol{m} = \int_{\mathbb{T}} g^2 w d\boldsymbol{m} < \infty.$$

Suppose now that $w^{-1} \in L^1$. Then obviously, $w^{-1} \in L^2(w)$. It is easy to see that

$$(w^{-1}, z^j)_{L^2(w)} = \begin{cases} 0, & j \neq 0, \\ 1, & j = 0, \end{cases}$$

and so $\mathbf{1} \notin \operatorname{span}_{L^2(w)}\{z^j : \ j \in \mathbb{Z}, \ j \neq 0\}$.

Since multiplication by z is a unitary operator on $L^2(w)$, it follows that $z^n \notin \operatorname{span}_{L^2(w)}\{z^j : \ j \in \mathbb{Z}, \ j \neq n\}$. ■

In §8.4 with a minimal stationary process $\{X_n\}_{n\in\mathbb{Z}}$ we have associated the interpolation error process $\{Y_n\}_{n\in\mathbb{Z}}$. The following theorem evaluates the spectral density of $\{Y_n\}_{n\in\mathbb{Z}}$.

Theorem 1.2. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be a minimal stationary process with spectral density w. Then the interpolation error process $\{Y_n\}_{n\in\mathbb{Z}}$ is also minimal and its spectral density is equal to w^{-1}.*

Proof. Applying the spectral representation Φ of the process $\{X_n\}_{n\in\mathbb{Z}}$ (see (8.1.5)), we can identify X_n with the function z^n in $L^2(w)$. Put $g_n = z^n w^{-1}$, $n \in \mathbb{Z}$. It is easy to see that $g_n \in L^2(w)$ and

$$(z^n, g_m)_{L^2(w)} = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases}$$

It is easy to see that $\Phi Y_n = g_n$, $n \in \mathbb{Z}$. Indeed, g_n is orthogonal to z^j with $j \neq n$ and $(z^n, g_n)_{L^2(w)} = (z^n, z^n)_{L^2(w)} = 1$.

We have

$$\begin{aligned} (Y_k, Y_n) &= (g_k, g_m)_{L^2(w)} \\ &= \int_{\mathbb{T}} z^k w^{-1} \bar{z}^m w^{-1} w \, d\boldsymbol{m} = \widehat{w^{-1}}(m-k), \quad k, \, m \in \mathbb{Z}. \end{aligned}$$

It follows that w^{-1} is the spectral density of the stationary process $\{Y_n\}_{n\in\mathbb{Z}}$. ■

Let us now compare the canonical correlations of the past and the future for both processes $\{X_n\}_{n\in\mathbb{Z}}$ and $\{Y_n\}_{n\in\mathbb{Z}}$. We prove that they are the same.

Let w be the spectral density of a minimal process $\{X_n\}_{n\in\mathbb{Z}}$. Let h be an outer function in H^2 such that $|h|^2 = w$. Consider the phase function $\bar{h}/h$ of $\{X_n\}_{n\in\mathbb{Z}}$ (see §8.5). It follows immediately from Theorem 1.2 that $h/\bar{h}$ is a phase function of $\{Y_n\}_{n\in\mathbb{Z}}$.

Theorem 1.3. *Let $|h|^2$ be the spectral density of a minimal stationary process, where h is an outer function in H^2. Then the moduli of the Hankel operators $H_{\bar{h}/h}$ and $H_{h/\bar{h}}$ are unitarily equivalent.*

Proof. Let $u = \bar{h}/h$. Then $\bar{u} = h/\bar{h}$. By Theorem 4.4.10, the Toeplitz operator T_u has dense range. Since the process is minimal, it follows from Theorem 1.1 that $1/h \in H^2$. Again, by Theorem 4.4.8 the Toeplitz operator $T_{\bar{u}}$ has dense range. Therefore

$$\{f \in H^2 : \ H_u^* H_u f = f\} = \{\mathbb{O}\}.$$

By Theorem 4.4.2, the operators $H_u^* H_u$ and $H_{\bar{u}}^* H_{\bar{u}}$ are unitarily equivalent, which implies the result. ∎

Corollary 1.4. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be a minimal stationary process and let $\{Y_n\}_{n\in\mathbb{Z}}$ be its interpolation error processes. Then both processes have the same canonical correlations between the past and the future.*

Proof. By Theorem 8.5.1, the canonical correlations of the past and the future of the process $\{X_n\}_{n\in\mathbb{Z}}$ are the singular values of $H_{\bar{h}/h}$. Similarly, by Theorems 1.2 and 8.5.1, the canonical correlations of the past and the future of the process $\{Y_n\}_{n\in\mathbb{Z}}$ are the singular values of the Hankel operator $H_{h/\bar{h}}$. The result follows now from Theorem 1.3. ∎

2. Angles between Past and Future

We consider here the class of stationary processes for which the future from time k and the past are at nonzero angle. We prove that the spectral density w of such a process can be represented as $|Q|^2 w_1$, where Q is a polynomial with zeros on $\mathbb{T}$ and w_1 is the spectral density of a minimal process for which the past before time zero and the future from time zero are at nonzero angle. This reduction to the case of minimal processes will be used in §3 to describe various regularity conditions in spectral terms. Finally, we prove in this section the Helson–Szegö theorem, which characterizes the stationary processes with nonzero angle between the past and the future.

Theorem 2.1. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be a stationary process with spectral density w such that for some $k \in \mathbb{Z}_+$ its past $\boldsymbol{G}_0$ and the future $\boldsymbol{G}^k$ from time k are at nonzero angle. Then there exist a polynomial Q, $\deg Q \le k$, with zeros on $\mathbb{T}$ and a minimal stationary process $\{X_n^\spadesuit\}_{n\in\mathbb{Z}}$ with spectral density w_1 such that $w = |Q|^2 w_1$, and the future and the past of $\{X_n^\spadesuit\}_{n\in\mathbb{Z}}$ are at nonzero angle.*

Proof. Let

$$\rho_j = \|\mathcal{P}_{\boldsymbol{G}_0}\mathcal{P}_{\boldsymbol{G}^j}\|, \quad j \in \mathbb{Z}_+.$$

Let h be an outer function in H^2 such that $|h|^2 = w$. Denote by u the phase function $\bar{h}/h$. By Theorem 8.5.1, $\rho_j = \|H_{z^j u}\|$, $j \in \mathbb{Z}_+$.

It is easy to see that $\|H_{\bar{z}u}\| = 1$. Indeed,

$$\|H_{\bar{z}u}h\|_{H^2_-} = \|\bar{z}\bar{h}\|_{H^2_-} = \|h\|_{H^2}.$$

By the hypotheses, $\|H_{z^k u}\| < 1$. Let m be the minimal nonnegative integer such that $\|H_{z^m u}\| < 1$ but $\|H_{z^{m-1}u}\| = 1$. Clearly, such an m exists.

Put $v = z^m u$. Then by Theorem 3.1.11, T_v is left invertible while $T_{\bar{z}v}$ is not left invertible. It follows now from Theorem 3.1.14 that T_v is invertible. By Corollary 3.2.2, $v = \bar{h}_1/h_1$ for some outer function h_1 such that $h_1 \in H^2$ and $1/h_1 \in H^2$.

We need the following lemma.

Lemma 2.2. *Let h and h_1 be outer functions in H^2 and $m \in \mathbb{Z}_+$ such that*

$$\frac{\bar{h}}{h} = \bar{z}^m \frac{\bar{h}_1}{h_1}$$

and $T_{\bar{h}_1/h_1}$ is invertible. Then there exists a polynomial Q of degree m with zeros on $\mathbb{T}$ such that $h = Qh_1$.

To prove Lemma 2.2, we need one more lemma.

Lemma 2.3. *Let Q be a polynomial of degree m with zeros on $\mathbb{T}$. Then*

$$\frac{\bar{Q}}{Q} = c\bar{z}^m,$$

where $|c| = 1$.

Proof of Lemma 2.3. It suffices to consider the case $Q = z - \tau$, $\tau \in \mathbb{T}$. We have

$$\frac{\bar{Q}}{Q} = \frac{\bar{z} - \bar{\tau}}{z - \tau} = -\bar{\tau}\bar{z}. \quad \blacksquare$$

Proof of Lemma 2.2. Since $T_{\bar{h}_1/h_1}$ is invertible, it follows that $\dim \operatorname{Ker} T_{\bar{z}^{m+1}\bar{h}_1/h_1} = m + 1$. We claim that

$$\operatorname{Ker} T_{\bar{z}^{m+1}\bar{h}_1/h_1} = \operatorname{span}\{z^j h_1 : \ 0 \le j \le m\}. \tag{2.1}$$

Indeed, it is easy to see that the right-hand side of (2.1) has dimension $m+1$ and is contained in the left-hand side.

Since $T_{\bar{z}^{m+1}\bar{h}_1/h_1} = T_{\bar{z}\bar{h}/h}$, it is easy to see that $h \in \operatorname{Ker} T_{\bar{z}^{m+1}\bar{h}_1/h_1}$. Hence, there exists a polynomial Q of degree at most m such that $h = Qh_1$.

It remains to prove that $\deg Q = m$ and all zeros of Q lie on $\mathbb{T}$. Since h is outer, it follows that Q has no zeros inside $\mathbb{D}$. Let $Q = Q_1 Q_2$, where Q_1 and Q_2 are polynomials such that Q_1 has zeros on $\mathbb{T}$ while Q_2 has zeros outside $\operatorname{clos}\mathbb{D}$. Let $j = \deg Q_1$. We have

$$\frac{\bar{h}}{h} = \frac{\bar{Q}_1}{Q_1} \cdot \frac{\bar{Q}_2}{Q_2} \cdot \frac{\bar{h}_1}{h_1} = c\bar{z}^j \frac{\bar{Q}_2}{Q_2} \cdot \frac{\bar{h}_1}{h_1}, \quad |c| = 1,$$

by Lemma 2.3. We have

$$\begin{aligned}\operatorname{ind} T_{\bar{h}/h} &= \operatorname{ind}(T_{\bar{z}^j} T_{\bar{Q}_2} T_{\bar{h}_1/h_1} T_{Q_2^{-1}}) = \\ &= j + \operatorname{ind} T_{\bar{Q}_2} + \operatorname{ind} T_{\bar{h}_1/h_1} + \operatorname{ind} T_{Q_2^{-1}} = j,\end{aligned}$$

since obviously, $T_{\bar{Q}_2}$ and $\operatorname{ind} T_{Q_2^{-1}}$ are invertible. It follows that $j = m$, and so Q_2 is a constant. ■

Let us complete the proof of Theorem 2.1. By Lemma 2.2, $h = Qh_1$ for a polynomial Q of degree m whose zeros lie on $\mathbb{T}$. Put $w_1 = |h_1|^2$. Then $w_1^{-1} \in L^1$, and so by Theorem 1.1, w_1 is the density of a minimal stationary process. Obviously, $w = |Q|^2 w_1$.

By Theorem 8.5.1, the cosine of the angle between $H^2_-(w_1)$ and $H^2(w_1)$ is equal to $\|H_{\bar{h}_1/h_1}\|$, which is less than 1, and so w_1 is the spectral density of a stationary process for which the past and the future are at nonzero angle. ■

Remark 1. Consider the phase function $v = \bar{h}_1/h_1$ of the process $\{X_n^{\spadesuit}\}_{n\in\mathbb{Z}}$ with density w_1. The equality

$$u = \bar{z}^m v \tag{2.2}$$

will be used in §3 to obtain spectral interpretations of regularity conditions.

Remark 2. Let us show that the representation $w = |Q|^2 w_1$ is unique modulo multiplicative constants. Indeed, suppose that Q and $\breve{Q}$ are polynomials with zeros on $\mathbb{T}$ such that $|Q|^2 w_1 = |\breve{Q}|^2 \breve{w}_1$ and w_1 and $\breve{w}_1$ are invertible in L^1. Then

$$\frac{|Q|^2}{|\breve{Q}|^2} = \frac{\breve{w}_1}{w_1} \in L^1.$$

Hence, $\breve{Q}$ divides Q. Similarly, Q divides $\breve{Q}$.

Let us now prove the Helson–Szegö theorem, which characterizes the processes with nonzero angle between the past and the future.

Theorem 2.4. *Let μ be a positive Borel measure on $\mathbb{T}$. The following are equivalent:*

(i) *the subspaces $H^2_-(\mu)$ and $H^2(\mu)$ are at nonzero angle;*

(ii) *μ is absolutely continuous with respect to $\boldsymbol{m}$ and its Radon–Nikodym derivative w admits a representation*

$$w = \exp(\xi + \tilde{\eta}),$$

where ξ and η are real functions in L^∞ and $\|\eta\|_{L^\infty} < \pi/2$.

Proof. Let us show that (i) implies (ii). Since

$$H^2_-(\mu) \cap H^2(\mu) = \{\mathbb{O}\},$$

by the Szegö–Kolmogorov alternative (see Appendix 2.4), μ is absolutely continuous with respect to $\boldsymbol{m}$ and its Radon–Nikodym derivative w satisfies $\log w \in L^1$.

Let h be an outer function in H^2 such that $|h|^2 = w$. By Theorem 8.5.1, $\|H_{\bar{h}/h}\| < 1$. Since $\|H_{\bar{z}\bar{h}/h}\| = 1$, it follows that $T_{\bar{h}/h}$ is invertible (see the proof of Theorem 2.1). (ii) follows now from Theorem 3.2.5.

Conversely, suppose that (ii) holds. Put

$$h = \exp\left(\frac{\xi + \tilde{\eta} + i\tilde{\xi} - i\eta}{2}\right).$$

Then $|h|^2 = \exp(\xi+\tilde{\eta})$. By Zygmund's theorem (see Appendix 2.1), $h \in H^2$. The result now follows from Theorem 3.2.5. ■

Recall that by Theorem 8.4.2, both (i) and (ii) in the statement of Theorem 2.4 are equivalent to the fact that the sequence $\{z^n\}_{n\in\mathbb{Z}}$ forms a basis in $L^2(\mu)$.

Finally, we obtain a characterization of processes with $\cos(\widehat{\boldsymbol{G}_0, \boldsymbol{G}^k}) < 1$.

Theorem 2.5. *Let μ be a positive Borel measure on $\mathbb{T}$ and $k \in \mathbb{Z}_+$. The following are equivalent:*

(i) *the subspaces $H^2_-(\mu)$ and $z^k H^2(\mu)$ are at nonzero angle;*

(ii) *μ is absolutely continuous with respect to $\boldsymbol{m}$ and its Radon–Nikodym derivative w admits a representation*

$$w = |Q|^2 \exp(\xi + \tilde{\eta}),$$

where ξ and η are real functions in L^∞ and $\|\eta\|_{L^\infty} < \pi/2$ and Q is a polynomial with zeros on $\mathbb{T}$ whose degree is at most k.

Proof. Suppose that (i) holds. Again, by the Szegö–Kolmogorov alternative, μ is absolutely continuous with respect to $\boldsymbol{m}$ and $w = d\mu/d\boldsymbol{m}$ satisfies $\log w \in L^1$. (ii) follows immediately from Theorems 2.1 and 2.4.

Conversely, suppose that (ii) holds. Put

$$h = Q\exp\left(\frac{\xi + \tilde{\eta} + i\tilde{\xi} - i\eta}{2}\right).$$

By Theorem 2.4 and Lemma 2.3, $\|H_{z^m\bar{h}/h}\| < 1$, where $m = \deg Q$, and so by Theorem 8.5.1, $\cos(\widehat{H^2_-(\mu), z^m H^2(\mu)}) < 1$. ■

3. Regularity Conditions in Spectral Terms

In this section we study various regularity conditions for stationary processes and we characterize the processes satisfying those conditions in spectral terms. First, we reformulate regularity conditions in terms of phase functions of processes. Then we describe the corresponding processes in terms of their spectral densities. We are going to essentially use the results of Chapter 7 (the recovery problem for unimodular functions and the description of arguments of unimodular functions), which in turn are based on Hankel operators. In the next section we study stronger regularity conditions.

Recall that for a stationary Gaussian process with spectral density w regularity coefficients ρ_j are defined by

$$\rho_j = \|\mathcal{P}_{H^2_-(w)}\mathcal{P}_{z^j H^2(w)}\|, \quad j \in \mathbb{Z}_+.$$

As we have already observed, ρ_j is the cosine of the angle between the past $H^2_-(w)$ and the future $z^j H^2(w)$ from time j.

In this section we deal only with regular stationary processes. Let h be an outer function in H^2 such that $|h|^2 = w$ and let $u \stackrel{\text{def}}{=} \bar{h}/h$ be the corresponding phase function.

Recall that a process is called completely regular if $\lim\limits_{j\to\infty} \rho_j = 0$. It is called completely regular of order α, $\alpha > 0$, if $\rho_j \le \text{const}(1+j)^{-\alpha}$. It is called p-regular, $p > 0$, if $P_{H^2_-(w)}P_{H^2(w)} \in \boldsymbol{S}_p$. Finally, it is called p-regular of order α if $\|P_{H^2_-(w)}P_{z^j H^2(w)}\|_{\boldsymbol{S}_p} \le \text{const}(1+j)^{-\alpha}$.

Theorem 3.1. *Let $w = |h|^2$ be the spectral density of a regular stationary process. The following assertions hold.*

(i) *The process is completely regular if and only if $\mathbb{P}_- u \in VMO$.*

(ii) *The process is completely regular of order α, $\alpha > 0$, if and only if $\mathbb{P}_- u \in \Lambda_\alpha$.*

(iii) *The process is p-regular, $0 < p < \infty$, if and only if $\mathbb{P}_- u \in B_p^{1/p}$.*

(iv) *The process is p-regular of order α, $0 < p < \infty$, $\alpha > 0$, if and only if $\mathbb{P}_- u \in B_{p\infty}^{1/p+\alpha}$.*

Proof. Let us first prove (i). By Theorem 8.5.1, we have

$$\begin{aligned}\lim_{j\to\infty} \rho_j &= \lim_{j\to\infty} \|H_{z^j u}\| = \lim_{j\to\infty} \operatorname{dist}_{L^\infty}(z^j u, H^\infty) \\ &= \lim_{j\to\infty} \operatorname{dist}_{L^\infty}(u, \bar{z}^j H^\infty) = \operatorname{dist}_{L^\infty}(u, H^\infty + C).\end{aligned}$$

Since $H^\infty + C$ is closed in L^∞ by Theorem 1.5.1, it follows that the process is completely regular if and only if $u \in H^\infty + C$. By Theorem 1.5.5, this is equivalent to the compactness of H_u, which in turn is equivalent by Theorem 1.5.8 to the condition $\mathbb{P}_- u \in VMO$.

To prove (ii), we note that the process is completely regular of order α if and only if

$$\|P_{H^2_-(w)}P_{z^j H^2(w)}\| = \operatorname{dist}_{L^\infty}(z^j u, H^\infty) \le \text{const}(1+j)^{-\alpha}, \quad j \in \mathbb{Z}_+.$$

It follows from (A2.11) of Appendix A2.6 that the last inequality is equivalent to the fact that $\mathbb{P}_- u \in \Lambda_\alpha$.

By Theorem 8.5.1, the process is p-regular if and only if $H_u \in \boldsymbol{S}_p$. By Theorems 6.1.1, 6.1.2, and 6.1.3, this is equivalent to the condition $\mathbb{P}_- u \in B_p^{1/p}$, which proves (iii).

Finally, by Theorems 8.5.1, 6.1.1, 6.1.2, and 6.1.3, the process is p-regular of order α if and only if

$$\|\mathbb{P}_- z^j u\|_{B_p^{1/p}} \le \operatorname{const}(1+j)^{-\alpha}.$$

It follows easily from (A2.12) of Appendix A2.6 that the last inequality is equivalent to the fact that $\mathbb{P}_- u \in B_{p\infty}^{1/p+\alpha}$. ■

Corollary 3.2. *A regular stationary process with spectral density w is completely regular if and only if the operator $P_{H^2_-(w)} P_{H^2(w)}$ is compact.*

Proof. We have shown in the proof of Theorem 3.1 that complete regularity is equivalent to the fact that H_u is compact. The result now follows from Theorem 8.5.1. ■

We can now see that all the above regularity conditions can be expressed in terms of a condition of the form $\mathbb{P}_- u \in X$, where X is a suitable space of functions on X. We introduce a more general notion. For a space X of functions on $\mathbb{T}$ we say that a stationary process is X*-regular* if $\mathbb{P}_- u \in X$.

We restrict ourselves here to the case when X is a decent space (i.e., X satisfies the axioms (A1)–(A4)), or $X = B_p^{1/p}$, $0 < p < \infty$, or $X = VMO$.

Theorem 3.3. *Let X be decent function space, or $X = B_p^{1/p}$, $0 < p < \infty$, or $X = VMO$. A stationary process with spectral density w is X-regular if and only if w admits a representation $w = |Q|^2 e^{\varphi}$, where Q is a polynomial with zeros on $\mathbb{T}$ and φ is a real function in X.*

Proof. Suppose that the process is X-regular. By Theorem 4.4.10 the Toeplitz operator T_u has dense range in H^2. Therefore we can solve the recovery problem for the unimodular function u (see Theorems 7.1.4 and 7.2.3), which yields $u \in X$.

Clearly, $X \subset VMO$, and so

$$\lim_{j\to\infty} \|P_{H^2_-(w)} P_{z^j H^2(w)}\| = \lim_{j\to\infty} \operatorname{dist}_{L^\infty}(u, \bar{z}^j H^\infty) = 0.$$

By Theorem 2.1, there exist a polynomial Q with zeros on $\mathbb{T}$ and an outer function h_1 invertible H^2 such that $h = Qh_1$ and the Toeplitz operator $T_{\bar{h}_1/h_1}$ is invertible. Moreover, $u = \bar{z}^m \bar{h}_1/h_1$, where $m = \deg Q$. Hence, $v \stackrel{\text{def}}{=} \bar{h}_1/h_1 \in X$. By Theorem 7.8.1, $v = e^{\mathrm{i}\psi}$, where ψ is a real function in X. By Corollary 3.2.2 and Lemma 3.2.3,

$$h_1 = C \exp \frac{\tilde{\psi} - \mathrm{i}\psi}{2}$$

for some $C \in \mathbb{R}$. We have

$$w = C^2 |Q|^2 e^{\tilde{\psi}},$$

which proves the part "only if" since $\tilde{\psi} \in X$.

Suppose now that $w = |Q|^2 e^{\varphi}$, where Q is a polynomial with zeros on $\mathbb{T}$ and φ is a real function in X. Put

$$h_1 = \exp \frac{\varphi + \mathrm{i}\tilde{\varphi}}{2}.$$

By Lemma 2.3,

$$u = c\bar{z}^m \frac{\bar{h}_1}{h_1},$$

where $m = \deg Q$ and c is a unimodular constant. It follows that $u \in X$, and so $\mathbb{P}_- u = c\bar{z}^m \mathbb{P}_- e^{-\mathrm{i}\tilde{\varphi}} \in X$ (see Corollary 7.8.2). ■

Now we are in a position to state the main result of this section.

Theorem 3.4. *Let w be the density of a regular stationary process. The following assertions hold.*

(i) *The process is completely regular if and only if w admits a representation $w = |Q|^2 e^{\varphi}$, where Q is a polynomial with zeros on $\mathbb{T}$ and φ is a real function in VMO.*

(ii) *The process is completely regular of order α, $\alpha > 0$, if and only if w admits a representation $w = |Q|^2 e^{\varphi}$, where Q is a polynomial with zeros on $\mathbb{T}$ and φ is a real function in Λ_α.*

(iii) *The process is p-regular, $0 < p < \infty$, if and only if w admits a representation $w = |Q|^2 e^{\varphi}$, where Q is a polynomial with zeros on $\mathbb{T}$ and φ is a real function in $B_p^{1/p}$.*

(iv) *The process is p-regular of order α, $0 < p < \infty$, $\alpha > 0$, if and only if w admits a representation $w = |Q|^2 e^{\varphi}$, where Q is a polynomial with zeros on $\mathbb{T}$ and φ is a real function in $B_{p\infty}^{1/p}$.*

Theorem 3.4 follows immediately from Theorem 3.3.

We can also consider the following regularity conditions:

$$\sum_{j \ge 1} j^s \rho_j^q < \infty, \quad s > -1, \ 1 \le q < \infty, \tag{3.1}$$

and

$$\sum_{j \ge 1} j^s \|P_{H^2_-(w)} P_{z^j H^2(w)}\|^q_{\boldsymbol{S}_p} < \infty, \quad s > -1, \ 1 \le q < \infty, \ 1 \le p < \infty. \tag{3.2}$$

As before it can be easily shown that a stationary process satisfies (3.1) if and only if the phase function u satisfies $\mathbb{P}_- u \in B_{\infty q}^{(s+1)/q}$ while (3.2) is equivalent to the condition $\mathbb{P}_- u \in B_{pq}^{(s+1)/q+1/p}$.

We can now apply Theorem 3.3 with $X = B_{\infty q}^{(s+1)/q}$ and $X = B_{pq}^{(s+1)/q+1/p}$ and obtain the following result.

Theorem 3.5. *Let $s > -1$, $1 \le q < \infty$, $1 \le p < \infty$, and let w be the spectral density of a stationary process. The following assertions hold:*

(i) *the process satisfies* (3.1) *if and only if $w = |Q|^2 e^{\varphi}$, where Q is a polynomial with zeros on $\mathbb{T}$ and φ is a real function in $B_{\infty q}^{(s+1)/q}$;*

(ii) *the process satisfies* (3.1) *if and only if $w = |Q|^2 e^{\varphi}$, where Q is a polynomial with zeros on $\mathbb{T}$ and φ is a real function in $B_{pq}^{(s+1)/q+1/p}$.*

Finally, note that in Theorem 3.3 X does not have to be a Banach space. In particular, Theorem 3.3 holds for Carleman classes considered in §7.4. The special case when X is the *Gevrey class* G_α, $0 < \alpha < \infty$, is most interesting. Recall (see §7.4) that a function $f \in C(\mathbb{T})$ belongs to G_α if and only if there exist K, $\delta > 0$, such that

$$|\hat{f}(n)| \le K \exp(-\delta |n|^{\alpha/(1+\alpha)}), \quad n \in \mathbb{Z}. \tag{3.3}$$

Theorem 3.3 with $X = G_\alpha$ allows us to obtain the following result.

Theorem 3.6. *Let $0 < \beta < 1$ and $1 \le q < \infty$ and let w be the spectral density of a stationary process. The following are equivalent:*

(i) *there exist positive numbers K and δ such that*

$$\rho_j \le K \exp(-\delta j^\beta), \quad j \in \mathbb{Z}_+;$$

(ii) *w admits a representation $w = |Q|^2 e^\varphi$, where Q is a polynomial with zeros on $\mathbb{T}$ and φ is a real function in $G_{\beta/(1-\beta)}$.*

Proof. Theorem 3.6 follows immediately from Theorem 3.3 and the following characterization of $(G_\alpha)_-$:

$$\mathbb{P}_- g \in G_\alpha \iff \exists K,\, \delta > 0: \ \operatorname{dist}_{L^\infty}(z^n g, H^\infty) \le K e^{-\delta n^{\frac{\alpha}{1+\alpha}}}, \quad n \in \mathbb{Z}_+. \tag{3.4}$$

It is easy to see that (3.4) follows from (3.3). ∎

4. Stronger Regularity Conditions

In this section we study strong regularity conditions for stationary processes. We consider the class of processes for which the regularity conditions decay exponentially. Then we consider the processes for which the kth canonical correlation of the future and the past is zero. Finally, we consider the processes such that the future from time m is independent on the past.

Let us first consider the case of exponential decay of the regularity coefficients ρ_n defined by (8.3.10). Recall that the space AC_δ, $0 < \delta < 1$, has been defined in §7.4.

Theorem 4.1. *Let w be the spectral density of a regular stationary processes and let $0 < \delta < 1$. The following are equivalent:*

(i) *for each $\gamma > \delta$ there exists $K > 0$ such that $\rho_n \le K\gamma^n$, $n \in \mathbb{Z}_+$;*

(ii) *$w \in AC_\delta$.*

Proof. It is easy to see that a function $f \in C(\mathbb{T})$ belongs to AC_δ if and only if for each $\gamma > \delta$ there exists $K > 0$ such that

$$|\hat{f}(j)| \le K\gamma^{|j|}, \quad j \in \mathbb{Z}. \tag{4.1}$$

It follows easily from (4.1) that if $u \in L^\infty$, then $\mathbb{P}_- u \in AC_\delta$ if and only if for each $\gamma > \delta$ there exists $K > 0$ such that

$$\operatorname{dist}_{L^\infty}(z^n u, H^\infty) \le K\gamma^j, \quad n \in \mathbb{Z}_+. \tag{4.2}$$

Let us first show that $|Q|^2 \in AC_\delta$ for any polynomial Q with zeros on $\mathbb{T}$. It is sufficient to consider the case $Q = z - \lambda$, $\lambda \in \mathbb{T}$. We have

$$|Q(\zeta)|^2 = (\zeta - \lambda)(\bar{\zeta} - \bar{\lambda}) = (\zeta - \lambda)(\frac{1}{\zeta} - \bar{\lambda}), \quad \zeta \in \mathbb{T},$$

and so $|Q|^2 \in AC_\delta$.

Suppose that (i) holds. Then the process is completely regular and by Theorem 3.4, $w = |Q|^2|h|^2$, where h is an outer function invertible in H^2 and Q is a polynomial with zeros on $\mathbb{T}$. Put $u = \bar{h}/h$. Clearly, u satisfies (4.2), and so $\mathbb{P}_- u \in AC_\delta$. By Theorem 7.4.17, $|h|^2 \in AC_\delta$, and so $w = |Q|^2|h|^2 \in AC_\delta$.

Suppose now that $w \in AC_\delta$.Then w can have only finitely many zeros on $\mathbb{T}$. Let Q be a polynomial with zeros on $\mathbb{T}$ such that $w = |Q|^2|h|^2$ for a function h invertible in H^∞. Clearly, $|h|^2 \in AC_\delta$. By Theorem 7.4.17, $\mathbb{P}_- \bar{h}/h \in AC_\delta$ and so $\bar{h}/h$ satisfies (4.2). Hence,

$$\begin{aligned} \rho_n &= \operatorname{dist}_{L^\infty}(z^n \frac{\bar{Q}}{Q}\frac{\bar{h}}{h}, H^\infty) \\ &= \operatorname{dist}_{L^\infty}(z^{n-\deg Q}\frac{\bar{h}}{h}, H^\infty) \le K\gamma^{n-\deg Q}, \quad n \ge \deg Q. \quad \blacksquare \end{aligned}$$

Consider now the class of stationary processes for which the kth canonical correlation of the past and the future is zero. Such processes are regular. Let w be the spectral density of a regular process. Clearly, the above condition means that the operator $P_{H^2_-(w)}P_{H^2(w)}$ on $L^2(w)$ has rank at most k. By Theorem 8.5.1, this is equivalent to the fact that the Hankel operator $H_{\bar{h}/h}$ has rank at most k, where h is an outer function in H^2 such that $|h|^2 = w$.

Theorem 4.2. *Let μ be the spectral measure of a stationary process and let $k \in \mathbb{Z}_+$. The following are equivalent:*

(i) $\operatorname{rank} P_{H^2_-(w)}P_{H^2(w)} = k$*;*

(ii) *w is a rational function with poles off $\mathbb{T}$ such that* $\deg w = 2k$*.*

Proof. Let h be an outer function such that $|h|^2 = w$. Suppose that $H_{\bar{h}/h}$ has finite rank. By Kronecker's theorem, the function $\mathbb{P}_- \bar{h}/h$ is rational. We have

$$\mathbb{P}_- w = \mathbb{P}_-|h|^2 = \mathbb{P}_-\left(h^2 \cdot \frac{\bar{h}}{h}\right) = \mathbb{P}_-\left(h^2 \cdot \mathbb{P}_-\frac{\bar{h}}{h}\right).$$

since $h^2 \in H^1$, it follows that $\mathbb{P}_- w$ is rational and

$$\deg \mathbb{P}_- w \le \deg \mathbb{P}_- \frac{\bar{h}}{h}.$$

Suppose now that w is rational and has no poles on $\mathbb{T}$. Then h is invertible in H^∞. We have

$$\mathbb{P}_- \frac{\bar{h}}{h} = \mathbb{P}_-\left(\frac{1}{h^2}|h|^2\right) = \mathbb{P}_-\left(\frac{1}{h^2}\mathbb{P}_-|h|^2\right).$$

Since $1/h^2 \in H^\infty$, it follows that $\mathbb{P}_-\bar{h}/h$ is rational and

$$\deg \mathbb{P}_- \frac{\bar{h}}{h} \le \deg \mathbb{P}_- w.$$

It remains to observe that since w is real, we have $\deg \mathbb{P}_- w = \deg \mathbb{P}_+ w$. ■

Finally, we consider the class of processes for which the future from time m is independent of the past.

Theorem 4.3. *Let w be the density of a regular stationary process and let $m \in \mathbb{Z}_+$. The following are equivalent:*
(i) $\rho_m = 0$;
(ii) $w = |Q|^2$, *where Q is a polynomial of degree at most m.*

Proof. Suppose that $w = |Q|^2$. Since $|\zeta - \lambda|^2 = |1 - \bar{\lambda}\zeta|^2$ for $\lambda \in \mathbb{D}$ and $\zeta \in \mathbb{T}$, we may assume without loss of generality that Q has no zeros in $\mathbb{D}$, and so Q is an outer function. It is easy to see that $z^m\bar{Q}/Q \in H^\infty$. Hence, $\rho_m = 0$.

Suppose now that $\rho_m = 0$. Let h be an outer function such that $|h|^2 = w$. Then $\mathbb{P}_- z^m\bar{h}/h = \mathbb{O}$. In other words, $f \stackrel{\text{def}}{=} z^m\bar{h}/h \in H^\infty$. Put $Q \stackrel{\text{def}}{=} z^m\bar{h} = hf \in H^2$. Obviously, Q is a polynomial of degree at most m and $|Q|^2 = |h|^2 = w$. ■

Concluding Remarks

Theorem 1.1 was proved in Kolmogorov [1]. Theorem 1.2 is well known. Theorem 1.3 is contained in Peller and Khrushchëv [1]. Corollary 1.4 was observed in Jewell and Bloomfield [1].

Theorem 2.1 is due to Helson and Sarason [1]. The proof given in §2 is a simplification of the proof given in Peller and Khrushchëv [1]. Theorem 2.4 is the famous Helson–Szegö theorem established in Helson and Szegö [1]. Theorem 2.4 is a combination of Theorems 2.1 and 2.4; see Helson and Sarason [1]. Note that the Helson–Szegö condition is equivalent to the *Muckenhoupt condition* A_2:

$$\sup_I \left(\frac{1}{\boldsymbol{m}(I)}\int_I w\,d\boldsymbol{m}\right)\left(\frac{1}{\boldsymbol{m}(I)}\int_I w^{-1}\,d\boldsymbol{m}\right).$$

In other words, the Riesz projection $\mathbb{P}_+$ is a bounded operator on $L^2(w)$ if and only if w satisfies the Muckenhoupt condition A_2; see Hunt, Muckenhoupt, and Wheeden [1].

The completely regular processes were characterized in Helson and Sarason [1] and later this description was refined in Sarason [2]. Of course, the space VMO was not mentioned in Sarason [2], since it did not exist at that time. After the work of Fefferman [1] and Sarason [4] on BMO and VMO it became possible to state the description of the completely regular stationary processes in terms of VMO. Note that before the work of Fefferman, Ibragimov had found necessary conditions and sufficient conditions of local character (see Ibragimov [1–4]). For some time it was not clear whether the necessary condition found in Ibragimov [4] is sufficient. Sarason [2] showed that this is not the case but Ibragimov's condition is still of interest. The proof given in §2 is based on the approach by Peller and Khrushchëv [1]. Let us mention here another approach given in Arocena, Cotlar, and Sadosky [1].

The completely regular processes of order α were described by Ibragimov [4]. Note that his proof was technically complicated. The p-regular processes were described in Ibragimov and Solev [1] for $p = 2$ (see also Solev [1]), Peller and Khrushchëv [1] for $1 \le p < \infty$, and Peller [8] for $0 < p < 1$. The p-regular processes of order α were described in Peller and Khrushchëv [1]. Note that in §3 we use the method of Peller and Khrushchëv [1], which gives a single simplified approach to all regularity conditions. Theorems 3.4 and 3.6 are also taken from Peller and Khrushchëv [1].

Theorem 4.1 is due to Ibragimov [3]. Theorem 4.2 was obtained in Yaglom [2]. Finally, Theorem 4.3 can be found in Ibragimov and Rozanov [1].

The corresponding problems for vectorial stationary processes are considerably more complicated. A sequence

$$X_n = \begin{pmatrix} X_{n,1} \\ \vdots \\ X_{n,d} \end{pmatrix}, \quad X_{n,j} \in \boldsymbol{G},$$

is called a *vectorial stationary Gaussian process* if the covariance matrix

$$Q_{nk} = \{\mathbb{E}X_{n,j_1}X_{k,j_2}\}_{1\le j_1,j_2\le d}$$

depends only on $n-k$. By analogy with the scalar case we can define the past and the future:

$$\begin{aligned} \boldsymbol{G}_0 &= \operatorname{span}\{X_{m,j} :\ m<0,\ 1\le j\le d\}, \\ \boldsymbol{G}^n &= \operatorname{span}\{X_{m,j} :\ m\ge n,\ 1\le j\le d\}. \end{aligned}$$

It is well known (see Rozanov [1]) that there exists a matrix-valued positive measure M on $\mathbb{T}$ such that $Q_{nk} = \hat{M}(n-k)$, $n,k\in\mathbb{Z}$. It is called the *spectral measure of the process.* As in the scalar case the process is called *regular* if $\bigcap\limits_{n\ge 0} \boldsymbol{G}^m = \{\mathbb{O}\}$. If the process is regular, the spectral measure M is absolutely continuous with respect to Lebesgue measure and its density W is called the *spectral density of the process.* For regular processes $\operatorname{rank} W(\zeta)$ is constant almost everywhere on $\mathbb{T}$ (see Rozanov [1]). The process is said to be of *full rank* if $\operatorname{rank} W(\zeta) = d$ for almost all $\zeta\in\mathbb{T}$. The process is regular and has full rank if and only if $\log\det W \in L^1$; see Rozanov [1]. Here we discuss only regular processes of full rank.

As in the scalar case one can identify the Gaussian space with the weighted space $L^2(W)$ of $\mathbb{C}^n$-valued function. Under this identification $\boldsymbol{G}^0$

becomes $H^2(W)$, the closure of the analytic polynomials in $L^2(W)$ while $\boldsymbol{G}_0$ becomes $H^2_-(W)$. By analogy with the scalar case we can define completely regular processes, completely regular processes of order α, p-regular processes, p-regular processes of order α, X-regular processes, etc.

By the Wiener–Masani theorem (Wiener and Masani [1]; see also Rozanov [1]) there are outer square matrix functions Ψ and $\Psi_\#$ such that $W = \Psi\Psi^* = \Psi_\#\Psi_\#^*$. Consider now the matrix function $U = \Psi_\#^*\Psi^{-1}$. It is unitary-valued and it is called a *phase function* of the process. It is defined up to constant unitary factors. Again, as in the scalar case $P_{\boldsymbol{G}_0}P_{\boldsymbol{G}^m} = \mathcal{V}_1^* H_{z^m U}\mathbb{P}_+\mathcal{V}_2$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are unitary operators (see Peller [17]). It was shown in Peller [26] that for a broad class of function spaces X a process with spectral density W is X-regular if and only if $W = Q^* W_1 Q$, where Q is a matrix polynomial such that all zeros of $\det Q$ lie on $\mathbb{T}$ and W_1 is a spectral density of a minimal X-regular processes (as in the scalar case minimality is equivalent to the fact that W_1^{-1} is integrable). This reduces the study of completely regular, p-regular, completely regular of order α, p-regular of order α processes to the case of minimal processes (i.e., processes with spectral densities invertible in L^1). The completely regular processes of order α were described by Ibragimov [5]: a process with spectral density W is a completely regular process of order α if and only if $W = Q^* W_1 Q$, where Q is a matrix polynomial as above, $W_1 \in \Lambda_\alpha$ (which means that all entries of W_1 belong to Λ_α) and $\det W_1$ does not vanish on $\mathbb{T}$. In Peller [17] and [26] the proof of Ibragimov was significantly simplified. The p-regular processes of order α were described in a similar way in Peller [26] (Λ_α has to be replaced with $B^{1/p+\alpha}_{p\infty}$).

However, it turns out that the problems to characterize the completely regular and p-regular processes are much more complicated. In Treil and Volberg [2] it was shown that a minimal vectorial process with spectral density W is completely regular if and only if

$$\lim_{\boldsymbol{m}(I)\to 0}\left\|\left(\frac{1}{\boldsymbol{m}(I)}\int_I W d\boldsymbol{m}\right)^{1/2}\left(\frac{1}{\boldsymbol{m}(I)}\int_I W^{-1} d\boldsymbol{m}\right)^{1/2}\right\| = 1,$$

where the limit is taken over subarcs I of $\mathbb{T}$. Note that this characterization does not generalize to the case of p-regular processes. In Peller [26] sufficient conditions of different character were found for complete regularity and p-regularity. The problem to characterize the p-regular processes is still open.

In this book we consider only stationary processes with discrete time. The case of continuous time is also very important, but the corresponding problems to characterize regularity conditions in spectral terms are still open. A continuous function $t \mapsto X_t$ from $\mathbb{R}$ to a Gaussian space $\boldsymbol{G}$ is called a *stationary Gaussian process with continuous time* if $\mathbb{E}X_t X_s$ depends only on $t - s$, $t, s \in \mathbb{R}$. By Bochner's theorem (see Akhiezer and Glazman [1], §70), there exists a finite positive measure μ (the spectral measure of the process) such that $\mathbb{E}X_t X_0 = \int\limits_{\mathbb{R}} e^{-\mathrm{i}tx} d\mu(x)$. In a similar way one can define the subspaces $\boldsymbol{G}_0$, the past, and $\boldsymbol{G}^t$, the future from time t, and consider similar regularity conditions. For example, the process is called completely regular if $\|P_{\boldsymbol{G}_0} P_{\boldsymbol{G}^t}\| \to 0$ as $t \to \infty$ and it is called p-regular if $P_{\boldsymbol{G}_0} P_{\boldsymbol{G}^t} \in \boldsymbol{S}_p$ if t is greater than a certain number t_0. However, such problems are much more complicated and still remain open. We mention here that unlike the case of discrete time it is not true that the process is completely regular if and only if $P_{\boldsymbol{G}_0} P_{\boldsymbol{G}^t}$ is compact for large values of t.

10

Spectral Properties of Hankel Operators

This chapter is an introduction to spectral properties of Hankel operators. Here we present certain selected results. Note that we do not include in this book some other known results on spectral properties of Hankel operators, and we give some references at the end of this chapter.

Certainly, to speak about spectral properties, we have to deal with operators that map a Hilbert space into itself. Given a function φ analytic in $\mathbb{D}$, we consider the Hankel matrix

$$\Gamma_\varphi = \{\hat{\varphi}(j+k)\}_{j,k\geq 0}. \tag{0.1}$$

By the Nehari theorem, Γ_φ determines a bounded linear operator on ℓ^2 if and only if there exists a function ψ in L^∞ such that

$$\hat{\psi}(k) = \hat{\varphi}(k), \quad k \in \mathbb{Z}_+, \tag{0.2}$$

and

$$\|\Gamma_\varphi\| = \inf \|\psi\|_{L^\infty}, \tag{0.3}$$

the infimum being taken over all functions $\psi \in L^\infty$ satisfying (0.2). For a function φ in L^∞ we define the Hankel matrix Γ_φ by (0.1). It follows easily from (0.3) that for $\varphi \in L^\infty$,

$$\|\Gamma_\varphi\| = \operatorname{dist}_{L^\infty}(\varphi, H^\infty_-),$$

where

$$H^\infty_- = \{f \in L^\infty : \ \hat{f}(k) = 0 \ \text{ for } k < 0\}.$$

We identify bounded Hankel matrices with operators on H^2 in the orthonormal basis z^k, $k \ge 0$. It is easy to see that

$$\Gamma_\varphi f = \mathbb{P}_+ \varphi \mathcal{J} f, \quad f \in H^2, \tag{0.4}$$

where the operator $\mathcal{J}$ on L^2 is defined by

$$(\mathcal{J} g)(z) = g(\bar{z}), \quad g \in L^2. \tag{0.5}$$

Indeed, it is easy to verify that

$$(\mathbb{P}_+ \varphi \mathcal{J} z^j, z^k) = \hat{\varphi}(j+k), \quad j, k \in \mathbb{Z}_+.$$

It is easy to verify that

$$\Gamma_\varphi^* = \Gamma_{\overline{\mathcal{J}\varphi}}.$$

It follows easily from formula (1.13) of Chapter 1 that

$$S^* \Gamma_\varphi = \Gamma_\varphi S = \Gamma_{S^* \mathbb{P}_+ \varphi}, \tag{0.6}$$

where S is unilateral shift, i.e., multiplication by z on H^2.

Let us show that the essential spectrum of a bounded Hankel operator always contains the origin.

Lemma 0.1. *Let Γ_φ be a bounded Hankel operator on H^2. Then $0 \in \sigma_{\mathrm{e}}(\Gamma_\varphi)$.*

Proof. Suppose that $0 \notin \sigma_{\mathrm{e}}(\Gamma_\varphi)$. Then there exist a bounded linear operator T and a compact operator K such that $T\Gamma_\varphi = I + K$. It follows from (0.6) that

$$T \Gamma_\varphi S^n \mathbf{1} = z^n + K z^n = T (S^*)^n \Gamma_\varphi \mathbf{1},$$

where $\mathbf{1}$ is the constant function identically equal to 1. Clearly, $\|K z^n\| \to 0$, since K is compact. It is also easy to see that $\|(S^*)^n \Gamma_\varphi \mathbf{1}\| \to 0$. However, $\|z^n\|_2 = 1$, and we get a contradiction. ■

In §1 we describe the essential spectrum of Hankel operators with piecewise continuous symbols. We also describe the spectrum of the Hilbert matrix. In §2 we study the Carleman operator introduced in §1.8. We prove that the Carleman operator has Lebesgue spectrum on $[0, \pi]$ of multiplicity 2. In the last section we show that there are no nonzero nilpotent Hankel operators. On the other hand, we show how to construct nonzero quasinilpotent Hankel operators. More generally, we consider in §3 a class of Hankel operators whose symbols have lacunary Fourier series and we develop a procedure to describe their spectra which leads to the construction of a nonzero quasinilpotent Hankel operator.

1. The Essential Spectrum of Hankel Operators with Piecewise Continuous Symbols

In this section we describe the essential spectrum of Hankel operators with piecewise continuous symbols. We also describe the spectrum of the Hilbert matrix.

We start with certain special Hankel operators. Let us introduce the following notation:

$$\mathbb{T}_+ \stackrel{\text{def}}{=} \{\zeta \in \mathbb{T} : \ \operatorname{Im} \zeta \geq 0\} \quad \text{and} \quad \mathbb{T}_- \stackrel{\text{def}}{=} \{\zeta \in \mathbb{T} : \ \operatorname{Im} \zeta < 0\}.$$

Consider the characteristic functions of $\mathbb{T}_+$ and $\mathbb{T}_-$:

$$\chi_+ \stackrel{\text{def}}{=} \chi_{\mathbb{T}_+} \quad \text{and} \quad \chi_- \stackrel{\text{def}}{=} \chi_{\mathbb{T}_-}.$$

If α and β are distinct points on the complex plane, we denote by $[\alpha, \beta]$ the interval of the straight line that joins α and β.

Lemma 1.1.

$$\sigma(\Gamma_{\bar{z}\chi_-}) = \sigma_{\mathrm{e}}(\Gamma_{\bar{z}\chi_-}) = \left[0, \frac{\mathrm{i}}{2}\right]$$

and

$$\sigma(\Gamma_{\bar{z}\chi_+}) = \sigma_{\mathrm{e}}(\Gamma_{\bar{z}\chi_+}) = \left[0, -\frac{\mathrm{i}}{2}\right].$$

Proof. Clearly, $\hat{\chi}_+(j) = -\hat{\chi}_-(j)$ for $j \neq 0$, and so

$$\Gamma_{\bar{z}\chi_+} = -\Gamma_{\bar{z}\chi_-}.$$

Thus it is sufficient to prove the result for $\Gamma\bar{z}\chi_+$. Using (0.4) and (0.5), we obtain for $f \in H^2$

$$\begin{aligned} -\Gamma^2_{\bar{z}\chi_+} f &= \Gamma_{\bar{z}\chi_+}\Gamma_{\bar{z}\chi_-} f = \mathbb{P}_+\bar{z}\chi_+\mathcal{J}(\Gamma_{\bar{z}\chi_-} f) \\ &= \mathbb{P}_+\bar{z}\chi_+\mathcal{J}(\mathbb{P}_+\bar{z}\chi_-\mathcal{J}f) = \mathbb{P}_+\bar{z}\chi_+\mathcal{J}\mathbb{P}_+\mathcal{J}\mathcal{J}(\bar{z}\chi_-\mathcal{J}f) \\ &= \mathbb{P}_+\bar{z}\chi_+z\mathbb{P}_-\bar{z}z\chi_+ f = \mathbb{P}_+\chi_+\mathbb{P}_-\chi_+ f \\ &= \mathbb{P}_+\chi_+ f - \mathbb{P}_+\chi_+\mathbb{P}_+\chi_+ f = T_{\chi_+} f - T^2_{\chi_+} f, \end{aligned}$$

and so

$$\Gamma^2_{\mathrm{i}\bar{z}\chi_+} = T_{\chi_+} - T^2_{\chi_+}.$$

By Theorem 3.3.1, $\sigma(T_{\chi_+}) = [0, 1]$. By the spectral mapping theorem,

$$\sigma\left(T_{\chi_+} - T^2_{\chi_+}\right) = \left[0, \frac{1}{4}\right].$$

It is easy to calculate the Fourier coefficients of χ_+ and find that

$$\Gamma_{\mathrm{i}\bar{z}\chi_+} = \frac{1}{\pi}\begin{pmatrix} 1 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & \cdots \\ 0 & \frac{1}{3} & 0 & \frac{1}{5} & 0 & \cdots \\ \frac{1}{3} & 0 & \frac{1}{5} & 0 & \frac{1}{7} & \cdots \\ 0 & \frac{1}{5} & 0 & \frac{1}{7} & 0 & \cdots \\ \frac{1}{5} & 0 & \frac{1}{7} & 0 & \frac{1}{9} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and so $\Gamma_{\mathrm{i}\bar{z}\chi_+} = \frac{1}{2\pi}\Gamma_{[\boldsymbol{m}_1]}$; see formula (7.9) of Chapter 1. By Corollary 1.7.6, $\Gamma_{\mathrm{i}\bar{z}\chi_+}$ is a nonnegative operator, and so

$$\sigma(\Gamma_{\mathrm{i}\bar{z}\chi_+}) = \mathrm{i}\sigma(\Gamma_{\bar{z}\chi_+}) = \left[0, \frac{1}{2}\right].$$

Since $\Gamma_{\mathrm{i}\bar{z}\chi_+}$ is self-adjoint, it is evident that $\sigma_{\mathrm{e}}(\Gamma_{\mathrm{i}\bar{z}\chi_+}) = [0, 1/2]$. ■

Remark. Recall (see §1.8) that the matrix of $2\pi\Gamma_{\mathrm{i}\bar{z}\chi_+}$ is the same as the matrix of the Carleman operator $\boldsymbol{\Gamma}$ in the basis $\{\mathcal{F}\xi_n\}_{n\ge 0}$, and so $2\pi\Gamma_{\mathrm{i}\bar{z}\chi_+}$ is unitarily equivalent to the Carleman operator. Thus

$$\sigma(\boldsymbol{\Gamma}) = \sigma_{\mathrm{e}}(\boldsymbol{\Gamma}) = [0, \pi].$$

Let us proceed now to the Hilbert matrix $\left\{\frac{1}{1+j+k}\right\}_{j,k\ge 0}$. In §1.1 we have seen that it is equal to Γ_{ψ_1}, where the function ψ_1 is defined by

$$\psi_1(e^{\mathrm{i}t}) = \mathrm{i}e^{-\mathrm{i}t}(\pi - t), \quad t \in [0, 2\pi).$$

We introduce the operators D_τ, $\tau \in \mathbb{T}$, defined by

$$(D_\tau f)(\zeta) = f(\tau\zeta), \quad \zeta \in \mathbb{T}.$$

For $\tau \in \mathbb{T}$ we consider the piecewise linear function ψ_τ on $\mathbb{T}$ defined by

$$\psi_\tau = D_{\bar{\tau}}\psi_1.$$

It is easy to see that ψ_τ has a jump discontinuity at τ and

$$\psi_\tau(\tau^+) - \psi_\tau(\tau^-) = \psi_1(1^+) - \psi_1(1^-) = 2\pi\mathrm{i}$$

(see the notation in §1.5). Put

$$\Gamma_\tau \stackrel{\mathrm{def}}{=} \Gamma_{\psi_\tau}, \quad \tau \in \mathbb{T}.$$

Clearly, $\Gamma_\tau = \left\{\frac{\bar{\tau}^{j+k}}{1+j+k}\right\}_{j,k\ge 0}$, Γ_1 is the Hilbert matrix, and

$$\Gamma_\tau^* = \Gamma_{\bar{\tau}}, \quad \tau \in \mathbb{T}.$$

It is also easy to verify that

$$D_{\bar{\tau}}\Gamma_1 D_{\bar{\tau}} = \Gamma_\tau. \tag{1.1}$$

Indeed, let $f \in H^2$. We have

$$\begin{aligned} D_{\bar{\tau}}\Gamma_1 D_{\bar{\tau}} f &= D_{\bar{\tau}}\Gamma_1 f(\bar{\tau}z) = D_{\bar{\tau}}\mathbb{P}_+\psi_1\mathcal{J}(f(\bar{\tau}z)) = D_{\bar{\tau}}\mathbb{P}_+\psi_1 f(\tau\bar{z}) \\ &= \mathbb{P}_+ D_{\bar{\tau}}(\psi_1 f(\tau\bar{z})) = \mathbb{P}_+\psi_1(\bar{\tau}z)f(\bar{z}) = \mathbb{P}_+\psi_\tau\mathcal{J}f = \Gamma_\tau f. \end{aligned}$$

We need several elementary properties of the operators Γ_τ.

Lemma 1.2. *If $\tau \in \mathbb{T}$ and $\tau \neq 1$, then $\Gamma_1 D_\tau \Gamma_1 \in \boldsymbol{S}_2$.*

Proof. We have

$$\begin{aligned}
\Gamma_1 D_\tau \Gamma_1 z^j &= \Gamma_1 \left(\sum_{k=0}^{\infty} \frac{\tau^k z^k}{1+j+k} \right) \\
&= \sum_{k=0}^{\infty} \frac{\tau^k}{1+j+k} \sum_{m=0}^{\infty} \frac{z^m}{1+k+m} = \sum_{m=0}^{\infty} a_{mj} z^j,
\end{aligned}$$

where

$$a_{mj} = \sum_{k=0}^{\infty} \frac{\tau^k}{(1+j+k)(1+k+m)}.$$

Put

$$s_k \stackrel{\text{def}}{=} 1 + \tau + \tau^2 + \cdots + \tau^k.$$

Clearly, $|s_k| \le 2|1-\tau|^{-1}$ and

$$\begin{aligned}
a_{mj} &= \frac{1}{(1+j)(1+m)} + \sum_{k=1}^{\infty} \frac{s_k - s_{k-1}}{(1+j+k)(1+k+m)} \\
&= \sum_{k=0}^{\infty} s_k \left(\frac{1}{(1+j+k)(1+k+m)} - \frac{1}{(2+j+k)(2+k+m)} \right)
\end{aligned}$$

Hence,

$$\begin{aligned}
|a_{mj}| &\le \frac{2}{|1-\tau|} \sum_{k=0}^{\infty} \left(\frac{1}{(1+j+k)(1+k+m)} - \frac{1}{(2+j+k)(2+k+m)} \right) \\
&= \frac{2}{|1-\tau|} \cdot \frac{1}{(1+m)(1+j)}.
\end{aligned}$$

It follows that

$$\sum_{m,j \in \mathbb{Z}_+} |a_{mj}|^2 < \infty,$$

and so $\Gamma_1 D_\tau \Gamma_1 \in \boldsymbol{S}_2$. ■

Lemma 1.3. *Let $\tau_1, \tau_2 \in \mathbb{T}$ and $\tau_1\tau_2 \neq 1$. Then $\Gamma_{\tau_1}\Gamma_{\tau_2} \in \boldsymbol{S}_2$.*

Proof. We have

$$\Gamma_{\tau_1}\Gamma_{\tau_2} = D_{\bar\tau_1}\Gamma_1 D_{\bar\tau_1} D_{\bar\tau_2}\Gamma_1 D_{\bar\tau_2} = D_{\bar\tau_1}\Gamma_1 D_{\overline{\tau_1\tau_2}}\Gamma_1 D_{\bar\tau_2} \in \boldsymbol{S}_2$$

by Lemma 1.2. ■

Lemma 1.4. $\Gamma_1 + \Gamma_{-1} = 2\pi\mathrm{i}\Gamma_{z\chi_+}$.

Proof. It is easy to verify that $\psi_1 + \psi_{-1} - 2\pi\mathrm{i}z\chi_+ \in H^\infty_-$, which implies the result. ■

We also need the following general fact.

Lemma 1.5. *Let T_1 and T_2 be bounded linear operators on Hilbert space such that the operators T_1T_2 and T_2T_1 are compact. Then*

$$\sigma_{\mathrm{e}}(T_1+T_2)\cup\{0\}=\sigma_{\mathrm{e}}(T_1)+\sigma_{\mathrm{e}}(T_2).$$

Proof. Suppose that T_1 and T_2 act on a Hilbert space $\mathcal{H}$. Let $\mathcal{B}$ the algebra of operators on $\mathcal{H}$ and let $\mathcal{C}$ be the ideal of compact operators. Consider the Calkin algebra $\mathcal{B}/\mathcal{C}$. It is a C^*-algebra. By the Gelfand–Naimark theorem (see Appendix 1.3), it is isomorphic to a C^*-algebra of operators on Hilbert space. Thus it is sufficient to prove that if R_1 and R_2 are operators on Hilbert space such that $R_1R_2=\mathbb{O}$, then $\sigma(R_1+R_2)\cup\{0\}=\sigma(R_1)\cup\sigma(R_2)$. This is an immediate consequence of the following lemma.

Lemma 1.6. *Let a and b be elements of a unital algebra A such that $ab=ba=0$. Then $\sigma(a+b)\cup\{0\}=\sigma(a)\cup\sigma(b)$.*

Proof. Let λ be a nonzero complex number and let I be the unit of A. We have

$$(a-\lambda I)(b-\lambda I)=\lambda^2I-\lambda(a+b)=\lambda(\lambda I-(a+b))=(b-\lambda I)(a-\lambda I).$$

Thus, $(\lambda I-(a+b))$ is invertible if and only if both $(a-\lambda I)$ and $(b-\lambda I)$ are invertible. On the other hand, since $ab=0$, 0 must belong to $\sigma(a)\cup\sigma(b)$. ■

The following result describes the spectrum and the essential spectrum of the Hilbert matrix Γ_1.

Theorem 1.7. $\sigma(\Gamma_1)=\sigma_{\mathrm{e}}(\Gamma_1)=\sigma(\Gamma_{-1})=\sigma_{\mathrm{e}}(\Gamma_{-1})=[0,\pi]$.

Proof. By Lemma 1.3, the operator $\Gamma_1\Gamma_{-1}$ is compact. Clearly, Γ_1 is self-adjoint. Since $\Gamma_{-1}=D_{-1}\Gamma_1D_{-1}$ and D_{-1} is a self-adjoint unitary operator, the operators Γ_1 and Γ_{-1} are unitarily equivalent. By Lemma 1.4, $\Gamma_1+\Gamma_{-1}=2\pi\mathrm{i}\Gamma_{z\chi_+}$, and by Lemma 1.1, $\sigma_{\mathrm{e}}(\Gamma_1+\Gamma_{-1})=[0,\pi]$.

By Lemma 1.5,

$$[0,\pi]=\sigma_{\mathrm{e}}(\Gamma_1+\Gamma_{-1})=\sigma_{\mathrm{e}}(\Gamma_1)\cup\sigma_{\mathrm{e}}(\Gamma_{-1})=\sigma_{\mathrm{e}}(\Gamma_1),$$

since Γ_1 and Γ_2 are unitarily equivalent. Clearly, $\sigma_{\mathrm{e}}(\Gamma_1)\subset\sigma(\Gamma_1)$. By Corollary 1.7.6, $\sigma(\Gamma_1)\subset\mathbb{R}_+$, and since $\|\Gamma_1\|\le\pi$ (see §1.1), it follows that $\sigma(\Gamma_1)=[0,\pi]$. ■

Given a bounded operator T on Hilbert space, we denote by $\check{T}$ the coset of T in the Calkin algebra $\mathcal{B}/\mathcal{C}$. We need the following general fact.

Lemma 1.8. *Let T_j, $j\ge0$, be bounded linear operators on Hilbert space such that the operators T_jT_k, T_kT_j, $T_j^*T_k$, and $T_kT_j^*$ are compact for $j\ne k$. Suppose that the series $\sum_{j\ge0}\check{T}_j$ converges unconditionally in the Calkin algebra $\mathcal{B}/\mathcal{C}$. Let T be a bounded linear operator such that $\check{T}=\sum_{j\ge0}\check{T}_j$. Then*

$$\|T\|_{\mathrm{e}}=\max_{j\ge0}\|T_j\|_{\mathrm{e}}$$

and

$$\sigma_{\rm e}(T)\cup\{\mathbb{O}\}=\bigcup_{j\ge0}\sigma_{\rm e}(T_j).$$

Proof. Again, we can apply the Gelfand–Naimark theorem and reduce the lemma to the following fact. Let R_j be bounded linear operators on Hilbert space such that $R_jR_k=R_kR_j=R_j^*R_k=R_kR_j^*=\mathbb{O}$ for $j\ne k$. Suppose that the series $\sum\limits_{j\ge0}R_j$ converges unconditionally in $\mathcal{B}$ and $R=\sum\limits_{j\ge0}R_j$. Then

$$\|R\|=\max_{j\ge0}\|R_j\| \tag{1.2}$$

and

$$\sigma(R)\cup\{\mathbb{O}\}=\bigcup_{j\ge0}\sigma(R_j). \tag{1.3}$$

Let us first prove (1.2). Clearly, it is sufficient to consider the case of two operators R_1 and R_2. Since

$$\|R_1+R_2\|^2=\|(R_1^*+R_2^*)(R_1+R_2)\|=\|R_1^*R_1+R_2^*R_2\|,$$

we may assume without loss of generality that R_1 and R_2 are self-adjoint and nonnegative. Then

$$\|R_j\|=\max\{\lambda:\ \lambda\in\sigma(R_j)\},\quad j=1,\,2,$$

and

$$\|R_1+R_2\|=\max\{\lambda:\ \lambda\in\sigma(R_1+R_2)\}.$$

Now (1.2) is a consequence of Lemma 1.6.

To prove (1.3), we consider the operators

$$Q_N\stackrel{\rm def}{=}\sum_{j=1}^N R_j\quad\text{and}\quad Q_N^{\#}\stackrel{\rm def}{=}\sum_{j>N}R_j.$$

By Lemma 1.6,

$$\sigma(Q_N)\cup\{0\}=\bigcup_{j=1}^N\sigma(R_j)$$

and

$$\sigma(R)\cup\{0\}=\sigma(Q_N)\cup\sigma(Q_N^{\#}).$$

On the other hand, by (1.2), $\sigma(Q_N^{\#})\subset[-\varepsilon_N,\varepsilon_N]$, where

$$\varepsilon_N\stackrel{\rm def}{=}\max_{j\ge N}\|R_j\|\to0\quad\text{as}\quad N\to\infty.$$

This proves (1.3). ■

To proceed to the main results of this section, we need one more lemma.

Lemma 1.9. *Suppose that* $\tau \in \mathbb{T}$ *and* $\operatorname{Im} \tau \neq 0$. *Then*

$$\sigma_{\mathrm{e}}(\alpha\Gamma_\tau + \beta\Gamma_{\bar{\tau}}) = -\sigma_{\mathrm{e}}(\alpha\Gamma_\tau + \beta\Gamma_{\bar{\tau}})$$

for any α, $\beta \in \mathbb{C}$.

Proof. By Lemma 1.3, Γ_τ^2 is compact. Thus by the Gelfand–Naimark theorem it is sufficient to prove the following result. Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$ such that $T^2 = \mathbb{O}$. Then

$$\sigma(\alpha T + \beta T^*) = -\sigma(\alpha T + \beta T^*). \tag{1.4}$$

Let $\mathcal{L} = \operatorname{Ker} T$. Then T has the following representation with respect to the decomposition $\mathcal{H} = \mathcal{L} \oplus \mathcal{L}^\perp$:

$$T = \begin{pmatrix} \mathbb{O} & Q \\ \mathbb{O} & \mathbb{O} \end{pmatrix}.$$

We have

$$\alpha T + \beta T^* = \begin{pmatrix} \mathbb{O} & \alpha Q \\ \beta Q^* & \mathbb{O} \end{pmatrix}.$$

Now (1.4) is a consequence of the following more general fact. If R_1 and R_2 are bounded linear operators, then the spectrum of

$$\begin{pmatrix} \mathbb{O} & R_1 \\ R_2 & \mathbb{O} \end{pmatrix}$$

is symmetric about the origin. Indeed, this can be seen from the following elementary identity:

$$\begin{pmatrix} I & \mathbb{O} \\ \mathbb{O} & -I \end{pmatrix}\begin{pmatrix} \lambda I & R_1 \\ R_2 & \lambda I \end{pmatrix}\begin{pmatrix} -I & \mathbb{O} \\ \mathbb{O} & I \end{pmatrix} = \begin{pmatrix} -\lambda I & R_1 \\ R_2 & -\lambda I \end{pmatrix}. \quad \blacksquare$$

Now we are going to evaluate the essential spectrum of Hankel operators with piecewise continuous symbols. Recall that the algebra of piecewise continuous functions PC consists of functions φ on $\mathbb{T}$ that have both the right limit $\varphi(\zeta^+)$ and the left limit $\varphi(\zeta^-)$ at each point ζ of $\mathbb{T}$ (see §1.5). As before, we assume that $\varphi(\zeta^+) = \varphi(\zeta)$ for any $\zeta \in \mathbb{T}$ and we denote by $\varkappa_\zeta(\varphi)$ the jump of φ at ζ:

$$\varkappa_\zeta(\varphi) = \varphi(\zeta) - \varphi(\zeta^-).$$

As we have already mentioned in §1.5, for any $\varepsilon > 0$ the set

$$\{\zeta \in \mathbb{T} : \ |\varkappa_\zeta(\varphi)| > \varepsilon\}$$

is finite.

Theorem 1.10. *Let* $\varphi \in PC$ *and let*

$$\Lambda \stackrel{\text{def}}{=} \{\zeta \in \mathbb{T} : \ \varkappa_\zeta(\varphi) \neq 0\}.$$

Then

$$\check{\Gamma}_\varphi = \frac{\mathrm{i}}{2\pi} \sum_{\zeta \in \Lambda} \varkappa_\zeta(\varphi) \check{\Gamma}_\zeta, \tag{1.5}$$

and if the set Λ *is infinite, the series on the right-hand side of* (1.5) *converges unconditionally in the Calkin algebra.*

Proof. Let us first prove that if Λ is infinite, then the series on the right-hand side of 1.5 converges unconditionally in the Calkin algebra. Let $\varepsilon > 0$. Suppose that Ω is a finite subset of the set

$$\mathbb{T}_+ \cap \{\zeta \in \mathbb{T} : \ |\varkappa_\zeta(\varphi)| \le \varepsilon\}.$$

Then

$$\left\| \sum_{\zeta\in\Omega} \left(\varkappa_\zeta(\varphi)\check{\Gamma}_\zeta + \varkappa_{\bar\zeta}(\varphi)\check{\Gamma}_{\bar\zeta}\right) \right\|_{\mathcal{B}/\mathcal{C}} \le 2\pi\varepsilon$$

by Lemma 1.8, which implies unconditional convergence.

Let us show that the series converges to $\check{\Gamma}_\varphi$. Put

$$\Lambda_0 = \{\zeta \in \mathbb{T} : \ |\varkappa_\zeta(\varphi)| \ge 1\}$$

and

$$\Lambda_n = \{\zeta \in \mathbb{T} : \ 2^{-n} \le |\varkappa_\zeta(\varphi)| < 2^{-n+1}\}, \quad n = 1, 2, 3, \cdots .$$

It is easy to see that for each $n \in \mathbb{Z}_+$ there exists a function f_n on $\mathbb{T}$ such that f_n is continuous on $\mathbb{T} \setminus \Lambda_n$, f_n has jump $\varkappa_\zeta(\varphi)$ at each point $\zeta \in \Lambda_n$, and

$$\|f_n\|_\infty = \frac{1}{2} \max_{\zeta\in\Lambda_n} |\varkappa_\zeta(\varphi)|.$$

Clearly,

$$f_n - \frac{\mathrm{i}}{2\pi} \sum_{\zeta\in\Lambda_n} \varkappa_\zeta(\varphi)\psi_\zeta \in C(\mathbb{T}),$$

since ψ_ζ has jump $2\pi\mathrm{i}$ at ζ. Let $f = \sum_{n\ge 0} f_n$. Then $\psi \in PC$ and $\varphi - f \in C(\mathbb{T})$.

We have

$$\begin{aligned} \check{\Gamma}_\varphi &= \check{\Gamma}_f = \sum_{n\ge0} \check{\Gamma}_{f_n} \\ &= \frac{\mathrm{i}}{2\pi} \sum_{n\ge0} \sum_{\zeta\in\Lambda_n} \varkappa_\zeta(\varphi)\check{\Gamma}_\zeta = \frac{\mathrm{i}}{2\pi} \sum_{\zeta\in\Lambda} \varkappa_\zeta(\varphi)\check{\Gamma}_\zeta. \quad \blacksquare \end{aligned}$$

Now we are in a position to describe the essential spectrum of Hankel operators with piecewise continuous symbols.

Theorem 1.11. *Let* $\varphi \in PC$. *Then*

$$\sigma_{\mathrm{e}}(\Gamma_\varphi) =$$
$$\left[0, \frac{\mathrm{i}\varkappa_1(\varphi)}{2}\right] \bigcup \left[0, \frac{\mathrm{i}\varkappa_{-1}(\varphi)}{2}\right] \bigcup_{\zeta\in\mathbb{T}\setminus\mathbb{R}} \left[-\frac{\mathrm{i}}{2}(\varkappa_\zeta(\varphi)\varkappa_{\bar\zeta}(\varphi))^{1/2}, \frac{\mathrm{i}}{2}(\varkappa_\zeta(\varphi)\varkappa_{\bar\zeta}(\varphi))^{1/2}\right].$$

Proof. By Lemmas 0.1 and 1.8, and Theorem 1.10, the result follows from the equalities

$$\sigma_{\mathrm{e}}(\Gamma_1) = \sigma_{\mathrm{e}}(\Gamma_{-1}) = [0, \pi] \tag{1.6}$$

and

$$\sigma_{\mathrm{e}}(\alpha\Gamma_\zeta + \beta\Gamma_{\bar{\zeta}}) = \left[-\pi(\alpha\beta)^{1/2}, \pi(\alpha\beta)^{1/2}\right], \quad \zeta \in \mathbb{T} \setminus \mathbb{R}. \tag{1.7}$$

Equality (1.6) is a part of Theorem 1.7. Let us prove (1.7). Put $\Gamma \stackrel{\text{def}}{=} \alpha\Gamma_\zeta + \beta\Gamma_{\bar{\zeta}}$. By Lemma 1.3, $\check{\Gamma}_\zeta^2 = \check{\Gamma}_{\bar{\zeta}}^2 = \mathbb{O}$. We have

$$\check{\Gamma}^2 = (\alpha\check{\Gamma}_\zeta + \beta\check{\Gamma}_{\bar{\zeta}})(\alpha\check{\Gamma}_\zeta + \beta\check{\Gamma}_{\bar{\zeta}}) = \alpha\beta(\check{\Gamma}_\zeta\check{\Gamma}_\zeta^* + \check{\Gamma}_\zeta^*\check{\Gamma}_\zeta).$$

Again, by Lemma 1.3,

$$(\check{\Gamma}_\zeta\check{\Gamma}_\zeta^*)(\check{\Gamma}_\zeta^*\check{\Gamma}_\zeta) = (\check{\Gamma}_\zeta^*\check{\Gamma}_\zeta)(\check{\Gamma}_\zeta\check{\Gamma}_\zeta^*) = \mathbb{O},$$

and so by Lemma 1.5,

$$\sigma_{\mathrm{e}}(\Gamma^2) = \sigma_{\mathrm{e}}(\Gamma_\zeta^*\Gamma_\zeta) \cup \sigma_{\mathrm{e}}(\Gamma_\zeta\Gamma_\zeta^*).$$

Since $\Gamma_\zeta = D_{\bar{\zeta}}\Gamma_1 D_{\bar{\zeta}}$, we have

$$\Gamma_\zeta^*\Gamma_\zeta = D_\zeta\Gamma_1^2 D_\zeta^* \quad \text{and} \quad \Gamma_\zeta\Gamma_\zeta^* = D_\zeta^*\Gamma_1^2 D_\zeta,$$

and so by Theorem 1.7, $\sigma_{\mathrm{e}}(\Gamma^2) = [0, \alpha\beta\pi^2]$. Finally, by Lemma 1.9, the essential spectrum of Γ is symmetric about the origin, which implies that

$$\sigma_{\mathrm{e}}(\Gamma) = \left[-\pi(\alpha\beta)^{1/2}, \pi(\alpha\beta)^{1/2}\right]. \quad \blacksquare$$

2. The Carleman Operator

In §1.8 we have introduced the Carleman operator $\boldsymbol{\Gamma}$, which is the integral operator on $L^2(\mathbb{R}_+)$ defined by

$$(\boldsymbol{\Gamma} f)(x) = \int_{\mathbb{R}_+} \frac{f(y)}{x+y} dy$$

on the dense subset of functions with compact support in $(0, \infty)$. Then $\boldsymbol{\Gamma}$ is a bounded self-adjoint operator, and we have shown in §1 that $\sigma(\boldsymbol{\Gamma}) = \sigma_{\mathrm{e}}(\boldsymbol{\Gamma}) = [0, \pi]$. We obtain a more precise result in this section. Namely, we prove that $\boldsymbol{\Gamma}$ has Lebesgue spectrum on $[0, \pi]$ of multiplicity 2.

Theorem 2.1. *The Carleman operator $\boldsymbol{\Gamma}$ is unitarily equivalent to multiplication by the function*

$$x \mapsto \frac{\pi}{\cosh(\pi^2 x)}, \quad x \in \mathbb{R},$$

on $L^2(\mathbb{R})$.

Proof. Consider the operator $V : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ defined by

$$(Vf)(t) = \sqrt{2}e^t f(e^{2t}), \quad t \in \mathbb{R}, \quad f \in L^2(\mathbb{R}_+).$$

It is easy to see that V is a unitary operator and

$$(V^*g)(x) = \frac{1}{\sqrt{2x}} g\left(\frac{\log x}{2}\right), \quad x > 0, \quad g \in L^2(\mathbb{R}).$$

Let us evaluate the operator $V\boldsymbol{\Gamma}V^*$ on $L^2(\mathbb{R})$. Let $g \in L^2(\mathbb{R})$. We have

$$(\boldsymbol{\Gamma}V^*g)(x) = \int_0^\infty \frac{\frac{1}{\sqrt{2y}} g\left(\frac{\log y}{2}\right)}{x+y} dy,$$

and so

$$\begin{aligned}
(V\boldsymbol{\Gamma}V^*g)(t) &= \int_0^\infty \frac{e^t}{e^{2t}+y} \frac{1}{\sqrt{y}} g\left(\frac{\log y}{2}\right) dy \\
&= \int_{-\infty}^\infty \frac{2e^t}{e^{2t}+e^{2s}} e^s g(s) ds \\
&= \int_{-\infty}^\infty \frac{1}{\cosh(t-s)} g(s) ds.
\end{aligned}$$

Thus Γ is unitarily equivalent to convolution on $L^2(\mathbb{R})$ with the function

$$s \mapsto \frac{1}{\cosh s},$$

and so Γ is unitarily equivalent to multiplication on $L^2(\mathbb{R})$ by the Fourier transform of this function. Thus it suffices to prove the following elementary lemma.

Lemma 2.2.

$$\int_{-\infty}^\infty \frac{1}{\cosh z} e^{-2\pi i z x} dz = \frac{\pi}{\cosh(\pi^2 x)}, \quad x \in \mathbb{R}.$$

Proof. We use contour integration. Clearly,

$$\int_{-\infty}^\infty \frac{1}{\cosh z} e^{-2\pi i z x} dz = \lim_{N \to \infty} \int_{-N}^N \frac{1}{\cosh z} e^{-2\pi i z x} dz.$$

It is easy to see that the function $(\cosh z)^{-1}$ has one simple pole at $\pi i/2$ in the rectangle with vertices $-N$, N, $N+\pi i$, $-N+\pi i$. Thus by the residue theorem we have

$$
\begin{aligned}
2\pi\mathrm{i}\,\mathrm{Res}_{\frac{\pi\mathrm{i}}{2}}\frac{e^{-2\pi\mathrm{i}z}}{\cosh z} &= \int_{-N}^{N}\frac{1}{\cosh z}e^{-2\pi\mathrm{i}zx}dz+\int_{N}^{N+\pi\mathrm{i}}\frac{1}{\cosh z}e^{-2\pi\mathrm{i}zx}dz\\
&+\int_{N+\pi\mathrm{i}}^{-N+\pi\mathrm{i}}\frac{1}{\cosh z}e^{-2\pi\mathrm{i}zx}dz+\int_{-N+\pi\mathrm{i}}^{-N}\frac{1}{\cosh z}e^{-2\pi\mathrm{i}zx}dz.
\end{aligned}
$$

It is easy to see that

$$\lim_{N\to\infty}\int_{N}^{N+\pi\mathrm{i}}\frac{1}{\cosh z}e^{-2\pi\mathrm{i}zx}dz=0$$

and

$$\int_{-N+\pi\mathrm{i}}^{-N}\frac{1}{\cosh z}e^{-2\pi\mathrm{i}zx}dz=0.$$

It is also easy to compute the residue of $(\cosh z)^{-1}$ at $\pi\mathrm{i}/2$:

$$\mathrm{Res}_{\frac{\pi\mathrm{i}}{2}}\frac{1}{\cosh z}=-\mathrm{i},$$

and so

$$2\pi\mathrm{i}\,\mathrm{Res}_{\frac{\pi\mathrm{i}}{2}}\frac{e^{-2\pi\mathrm{i}zx}}{\cosh z}=2\pi e^{\pi^2 x}$$

Using the substitution $z\mapsto z+\pi\mathrm{i}$, we obtain

$$
\begin{aligned}
\int_{N+\pi\mathrm{i}}^{-N+\pi\mathrm{i}}\frac{1}{\cosh z}e^{-2\pi\mathrm{i}zx}dz &= \int_{-N}^{N}\frac{1}{\cosh z}e^{-2\pi\mathrm{i}(z+\pi\mathrm{i})x}dz\\
&= e^{2\pi^2 x}\int_{-N}^{N}\frac{1}{\cosh z}e^{-2\pi\mathrm{i}zx}dz.
\end{aligned}
$$

Thus

$$\int_{-\infty}^{\infty}\frac{1}{\cosh z}e^{-2\pi\mathrm{i}zx}dz=\frac{2\pi e^{\pi^2 x}}{1+e^{2\pi^2 x}}=\frac{\pi}{\cosh(\pi^2 x)}.\quad\blacksquare$$

Theorem 2.3. *The Carleman operator* $\boldsymbol{\Gamma}$ *is unitarily equivalent to multiplication by the independent variable on the space* $L^2([0,\pi],\mathbb{C}^2)$ *of* $\mathbb{C}^2$*-valued functions on* $[0,\pi]$.

In other words, Theorem 2.3 says that $\boldsymbol{\Gamma}$ has Lebesgue spectrum on $[0,\pi]$ of multiplicity 2.

Proof. Let

$$\varphi(x)\stackrel{\text{def}}{=}\frac{\pi}{\cosh(\pi^2 x)},\quad x\in\mathbb{R}.$$

By Theorem 2.1, we have to show that multiplication by φ on $L^2(\mathbb{R})$ has Lebesgue spectrum on $[0,\pi]$ of multiplicity 2. Since the function φ is even, it is sufficient to show that multiplication by φ on $L^2(\mathbb{R}_+)$ is unitarily equivalent to the operator A of multiplication by the independent variable on $L^2[0,\pi]$. Consider the operator $U : L^2[0,\pi] \to L^2(\mathbb{R}_+)$ defined by

$$(Uf)(s) = |\varphi'(s)|^{1/2} f(\varphi(s)), \quad f \in L^2[0,\pi].$$

It is easy to verify that U is a unitary operator from $L^2[0,\pi]$ onto $L^2(\mathbb{R}_+)$ and UAU^* is multiplication by φ on $L^2(\mathbb{R}_+)$. This completes the proof. ■

3. Quasinilpotent Hankel Operators

In this section we construct nonzero quasinilpotent Hankel operators. The existence of such operators is not obvious at all. To be more precise, we consider a class of Hankel operators whose symbols have lacunary Fourier series and find a procedure how to evaluate the spectra of such operators. In particular, we will be able to find among them nonzero quasinilpotent Hankel operators.

However, we start this section with the problem of whether there are nonzero nilpotent Hankel operators.

Theorem 3.1. *Let φ be a function in L^∞ such that Γ_φ is nilpotent. Then $\Gamma_\varphi = \mathbb{O}$.*

We introduce the following notation. For a function g in L^1 we put

$$g_\#(z) = \overline{g(\bar{z})}.$$

Proof. Suppose that $\Gamma_\varphi \neq \mathbb{O}$. Then $\operatorname{Ker}\Gamma_\varphi$ is a nontrivial invariant subspace of multiplication by z on H^2, and so by Beurling's theorem, $\operatorname{Ker}\Gamma_\varphi = \vartheta H^2$, where ϑ is a nonconstant inner function (see Appendix 2.2). We have

$$\mathbb{O} = \Gamma_\varphi \vartheta f = \mathbb{P}_+ \varphi \mathcal{J}(\vartheta f) = \mathbb{P}_+ \varphi (\mathcal{J}\vartheta)(\mathcal{J} f) = \Gamma_{\varphi \mathcal{J}\vartheta} f, \quad f \in H^2.$$

Hence, $\varphi \mathcal{J}\vartheta \in H^\infty_-$, and so there exists $\psi \in H^\infty$ such that $\varphi \mathcal{J}\vartheta = \bar{z}\bar{\psi}$. It follows that $\varphi = \bar{z}\vartheta_\# \bar{\psi}$. We may assume that $\vartheta_\#$ and ψ are coprime, i.e., they have no common nonconstant inner factor.

Let $f \in H^2$. We have

$$\Gamma_\varphi f = \mathbb{P}_+ \bar{z}\vartheta_\# \bar{\psi} \mathcal{J} f = \mathbb{P}_+ \bar{z}\vartheta_\# \mathcal{J}((\mathcal{J}\bar{\psi}) f) = \mathbb{P}_+ \bar{z}\vartheta_\# \mathcal{J}(\psi_\# f) = \Gamma_{\bar{z}\vartheta_\#} \psi_\# f.$$

It is easy to see that $\Gamma_{\bar{z}\vartheta_\#}$ is a partial isometry with initial space $K_\vartheta = H^2 \ominus \vartheta H^2$ and final space $K_{\vartheta_\#} = H^2 \ominus \vartheta_\# H^2$ (cf. Theorem 1.2.5). Indeed, if $g \in H^2$, then

$$\Gamma_{\bar{z}\vartheta_\#} \vartheta g = \mathbb{P}_+ \bar{z}\vartheta_\# \mathcal{J}(\vartheta g) = \mathbb{P}_+ \bar{z}\vartheta_\# (\mathcal{J}\vartheta)(\mathcal{J} g) = \mathbb{P}_+ \bar{z}(\mathcal{J} g) = \mathbb{O}.$$

On the other hand, if $g \in K_\vartheta$, then

$$\Gamma_{\bar{z}\vartheta_\#} g = \mathbb{P}_+ \bar{z}\vartheta_\# \mathcal{J} g = \bar{z}\vartheta_\# \mathcal{J} g, \tag{3.1}$$

since $\mathcal{J}g \perp z(\mathcal{J}\vartheta)H^2_-$, and so $\bar{z}\vartheta_\#\mathcal{J}g \perp H^2_-$. We have

$$\bar{z}\vartheta_\#(H^2 \ominus \vartheta H^2) = \vartheta_\# H^2_- \ominus H^2_- = H^2 \ominus \vartheta_\# H^2 = K_{\vartheta_\#},$$

which together with (3.1) proves that the final space of $\Gamma_{\bar{z}\vartheta_\#}$ is $K_{\vartheta_\#}$.

Since Γ_φ is quasinilpotent, it follows that there exists a nonzero function g in $K_{\vartheta_\#}$ such that $\psi_\# g \in \vartheta H^2$. Since $\psi_\#$ and ϑ are coprime, it follows that ϑ divides g, and so $g = \vartheta h$ with $h \in H^2$. Since $g \in K_{\vartheta_\#}$, we have $\mathbb{P}_+\bar{\vartheta}_\#\vartheta h = \mathbb{O}$, which means that $\operatorname{Ker} T_{\bar{\vartheta}_\#\vartheta} \neq \{\mathbb{O}\}$. However,

$$\operatorname{Ker} T^*_{\bar{\vartheta}_\#\vartheta} = \operatorname{Ker} T_{\vartheta_\#\bar{\vartheta}} = \operatorname{Ker} T_{(\bar{\vartheta}_\#\vartheta)_\#} = \{f_\# :\ f \in \operatorname{Ker} T_{\bar{\vartheta}_\#\vartheta}\}.$$

Thus both $\operatorname{Ker} T^*_{\bar{\vartheta}_\#\vartheta}$ and $\operatorname{Ker} T_{\bar{\vartheta}_\#\vartheta}$ are nontrivial, which contradicts Theorem 3.1.4. ■

It turns out, however, that nonzero quasinilpotent Hankel operators do exist. We are going to prove that the operator with the following Hankel matrix is compact and quasinilpotent:

$$\Gamma_\heartsuit = \begin{pmatrix} \mathrm{i} & \frac12 & 0 & \frac14 & 0 & \cdots \\ \frac12 & 0 & \frac14 & 0 & 0 & \cdots \\ 0 & \frac14 & 0 & 0 & 0 & \cdots \\ \frac14 & 0 & 0 & 0 & \frac18 & \cdots \\ 0 & 0 & 0 & \frac18 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.2}$$

We consider a more general situation of Hankel operators of the form

$$\Gamma = \begin{pmatrix} \alpha_0 & \alpha_1 & 0 & \alpha_2 & 0 & \cdots \\ \alpha_1 & 0 & \alpha_2 & 0 & 0 & \cdots \\ 0 & \alpha_2 & 0 & 0 & 0 & \cdots \\ \alpha_2 & 0 & 0 & 0 & \alpha_3 & \cdots \\ 0 & 0 & 0 & \alpha_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\{\alpha_k\}_{k\ge0}$ is a sequence of complex numbers. In other words $\Gamma = \{\gamma_{j+k}\}_{j,k\ge0}$, where

$$\gamma_j = \begin{cases} \alpha_k, & j = 2^k - 1, \quad k \in \mathbb{Z}_+, \\ 0, & j \neq 2^k - 1, \quad k \in \mathbb{Z}_+. \end{cases}$$

We evaluate the norm of Γ and give a certain description of its spectrum.

By Paley's theorem,

$$\left\{ \{\hat{f}(2^k - 1)\}_{k\ge0} :\ f \in H^1 \right\} = \ell^2$$

(see Zygmund [1], Ch.XII, §7). It follows that that $\sum_{k\ge0} \alpha_k z^{2^k-1} \in BMOA$ if and only if $\{\alpha_k\}_{k\ge0} \in \ell^2$. So by Theorem 1.1.2, Γ is a matrix of a bounded operator if and only if $\{\alpha_k\}_{k\ge0} \in \ell^2$; moreover, $\|\Gamma\|$ is equivalent to $\|\{\alpha_k\}_{k\ge0}\|_{\ell^2}$. It is also clear that for $\{\alpha_k\}_{k\ge0} \in \ell^2$ the function

$\sum_{k\geq 0} \alpha_k z^{2^k-1}$ belongs to $VMOA$, and so Γ is bounded if and only if it is compact.

We associate with Γ the sequence $\{\mu_k\}_{k\geq 0}$ defined by

$$\mu_0 = 0, \quad \mu_{k+1} = \frac{1}{2}\left(\mu_k + 2|\alpha_{k+1}|^2 + \left(\mu_k^2 + 4|\alpha_{k+1}|^2\right)^{1/2}\right), \quad k \in \mathbb{Z}_+. \tag{3.3}$$

The following theorem evaluates the norm of Γ.

Theorem 3.2. *If $\{\alpha_k\}_{k\geq 0} \in \ell^2$, then the sequence $\{\mu_k\}_{k\geq 0}$ converges and*

$$\|\Gamma\|^2 = \lim_{k\to\infty} \mu_k.$$

To describe the spectrum of Γ consider the class Λ of sequences of complex numbers $\{\lambda_j\}_{j\geq 0}$ that satisfy the equalities

$$\lambda_0 = \alpha_0, \quad (\lambda_j - \lambda_{j-1})\lambda_j = \alpha_j^2, \quad j \geq 1. \tag{3.4}$$

Theorem 3.3. *Suppose that $\{\alpha_j\}_{j\geq 0} \in \ell^2$. Any sequence $\{\lambda_j\}_{j\geq 0}$ in Λ converges. The spectrum $\sigma(\Gamma)$ consists of 0 and the limits of such sequences.*

To prove Theorems 3.2 and 3.3, we consider finite submatrices of Γ. Let $\mathcal{L}_k$ be the linear span of the basis vectors e_j, $j = 0, 1, \cdots, 2^k - 1$, and let P_k be the orthogonal projection from ℓ^2 onto $\mathcal{L}_k$. Consider the operator $\Gamma_k \stackrel{\text{def}}{=} P_k \Gamma \big| \mathcal{L}_k$ and identify it with its $2^k \times 2^k$ matrix. Put $\tilde{\Gamma}_k \stackrel{\text{def}}{=} \Gamma_k P_k$, i.e.,

$$\tilde{\Gamma}_k = \begin{pmatrix} \Gamma_k & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}.$$

It is easy to see that $\|\Gamma_k\| = \|\tilde{\Gamma}_k\|$ and $\sigma(\tilde{\Gamma}_k) = \sigma(\Gamma_k) \cup \{0\}$. Clearly,

$$\Gamma_{k+1} = \begin{pmatrix} \Gamma_k & \alpha_{k+1} J_k \\ \alpha_{k+1} J_k & \mathbb{O} \end{pmatrix},$$

where J_k is the $2^k \times 2^k$ matrix given by

$$J_k = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We need the following well-known fact from linear algebra (see Gantmakher [1], Ch. 2, §5.3):

Let N be a block matrix of the form

$$N = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A and D are square matrices and D is invertible. Then

$$\det N = \det D \det(A - BD^{-1}C). \tag{3.5}$$

Proof of Theorem 3.2. Since $J_k^* = J_k$ and J_k^2 is the identity matrix of size $2^k \times 2^k$ (which we denote by I_{2^k}), we have

$$\Gamma_{k+1}^*\Gamma_{k+1} = \begin{pmatrix} \Gamma_k^*\Gamma_k + |\alpha_{k+1}|^2 I_{2^k} & \alpha_{k+1}\tilde{\Gamma}_k^* J_k \\ \bar{\alpha}_{k+1} J_k \tilde{\Gamma}_k & |\alpha_{k+1}|^2 I_{2^k} \end{pmatrix}. \tag{3.6}$$

Applying formula (3.5) to the matrix $\Gamma_{k+1}^*\Gamma_{k+1} - \lambda I_{2^{k+1}}$, $\lambda \neq |\alpha_{k+1}|^2$, we obtain

$$\det\left(\Gamma_{k+1}^*\Gamma_{k+1} - \lambda I_{2^{k+1}}\right) = \rho^{2^k} \det\left(-\frac{\lambda}{\rho}\Gamma_k^*\Gamma_k + \rho I_{2^k}\right), \quad \rho \stackrel{\text{def}}{=} |\alpha_{k+1}|^2 - \lambda. \tag{3.7}$$

Since Γ is a bounded operator, we have $\|\Gamma\| = \lim\limits_{k\to\infty} \|\Gamma_k\|$. Therefore it is sufficient to show that $\mu_k = \|\Gamma_k\|^{1/2}$ or, which is the same, that μ_k is the largest eigenvalue of $\Gamma_k^*\Gamma_k$. We proceed by induction on k. For $k = 0$ the assertion is obvious.

If $\Gamma_k = \mathbb{O}$, the assertion is obvious. Otherwise, it follows easily from (3.6) that $\|\Gamma_{k+1}^*\Gamma_{k+1}\| > |\alpha_{k+1}|^2$.

It is easy to see from (3.7) that $\lambda \neq |\alpha_{k+1}|^2$ is an eigenvalue of $\Gamma_{k+1}^*\Gamma_{k+1}$ if and only if ρ^2/λ is an eigenvalue of $\Gamma_k^*\Gamma_k$. Put

$$\mu = \rho^2/\lambda = (|\alpha_{k+1}|^2 - \lambda)^2/\lambda.$$

If μ is an eigenvalue of $\Gamma_k^*\Gamma_k$, it generates two eigenvalues of $\Gamma_{k+1}^*\Gamma_{k+1}$:

$$\frac{1}{2}\left(\mu + 2|\alpha_{k+1}|^2 + (\mu^2 + 4|\alpha_{k+1}|^2)^{1/2}\right)$$

and

$$\frac{1}{2}\left(\mu + 2|\alpha_{k+1}|^2 - (\mu^2 + 4|\alpha_{k+1}|^2)^{1/2}\right).$$

Clearly, to get the largest eigenvalue of $\Gamma_{k+1}^*\Gamma_{k+1}$, we have to put $\mu = \mu_k$ and choose the first of the above eigenvalues. This proves that μ_{k+1} defined by (3.3) is the largest eigenvalue of $\Gamma_{k+1}^*\Gamma_{k+1}$. ■

To prove Theorem 3.3 we need the following lemmas.

Lemma 3.4. *Let Λ_k be the set of kth terms of sequences in Λ, i.e.,*

$$\Lambda_k = \{\lambda_k : \{\lambda_j\}_{j\ge0} \in \Lambda\}.$$

If $\{\zeta_j\}_{j\ge0}$ is an arbitrary sequence satisfying $\zeta_j \in \Lambda_j$, then it converges if and only if $\lim\limits_{j\to\infty} \zeta_j = 0$ or there exists a sequence $\{\lambda_j\}_{j\ge0} \in \Lambda$ such that $\zeta_j = \lambda_j$ for sufficiently large j.

Lemma 3.5. *Let A be a compact operator on Hilbert space and let $\{A_j\}_{j\ge0}$ be a sequence of bounded linear operators such that $\lim\limits_{j\to\infty} \|A - A_j\| = 0$. Then the spectrum $\sigma(A)$ consists of limits of all convergent sequences $\{\nu_j\}_{j\ge0}$ such that $\nu_j \in \sigma(A_j)$.*

Let us first deduce Theorem 3.3 from Lemmas 3.4 and 3.5.

Proof of Theorem 3.3. Since Γ is compact, $0 \in \sigma(\Gamma)$.

For $\lambda \in \mathbb{C} \setminus \{0\}$ we apply formula (3.5) to the matrix $\Gamma_{k+1} - \lambda I_{2^{k+1}}$ and obtain

$$\det(\Gamma_{k+1} - \lambda I_{2^{k+1}}) = (-\lambda)^{2^k} \det\left(\Gamma_k - \left(\lambda - \frac{\alpha_{k+1}^2}{\lambda}\right) I_{2^k}\right). \tag{3.8}$$

Obviously,

$$0 \in \sigma(\Gamma_k) \quad \text{if and only if} \quad \alpha_k = 0.$$

Together with (3.8) this implies that $\lambda \in \sigma(\Gamma_k)$ if and only if there exists $\lambda' \in \sigma(\Gamma_{k-1})$ such that $(\lambda - \lambda')\lambda = \alpha_k^2$.

Let λ be a nonzero point of spectrum of Γ. Then by Lemma 3.5, there exists a sequence $\{\nu_j\}_{j\ge0}$ such that $\nu_j \to \lambda$ as $j \to \infty$ and $\nu_j \in \sigma(\tilde{\Gamma}_j)$. Since $\lambda \ne 0$ we may assume without loss of generality that $\nu_j \in \sigma(\Gamma_j)$. It now follows from Lemma 3.4 that there exists a sequence $\{\lambda_j\}_{j\ge0}$ in Λ such that $\lambda_j = \nu_j$ for sufficiently large j, and so $\lambda = \lim\limits_{j\to\infty} \lambda_j$.

Conversely, let $\{\lambda_j\}_{j\ge0} \in \Lambda$. By Lemma 3.4, $\{\lambda_j\}_{j\ge0}$ converges to a point $\lambda \in \mathbb{C}$. As we have already observed, $\lambda_j \in \sigma(\Gamma_j)$ and so by Lemma 3.5, $\lambda \in \sigma(\Gamma)$. ■

Proof of Lemma 3.4. Let $\{\lambda_j\}_{j\ge0} \in \Lambda$. Then $|\lambda_j| \le |\lambda_{j-1}| + |\alpha_j|^2/|\lambda_j|$, $j \ge 1$. It follows that

$$|\lambda_j| \le \max\{\varepsilon, |\lambda_{j-1}| + |\alpha_j|^2/\varepsilon\} \tag{3.9}$$

for any $\varepsilon > 0$. Let us show that either $\lambda_j \to 0$ as $j \to \infty$ or $|\lambda_j| \ge \delta$ for some $\delta > 0$ for sufficiently large j.

To prove this, let us show that if $\varepsilon > 0$ and $\liminf\limits_{j\to\infty} |\lambda_j| < \varepsilon$, then $\limsup\limits_{j\to\infty} |\lambda_j| \le 2\varepsilon$. Assume the contrary, i.e., $\liminf\limits_{j\to\infty} |\lambda_j| < \varepsilon$ and $\limsup\limits_{j\to\infty} |\lambda_j| > 2\varepsilon$ for some $\varepsilon > 0$. It follows that for any $N \in \mathbb{Z}_+$ there exist positive integers m and n such that $N \le m < n$, $|\lambda_{m-1}| < \varepsilon$, $|\lambda_j| \ge \varepsilon$ for $m \le j \le n$, and $|\lambda_n| \ge 2\varepsilon$. It follows from (3.9) that $|\lambda_j| \le |\lambda_{j-1}| + |\alpha_j|^2/\varepsilon$, $m \le j \le n$. Therefore

$$|\lambda_n| \le |\lambda_{m-1}| + \frac{1}{\varepsilon} \sum_{j=m}^{n} |\alpha_j|^2.$$

Since $\{\alpha_k\}_{k\ge0} \in \ell^2$, we can choose N so large that $\frac{1}{\varepsilon} \sum\limits_{j=m}^{\infty} |\alpha_j|^2 < \varepsilon$, which contradicts the inequality $|\lambda_n| \ge 2\varepsilon$.

If $|\lambda_j| \ge \delta > 0$ for large values of j, then by (3.4),

$$|\lambda_j - \lambda_{j-1}| \le \frac{|\alpha_j|^2}{\delta}.$$

Therefore $\{\lambda_j\}_{j\ge0}$ converges.

Suppose now that $\{\lambda_j\}_{j\ge0}$ and $\{\nu_j\}_{j\ge0}$ are sequences in Λ that have nonzero limits. Then for sufficiently large j

$$|\lambda_{j-1}-\nu_{j-1}| = \left|(\lambda_j-\nu_j)\left(1+\frac{\alpha_j^2}{\lambda_j\nu_j}\right)\right| \le |\lambda_j-\nu_j|(1+d|\alpha_j|^2)$$

for some $d>0$. Iterating this inequality, we obtain

$$|\lambda_{j-1}-\nu_{j-1}| \le \left|\lim_{j\to\infty}\lambda_j - \lim_{j\to\infty}\nu_j\right| \cdot \prod_{m=j}^{\infty}(1+d|\alpha_m|^2)$$

(the infinite product on the right-hand side converges since $\{\alpha_j\}_{j\ge0}\in\ell^2$). Therefore if $\lim\limits_{j\to\infty}\lambda_j = \lim\limits_{j\to\infty}\nu_j$, then $\lambda_j=\nu_j$ for sufficiently large j.

For $\varepsilon>0$ we consider the set of sequences $\{\lambda_j\}_{j\ge0}$ in Λ such that $\sup\{|\lambda_j| : j\ge k\}\ge\varepsilon$ for any positive integer k. Let us show that the number of such sequences is finite. Suppose that $\{\lambda_j\}_{j\ge0}\in\Lambda$ is such a sequence. As we have observed in the beginning of the proof, there exist $\delta>0$ and $j_0\in\mathbb{Z}_+$ such that $|\lambda_j|\ge\delta$ for sufficiently large j. Clearly, $|\alpha_j|<\delta$ for sufficiently large j. It follows that if j is sufficiently large, then λ_j is uniquely determined by λ_{j-1} by the conditions

$$(\lambda_j-\lambda_{j-1})\lambda_j=\alpha_j^2, \quad |\lambda_j|\ge\delta.$$

Hence, there are only finitely many possibilities to get such sequences.

Now let$\{\zeta_j\}_{j\ge0}$ be a converging sequence such that $\zeta_j\in\Lambda_j$, $j\ge0$, and $\lim\limits_{j\to\infty}\zeta_j\ne0$. Then as we have already proved, there are finitely many sequences $\left\{\lambda_j^{(s)}\right\}_{j\ge0}\in\Lambda$, $s=1,\cdots,m$, such that $\zeta_j\in\left\{\lambda_j^{(1)},\cdots,\lambda_j^{(m)}\right\}$ for sufficiently large j and the sequences $\left\{\lambda_j^{(s)}\right\}_{j\ge0}$ have distinct limits. It follows that there exists an s, $1\le s\le m$, such that $\zeta_j=\lambda_j^{(s)}$ for sufficiently large j. ∎

Proof of Lemma 3.5. Since the set of invertible operators is open in the norm topology, it follows that $\sigma(A)$ contains the set of limits of converging sequences $\{\nu_j\}_{j\ge0}$ such that $\nu_j\in\sigma(A_j)$. Let us prove the opposite inclusion.

Suppose that $\lambda_0\in\sigma(A)$. If λ_0 is not a limit of any such sequence, then there is a subsequence $\{A_{j_k}\}_{k\ge0}$ such that $\operatorname{dist}(\lambda_0,\sigma(A_{j_k}))\ge\delta_0$ for some positive δ_0. Since A is compact, there exists a $\delta\in(0,\delta_0)$ such that the circle of radius δ with center at λ_0 does not intersect $\sigma(A)$. For arbitrary vectors x, y we put

$$\Psi_{x,y}(\zeta)=(x,(A-(\lambda_0+\delta\zeta)I)^{-1}y),$$
$$\Psi_{x,y}^{(k)}(\zeta)=(x,(A_{j_k}-(\lambda_0+\delta\zeta)I)^{-1}y), \quad \zeta\in\mathbb{T}.$$

Obviously, the functions $\Psi_{x,y}^{(k)}$ extend analytically to a neighborhood of the closed unit disk and $\Psi_{x,y}^{(k)}\to\Psi_{x,y}$ uniformly on $\mathbb{T}$. It follows that $\Psi_{x,y}$

extends to a function in C_A. Obviously,

$$\sup\{|\Psi_{x,y}(\zeta)|:\ \zeta \in \mathbb{T}\} \le \text{const}\,\|x\|\cdot\|y\|,$$

which implies that

$$\sup\{|\Psi_{x,y}(\zeta)|:\ |\zeta| \le 1\} \le \text{const}\,\|x\|\cdot\|y\|.$$

This contradicts the fact that $\lambda_0 \in \sigma(A)$. ■

Let us now proceed to the operator $\Gamma_{\heartsuit}$ defined by (3.2). In other words we consider the operator Γ with

$$\alpha_0 = \mathrm{i}, \quad \alpha_j = \frac{1}{2^j}, \quad j \ge 1.$$

Theorem 3.6. *$\Gamma_{\heartsuit}$ is a compact quasinilpotent operator.*

Proof. It is easy to see by induction that if $\{\lambda_j\}_{j\ge0}$ satisfies (3.4), then $\lambda_j = 2^{-j}\mathrm{i}$, and so by Theorem 3.3, $\sigma(\Gamma_{\#}) = \{0\}$. We have already seen that bounded Hankel operators of this form are always compact. ■

Remark. We can consider a more general situation when $\Gamma = \{\gamma_{j+k}\}_{j,k\ge0}$ with

$$\gamma_j = \begin{cases} \alpha_k, & j = n_k - 1, \quad j \in \mathbb{Z}_+, \\ 0 & j \ne n_k - 1, \quad k \in \mathbb{Z}_+, \end{cases}$$

where $\{n_k\}_{k\ge0}$ is a sequence of natural numbers such that $n_{k+1} \ge 2n_k$, $k \ge 0$, and $\{\alpha_k\}_{k\ge0} \in \ell^2$. It is easy to see that the same results hold and the same proofs also work in this situation, which allows one to construct other quasinilpotent Hankel operators. In particular, it is easy to see that the following Hankel operator is quasinilpotent:

$$\begin{pmatrix} 0 & \mathrm{i} & 0 & \frac12 & 0 & \cdots \\ \mathrm{i} & 0 & \frac12 & 0 & 0 & \cdots \\ 0 & \frac12 & 0 & 0 & 0 & \cdots \\ \frac12 & 0 & 0 & 0 & \frac14 & \cdots \\ 0 & 0 & 0 & \frac14 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The following fact is an amusing consequence of the generalization of Theorem 3.2 mentioned in the above remark.

Corollary 3.7. *Let $\{n_k\}_{k\ge0}$ be a sequence of natural numbers such that $n_{k+1} \ge 2n_k$, $k \ge 0$, and $\{a_k\}_{k\ge0} \in \ell^2$. Let φ be the function on $\mathbb{T}$ defined by*

$$\varphi(z) = \sum_{k\ge0} \alpha_k z^{n_k - 1}.$$

Then

$$\inf\{\|\varphi - f\|_\infty:\ f \in \mathbb{P}_-BMO,\ \varphi - f \in L^\infty\}$$

depends only on the moduli of the α_k, $k \ge 0$.

Proof. This is an immediate consequence of Theorem 3.2. ■

Concluding Remarks

The essential spectrum of Hankel operators with piecewise continuous symbols was described by Power [1]; see also Power [2]. An important role in the proof is played by the Hilbert matrix whose spectrum was described in Magnus [1] (see Theorem 1.7).

The results of §2 are due to Carleman [1]; see also Power [2].

Theorem 3.1 is due to Power [3]. The problem of whether there exist nonzero quasinilpotent Hankel operators was posed in Power [3]. It was solved by Megretskii [1]. His solution is presented in §3. Lemma 3.5 is well known; see Newburg [1].

We mention here the paper Rosenblum [1] in which it was shown that the Hilbert matrix has simple Lebesgue spectrum on the interval $[0,\pi]$. In other words, it is unitarily equivalent to multiplication by the independent variable on the space $L^2[0,\pi]$ (with respect to Lebesgue measure).

Interesting results on spectral properties of self-adjoint Hankel operators were obtained by Howland [2–4]. In particular, in Howland [2] the following result was obtained. Suppose that φ is a real piecewise continuous function with finitely many jumps and such that it is of class C^2 on the complement of the jump points of f. If ξ is a jump point of f, we put

$$I(\xi)=\begin{cases}\big[-\frac{1}{2}|\varkappa_\xi(f)|,\frac{1}{2}|\varkappa_\xi(f)|\big], & \operatorname{Im}\xi\neq 0,\\ \big[0,\frac{1}{2}\varkappa_\xi(f)\big], & \xi=\pm1,\ \varkappa_\xi(f)>0,\\ \big[\frac{1}{2}\varkappa_\xi(f),0\big], & \xi=\pm1,\ \varkappa_\xi(f)<0,\end{cases}$$

where $\varkappa_\xi(f)=f(\xi^+)-f(\xi^-)$. It was proved in Howland [2] that the absolutely continuous part of Γ_f is unitarily equivalent to the orthogonal sum of multiplications by the independent variable on $I(\xi)$.

In Howland [3] the author studied integral operators on $L^2(\mathbb{R}_+)$ of the form

$$(Qf)(x)=\int_0^\infty\frac{\varphi(x)\bar\varphi(y)}{x+y}dy.$$

In the case when the limits $a=\lim\limits_{x\to0+}|\varphi(x)|$ and $b=\lim\limits_{x\to\infty}|\varphi(x)|$ exist it was shown in Howland [3] under mild conditions on φ that Q has no singular continuous part, its absolutely continuous part is the orthogonal sum of multiplications by the independent variable on $[0,\pi a^2]$ and $[0,\pi b^2]$, and the nonzero eigenvalues can accumulate only at 0. On the other hand, Q is unitarily equivalent to the Hankel operator $\boldsymbol{\Gamma}_k$, where

$$k(x)=\int_0^\infty e^{-xs}|\varphi(x)|^2ds.$$

In Howland [4] the author studied Hankel operators $\boldsymbol{\Gamma}_k$ on $L^2(\mathbb{R}_+)$. He proved that if k is of class C^2 on $(0,\infty)$, the limits $a=\lim\limits_{x\to0}xk(x)$ and

$b = \lim_{x\to\infty} xk(x)$ exist, then under certain mild assumptions $\boldsymbol{\Gamma}_k$ has no singular continuous part and its absolutely continuous part is unitarily equivalent to the orthogonal sum of multiplications by the coordinate function on $L^2[0,\pi a]$ and $L^2[0,\pi b]$.

The spectral structure of self-adjoint Hankel operators was completely described in Megretskii, Peller, and Treil [1] and [2]. We present these results in Chapter 12.

Finally, we mention here the paper Martinez-Avendaño and Treil [1], in which it was shown that the spectrum of a Hankel operator can be an arbitrary compact set in $\mathbb{C}$ that contains the origin.

11

Hankel Operators in Control Theory

This chapter is a brief introduction to control theory written for mathematicians. To be more precise, we consider here several problems in control theory that involve Hankel operators. For readers interested in a more detailed study we recommend the books Doyle [1], Francis [1], Fuhrmann [4], Helton [3], Doyle, Francis, and Tannnenbaum [1], Dahleh and Diaz-Bobillo [1], and Chui and Chen [1].

In §1 we introduce the notion of a plant and its transfer function. In §2 we consider realizations of transfer functions with discrete time while in §3 we consider the case of continuous time. Next, in §4 we study the model reduction problem. The rest of the chapter is devoted to the problem of robust stabilization. In §5 we describe the problem and state a necessary and sufficient condition for a feedback stabilizer to be robust (Theorem 5.1). We obtain in §6 a doubly coprime factorization of rational matrix functions. Section 7 is devoted to the proof of Theorem 5.1 stated in §5. In §8 we obtain a parametrization formula for all feedback stabilizers of a given plant. Finally, in §9 we reduce the problem of robust stabilization to the so-called model matching problem, which in an important special case reduces to the Nehari problem.

Note that realizations of linear systems are used essentially in Chapter 12 to solve the inverse spectral problem for self-adjoint Hankel operators.

1. Transfer Functions

The main object we are going to study in this section is a *plant* (or a *black box*, or a *system*). A plant G with discrete time is a map on a space of (two-sided) sequences (scalar or vector). We say that a plant is *single input single output* (or briefly, SISO) if it maps scalar sequences into scalar sequences. If G is defined on a space of $\mathbb{C}^n$-valued sequences and maps them to $\mathbb{C}^m$-valued sequences and both m and n are greater than 1, we say that G is a *multiple input multiple output* (or briefly, MIMO) plant. Certainly, there are single input multiple output and multiple input single output plants. One can also consider plants acting between spaces of sequences that take values in infinite-dimensional spaces (such as ℓ^2). We interpret the index set $\mathbb{Z}$ as discrete time.

A plant with continuous time is defined on a space functions (scalar or vector) on $\mathbb{R}$ and maps it to another space functions. Similarly, we can consider the case of single input single output or multiple input multiple output. Here we interpret $\mathbb{R}$ as continuous time.

We are going to represent the plant G with the following diagram:

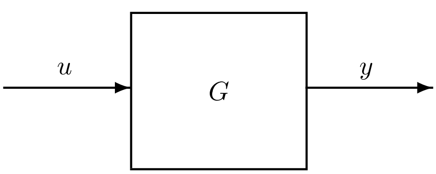

FIGURE 1. Plant

Here G takes an input sequence $u = \{u_n\}$ (or a function u on $\mathbb{R}$ in the case of continuous time) into an output sequence $y = \{y_n\}$ (a function y on $\mathbb{R}$, respectively).

We consider the following properties of plants:

1. Linearity. By this we simply mean that G is a linear operator.

2. Time invariance. This means that if the input u is shifted by m, then the output y is also shifted by m.

3. Causality. This property means that if two inputs are identical before time m, then the corresponding outputs are also identical before time m.

All plants considered here are supposed to possess all three above properties. We consider here one more property that some plants have, and others do not have.

4. Stability. A plant is called *stable* if it takes inputs of finite energy (which is the norm of u in (scalar or vector) ℓ^2 or L^2 norm) into outputs of finite energy. In other words this means that G is bounded.

Suppose now that G a SISO plant with discrete time. We can identify sequences $\{x_n\}_{n\in\mathbb{Z}}$ with zero past (i.e., such that $x_n = 0$ for $n < 0$) with one-sided sequences $\{x_n\}_{n\in\mathbb{Z}_+}$. It follows from causality that for such inputs the output sequences also have zero past. Consider the input sequence $v = \{v_n\}_{n\ge 0}$ with $v_0 = 1$ and $v_n = 0$ for $n > 0$. Let $\{g_n\}_{n\ge 0}$ be the corresponding output. Consider the formal power series $\boldsymbol{G}(z) = \sum\limits_{n\ge 0} g_n z^n$. It follows from the above properties 1–3 that for any finitely supported sequence $u = \{u_n\}_{n\ge 0}$ the output sequence $y = \{y_n\}_{n\ge 0}$ is given by

$$\check{y}(z) = \boldsymbol{G}(z)\check{u}(z), \quad \text{where} \quad \check{u}(z) \stackrel{\text{def}}{=} \sum_{n\ge 0} u_n z^n \quad \text{and} \quad \check{y}(z) \stackrel{\text{def}}{=} \sum_{n\ge 0} y_n z^n. \tag{1.1}$$

In control theory one usually considers only plants G such that

$$G\ell^2 \subset \left\{ \{y_n\}_{n\ge 0} : \sum_{n\ge 0} |y_n M^{-n}|^2 < \infty \right\}$$

for some $M > 0$. It is easy to see that this is equivalent to the fact that the power series $\boldsymbol{G}(z) = \sum\limits_{n\ge 0} g_n z^n$ converges in a disk of positive radius, the function $\boldsymbol{G}(z) = \sum\limits_{n\ge 0} g_n z^n$ is called the *transfer function of the plant* G.

If G is a stable plant, then multiplication by $\boldsymbol{G}$ extends to a bounded operator on H^2, and so $\boldsymbol{G}$ is a analytic in $\mathbb{D}$ and bounded there, i.e., $\boldsymbol{G} \in H^\infty$. Clearly, the converse is also true. Thus a plant G has the above properties 1–4 if and only if there exists $\boldsymbol{G} \in H^\infty$ such that (1.1) holds for any input sequence $u = \{u_n\}_{n\ge 0} \in \ell^2$, where $y = \{y_n\}_{n\ge 0}$ is the output sequence.

Note that in control theory the most important case is when the transfer function is rational. Indeed, very often the input and output sequences are connected with each other by

$$p(S)u = q(S)y,$$

where p and q are polynomials and S is the shift operator on the space of sequences. If this is the case, we have

$$p(z)\check{u} = q(z)\check{y}(z),$$

and so the transfer function of such a plant is the rational function p/q. Note, however, that certain problems in control theory lead to nonrational transfer functions (see the case of continuous time below).

If the input sequences and the output sequences take values in Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the transfer function $\boldsymbol{G}$ is a formal power series whose coefficients are operators from $\mathcal{H}$ to $\mathcal{K}$. In particular, if $\mathcal{H} = \mathbb{C}^n$ and $\mathcal{K} = \mathbb{C}^m$, the coefficients of the formal power series $\boldsymbol{G}$ can be interpreted as $m \times n$ matrices. As in the case of SISO systems, the system is stable if and only if $\boldsymbol{G} \in H^\infty(\mathcal{B}(\mathcal{H}, \mathcal{K}))$.

Consider now the case of SISO systems with continuous time. Again we identify in a natural way $L^2(\mathbb{R}_+)$ with the subspace of L^2 of functions vanishing on $(-\infty, 0)$. In control theory it is natural to assume that there exists a number $\sigma \in \mathbb{R}$ such that the plant G maps $L^2(\mathbb{R}_+)$ into the space $L^2_\sigma(\mathbb{R}_+)$,

$$L^2_\sigma(\mathbb{R}_+) \stackrel{\text{def}}{=} \left\{ f : \int_0^\infty \left| e^{-\sigma t} f(t) \right|^2 dt < \infty \right\}.$$

If f is a function in $L^2_\sigma(\mathbb{R}_+)$, its Laplace transform $\boldsymbol{\mathcal{L}} f$,

$$(\boldsymbol{\mathcal{L}} f)(\zeta) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t) e^{-\zeta t} dt,$$

is a function analytic in the half-plane $\{\zeta \in \mathbb{C} : \operatorname{Re} \zeta > \sigma\}$. By the Paley–Wiener theorem (see Appendix 2.1), $\boldsymbol{\mathcal{L}}$ maps unitarily $L^2(\mathbb{R}_+)$ onto the Hardy class $H^2(\mathbb{C}^+)$. Note that in control theory it is common to consider Laplace transform rather than Fourier transform and to consider the Hardy classes $H^2(\mathbb{C}^-)$ and $H^2(\mathbb{C}^+)$ of functions on the left and right half-planes $\mathbb{C}^- \stackrel{\text{def}}{=} \{\zeta \in \mathbb{C} : \operatorname{Re} \zeta < 0\}$ and $\mathbb{C}^+ \stackrel{\text{def}}{=} \{\zeta \in \mathbb{C} : \operatorname{Re} \zeta > 0\}$ rather than the lower and the upper half-planes. It follows immediately from the Paley–Wiener theorem that $\boldsymbol{\mathcal{L}}$ maps unitarily $L^2_\sigma(\mathbb{R}_+)$ onto the Hardy class H^2 of functions on the half-plane $\{\zeta \in \mathbb{C} : \operatorname{Re} \zeta > \sigma\}$.

Consider now the operator $\mathcal{G} \stackrel{\text{def}}{=} \boldsymbol{\mathcal{L}} G \boldsymbol{\mathcal{L}}^{-1}$ as an operator from $H^2(\mathbb{C}^+)$ to $H^2(\{\zeta \in \mathbb{C} : \operatorname{Re} \zeta > \sigma\})$. It is easy to verify that time invariance means that

$$\mathcal{G}(e^{-az}\varphi) = e^{-az}\mathcal{G}\varphi, \quad \varphi \in H^2(\mathbb{C}^+), \quad a > 0.$$

It follows that if φ and ψ are linear combinations of functions e^{-az}, $a > 0$, then

$$\psi \mathcal{G} \varphi = \varphi \mathcal{G} \psi,$$

and so, if φ is a linear combination of functions e^{-az}, $a > 0$, then

$$\mathcal{G}\varphi = \boldsymbol{G}\varphi, \tag{1.2}$$

where

$$\boldsymbol{G}(\zeta) = \frac{(\mathcal{G}e^{-z})(\zeta)}{e^{-\zeta}}, \quad \operatorname{Re} \zeta > \sigma.$$

Clearly, the function $\boldsymbol{G}$ is analytic in $\{\zeta \in \mathbb{C} : \operatorname{Re} \zeta > \sigma\}$. It is called the *transfer function* of the plant G. Since the set of such functions φ is dense in $H^2(\mathbb{C}^+)$, (1.2) extends to arbitrary functions φ in $H^2(\mathbb{C}^+)$.

Now it is easy to see that G is a stable system if and only if the transfer function $\boldsymbol{G}$ is bounded and analytic in $\mathbb{C}^+$.

As in the case of discrete time the most important case in control theory is when the transfer function is rational, since very often the input and output functions are related to each other via the formula

$$p(D)u = q(D)y,$$

where D is the differentiation operator, and p and q are polynomials. If we pass to Laplace transform, we see that in this case the transfer function is rational.

Example. Consider now an important example of a time-delay system in which the transfer function is nonrational. Suppose that the output function y depends on the input function u via the formula

$$y'(t) + y(t-1) = u(t).$$

If we pass to Laplace transform, we obtain

$$\zeta(\boldsymbol{\mathcal{L}}y)(\zeta) + e^{-\zeta}(\boldsymbol{\mathcal{L}}y)(\zeta) = (\boldsymbol{\mathcal{L}}u)(\zeta),$$

and so the transfer function is

$$\frac{1}{z+e^{-z}},$$

which is nonrational.

As in the case of discrete time one can also consider vector-valued input and output functions and deal with operator-valued (or matrix-valued) transfer functions.

As an example of a physical system one can consider a flying airplane. Its position is influenced by control input signals. Certainly, this dependence is nonlinear. However, if we consider only small deviations from a certain equilibrium state, we can approximate nonlinear functions by linear ones and study the resulting linear system.

2. Realizations with Discrete Time

In this section we introduce the notion of a realization of a transfer function in the case of discrete time. We define the Hankel operator associated with a given realization (linear system). We introduce the important notion of a balanced realization and we show that a transfer function has balanced realization if and only if it generates a bounded Hankel operator. Finally, we establish the uniqueness of a balanced realization up to equivalence.

Let $\mathcal{K}$, $\mathcal{E}$, and $\mathcal{H}$ be Hilbert spaces. Suppose that $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $C \in \mathcal{B}(\mathcal{H},\mathcal{E})$, and $D \in \mathcal{B}(\mathcal{K},\mathcal{E})$. Consider the *linear dynamical system* with discrete time

$$\begin{cases} x_{k+1} = Ax_k + Bu_k, \\ y_k = Cx_k + Du_k, \end{cases} \qquad u_k \in \mathcal{K}, \quad x_k \in \mathcal{H}, \quad \text{and} \quad y_k \in \mathcal{E}. \tag{2.1}$$

This system takes an input signal $\{u_k\}_{k\in\mathbb{Z}}$ to the output signal $\{y_k\}_{k\in\mathbb{Z}}$. $\mathcal{H}$ is called the *state space* $\mathcal{K}$ is called the *input space*, and $\mathcal{E}$ is called the *output space*. We use the notation $\{A, B, C, D\}$ for the linear system (2.1).

Suppose that $\{u_k\}_{k\in\mathbb{Z}}$ is a finitely supported sequence such that $u_k = \mathbb{O}$ and $x_k = \mathbb{O}$ for $k < 0$. Clearly, $y_k = \mathbb{O}$ for $k < 0$ and $x_0 = \mathbb{O}$. This linear system determines a plant that has the properties of linearity, causality, and time invariance. Let us compute the transfer function of the system.

Consider the power series

$$\check{u}(z) \stackrel{\text{def}}{=} \sum_{j\ge 0} z^j u_j, \quad \check{x}(z) \stackrel{\text{def}}{=} \sum_{j\ge 0} z^j x_j, \quad \text{and} \quad \check{y}(z) \stackrel{\text{def}}{=} \sum_{j\ge 0} z^j y_j.$$

Now we can rewrite (2.1) as follows:

$$\begin{cases} \frac{\check{x}(z)}{z} = A\check{x}(z) + B\check{u}(z), \\ \check{y}(z) = C\check{x}(z) + D\check{u}(z). \end{cases}$$

It follows that for sufficiently small ζ

$$\check{y}(\zeta) = (\zeta C(I-\zeta A)^{-1}B + D)\check{u}(\zeta), \tag{2.2}$$

i.e., $\boldsymbol{G} \stackrel{\text{def}}{=} zC(I-zA)^{-1}B + D$ is the transfer function of the plant, or the *transfer function of the system.*

Let now G be a plant. We say that the linear dynamical system (2.1) is a *realization of the plant* G with transfer function $\boldsymbol{G}$ (or a *realization of the transfer function* $\boldsymbol{G}$) if $\boldsymbol{G} = zC(I-zA)^{-1}B + D$.

As before, we say that the system (2.1) is *single input single output (or SISO)* if $\dim\mathcal{K} = \dim\mathcal{E} = 1$.

Let us illustrate now how linear dynamical systems lead to Hankel operators. Suppose now that $\{u_k\}_{k\in\mathbb{Z}}$ is a finitely supported sequence and we cut the input signal at time 0, i.e., we annihilate u_k for $k \ge 0$. Suppose also that $x_k = \mathbb{O}$ when $k < \min\{j : \ u_j \ne \mathbb{O}\}$. Let us study the behavior of the output signal $\{y_k\}$ for $k \ge 0$. Clearly, the y_k for $k \ge 0$ do not depend on D, and so we may assume that $D = \mathbb{O}$. Consider the formal Laurent series

$$\check{u}(z) \stackrel{\text{def}}{=} \sum_{j<0} z^j u_j \quad \text{and} \quad \check{y}(z) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} z^j y_j.$$

It follows from (2.2) and time invariance that

$$\check{y}(z) = (zC(I-zA)^{-1}B)\check{u}(z)$$

(since among the u_k only finitely many terms may be nonzero, the product of the analytic function $zC(I-zA)^{-1}B$ and the Laurent series $\check{u}(z)$ makes sense).

Since we are interested only in the terms y_k with $k \ge 0$, we can disregard the y_k with $k < 0$. Thus we obtain a Hankel-like operator that takes the sequence $\{u_k\}_{k<0}$ to the sequence $\{y_k\}_{k\ge 0}$. To be more precise, this operator can be described with the help of the block Hankel matrix $\{CA^{j+k}B\}_{j,k\ge 0}$:

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} CAB & CA^2B & CA^3B & \cdots \\ CA^2B & CA^3B & CA^4B & \cdots \\ CA^3B & CA^4B & CA^5B & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} u_{-1} \\ u_{-2} \\ u_{-3} \\ \vdots \end{pmatrix}.$$

We say that this block Hankel matrix *is associated with the system* (2.1). Certainly, the matrix $\{CA^{j+k}B\}_{j,k\geq 0}$ does not have to be bounded. However, if it is bounded, then

$$\sum_{k\geq 0} z^k y_k = H^*_{\boldsymbol{G}^*} \sum_{k<0} z^k u_k,$$

where $\boldsymbol{G}$ is the transfer function of the system and $H_{\boldsymbol{G}^*}$ is the Hankel operator from $H^2(\mathcal{E})$ to $H^2_-(\mathcal{K})$.

The sequence $\{CA^nB\}_{n\geq 0}$ is called the *impulse response sequence* of the system $\{A, B, C, D\}$.

The system (2.1) is said to be *controllable* if

$$\operatorname{span}\{A^nBu : \ u \in \mathcal{K}, \ n \in \mathbb{Z}_+\} = \mathcal{H}.$$

Controllability can be interpreted as follows. Suppose that $x_0 = \mathbb{O}$, and we have an input sequence $u_0, u_1, u_2, \cdots$. Then

$$x_{n+1} = A^nBu_0 + A^{n-1}Bu_1 + \cdots + Bu_n.$$

Thus controllability means that for any $x \in \mathcal{H}$ and any $\varepsilon > 0$ there exist $n \in \mathbb{Z}_+$ and inputs $u_0, u_1, \cdots, u_n$ such that starting from state $x_0 = \mathbb{O}$, the system reaches state x_{n+1} within the ε-neighborhood of x.

The system (2.1) is said to be *observable* if

$$\{x \in \mathcal{H} : \ CA^nx = \mathbb{O} \ \text{ for any } \ n \in \mathbb{Z}_+\} = \{\mathbb{O}\}.$$

If $x \in \mathcal{H}$ and $x \in \operatorname{Ker} CA^n$ for any $n \in \mathbb{Z}_+$, this means if the system starts from state $x_0 = x$ and the intput sequence $u_0, u_1, u_2, \cdots$ is zero, the output sequence $y_0, y_1, y_2, \cdots$ is zero, and so the sequence $x_0, x_1, x_2, \cdots$ cannot be observed.

The system (2.1) is called *minimal*, if it is both controllable and observable. If the system is not minimal, this means that the state space is too large.

We associate with the system the *controllability Gramian*

$$W_c \stackrel{\text{def}}{=} \sum_{j\geq 0} A^jBB^*(A^*)^j \tag{2.3}$$

and the *observability Gramian*

$$W_o \stackrel{\text{def}}{=} \sum_{j\geq 0} (A^*)^jC^*CA^j \tag{2.4}$$

provided the corresponding series converge in the weak operator topology. It is easy to see that in this case the system is controllable (observable) if and only if $\operatorname{Ker} W_c = \{\mathbb{O}\}$ ($\operatorname{Ker} W_o = \{\mathbb{O}\}$). The system is called *balanced* if both the controllability and observability Gramians exist and $W_c = W_o$. We also say that the system is a *balanced realization* of the transfer function $\boldsymbol{G}$.

Theorem 2.1. *Suppose that the series that define the controllability and observability Gramians* (2.3) *and* (2.4) *of the system* (2.1) *converge in the weak operator topology. Then the block Hankel matrix* $\Gamma = \{CA^{j+k}B\}_{j,k\ge 0}$ *associated with the system* (2.1) *determines a bounded Hankel operator from* $\ell^2(\mathcal{K})$ *to* $\ell^2(\mathcal{E})$.

Proof. Consider the operators $V_{\rm c} : \mathcal{H} \to \ell^2(\mathcal{K})$ and $V_{\rm o} : \mathcal{H} \to \ell^2(\mathcal{E})$ defined by

$$V_{\rm c}x = \{B^*(A^*)^n x\}_{n\ge 0} \quad \text{and} \quad V_{\rm o}x = \{CA^n x\}_{n\ge 0}, \quad x \in \mathcal{H}. \tag{2.5}$$

It is easy to see that

$$V_{\rm c}^*V_{\rm c} = \sum_{n\ge 0} A^n BB^*(A^*)^n = W_{\rm c}$$

and

$$V_{\rm o}^*V_{\rm o} = \sum_{n\ge 0} (A^*)^n C^*CA^n = W_{\rm o},$$

and $V_{\rm c}$ and $V_{\rm o}$ are bounded operators. It is also easy to see that

$$\Gamma = V_{\rm o}V_{\rm c}^*, \tag{2.6}$$

which implies the boundedness of Γ. ∎

In particular, the conclusion of the theorem holds for balanced linear systems. Thus for a transfer function $\boldsymbol{G}$ to have a balanced realization it is necessary that there exists a function $\Psi \in L^\infty(\mathcal{B}(\mathcal{K},\mathcal{E}))$ such that $\hat{\Psi}(j) = \hat{\boldsymbol{G}}(j)$ for $j > 0$. We are going to show that the converse also holds.

Theorem 2.2. *Under the hypotheses of Theorem 2.1 the restriction to* $(\operatorname{Ker}\Gamma)^\perp$ *of the modulus* $|\Gamma|$ *of the Hankel operator*

$$\Gamma = \{CA^{j+k}B\}_{j,k\ge 0} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{E})$$

is unitarily equivalent to $\left(W_{\rm o}^{1/2}W_{\rm c}W_{\rm o}^{1/2}\right)^{1/2}$.

Proof. Consider the operators $V_{\rm c}$ and $V_{\rm o}$ defined by (2.5) and consider their polar decompositions:

$$V_{\rm c} = U_{\rm c}W_{\rm c}^{1/2}, \quad V_{\rm o} = U_{\rm o}W_{\rm o}^{1/2}.$$

It is easy to see that

$$\operatorname{Ker} V_{\rm c} = \operatorname{Ker} W_{\rm c} = \{\mathbb{O}\} \quad \text{and} \quad \operatorname{Ker} V_{\rm o} = \operatorname{Ker} W_{\rm o} = \{\mathbb{O}\}.$$

Hence, both $U_{\rm c}$ and $U_{\rm o}$ are isometries, $\operatorname{Range} U_{\rm c} = \operatorname{clos}\operatorname{Range} V_{\rm c}$, and $\operatorname{Range} U_{\rm o} = \operatorname{clos}\operatorname{Range} V_{\rm o}$. We have by (2.6)

$$|\Gamma|^2 = \Gamma^*\Gamma = V_{\rm c}V_{\rm o}^*V_{\rm o}V_{\rm c}^* = U_{\rm c}W_{\rm o}^{1/2}W_{\rm c}W_{\rm o}^{1/2}U_{\rm c}^*.$$

Since $\operatorname{Ker} U_{\rm c}^* = (\operatorname{Range} V_{\rm c}^*)^\perp = \operatorname{Ker} V_{\rm c}^* = \operatorname{Ker}\Gamma$, it follows that $W_{\rm o}^{1/2}W_{\rm c}W_{\rm o}^{1/2}$ is unitarily equivalent to $|\Gamma|^2\big|(\operatorname{Ker}\Gamma)^\perp$, which implies the result. ∎

Corollary 2.3. *If* (2.1) *is a balanced system, then* $|\Gamma|\big|(\operatorname{Ker}\Gamma)^\perp$ *is unitarily equivalent to* $W_{\rm c} = W_{\rm o}$.

Corollary 2.4. *Under the hypotheses of Theorem 2.2*

$$\dim \mathcal{H} = \operatorname{rank} \Gamma.$$

Proof. It follows from Theorem 2.2 that $\operatorname{rank} \Gamma = \operatorname{rank} W_{\mathrm{o}}^{1/2} W_{\mathrm{c}} W_{\mathrm{o}}^{1/2}$. The result follows from the fact that $\operatorname{Ker} W_{\mathrm{c}} = \operatorname{Ker} W_{\mathrm{o}} = \{\mathbb{O}\}$. ■

It is easy to see that if the system (2.1) is a realization of a transfer function $\boldsymbol{G}$, then $D = \boldsymbol{G}(0)$, and without loss of generality we may assume that $\boldsymbol{G}(0) = \mathbb{O}$. If $D = \mathbb{O}$ in (2.1), we use the notation $\{A, B, C\}$ for the system (2.1).

Suppose that $\boldsymbol{G}$ is a $\mathcal{B}(\mathcal{K}, \mathcal{E})$-valued transfer function that produces bounded Hankel operator $\{\hat{\boldsymbol{G}}(j+k+1)\}_{j,k\geq 0}$ from $\ell^2(\mathcal{K})$ to $\ell^2(\mathcal{E})$. First we construct the so-called backward shift realization of $\boldsymbol{G}$. We put $\mathcal{H} = H^2(\mathcal{E})$ and define the operators $A : \mathcal{H} \to \mathcal{H}$, $B : \mathcal{K} \to \mathcal{H}$, and $C : \mathcal{H} \to \mathcal{E}$ by

$$\begin{aligned}
Bu &= S_{\mathcal{E}}^*(\boldsymbol{G}u) = \frac{1}{z}\boldsymbol{G}u, \quad u \in \mathcal{K}, \\
Af &= S_{\mathcal{E}}^* f = \frac{1}{z}(f - f(0)), \quad f \in H^2(\mathcal{E}), \\
Cf &= f(0), \quad f \in H^2(\mathcal{E}).
\end{aligned}$$

Then the transfer function of the system $\{A, B, C\}$ is $zC(I - zA)^{-1}B$. Let $u \in \mathcal{K}$. We have

$$\begin{aligned}
zC(I - zA)^{-1}Bu &= (z(I - zS_{\mathcal{E}}^*)^{-1} S_{\mathcal{E}}^*(\boldsymbol{G}u))(0) \\
&= \sum_{n\geq 0} z^{n+1} (S_{\mathcal{E}}^*)^{n+1} (\boldsymbol{G}u)(0) \\
&= \sum_{n\geq 1} z^n \hat{\boldsymbol{G}}(n)u = \boldsymbol{G}(z)u, \quad u \in \mathcal{K}.
\end{aligned}$$

Thus we have found a realization of $\boldsymbol{G}$. This realization is called the *backward shift realization*. Its disadvantage is that the state space $\mathcal{H}$ is very large. It is always infinite-dimensional. On the other hand, if $\boldsymbol{G}$ is rational it follows from Corollary 2.4 that the state space of a balanced realization must be finite-dimensional.

It turns out that we can replace the state space in the backward shift realization with a smaller one. Namely, we can define the new state space $\mathcal{H}$ as follows:

$$\mathcal{H} \stackrel{\text{def}}{=} \operatorname{span}\{(S_{\mathcal{E}}^*)^n (\boldsymbol{G}u) : \ u \in \mathcal{K}, \ n \geq 1\}. \tag{2.7}$$

Then we can define the operators $A : \mathcal{H} \to \mathcal{H}$, $B : \mathcal{K} \to \mathcal{H}$, and $C : \mathcal{H} \to \mathcal{E}$ by

$$Bu = S_{\mathcal{E}}^*(\boldsymbol{G}u), \quad u \in \mathcal{K}, \qquad Af = S_{\mathcal{E}}^* f, \quad f \in \mathcal{H}, \qquad Cf = f(0), \quad f \in \mathcal{H}.$$

Clearly, we obtain another realization of $\boldsymbol{G}$ with state space (2.7). It is called the *restricted backward shift realization*. Unlike the backward shift realization, the restricted backward shift realization is minimal. Indeed, it is controllable by the definition of $\mathcal{H}$ and it is obviously, observable.

Let us compute the observability and controllability Gramians for the restricted backward shift realization. Consider the operator $V_c^* : \ell^2(\mathcal{K}) \to \mathcal{H}$, where V_c is defined by (2.5). It is easy to see that

$$V_c^*\{u_j\}_{j\ge 0} = \sum_{n\ge 0}(S_{\mathcal{E}}^*)^{n+1}(\boldsymbol{G}u_n) = \Gamma_{\frac{1}{z}\boldsymbol{G}}\{u_j\}_{j\ge 0}, \quad \{u_j\}_{j\ge 0} \in \ell^2(\mathcal{K}),$$

where $\Gamma_{\frac{1}{z}\boldsymbol{G}}$ is the Hankel operator with block matrix $\{\boldsymbol{G}(j+k+1)\}_{j,k\ge 0}$. Thus the boundedness of this matrix $\Gamma_{\frac{1}{z}\boldsymbol{G}}$ implies that W_c is bounded. It is easy to see that $W_c = \left|\Gamma_{\frac{1}{z}\boldsymbol{G}}\right|^2 \big| \mathcal{H}$. On the other hand, it is easy to see that the operator V_o defined by (2.5) is an isometry, and so $W_o = I$. Minimal realizations for which W_c exists and $W_o = I$ are called *output normal.*

To construct a balanced realization, we use the restricted backward realization we have just constructed and modify the above operators A, B, and C.

Theorem 2.5. *Let $\mathcal{K}$ and $\mathcal{E}$ be Hilbert spaces and let $\boldsymbol{G}$ be a $\mathcal{B}(\mathcal{K},\mathcal{E})$-valued function analytic in $\mathbb{D}$ such that the block Hankel matrix $\{\hat{\boldsymbol{G}}(j+k+1)\}_{j,k\ge 0}$ determines a bounded operator. Then there exist a Hilbert space $\mathcal{H}$, bounded linear operators $A_b : \mathcal{H} \to \mathcal{H}$, $B_b : \mathcal{K} \to \mathcal{H}$, $C_b : \mathcal{H} \to \mathcal{E}$, and $D_b : \mathcal{K} \to \mathcal{E}$ such that $\|A_b\| \le 1$ and the system $\{A_b, B_b, C_b, D_b\}$ is a balanced realization of $\boldsymbol{G}$.*

As we have already observed, $D_b = \boldsymbol{G}(0)$ and we may assume that $\boldsymbol{G}(0) = \mathbb{O}$. We start with the output normal realization $\{A, B, C\}$ constructed above, and we use the same spaces $\mathcal{K}$, $\mathcal{E}$, and $\mathcal{H}$. Suppose that K is a bounded invertible operator on $\mathcal{H}$. Consider the operators

$$A_b \stackrel{\text{def}}{=} K^{-1}AK, \quad B_b \stackrel{\text{def}}{=} K^{-1}B, \quad \text{and} \quad C_b \stackrel{\text{def}}{=} CK. \tag{2.8}$$

Let us evaluate the transfer function of the system $\{A_b, B_b, C_b\}$:

$$zC_b(I - zA_b)^{-1}B_b = zCK(I - zK^{-1}AK)^{-1}K^{-1}B = zC(I - zA)^{-1}B.$$

Thus the new system $\{A_b, B_b, C_b\}$ has the same transfer function. Let us compute now the observability Gramian W_{bo} and the controllability Gramian W_{bc} of the new system. We have

$$\begin{aligned} W_{bc} &= \sum_{j\ge 0} A_b^j B_b B_b^* (A_b^*)^j \\ &= \sum_{j\ge 0} K^{-1}A^jKK^{-1}BB^*(K^*)^{-1}K^*(A^*)^j(K^*)^{-1} \\ &= \sum_{j\ge 0} K^{-1}A^jBB^*(A^*)^j(K^*)^{-1} = K^{-1}W_c(K^*)^{-1}. \end{aligned} \tag{2.9}$$

Similarly,

$$W_{bo} = \sum_{j\ge 0}(A_b^*)^jC_b^*C_bA_b^j = K^*W_oK = K^*K. \tag{2.10}$$

Recall that W_c and $W_o = I$ are the controllability and the observability Gramians of the output normal system $\{A, B, C\}$. Thus the system $\{A_b, B_b, C_b\}$ is minimal, and it is controllable if and only if

$$K^{-1} W_c (K^*)^{-1} = K^* K.$$

We can now put $K = W_c^{1/4}$, which guarantees the above identity. The problem is that W_c does not have to be invertible. It turns out, however, that if we consider the operator $K^{-1} = W_c^{-1/4}$ defined on the dense subset $\operatorname{Range} W_c^{1/4}$ of $\mathcal{H}$, then the operators A_b, and B_b defined by (2.8) are bounded operators, A_b is a contraction, and $\{A_b, B_b, C_b\}$ is a balanced realization of our transfer function $\boldsymbol{G}$. Let us prove this.

Lemma 2.6. *Let $K = W_c^{1/4}$. Then the operator $A_b \stackrel{\text{def}}{=} K^{-1} A K$ is defined on the whole space $\mathcal{H}$ and is a contraction on $\mathcal{H}$.*

Proof. It is easy to see that

$$W_c = \sum_{n \ge 0} A^n B B^* (A^*)^n \ge \sum_{n \ge 0} A^{n+1} B B^* (A^*)^{n+1} = A W_c A^*.$$

Since $\|A\| \le 1$ by construction, we have

$$K^4 \ge A K^4 A^* \ge A K^2 A^* A K^2 A^* = (A K^2 A^*)^2.$$

Applying the Heinz inequality (see Appendix 1.7), we obtain

$$AK(AK)^* = A K^2 A^* \le K^2.$$

By Lemma 2.1.2, there exists a contraction Q on $\mathcal{H}$ such that

$$AK = KQ,$$

and so $K^{-1} A K = Q$ is defined on the whole space $\mathcal{H}$, and $\|K^{-1} A K\| \le 1$. ■

Lemma 2.7. *The operator $B_b \stackrel{\text{def}}{=} K^{-1} B$ is defined on the whole space $\mathcal{K}$ and it is bounded.*

Proof. Let $U_c : \mathcal{H} \to \ell^2(\mathcal{K})$ be the isometry defined in the proof of Theorem 2.2. Define the bounded operator $B_\# : \mathcal{K} \to \mathcal{H}$ by

$$B_\# u = K U_c^* \begin{pmatrix} u \\ \mathbb{O} \\ \mathbb{O} \\ \vdots \end{pmatrix}, \quad u \in \mathcal{K}.$$

We have

$$K B_\# u = W_c^{1/2} U_c^* \begin{pmatrix} u \\ \mathbb{O} \\ \mathbb{O} \\ \vdots \end{pmatrix} = V_c^* \begin{pmatrix} u \\ \mathbb{O} \\ \mathbb{O} \\ \vdots \end{pmatrix} = Bu, \quad u \in \mathcal{K},$$

where V_c is defined by (2.5). It follows that $B_b = B_\#$. ■

Proof of Theorem 2.5. Let us compute the transfer function of $\{A_{\mathrm{b}}, B_{\mathrm{b}}, C_{\mathrm{b}}\}$. It is easy to see that the operator A_{b}^n for $n \in \mathbb{Z}_+$ is well defined by $A_{\mathrm{b}}^n = K^{-1}A^nK$ and

$$\begin{aligned}
zC_{\mathrm{b}}(I - zA_{\mathrm{b}})^{-1}B_{\mathrm{b}} &= zCK(I - zK^{-1}AK)^{-1}K^{-1}B \\
&= zCK\sum_{n\ge 0} z^n(K^{-1}AK)^nK^{-1}B \\
&= zCK\sum_{n\ge 0} z^nK^{-1}A^nKK^{-1}B \\
&= zC\left(\sum_{n\ge 0} z^nA^n\right)B = zC(I - zA)^{-1}B.
\end{aligned}$$

Thus the new system $\{A_{\mathrm{b}}, B_{\mathrm{b}}, C_{\mathrm{b}}\}$ has the same transfer function as the system $\{A, B, C\}$. Let us show that $\{A_{\mathrm{b}}, B_{\mathrm{b}}, C_{\mathrm{b}}\}$ is a balanced system. It is easy to see that the operator A_{b}^* is defined by $A_{\mathrm{b}}^*x = KA^*K^{-1}x$ for $x \in \operatorname{Range} K$ and extends by continuity to the whole space $\mathcal{H}$. It is easy to see that the operator $(A_{\mathrm{b}}^*)^n$ for $n \in \mathbb{Z}_+$ is defined on $\operatorname{Range} K$ by $(A_{\mathrm{b}}^*)^n = K(A^*)^nK^{-1}$ and also extends by continuity to the whole space $\mathcal{H}$. It is easy to see now that (2.9) and (2.10) show that $W_{\mathrm{bc}} = W_{\mathrm{bo}} = K^2$, which completes the proof. ■

We proceed now to the problem of uniqueness. Suppose we have two linear systems $\{A_i, B_i, C_i, D_i\}$, $i = 1, 2$, with the same input space $\mathcal{K}$, the same output space $\mathcal{E}$ and possibly different state spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. We say that they are *equivalent* if $D_1 = D_2$ and there is a unitary operator $\mathcal{V}$ from $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that

$$A_1 = \mathcal{V}^*A_2\mathcal{V}, \quad B_1 = \mathcal{V}^*B_2, \quad \text{and} \quad C_1 = C_2\mathcal{V}.$$

Clearly, equivalent linear systems have the same transfer functions.

First we are going to show that all output normal realizations of the same transfer function are equivalent. As usual, we may assume that $D_1 = D_2 = \mathbb{O}$. Note that if $\{A, B, C\}$ is an output normal system, them by Theorem 2.1 the Hankel operator associated with it is bounded, and so its transfer function has the restricted backward shift realization.

Theorem 2.8. *An output normal system is equivalent to the restricted backward shift realization of its transfer function.*

Proof. Suppose that $\{A, B, C\}$ is an output normal system (we assume that $D = \mathbb{O}$) with transfer function $\boldsymbol{G}(z) = zC(I - zA)^{-1}B$. By Theorem 2.1, the Hankel operator associated with the system $\{A, B, C\}$ is bounded, and so the restricted backward shift realization of $\boldsymbol{G}$ exists.

Let $\{A_\#, B_\#, C_\#\}$ be the restricted backward shift realization of $\boldsymbol{G}$, i.e., the state space $\mathcal{H}_\#$ of $\{A_\#, B_\#, C_\#\}$ is the minimal $S_{\mathcal{E}}^*$-invariant subspace of $H^2(\mathcal{E})$ containing the vectors $S_{\mathcal{E}}^*\boldsymbol{G}u$, $u \in \mathcal{K}$, $A_\#f = S_{\mathcal{E}}^*f$, $f \in \mathcal{H}_\#$, $B_\#u = S_{\mathcal{E}}^*(\boldsymbol{G}u)$, $u \in \mathcal{K}$, and $C_\#f = f(0)$, $f \in \mathcal{H}$.

Consider the operator $V_o : \mathcal{H} \to H^2(\mathcal{E})$ defined by (2.5) (we identify $\ell^2(\mathcal{E})$ with $H^2(\mathcal{E})$). Let us show that $\operatorname{Range} V_o = \mathcal{H}_\#$. Identifying $\ell^2(\mathcal{K})$ with $H^2(\mathcal{K})$, we obtain for $u \in \mathcal{K}$

$$\begin{aligned} V_o V_c^* z^j u &= V_o A^j B u = \sum_{n\ge 0} z^n C A^{j+n} B u \\ &= (S_\mathcal{E}^*)^j \sum_{n\ge 0} z^n C A^n B u = (S_\mathcal{E}^*)^{j+1}(\boldsymbol{G}u) \in \mathcal{H}_\#. \end{aligned}$$

Thus $\operatorname{Range} V_o \subset \mathcal{H}_\#$. Since the system $\{A, B, C\}$ is controllable, it is easy to see that $\operatorname{Range} V_o$ is dense in $\mathcal{H}_\#$. Finally, $\{A, B, C\}$ is output normal, and so V_o is an isometry that implies that $\operatorname{Range} V_o = \mathcal{H}_\#$.

Let $\mathcal{V}$ be the unitary operator from $\mathcal{H}$ onto $\mathcal{H}_\#$ defined by $\mathcal{V}x = V_o x$, $x \in \mathcal{H}$. We have

$$\begin{aligned} \mathcal{V}Ax &= V_o A x = \sum_{n\ge 0} z^n C A^{n+1} x \\ &= S_\mathcal{E}^* \sum_{n\ge 0} z^n C A^n x = S_\mathcal{E}^* V_o x = A_\# \mathcal{V} x, \quad x \in \mathcal{H}. \end{aligned}$$

Next,

$$C_\# \mathcal{V} x = C_\# \sum_{n\ge 0} z^n C A^n x = \left(\sum_{n\ge 0} z^n C A^n x \right)(0) = Cx, \quad x \in \mathcal{H}.$$

Finally,

$$B_\# u = S_\mathcal{E}^*(\boldsymbol{G})u = \sum_{n\ge 0} z^n C A^n B u = \mathcal{V} B u, \quad u \in \mathcal{K}.$$

This completes the proof of the fact that the systems $\{A, B, C\}$ and $\{A_\#, B_\#, C_\#\}$ are equivalent. ■

Now we are able to prove the uniqueness result for balanced realizations.

Theorem 2.9. *Let $\{A_i, B_i, C_i\}$, $i = 1, 2$, be balanced realizations of the same transfer function. Then they are equivalent.*

Proof. Let $\mathcal{K}$ be the input space, $\mathcal{E}$ the output space, and $\mathcal{H}_i$ the state space of $\{A_i, B_i, C_i\}$. Let W_i be the observability Gramian (and the controllability Gramian) of $\{A_i, B_i, C_i\}$. We have

$$W_i = \sum_{n\ge 0} (A_i^*)^n C_i^* C_i A_i^n \ge \sum_{n\ge 1} (A_i^*)^n C_i^* C_i A_i^n = A_i^* W_i A_i.$$

By Lemma 2.1.2, there exist contractions $\check{A}_i$ on $\mathcal{H}_i$ such that

$$\check{A}_i W_i^{1/2} = W_i^{1/2} A_i, \quad i = 1, 2.$$

Put

$$\check{B}_i = W_i^{1/2} B_i \quad \text{and} \quad \check{C}_i = C_i W_i^{-1/2}, \quad i = 1, 2.$$

The operators $\check{C}_i$ are defined on the dense subset $\operatorname{Range} W_i^{1/2}$ of $\mathcal{H}_i$. Let us show that $\check{C}_i$ extends by continuity to $\mathcal{H}_i$. Consider the operator

$V_{oi} : \mathcal{H}_i \to \ell^2(\mathcal{K})$, $V_{oi}x = \{C_i A_i^n x\}_{n\ge0}$, $x \in \mathcal{H}_i$ (see (2.5)). Put $x = W_i^{1/2}y$. We have

$$\|\check{C}_i x\| = \|C_i y\| \le \left(\sum_{n\ge0}\|C_i A_i^n y\|^2\right)^{1/2} = \|V_{oi}y\| = \|W_i^{1/2}y\| = \|x\|,$$

and so $\check{C}_i$ extends by continuity to $\mathcal{H}_i$.

It is easy to verify that the transfer functions of $\{A_i, B_i, C_i\}$ and $\{\check{A}_i, \check{B}_i, \check{C}_i\}$ coincide. It is also easy to check that the system $\{\check{A}_i, \check{B}_i, \check{C}_i\}$, $i = 1, 2$, is output normal and its controllability Gramian is W_i^2. Thus by Theorem 2.8, they are equivalent, i.e., there exist a unitary operator $\mathcal{V} : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\check{A}_1 = \mathcal{V}^*\check{A}_2\mathcal{V}, \quad \check{B}_1 = \mathcal{V}^*\check{B}_2, \quad \text{and} \quad \check{C}_1 = \check{C}_2\mathcal{V}.$$

Clearly, $W_1^2 = \mathcal{V}^*W_2^2\mathcal{V}$, and so $W_1^{1/2} = \mathcal{V}^*W_2^{1/2}\mathcal{V}$. It follows that

$$\begin{aligned} W_1^{1/2}A_1W_1^{-1/2} &= \check{A}_1 = \mathcal{V}^*\check{A}_2\mathcal{V} \\ &= \mathcal{V}^*W_2^{1/2}A_2W_2^{-1/2}\mathcal{V} = W_1^{1/2}\mathcal{V}^*A_2\mathcal{V}W_1^{-1/2}. \end{aligned}$$

Hence, $A_1 = \mathcal{V}^*A\mathcal{V}$. Similarly, $B_1 = \mathcal{V}^*B_2$ and $C_1 = C_2\mathcal{V}$. ■

Remark. Suppose now that $\boldsymbol{G}$ is a transfer function, i.e., $\boldsymbol{G}$ function analytic in a neighborhood of the origin which takes values in the space of bounded operators from a Hilbert space $\mathcal{K}$ to a Hilbert space $\mathcal{E}$. Let us construct a minimal realization of $\boldsymbol{G}$. Let r be a positive number such that $\boldsymbol{G}$ is a bounded analytic function in the disk of radius r centered at the origin. Consider the auxiliary function $\boldsymbol{G}_\#$ defined by $\boldsymbol{G}_\#(\zeta) = \boldsymbol{G}(r\zeta)$. By Theorem 2.5, there exists a balanced realization $\{A, B, C, D\}$ of $\boldsymbol{G}_\#$. It is easy to see that $\{r^{-1}A, B, r^{-1}C, D\}$ is a minimal realization of $\boldsymbol{G}$.

3. Realizations with Continuous Time

Here we consider linear systems (or realizations) with continuous time. We introduce the Hankel operator associated with a linear system and define the important class of balanced realizations. However, unlike the case of discrete time the class of linear systems defined in terms of bounded operators is not sufficient to obtain realizations of all transfer functions that produce bounded Hankel operators.

Let $\mathcal{K}$, $\mathcal{E}$, and $\mathcal{H}$ be Hilbert spaces. Suppose that $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $C \in \mathcal{B}(\mathcal{H}, \mathcal{E})$, and $D \in \mathcal{B}(\mathcal{K}, \mathcal{E})$. Consider the *linear dynamical system* with continuous time

$$\begin{cases} x'(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \tag{3.1}$$

Here u is an input signal, i.e., a function with values in the *input space* $\mathcal{K}$, y is the output signal that is a function with values in the *output space* $\mathcal{E}$, x is a function with values in the *state space* $\mathcal{H}$. As in the case of discrete time we use the notation $\{A, B, C, D\}$ for the system (3.1). If $D = \mathbb{O}$, we write simply $\{A, B, C\}$.

Suppose that u and x vanish on $(-\infty, 0)$. Clearly, the system (3.1) determines a plant that is linear, causal, and time-invariant. Let us compute the transfer function of the system. Let u be a smooth function with compact support in $(0, \infty)$. Then the functions x and y are also smooth and possess Laplace transform in some half-plane $\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > \sigma\}$. Let us pass to Laplace transform. We have from (3.1)

$$\begin{cases} \zeta(\boldsymbol{\mathcal{L}}x)(\zeta) = A\big((\boldsymbol{\mathcal{L}}x)(\zeta)\big) + B\big((\boldsymbol{\mathcal{L}}u)(\zeta)\big), \\ (\boldsymbol{\mathcal{L}}y)(\zeta) = C\big((\boldsymbol{\mathcal{L}}x)(\zeta)\big) + D\big((\boldsymbol{\mathcal{L}}u)(\zeta)\big), \end{cases} \quad \operatorname{Re}\zeta > \sigma.$$

It follows that

$$(\boldsymbol{\mathcal{L}}y)(\zeta) = \big(C(\zeta I - A)^{-1}B + D\big)\big((\boldsymbol{\mathcal{L}}u)(\zeta)\big), \quad \operatorname{Re}\zeta > \sigma.$$

Thus $C(zI - A)^{-1}B + D$ is the transfer function of the system.

If G is a plant with continuous time whose transfer function is $\boldsymbol{G}$, we say that the system (3.1) is a *realization of the plant* G (or a *realization of the transfer function* $\boldsymbol{G}$) if $\boldsymbol{G}(\zeta) = C(\zeta I - A)^{-1}B + D$ for $\operatorname{Re}\zeta$ sufficiently large.

As usual we say that the system (3.1) is single input single output (or SISO) if $\dim\mathcal{K} = \dim\mathcal{E} = 1$.

As in the case of discrete time, the study of systems (3.1) naturally leads to Hankel operators. Suppose that u is smooth and has compact support in $(-\infty, 0)$ and $x(t) = 0$ for $t < \operatorname{supp} u$, i.e., cut the input signal at $t = 0$, and we study the output signal y for positive values of t. Clearly, if we are interested only in the behavior of the output signal for $t > 0$, D does not play any role, and we can assume that $D = \mathbb{O}$. Let M be a positive number such that $\operatorname{supp} u \subset (-M, 0)$. Consider the translations u_M and y_M of u and y,

$$u_M(t) = u(t - M), \quad y_M(t) = y(t - M), \quad t \in \mathbb{R}.$$

It is easy to see that

$$(\boldsymbol{\mathcal{L}}y_M)(\zeta) = \big(C(\zeta I - A)^{-1}B\big)\big((\boldsymbol{\mathcal{L}}u_M)(\zeta)\big), \quad \operatorname{Re}\zeta > \sigma.$$

Consider the operator function Ξ on $\mathbb{R}$ defined by $\Xi(t) = Ce^{tA}B$. Its Laplace transform can be computed very easily:

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\zeta t} C e^{tA} B dt = \frac{1}{\sqrt{2\pi}} C(\zeta I - A)^{-1}B,$$

if $\operatorname{Re}\zeta$ is sufficiently large, i.e.,

$$(\boldsymbol{\mathcal{L}}\Xi)(\zeta) = \frac{1}{\sqrt{2\pi}} C(\zeta I - A)^{-1}B = \frac{1}{\sqrt{2\pi}} \boldsymbol{G}(\zeta).$$

Thus y_M is the convolution of the functions Ξ and u_M, i.e.,

$$y_M(s) = \int_0^s Ce^{(s-t)A}Bu_M(t)dt,$$

and it is easy to see that

$$y(s) = \int_0^\infty Ce^{(t+s)A}Bu(-t)dt = \int_0^\infty \Xi(s+t)v(t)dt \stackrel{\text{def}}{=} (\boldsymbol{\Gamma}_\Xi v)(s), \quad s > 0.$$

where $v(t) \stackrel{\text{def}}{=} u(-t)$, $t \in \mathbb{R}$. The Hankel operator $\boldsymbol{\Gamma}_\Xi$ is defined on the set of smooth functions with compact support in $(0, \infty)$ (see §1.8 and the Remark at the end of §2.2). The function Ξ is called the *impulse response function* of the system $\{A, B, C\}$.

We say that $\boldsymbol{\Gamma}_\Xi$ is the Hankel operator associated with the system $\{A, B, C\}$. It does not have to be a bounded operator from $L^2(\mathbb{R}_+, \mathcal{K})$ to $L^2(\mathbb{R}_+, \mathcal{E})$ (we use this notation for the L^2 spaces of $\mathcal{K}$-valued and $\mathcal{E}$-valued functions on $\mathbb{R}_+$). However, as in the case of discrete time, $\boldsymbol{\Gamma}_\Xi$ is bounded in important special cases.

Example. Consider the linear ordinary differential equation with constant coefficients

$$y''(t) + ay'(t) + by(t) = u(t).$$

Here u is an input function and y is an output function. Let us construct a realization of this SISO system. The input space and the output space are equal to $\mathbb{C}$, while the state space is $\mathbb{C}^2$. Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$. Consider the following realization

$$\begin{cases} \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t), \\ y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}. \end{cases}$$

It is easy to see that the transfer function is the scalar rational function $(z^2 + za + b)^{-1}$. The system is stable if and only if both zeros of this polynomial have negative real parts.

As in the case of discrete time, we say that the system (3.1) is *controllable* if

$$\operatorname{span}\{e^{tA}Bu : \ u \in \mathcal{K}, \ t > 0\} = \mathcal{H}.$$

It is called *observable* if

$$\{x \in \mathcal{H} : \ Ce^{tA}x = \mathbb{O} \ \text{ for any } \ t > 0\} = \{\mathbb{O}\}.$$

The system is called *minimal* if it is both controllable and observable.

The *controllability Gramian* $W_{\rm o}$ and the *observability Gramian* $W_{\rm c}$ are defined by

$$W_{\rm c} = \int_0^\infty e^{tA}BB^*e^{tA^*}dt, \quad W_{\rm o} = \int_0^\infty e^{tA^*}C^*Ce^{tA}dt$$

if the integrals converge in the weak operator topology. The system is called *balanced* if both $W_{\rm o}$ and $W_{\rm c}$ exist and $W_{\rm o} = W_{\rm c}$.

The following result is an analog of Theorems 2.1 and 2.2.

Theorem 3.1. *Suppose that the series that define the controllability and observability Gramians $W_{\rm c}$ and $W_{\rm o}$ of the system* (3.1) *converge in the weak operator topology. Then the Hankel operator $\boldsymbol{\Gamma}_\Xi$ associated with the system* (3.1) *is a bounded operator from $L^2(\mathbb{R}_+, \mathcal{K})$ to $L^2(\mathbb{R}_+, \mathcal{E})$. The restriction of $|\boldsymbol{\Gamma}_\Xi|$ to $(\operatorname{Ker}\boldsymbol{\Gamma}_\Xi)^\perp$ is unitarily equivalent to $\left(W_{\rm o}^{1/2}W_{\rm c}W_{\rm o}^{1/2}\right)^{1/2}$.*

Proof. The proof of Theorem 3.1 is exactly the same as the proof of Theorems 2.1 and 2.2. Indeed, we can introduce the operators $V_{\rm c} : \mathcal{H} \to \Lambda^2(\mathbb{R}_+\mathcal{K})$ and $V_{\rm o} : \mathcal{H} \to \Lambda^2(\mathbb{R}_+, \mathcal{E})$,

$$(V_{\rm c}x)(t) = B^*e^{tA^*} \quad \text{and} \quad V_{\rm o}x = Ce^{tA}x, \quad x \in \mathcal{H}, \quad t \in \mathbb{R}_+. \tag{3.2}$$

Then as in the case of discrete time it is easy to see that

$$\boldsymbol{\Gamma}_\Xi = V_{\rm o}^*V_{\rm c}.$$

Again, as in the case of discrete time it is easy to see that $|\boldsymbol{\Gamma}_\Xi|^2 = U_{\rm c}W_{\rm o}^{1/2}W_{\rm c}W_{\rm o}^{1/2}U_{\rm c}^*$, where $U_{\rm c}$ is the partially isometric factor in the polar decomposition of $V_{\rm c}$. As in the proof of Theorem 2.2 this implies the result. ■

Suppose now that $\boldsymbol{G}$ is a $\mathcal{B}(\mathcal{K}, \mathcal{E})$-valued function analytic in $\mathbb{C}^+$ such that the limits $\boldsymbol{G}(y) \stackrel{\text{def}}{=} \lim\limits_{x\to 0} \boldsymbol{G}(x + \mathrm{i}y)$ exist almost everywhere in the weak operator topology and the Hankel-like operator

$$f \mapsto P_{H^2(\mathbb{C}^+)}\boldsymbol{G}f, \quad f \in H^2(\mathbb{C}^-),$$

is bounded from $H^2(\mathbb{C}^-)$ to $H^2(\mathbb{C}^+)$. It turns out that unlike the case of discrete time, it is not always possible to find a balanced realization of $\boldsymbol{G}$ of the form (3.1) with bounded operators A, B, and C (see §12.7). One can consider the conformal map $\iota : \mathbb{C}^+ \to \mathbb{D}$,

$$\iota(\zeta) = \frac{\zeta - 1}{\zeta + 1}, \quad \operatorname{Re}\zeta > 0,$$

and define the function $\boldsymbol{G}_{\rm d}$ on $\mathbb{D}$ by

$$\boldsymbol{G}_{\rm d} = \boldsymbol{G} \circ \iota^{-1}.$$

Then as in the case of scalar functions (see §1.8 and the Remark at the end of §2.2), the block Hankel matrix $\{\hat{\boldsymbol{G}}_{\rm d}(j+k)\}_{j,k\geq 0}$ is bounded. By Theorem

2.5, there exists a balanced realization $\{A_{\rm d}, B_{\rm d}, C_{\rm d}, D_{\rm d}\}$ of $\boldsymbol{G}_{\rm d}$ with discrete time. Suppose that $A_{\rm d} - I$ is invertible. Consider the following operators:

$$\begin{array}{ll} A = (A_{\rm d} - I)^{-1}(A_{\rm d} + I), & B = \sqrt{2}(A_{\rm d} - I)^{-1}B_{\rm d}, \\ C = -\sqrt{2}C_{\rm d}(A_{\rm d} - I)^{-1}, & D = D_{\rm d} - C_{\rm d}(A_{\rm d} - I)^{-1}B_{\rm d}. \end{array}$$

Then $\{A, B, C, D\}$ is a balanced realization of $\boldsymbol{G}$ with continuous time. Indeed, let us compute the transfer function of the system $\{A, B, C, D\}$ with continuous time. Put $w = (\zeta - 1)(\zeta + 1)^{-1}$, $\zeta \in \mathbb{C}^+$. We have

$$\begin{aligned} & C(\zeta I - A)^{-1}B + D \\ & = -2C_{\rm d}(A_{\rm d} - I)^{-2}\big(\zeta I - (A_{\rm d} - I)^{-1}(A_{\rm d} + I)\big)^{-1}B_{\rm d} + D \\ & = -2C_{\rm d}(A_{\rm d} - I)^{-1}\big(-(\zeta + 1)I + (\zeta - 1)A_{\rm d}\big)^{-1}B_{\rm d} + D \\ & = -2(\zeta + 1)^{-1}(I - A_{\rm d})^{-1}(I - wA_{\rm d})^{-1}B_{\rm d} + D \\ & = wC_{\rm d}(I - wA_{\rm d})^{-1}B_{\rm d} + C_{\rm d}(A_{\rm d} - I)^{-1}B_{\rm d} + D \\ & = wC_{\rm d}(I - wA_{\rm d})^{-1}B_{\rm d} + D_{\rm d} = \boldsymbol{G}_{\rm d}(w). \end{aligned}$$

Now let us show that the system $\{A, B, C, D\}$ is balanced.

Consider the operator $V_{\rm o}$ defined by (3.2) and consider the corresponding operator $V_{\rm od}$ for the system $\{A_{\rm d}, B_{\rm d}, C_{\rm d}, D_{\rm d}\}$ with discrete time:

$$V_{\rm od}x = \{C_{\rm d}A_{\rm d}^n x\}_{n\ge 0}, \quad x \in \mathcal{H}.$$

Consider the operator $\mathcal{V}$ from $H^2(\mathcal{H})$ to $H^2(\mathbb{C}^+, \mathcal{H})$ defined by

$$(\mathcal{V}f)(\zeta) = \pi^{-1/2}\frac{1}{\zeta + 1}(f \circ \iota)(\zeta) = \pi^{-1/2}\frac{1}{\zeta + 1}f\left(\frac{\zeta - 1}{\zeta + 1}\right), \quad \zeta \in \mathbb{C}^+.$$

Then $\mathcal{V}$ is the unitary operator from $H^2(\mathcal{H})$ onto $H^2(\mathbb{C}^+, \mathcal{H})$ (see Appendix 2.1, where the similar operator $\mathcal{U}$ from H^2 onto $H^2(\mathbb{C}_+)$ is considered).

We now define the unitary operator $\mathcal{W} : \ell^2(\mathcal{H}) \to L^2(\mathbb{R}_+, \mathcal{H})$ by

$$\mathcal{W}\{x_n\}_{n\ge 0} = \boldsymbol{\mathcal{L}}^{-1}\mathcal{V}\left(\sum_{n\ge 0} z^n x_n\right).$$

Here $\boldsymbol{\mathcal{L}}$ is the Laplace transform on the space $L^2(\mathbb{R}_+, \mathcal{H})$ of $\mathcal{H}$-valued functions which is defined in the same way as for scalar functions.

We claim that

$$V_{\rm o} = \mathcal{W}V_{\rm od}. \tag{3.3}$$

Indeed, we have

$$\begin{aligned} (\boldsymbol{\mathcal{L}}V_{\rm o}x)(\zeta) &= \frac{1}{\sqrt{2\pi}}C\int_0^\infty e^{-\zeta t}e^{tA}x\,dt \\ &= \frac{1}{\sqrt{2\pi}}C(\zeta I - A)^{-1}x, \quad x \in \mathcal{H}, \quad \operatorname{Re}\zeta > 0, \end{aligned}$$

(since A_d is a contraction (see Theorems 2.5 and 2.9), it follows that $\|e^{tA}\| \le 1$, $t > 0$, which implies the convergence of the integral).

On the other hand,

$$\sum_{n\ge 0} z^n C_\mathrm{d} A_\mathrm{d}^n x = C_\mathrm{d}(I - zA_\mathrm{d})^{-1}x, \quad x \in \mathcal{H}.$$

Thus (3.3) is equivalent to the equality

$$\frac{1}{\sqrt{2\pi}} C(\zeta I - A)^{-1} = \frac{1}{\sqrt{\pi}}(\zeta+1)^{-1} C_\mathrm{d}\left(I - \frac{\zeta-1}{\zeta+1}A_\mathrm{d}\right)^{-1}, \quad \operatorname{Re}\zeta > 0.$$

We have

$$\begin{aligned}\frac{1}{\sqrt{2}} C(\zeta I - A)^{-1} &= -C_\mathrm{d}(A_\mathrm{d} - I)^{-1}\big(\zeta I - (A_\mathrm{d} - I)^{-1}(A_\mathrm{d} + I)\big)^{-1} \\ &= -C_\mathrm{d}\big(-(\zeta+1)I + (\zeta-1)A_\mathrm{d}\big)^{-1} \\ &= (\zeta+1)^{-1} C_\mathrm{d}\left(I - \frac{\zeta-1}{\zeta+1}A_\mathrm{d}\right)^{-1},\end{aligned}$$

which completes the proof of (3.3).

Consider now the operator V_c defined by (3.2) and the corresponding operator V_cd for the system $\{A_\mathrm{d}, B_\mathrm{d}, C_\mathrm{d}, D_\mathrm{d}\}$ with discrete time:

$$V_\mathrm{cd} x = \{B_\mathrm{d}^*(A_\mathrm{d}^*)^n x\}_{n\ge 0}, \quad x \in \mathcal{H}.$$

In a similar way it is easy to show that

$$V_\mathrm{c} = -\mathcal{W} V_\mathrm{cd}. \tag{3.4}$$

Since the system $\{A_\mathrm{d}, B_\mathrm{d}, C_\mathrm{d}, D_\mathrm{d}\}$ is balanced, it follows from (3.3) and (3.4) that the system $\{A, B, C, D\}$ is balanced.

However, if $I - A_\mathrm{d}$ is not invertible, the situation is more complicated. In the general case the operator A does not have to be bounded, it is the generator of a contractive semigroup, B is an operator from $\mathcal{K}$ to the dual space $(\mathcal{D}(A^*))^*$ to the domain of A^* (note that $\mathcal{H}$ is naturally imbedded in $(\mathcal{D}(A^*))^*$), and C is a densely defined functional on the domain $\mathcal{D}(A)$ of A.

We are going to consider in this book only realizations with bounded operators A, B, and C. We call such realizations (or such linear systems) *proper*, while realizations for which the operator $I - A_\mathrm{d}$ is not invertible will be referred to as *generalized realizations*.

We refer the reader to Ober and Montgomery-Smith [1], where the general case is considered.

Remark. Note that if the transfer function $\boldsymbol{G}$ is a stable rational function, then it admits a proper balanced realization. Indeed, in this case the corresponding function $\boldsymbol{G}_\mathrm{d}$ in $\mathbb{D}$ is rational and has poles outside the closed unit disc, and so the operator A_d has spectrum in the open unit disk. Thus

$A_{\mathrm{d}} - I$ is invertible, which implies that A, B, and C are bounded operators. It follows now from Corollary 2.4 that if $\boldsymbol{G}$ is a stable rational transfer function (i.e., $\boldsymbol{G}$ has no poles in the closed right half-plane), then the dimension of the state space of a balanced realization of $\boldsymbol{G}$ is equal to the McMillan degree of $\boldsymbol{G}$.

Let us prove now that any *proper* rational matrix function (i.e., such a rational matrix function that has no pole at ∞) has a minimal realization. Clearly, any rational transfer function is proper.

Corollary 3.2. *Let $\boldsymbol{G}$ be a proper rational matrix function. Then $\boldsymbol{G}$ has a minimal realization.*

Proof. Let a be a positive number such that $\boldsymbol{G}$ has no poles in the half-plane $\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta \geq a\}$. Consider the auxiliary transfer function $\boldsymbol{G}_\#$ defined by $\boldsymbol{G}_\#(\zeta) = \boldsymbol{G}(\zeta + a)$. Clearly, $\boldsymbol{G}_\#$ is a stable rational function, and so it has a proper balanced realization $\{A, B, C, D\}$. It is easy to see now that $\{A + aI, B, C, D\}$ is a minimal realization of $\boldsymbol{G}$. ■

4. Model Reduction

Suppose that we have a plant that admits a realization in a finite-dimensional state space. Such a system can be designed by engineers if the dimension of the state space is not too large. The dimension of the state space reflects the complexity of the system. The larger the dimension is, the more difficult it is to design the system.

Let $\boldsymbol{G}$ be the transfer function of our plant. If it is a stable rational function, it admits a balanced realization such that the dimension of the state space is equal to the McMillan degree of $\boldsymbol{G}$ (see Corollary 2.4 and the Remark at the end of §3). If $\boldsymbol{G}$ is not rational and it has a balanced realization, then the state space is infinite-dimensional. In general, if the dimension of the state space is too large, it would be helpful to design another system that has a realization in a space of smaller dimension and whose properties do not differ much from the properties of the initial system. This is exactly the *model reduction problem.*

Suppose that $\boldsymbol{G}$ is the initial transfer function and $\boldsymbol{G}_\#$ is the transfer function of the realization designed to model the initial system. How do we measure the deviation of $\boldsymbol{G}_\#$ from $\boldsymbol{G}$? To be definite, consider the case of discrete time. If $\boldsymbol{G}$ is a stable transfer function, we can consider the norm of the linear operator that takes the input sequence $\{u_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathcal{K})$ to the output sequence $\{y_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathcal{E})$:

$$\sup\left\{\|\{y_n\}_{n\in\mathbb{Z}}\|_{\ell^2(\mathcal{E})} : \|\{u_n\}_{n\in\mathbb{Z}}\|_{\ell^2(\mathcal{K})} \leq 1\right\}.$$

Clearly, this norm is equal to $\|\boldsymbol{G}\|_{H^\infty(\mathcal{B}(\mathcal{K},\mathcal{E}))}$. This norm on the space of stable transfer functions is physically well-motivated. We say that systems with transfer functions $\boldsymbol{G}$ and $\boldsymbol{G}_\#$ have close performances if the norm $\|\boldsymbol{G} - \boldsymbol{G}_\#\|_{H^\infty(\mathcal{B}(\mathcal{K},\mathcal{E}))}$ is small.

Thus we arrive at the following problem. Given a stable transfer function $\boldsymbol{G} \in H^\infty(\mathcal{K}, \mathcal{E})$ and a positive integer m, find a rational transfer function $\boldsymbol{G}_\# \in H^\infty(\mathcal{K}, \mathcal{E})$ such that the McMillan degree of $\boldsymbol{G}_\#$ is at most m and $\|\boldsymbol{G} - \boldsymbol{G}_\#\|_{H^\infty(\mathcal{B}(\mathcal{K},\mathcal{E}))}$ is small.

Clearly, this is the problem of rational approximation of bounded analytic operator functions by rational operator functions of McMillan degree at most m. In particular, we can consider the problem of finding

$$\inf\left\{\|\boldsymbol{G} - \boldsymbol{G}_\#\|_{H^\infty(\mathcal{B}(\mathcal{K},\mathcal{E}))} : \ \deg \boldsymbol{G}_\# \le m\right\}. \tag{4.1}$$

We can mention here the results of Pekarskii [2] which estimate the rate of decay of the numbers in (4.1) in terms of properties of $\boldsymbol{G}$ (in the scalar case). We also mention here the results of Glover [2], which estimate the deviations in (4.1) in terms of the singular values of the block Hankel operator $\Gamma_{\boldsymbol{G}} = \{\hat{\boldsymbol{G}}(j+k)\}_{j,k\ge 0}$.

Another way to measure the closeness of two transfer functions is to consider the "Hankel norm" on the space of stable transfer functions:

$$\begin{aligned}\|\boldsymbol{G}\|_{\mathcal{H}} &\stackrel{\text{def}}{=} \|\Gamma_{\boldsymbol{G}}\| \\ &= \inf\left\{\|\boldsymbol{G} - Q\|_{L^\infty(\mathcal{B}(\mathcal{K},\mathcal{E}))} : \ Q \in L^\infty(\mathcal{B}(\mathcal{K},\mathcal{E})), \ \hat{Q}(j) = 0 \text{ for } j < 0\right\}.\end{aligned}$$

Recall (see §2) that the Hankel norm is the supremum of the energy of the future output $\{y_n\}_{n\ge 0}$ over the past inputs $\{u_n\}_{n<0}$ of energy at most 1. If we measure closeness in the Hankel norm, we say that the performances of systems with transfer functions $\boldsymbol{G}$ and $\boldsymbol{G}_\#$ are close if the systems on past input signals give close future outputs. This measure of performance is also physically well motivated. However, unlike the H^∞-norm the problem of rational approximation in the Hankel norm admits a satisfactory solution.

Suppose that we have a system with transfer function $\boldsymbol{G}$ and we want to approximate its performance in the Hankel norm by a system of order at most m. By the Adamyan–Arov–Krein theorem (and its generaliztion to the case of vectorial Hankel operators),

$$\inf\left\{\|\boldsymbol{G} - \boldsymbol{G}_\#\|_{\mathcal{H}} : \ \deg \boldsymbol{G}_\# \le n\right\} = s_m(\Gamma_{\boldsymbol{G}}) \tag{4.2}$$

(see §§4.1–4.3). One can find constructively a best approximation; see §4.1 for scalar functions $\boldsymbol{G}$ and §14.17 for matrix functions. We also mention here the results of §6.6 where the rate of decay of the numbers in (4.2) is estimated in terms of properties of the function $\boldsymbol{G}$ (in the scalar case).

5. Robust Stabilization

Not all physical systems are stable. However, unstable systems cause real problems. To stabilize an unstable system, one can use a *feedback controller*, which is also a plant.

Consider the feedback system represented by the diagram on Fig. 2. Here G is an unstable plant that we want to stabilize by a feedback controller K. u is an input signal. We also add to the system another input signal signal v, which can be considered as noise in the sensor that measures the output signal y. To be definite, we consider the case of continuous time.

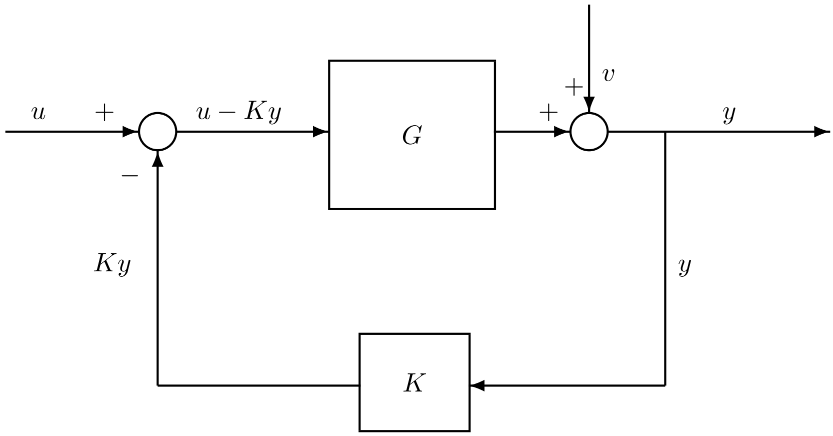

FIGURE 2. Feedback system

We make the following assumptions on G. We assume that the transfer function $\boldsymbol{G}$ of G is a *strictly proper* rational matrix function, i.e., $\boldsymbol{G}$ vanishes at ∞. K is a feedback controller that has to be designed to stabilize the system. We also assume that the transfer function $\boldsymbol{K}$ of K is a proper rational matrix function. The output signal y of the system goes to the feedback controller and the output signal Ky of the controller is subtracted from the input signal u of the system.

For the rest of the chapter we are going to use the following notation. If X is a plant, we denote by $\boldsymbol{X}$ its transfer function. If $\boldsymbol{X}$ is a proper rational matrix function, we denote by X a plant whose transfer function is $\boldsymbol{X}$.

Let us compute the transfer function from u to y. Put $v = \mathbb{O}$. Clearly, we have $G(u - Ky) = y$, and so

$$y = (I + GK)^{-1}Gu$$

or, in other words,

$$(\boldsymbol{\mathcal{L}}y)(z) = (\boldsymbol{I} + \boldsymbol{G}\boldsymbol{K})^{-1}\boldsymbol{G}(\boldsymbol{\mathcal{L}}u)(z),$$

i.e., the transfer function from u to y is $(\boldsymbol{I} + \boldsymbol{G}\boldsymbol{K})^{-1}\boldsymbol{G}$. Thus if $v = \mathbb{O}$, the feedback system is stable if the function $(\boldsymbol{I} + \boldsymbol{G}\boldsymbol{K})^{-1}\boldsymbol{G}$ is bounded in $\mathbb{C}^+$. It is easy to see that under our assumptions $\boldsymbol{I} + \boldsymbol{G}\boldsymbol{K}$ is a *nonsingular* rational matrix function, i.e., the function $(\boldsymbol{I}+\boldsymbol{GK})^{-1}$ exists as a rational matrix function. Indeed, at ∞ the determinant of $\boldsymbol{I}+\boldsymbol{GK}$ is 1.

Consider now the signal that goes to the plant G. It is

$$\begin{aligned} u - Ky &= u - K(I+GK)^{-1}Gu \\ &= u - (I+KG)^{-1}KGu = (I+KG)^{-1}u. \end{aligned}$$

If the transfer function $(\boldsymbol{I}+\boldsymbol{KG})^{-1}$ is unstable, the feedback system is internally unstable. Again, it is easy to see that the rational matrix function $(\boldsymbol{I}+\boldsymbol{KG})$ is nonsingular.

Let us compute now the transfer function from v to y. Put $u=\mathbb{O}$. We have $-GKy+v=y$, and so $y=(I+GK)^{-1}v$, i.e., the transfer function from v to y is $(\boldsymbol{I}+\boldsymbol{GK})^{-1}$. Finally, we compute the transfer function from v to the output signal from the controller K. This signal is $Ky=K(I+GK)^{-1}v$ (under the assumption that $u=\mathbb{O}$). Thus the transfer function is $\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{GK})^{-1}$.

The system is called *internally stable* if all four transfer functions $(\boldsymbol{I}+\boldsymbol{GK})^{-1}\boldsymbol{G}$, $(\boldsymbol{I}+\boldsymbol{KG})^{-1}$, $(\boldsymbol{I}+\boldsymbol{GK})^{-1}$, and $\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{GK})^{-1}$ are stable.

In §8 we obtain a formula that parametrizes all rational controllers K that internally stabilize the system. However, for practical purposes a controller K that stabilizes G does not necessarily work satisfactorily. A possible problem can occur due to the fact that a mathematical model does not describe the behavior of the system precisely. It is reasonable to make an assumption that the real plant is a small perturbation of our mathematical model. For a feedback controller to work satisfactorily, it has to be *robust*. In other words, it has to stabilize all plants whose transfer functions are small perturbations (say, in the L^∞ norm) of $\boldsymbol{G}$. Thus we arrive at the problem of *robust stabilization* (see Fig. 3). We assume that the real transfer function is $\boldsymbol{G}+\boldsymbol{\Delta G}$, it has the same number of poles in $\operatorname{clos}\mathbb{C}^+$ as $\boldsymbol{G}$ (in the sense of McMillan degree), and $\|\boldsymbol{\Delta G}\|_{L^\infty}\le\varepsilon$, where ε is a given positive number. Here

$$\|\boldsymbol{\Delta G}\|_{L^\infty} \stackrel{\text{def}}{=} \operatorname{ess\,sup}\{\|\boldsymbol{\Delta G}(\mathrm{i}y)\|:\ y\in\mathbb{R}\}.$$

We denote by $\mathcal{G}_\varepsilon$ the class of all such systems of the form $G+\Delta G$.

Theorem 5.1. *Suppose that $\boldsymbol{G}$ is a strictly proper rational matrix function and $\varepsilon>0$. A feedback controller K with a proper rational transfer function $\boldsymbol{K}$ stabilizes internally all systems in $\mathcal{G}_\varepsilon$ if and only if it internally stabilizes the system G and*

$$\|\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{GK})^{-1}\|_{L^\infty}<\varepsilon^{-1}.$$

To prove Theorem 5.1, we need results on coprime factorization of rational matrix functions, which are given in §6. We prove Theorem 5.1 in §7. In §8 we parametrize all feedback controllers with proper rational transfer functions that internally stabilize the system. Finally, in §9 we reduce the problem of robust stabilization to the Nehari problem.

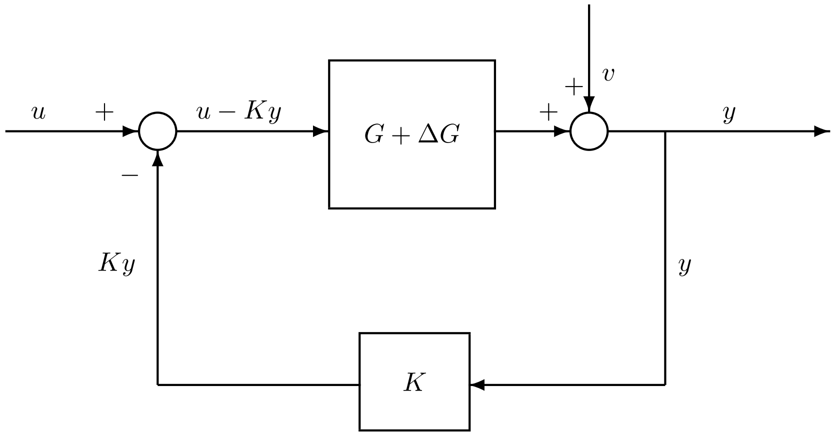

FIGURE 3. Robust stabilization

6. Coprime Factorization

In this section we obtain results on coprime factorization of rational matrix functions which will be used in §7 and §8.

Let $\boldsymbol{G}$ be a matrix rational function. We say that a factorization $\boldsymbol{G} = \boldsymbol{N}\boldsymbol{M}^{-1}$ is a *right coprime factorization over* $H^\infty(\mathbb{C}^+)$ if both $\boldsymbol{M}$ and $\boldsymbol{N}$ are rational matrix functions bounded in $\mathbb{C}^+$ (certainly, $\boldsymbol{M}$ is a square matrix function) and the matrix function $\begin{pmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{pmatrix}$ is left invertible in the space of bounded rational matrix functions, i.e., there exists a rational matrix function $\boldsymbol{R}$ in $H^\infty(\mathbb{C}^+)$ such that

$$\boldsymbol{R}\begin{pmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{pmatrix} = \boldsymbol{I}.$$

(A matrix function is said to belong to $H^\infty(\mathbb{C}^+)$ if all its entries belong to $H^\infty(\mathbb{C}^+)$.) Similarly, a factorization $\boldsymbol{G} = \boldsymbol{M}_\#^{-1}\boldsymbol{N}_\#$ is called a *left coprime factorization over* $H^\infty(\mathbb{C}^+)$ if both $\boldsymbol{M}_\#$ and $\boldsymbol{N}_\#$ are rational matrix functions in $H^\infty(\mathbb{C}^+)$ and the matrix function $\begin{pmatrix} \boldsymbol{N}_\# & \boldsymbol{M}_\# \end{pmatrix}$ is right invertible in the space of bounded rational matrix functions in $\mathbb{C}^+$, i.e., there exists a rational matrix function $\boldsymbol{R}$ in $H^\infty(\mathbb{C}^+)$ such that

$$\begin{pmatrix} \boldsymbol{N}_\# & \boldsymbol{M}_\# \end{pmatrix}\boldsymbol{R} = \boldsymbol{I}.$$

Finally, we say that $\boldsymbol{G} = \boldsymbol{N}\boldsymbol{M}^{-1} = \boldsymbol{M}_\#^{-1}\boldsymbol{N}_\#$ is a *doubly coprime factorization of* $\boldsymbol{G}$ over $H^\infty(\mathbb{C}^+)$ if there exist rational matrix functions $\boldsymbol{X}$, $\boldsymbol{Y}$,

$\boldsymbol{X}_{\#}$ and $\boldsymbol{Y}_{\#}$, in $H^\infty(\mathbb{C}^+)$ such that

$$\begin{pmatrix} \boldsymbol{X}_{\#} & -\boldsymbol{Y}_{\#} \\ -\boldsymbol{N}_{\#} & \boldsymbol{M}_{\#} \end{pmatrix} \begin{pmatrix} \boldsymbol{M} & \boldsymbol{Y} \\ \boldsymbol{N} & \boldsymbol{X} \end{pmatrix} = \boldsymbol{I}. \tag{6.1}$$

Similarly, one can introduce the notions of left coprime, right coprime, and doubly coprime factorizations over the space H^∞ of bounded analytic functions in $\mathbb{D}$.

Theorem 6.1. *Each rational matrix function has doubly coprime factorization over $H^\infty(\mathbb{C}^+)$.*

Proof. Let $\boldsymbol{G}$ be a rational matrix function. Without loss of generality we may assume that $\boldsymbol{G}$ is proper, i.e., has no pole at infinity (otherwise we can apply a suitable conformal map of $\mathbb{C}^+$ onto itself).

By Corollary 3.2, there exists a minimal realization $\{A, B, C, D\}$ of $\boldsymbol{G}$ (see the Remark at the end of §3), i.e., $\boldsymbol{G}(z) = C(z - A)^{-1}B + D$, where A, B, C, and D are finite matrices. Consider the corresponding linear system

$$\begin{cases} x'(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases}$$

We are going to use the following fact, which can be found in Gohberg, Lancaster, and Rodman [1], Theorem 6.5.1: if $\{A, B, C, D\}$ is a controllable realization, then there exists a matrix Ξ such that all eigenvalues of the matrix $A_{[\Xi]} \stackrel{\text{def}}{=} A + B\Xi$ have negative real parts (Ξ has size such that $B\Xi$ is a square matrix). In fact, by choosing a suitable matrix Ξ one can make the eigenvalues of $A + B\Xi$ arbitrary. Put $C_{[\Xi]} = C + D\Xi$ and define the function v by $v(t) = u(t) - \Xi x(t),\ t \in \mathbb{R}$. We have

$$\begin{cases} x'(t) = A_{[\Xi]}x(t) + Bv(t), \\ u(t) = \Xi x(t) + v(t), \\ y(t) = C_{[\Xi]}x(t) + Dv(t). \end{cases}$$

Denote by $\boldsymbol{M}$ the transfer function from v to u and by $\boldsymbol{N}$ the transfer function from v to y, i.e.,

$$\boldsymbol{M}(z) = \Xi(zI - A_{[\Xi]})^{-1}B + I \quad \text{and} \quad \boldsymbol{N}(z) = C_{[\Xi]}(zI - A_{[\Xi]})^{-1}B + D.$$

Clearly, both $\boldsymbol{M}$ and $\boldsymbol{N}$ are in $H^\infty(\mathbb{C}^+)$. Since $\boldsymbol{G}$ is the transfer function from u to y, it follows that $\boldsymbol{G}\boldsymbol{M} = \boldsymbol{N}$, i.e., $\boldsymbol{G} = \boldsymbol{N}\boldsymbol{M}^{-1}$.

Similarly, since the realization $\{A, B, C, D\}$ is observable, by Theorem 6.5.1 of Gohberg, Lancaster, and Rodman [1], there exists a matrix Ω such that the matrix $A^{[\Omega]} \stackrel{\text{def}}{=} A + \Omega C$ has all eigenvalues in the open left half-plane. Put $B^{[\Omega]} = B + \Omega D$ and consider the rational matrix functions

$$\boldsymbol{M}_{\#}(z) = C(zI - A^{[\Omega]})^{-1}\Omega + I \quad \text{and} \quad \boldsymbol{N}_{\#}(z) = C(zI - A^{[\Omega]})^{-1}B^{[\Omega]} + D.$$

If we pass to the transposed matrix function $\boldsymbol{G}^{\mathrm{t}}$, we deduce from above that $\boldsymbol{G} = \boldsymbol{M}_{\#}^{-1}\boldsymbol{N}_{\#}$.

We now define $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{X}_{\#}$, and $\boldsymbol{Y}_{\#}$ by

$$\boldsymbol{X}(z) = -C_{[\Xi]}(zI - A_{[\Xi]})^{-1}\Omega + I, \quad \boldsymbol{Y}(z) = -\Xi(zI - A_{[\Xi]})^{-1}\Omega,$$

$$\boldsymbol{X}_\#(z) = -\Xi\big(zI - A^{[\Omega]}\big)^{-1}B^{[\Omega]} + I, \quad \text{and} \quad \boldsymbol{Y}_\#(z) = -\Xi\big(zI - A^{[\Omega]}\big)^{-1}\Omega.$$

Let us prove (6.1). The equality $\boldsymbol{N}_\#\boldsymbol{M} = \boldsymbol{M}_\#\boldsymbol{N}$ is an obvious consequence of $\boldsymbol{G} = \boldsymbol{N}\boldsymbol{M}^{-1} = \boldsymbol{M}_\#^{-1}\boldsymbol{N}_\#$. Let us show that $\boldsymbol{X}_\#\boldsymbol{M} - \boldsymbol{Y}_\#\boldsymbol{N} = \boldsymbol{I}$. We have

$$\begin{aligned}
&(\boldsymbol{X}_\#\boldsymbol{M} - \boldsymbol{Y}_\#\boldsymbol{N})(z)\\
&= (-\Xi(zI - A^{[\Omega]})^{-1}B^{[\Omega]} + I)(\Xi(zI - A_{[\Xi]})^{-1}B + I)\\
&+ (\Xi(zI - A^{[\Omega]})^{-1}\Omega)(C_{[\Xi]}(zI - A_{[\Xi]})^{-1}B + D)\\
&= \Xi(zI - A^{[\Omega]})^{-1}(\Omega C_{[\Xi]} - B^{[\Omega]}\Xi)(zI - A_{[\Xi]})^{-1}B\\
&- \Xi(zI - A^{[\Omega]})^{-1}B^{[\Omega]} + \Xi(zI - A_{[\Xi]})^{-1}B\\
&+ I + (\Xi(zI - A^{[\Omega]})^{-1}\Omega)D\\
&= \Xi(zI - A^{[\Omega]})^{-1}(\Omega C - B\Xi)(zI - A_{[\Xi]})^{-1}B\\
&- \Xi(zI - A^{[\Omega]})^{-1}B + \Xi(zI - A_{[\Xi]})^{-1}B + I = I
\end{aligned}$$

Again, passing to the transposed functions, we get $\boldsymbol{M}_\#\boldsymbol{X} - \boldsymbol{N}_\#\boldsymbol{Y} = \boldsymbol{I}$. It remains to show that $\boldsymbol{X}_\#\boldsymbol{Y} = \boldsymbol{Y}_\#\boldsymbol{X}$. We have

$$\begin{aligned}
(\boldsymbol{X}_\#\boldsymbol{Y} - \boldsymbol{Y}_\#\boldsymbol{X})(z) &= (\Xi(zI - A^{[\Omega]})^{-1}B^{[\Omega]} - I)\Xi(zI - A_{[\Xi]})^{-1}\Omega\\
&- \Xi(zI - A^{[\Omega]})^{-1}\Omega(C_{[\Xi]}(zI - A_{[\Xi]})^{-1}\Omega - I)\\
&= \Xi(zI - A^{[\Omega]})^{-1}(B\Xi - \Omega C)(zI - A_{[\Xi]})^{-1}\Omega\\
&- \Xi(zI - A_{[\Xi]})^{-1}\Omega + \Xi(zI - A^{[\Omega]})^{-1}\Omega = \mathbb{O}. \quad \blacksquare
\end{aligned}$$

We conclude this section with the following elementary result, which will be used in the next section.

Theorem 6.2. *Let $\boldsymbol{G}$ be a rational matrix function that has no poles on $\mathbb{T}$. Let*

$$\boldsymbol{G} = \boldsymbol{N}\boldsymbol{M}^{-1} = \boldsymbol{M}_\#^{-1}\boldsymbol{N}_\#$$

be right coprime and left coprime factorizations of $\boldsymbol{G}$ over H^∞. Then

$$\operatorname{rank} H_{\boldsymbol{G}} = \operatorname{wind}\det\boldsymbol{M} = \operatorname{wind}\det\boldsymbol{M}_\#.$$

Proof. We prove the equality $\operatorname{rank} H_{\boldsymbol{G}} = \operatorname{wind}\det\boldsymbol{M}_\#$. To prove the other equality, it suffices to pass to transposition. Let us show that

$$\operatorname{rank} H_{\boldsymbol{G}} = \operatorname{rank} H_{\boldsymbol{M}_\#^{-1}}.$$

Indeed, $H_{\boldsymbol{G}} = H_{\boldsymbol{M}_\#^{-1}\boldsymbol{N}_\#}$ is the product of $H_{\boldsymbol{M}_\#}$ and multiplication by $\boldsymbol{N}_\#$, and so $\operatorname{rank} H_{\boldsymbol{G}} \le \operatorname{rank} H_{\boldsymbol{M}_\#^{-1}}$.

To prove the opposite inequality, we consider matrix functions $\boldsymbol{X}$ and $\boldsymbol{Y}$ in H^∞ such that $-\boldsymbol{N}_\#\boldsymbol{Y}+\boldsymbol{M}_\#\boldsymbol{X}=\boldsymbol{I}$. Then $-\boldsymbol{G}\boldsymbol{Y}+\boldsymbol{X}=\boldsymbol{M}_\#^{-1}$, and so

$$\operatorname{rank} H_{\boldsymbol{M}_\#^{-1}}=\operatorname{rank} H_{\boldsymbol{G}\boldsymbol{Y}}\le\operatorname{rank} H_{\boldsymbol{G}}.$$

Let $\boldsymbol{B}$ be a Blaschke–Potapov product of degree $\operatorname{rank} H_{\boldsymbol{M}_\#^{-1}}$ such that $\boldsymbol{M}_\#^{-1}\boldsymbol{B}\in H^\infty$ (see Theorem 2.5.2). Clearly, $\det\boldsymbol{M}_\#^{-1}$ has no zeros in $\operatorname{clos}\mathbb{D}$. It is easy to see from the construction of $\boldsymbol{B}$ given in the proof of Lemma 2.5.1 that $\boldsymbol{M}_\#^{-1}\boldsymbol{B}$ is invertible in H^∞. Thus

$$0=\operatorname{wind}\det\boldsymbol{M}_\#^{-1}\boldsymbol{B}=\operatorname{wind}\det\boldsymbol{B}-\operatorname{wind}\det\boldsymbol{M}_\#.$$

The result follows from the fact that

$$\operatorname{wind}\det\boldsymbol{B}=\deg\boldsymbol{B},$$

which is an immediate consequence of the definition of Blaschke–Potapov products (see §2.5). ■

7. Proof of Theorem 5.1

For convenience, in this section we assume that we deal with discrete time systems. In other words, a rational transfer function is stable if it belongs to the space H^∞ (i.e., has no poles in $\operatorname{clos}\mathbb{D}$). We say that a rational matrix function $\boldsymbol{K}$ internally stabilizes $\boldsymbol{G}$ if the four transfer functions $(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1}$, $(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1}\boldsymbol{G}$, $\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1}$, and $(\boldsymbol{I}+\boldsymbol{K}\boldsymbol{G})^{-1}$ are stable.

Clearly, with the help of a conformal map from $\mathbb{C}^+$ onto $\mathbb{D}$, we can reduce Theorem 5.1 to the following result.

Theorem 7.1. *Suppose that $\boldsymbol{G}$ is a rational matrix function and $\varepsilon>0$. Let $\boldsymbol{\mathcal{G}}_\varepsilon$ be the set of rational matrix functions of the form $\boldsymbol{G}+\boldsymbol{\Delta G}$ such that $\|\boldsymbol{\Delta G}\|_{L^\infty(\mathbb{T})}\le\varepsilon$, and $\boldsymbol{G}$ and $\boldsymbol{G}+\boldsymbol{\Delta G}$ have the same number of poles in $\operatorname{clos}\mathbb{D}$ (in the sense of McMillan degree). A rational matrix function $\boldsymbol{K}$ internally stabilizes all transfer functions in $\boldsymbol{\mathcal{G}}_\varepsilon$ if and only if it internally stabilizes $\boldsymbol{G}$ and $\|\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1}\|_{L^\infty}<\varepsilon^{-1}$.*

Let ρ be a number such that $\rho>1$ and the function $\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K}$ has no zeros and no poles in $\{\zeta\in\mathbb{C}:\ 1<|\zeta|\le\rho\}$ while the functions $\boldsymbol{G}$ and $\boldsymbol{K}$ have no poles in $\{\zeta\in\mathbb{C}:\ 1<|\zeta|\le\rho\}$.

Lemma 7.2. *$\boldsymbol{K}$ stabilizes $\boldsymbol{G}$ internally if and only if*

$$\operatorname{wind}\det\big((\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})(z/\rho)\big)=-\big(\deg\mathbb{P}_-\boldsymbol{G}(z/\rho)+\deg\mathbb{P}_-\boldsymbol{K}(z/\rho)\big).\quad(7.1)$$

Clearly, it suffices to prove (7.1) for some $\rho>1$ sufficiently close to 1. Recall that $\deg\mathbb{P}_-\boldsymbol{G}(z/\rho)=\operatorname{rank} H_{\boldsymbol{G}(z/\rho)}$ and $\deg\mathbb{P}_-\boldsymbol{K}(z/\rho)=\operatorname{rank} H_{\boldsymbol{K}(z/\rho)}$.

Proof of Lemma 7.2. Consider a left coprime factorization $\boldsymbol{G}=\boldsymbol{M}_\#^{-1}\boldsymbol{N}_\#$ of $\boldsymbol{G}$ and a right coprime factorization $\boldsymbol{K}=\boldsymbol{U}\boldsymbol{V}^{-1}$ of

$\boldsymbol{K}$. By changing ρ if necessary, we can assume that the functions $\boldsymbol{M}_{\#}$, $\boldsymbol{N}_{\#}$, $\boldsymbol{U}$, and $\boldsymbol{V}$ have no poles in the disk $\{\zeta \in \mathbb{C} : |\zeta| \le \rho\}$. We have

$$\boldsymbol{I} + \boldsymbol{G}\boldsymbol{K} = \boldsymbol{M}_{\#}^{-1}(\boldsymbol{M}_{\#}\boldsymbol{V} + \boldsymbol{N}_{\#}\boldsymbol{U})\boldsymbol{V}^{-1}.$$

Hence,

$$\begin{aligned} \operatorname{wind}\det\big((\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})(z/\rho)\big) &= \operatorname{wind}\det\big((\boldsymbol{M}_{\#}\boldsymbol{V}+\boldsymbol{N}_{\#}\boldsymbol{U})(z/\rho)\big) \\ &- \operatorname{wind}\det(\boldsymbol{M}_{\#}(z/\rho)) - \operatorname{wind}\det(\boldsymbol{V}(z/\rho)). \end{aligned}$$

By the argument principle, $\operatorname{wind}\det(\boldsymbol{M}_{\#}(z/\rho))$ is the number of zeros of $\det(\boldsymbol{M}_{\#}(z/\rho))$ in $\{\zeta \in \mathbb{C} : |\zeta| \le \rho\}$ (counted with multiplicities). By Theorem 6.2,

$$\operatorname{wind}\det(\boldsymbol{M}_{\#}(z/\rho)) = \operatorname{rank} H_{\boldsymbol{G}(z/\rho)}$$

and

$$\operatorname{wind}\det(\boldsymbol{V}(z/\rho)) = \operatorname{rank} H_{\boldsymbol{K}(z/\rho)}.$$

Hence, it remains to show that the system is internally stable if and only if $\operatorname{wind}\det\big((\boldsymbol{M}_{\#}\boldsymbol{V} + \boldsymbol{N}_{\#}\boldsymbol{U})(z/\rho)\big) = 0$. In other words, this is equivalent to that fact that $\big((\boldsymbol{M}_{\#}\boldsymbol{V} + \boldsymbol{N}_{\#}\boldsymbol{U})(z/\rho)\big)^{-1}$ is stable. (Obviously, $(\boldsymbol{M}_{\#}\boldsymbol{V} + \boldsymbol{N}_{\#}\boldsymbol{U})$ is nonsingular if and only if $\boldsymbol{I} + \boldsymbol{G}\boldsymbol{K}$ is.) Note also that $(\boldsymbol{M}_{\#}\boldsymbol{V} + \boldsymbol{N}_{\#}\boldsymbol{U})^{-1}$ is stable if and only if $\big((\boldsymbol{M}_{\#}\boldsymbol{V} + \boldsymbol{N}_{\#}\boldsymbol{U})(z/\rho)\big)^{-1}$ is stable for some $\rho > 1$.

It is easy to see that the following equality holds

$$\boldsymbol{K}(\boldsymbol{I} + \boldsymbol{G}\boldsymbol{K})^{-1}\boldsymbol{G} = \boldsymbol{I} - (\boldsymbol{I} + \boldsymbol{K}\boldsymbol{G})^{-1}.$$

Hence, $(\boldsymbol{I} + \boldsymbol{K}\boldsymbol{G})^{-1}$ is stable if and only if $\boldsymbol{K}(\boldsymbol{I} + \boldsymbol{G}\boldsymbol{K})^{-1}\boldsymbol{G}$ is.

We have

$$(\boldsymbol{I} + \boldsymbol{G}\boldsymbol{K})^{-1} = \boldsymbol{V}(\boldsymbol{M}_{\#}\boldsymbol{V} + \boldsymbol{N}_{\#}\boldsymbol{U})^{-1}\boldsymbol{M}_{\#}, \tag{7.2}$$

$$(\boldsymbol{I} + \boldsymbol{G}\boldsymbol{K})^{-1}\boldsymbol{G} = \boldsymbol{V}(\boldsymbol{M}_{\#}\boldsymbol{V} + \boldsymbol{N}_{\#}\boldsymbol{U})^{-1}\boldsymbol{N}_{\#}, \tag{7.3}$$

$$\boldsymbol{K}(\boldsymbol{I} + \boldsymbol{G}\boldsymbol{K})^{-1} = \boldsymbol{U}(\boldsymbol{M}_{\#}\boldsymbol{V} + \boldsymbol{N}_{\#}\boldsymbol{U})^{-1}\boldsymbol{M}_{\#}, \tag{7.4}$$

and

$$\boldsymbol{K}(\boldsymbol{I} + \boldsymbol{G}\boldsymbol{K})^{-1}\boldsymbol{G} = \boldsymbol{U}(\boldsymbol{M}_{\#}\boldsymbol{V} + \boldsymbol{N}_{\#}\boldsymbol{U})^{-1}\boldsymbol{N}_{\#}. \tag{7.5}$$

Clearly, if $(\boldsymbol{M}_{\#}\boldsymbol{V} + \boldsymbol{N}_{\#}\boldsymbol{U})^{-1}$ is stable, then all functions in (7.2)–(7.5) are stable. Conversely, suppose that the functions in (7.2)–(7.5) are stable. Let $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{\Phi}$, and $\boldsymbol{\Psi}$ are stable rational matrix functions such that

$$-\boldsymbol{N}_{\#}\boldsymbol{Y} + \boldsymbol{M}_{\#}\boldsymbol{X} = \boldsymbol{I} \quad \text{and} \quad \boldsymbol{\Phi}\boldsymbol{U} + \boldsymbol{\Psi}\boldsymbol{V} = \boldsymbol{I}.$$

We have

$$\boldsymbol{\Psi}(\boldsymbol{I} + \boldsymbol{G}\boldsymbol{K})^{-1} + \boldsymbol{\Phi}\boldsymbol{K}(\boldsymbol{I} + \boldsymbol{G}\boldsymbol{K})^{-1} = (\boldsymbol{M}_{\#}\boldsymbol{V} + \boldsymbol{N}_{\#}\boldsymbol{U})^{-1}\boldsymbol{M}_{\#} \in H^\infty$$

and

$$\boldsymbol{\Psi}(\boldsymbol{I} + \boldsymbol{G}\boldsymbol{K})^{-1}\boldsymbol{G} + \boldsymbol{\Phi}\boldsymbol{K}(\boldsymbol{I} + \boldsymbol{G}\boldsymbol{K})^{-1}\boldsymbol{G} = (\boldsymbol{M}_{\#}\boldsymbol{V} + \boldsymbol{N}_{\#}\boldsymbol{U})^{-1}\boldsymbol{N}_{\#} \in H^\infty.$$

Hence,

$$
\begin{aligned}
&-(\boldsymbol{M}_{\#}\boldsymbol{V}+\boldsymbol{N}_{\#}\boldsymbol{U})^{-1}\boldsymbol{N}_{\#}\boldsymbol{Y}+(\boldsymbol{M}_{\#}\boldsymbol{V}+\boldsymbol{N}_{\#}\boldsymbol{U})^{-1}\boldsymbol{M}_{\#}\boldsymbol{X}\\
&=(\boldsymbol{M}_{\#}\boldsymbol{V}+\boldsymbol{N}_{\#}\boldsymbol{U})^{-1}\in H^{\infty},
\end{aligned}
$$

which completes the proof. ∎

Proof of Theorem 7.1. Let $\boldsymbol{G}_1\in\boldsymbol{\mathcal{G}}_\varepsilon$ and $\boldsymbol{G}_1=\boldsymbol{G}+\boldsymbol{\Delta G}$. We have

$$
\begin{aligned}
\boldsymbol{I}+\boldsymbol{G}_1\boldsymbol{K} &= \boldsymbol{I}+\boldsymbol{G}\boldsymbol{K}+(\boldsymbol{\Delta G})\boldsymbol{K}\\
&= \big(\boldsymbol{I}+(\boldsymbol{\Delta G})\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1}\big)(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K}). \qquad (7.6)
\end{aligned}
$$

Suppose that $\boldsymbol{K}$ internally stabilizes $\boldsymbol{G}$, $\rho>1$, and ρ is sufficiently close to 1. We have

$$
\begin{aligned}
\operatorname{wind}\det\big((\boldsymbol{I}+\boldsymbol{G}_1\boldsymbol{K})(z/\rho)\big) &= \operatorname{wind}\det\big(\boldsymbol{I}+(\boldsymbol{\Delta G})\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1}(z/\rho)\big)\\
&+ \operatorname{wind}\det\big((\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})(z/\rho)\big). \qquad (7.7)
\end{aligned}
$$

By Lemma 7.2,

$$
\operatorname{wind}\det\big((\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})(z/\rho)\big)=-\big(\deg\mathbb{P}_-\boldsymbol{G}(z/\rho)+\deg\mathbb{P}_-\boldsymbol{K}(z/\rho)\big).
$$

The internal stability of the perturbed system with plant $\boldsymbol{G}_1$ and controller $\boldsymbol{K}$ is equivalent to

$$
\operatorname{wind}\det\big((\boldsymbol{I}+\boldsymbol{G}_1\boldsymbol{K})(z/\rho)\big)=-\big(\deg\mathbb{P}_-\boldsymbol{G}_1(z/\rho)+\deg\mathbb{P}_-\boldsymbol{K}(z/\rho)\big).
$$

Since $\boldsymbol{G}$ and $\boldsymbol{G}_1$ have the same number of unstable poles, the internal stability of the perturbed system is equivalent to the equality

$$
\operatorname{wind}\det\big((\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})(z/\rho)\big)=\operatorname{wind}\det\big((\boldsymbol{I}+\boldsymbol{G}_1\boldsymbol{K})(z/\rho)\big).
$$

By (7.7), this is equivalent to

$$
\operatorname{wind}\det\big(\boldsymbol{I}+(\boldsymbol{\Delta G})\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1}(z/\rho)\big)=0. \qquad (7.8)
$$

Suppose that $\|\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1}\|_{L^\infty}<\varepsilon^{-1}$ and $\|\boldsymbol{\Delta G}\|_{L^\infty}\le\varepsilon$. Clearly, $\big(\boldsymbol{I}+t(\boldsymbol{\Delta G})\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1}\big)(\zeta)$ is invertible for all $\zeta\in\mathbb{T}$ and all $t\in[0,1]$ and the function $\det\big(\boldsymbol{I}+t(\boldsymbol{\Delta G})\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1}\big):\mathbb{T}\to\mathbb{C}\setminus\{0\}$ depends on t continuously. It follows that $\operatorname{wind}\det\big(\boldsymbol{I}+t(\boldsymbol{\Delta G})\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1}\big)$ does not depend on t. Thus $\operatorname{wind}\det\big(\boldsymbol{I}+(\boldsymbol{\Delta G})\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1}\big)=0$. This implies (7.8) for ρ sufficiently close to 1.

Conversely, suppose that $\|\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1}\|_{L^\infty}\ge\varepsilon^{-1}$. Let us find a perturbation $\boldsymbol{\Delta G}$ such that $\boldsymbol{G}_1\stackrel{\text{def}}{=}\boldsymbol{G}+\boldsymbol{\Delta G}$ has the same number poles in $\operatorname{clos}\mathbb{D}$ as $\boldsymbol{G}$ and $\|\boldsymbol{\Delta G}\|_{L^\infty}\le\varepsilon$, but the system with plant $\boldsymbol{G}_1$ and controller $\boldsymbol{K}$ is not internally stable. Let ζ_0 be a point on $\mathbb{T}$ such that $\sigma\stackrel{\text{def}}{=}\|\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1}(\zeta_0)\|\ge\varepsilon^{-1}$. Let f and g be unit vectors such that $\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1}(\zeta_0)f=\sigma g$. Consider the following constant matrix function $\boldsymbol{\Delta G}$:

$$
\boldsymbol{\Delta G}(\zeta)x=-\sigma^{-1}(x,g)f.
$$

Clearly, $\|\boldsymbol{\Delta G}\|=\sigma^{-1}\le\varepsilon$. It is easy to see that

$$
\big(\big(\boldsymbol{I}+(\boldsymbol{\Delta G})\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1}\big)(\zeta_0)\big)f=\mathbb{O},
$$

and so $\big(\boldsymbol{I} + (\boldsymbol{\Delta G})\boldsymbol{K}(\boldsymbol{I} + \boldsymbol{GK})^{-1}\big)$ is not invertible at ζ_0. It now follows from (7.6) that the transfer function $\boldsymbol{I} + \boldsymbol{G}_1\boldsymbol{K}$ is unstable. ■

8. Parametrization of Stabilizing Controllers

Now we are ready to parametrize all stabilizing controllers whose transfer functions are *proper* rational matrix functions (i.e., rational matrix functions that have no pole at infinity). Suppose that the transfer function $\boldsymbol{G}$ of the plant G is a rational matrix function that is *strictly proper* (i.e., is zero at infinity). By Lemma 6.1, $\boldsymbol{G}$ admits a doubly coprime factorization, i.e.,

$$\boldsymbol{G} = \boldsymbol{N}\boldsymbol{M}^{-1} = \boldsymbol{M}_{\#}^{-1}\boldsymbol{N}_{\#}, \quad \boldsymbol{N}, \boldsymbol{M}, \boldsymbol{N}_{\#}, \boldsymbol{M}_{\#} \in H^{\infty}(\mathbb{C}^{+}),$$

and (6.1) holds for rational matrix functions $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{X}_{\#}$, and $\boldsymbol{Y}_{\#}$ in $H^{\infty}(\mathbb{C}^{+})$. We are interested in feedback controllers K with proper rational transfer function $\boldsymbol{K}$ that internally stabilizes the system. In this case we say that the transfer function $\boldsymbol{K}$ internally stabilizes the system.

Theorem 8.1. *A proper rational matrix function $\boldsymbol{K}$ internally stabilizes the system if and only if*

$$\boldsymbol{K} = (\boldsymbol{MQ} - \boldsymbol{Y})(\boldsymbol{X} - \boldsymbol{NQ})^{-1} = (\boldsymbol{X}_{\#} - \boldsymbol{QN}_{\#})^{-1}(\boldsymbol{QM}_{\#} - \boldsymbol{Y}_{\#}), \quad (8.1)$$

where $\boldsymbol{Q}$ is a matrix function in $H^{\infty}(\mathbb{C}^{+})$.

Clearly, the size of the matrix function $\boldsymbol{Q}$ is such that (8.1) makes sense.

Lemma 8.2. *Suppose that $\boldsymbol{K}$ is a proper rational matrix function and*

$$\boldsymbol{K} = \boldsymbol{U}\boldsymbol{V}^{-1} = \boldsymbol{V}_{\#}^{-1}\boldsymbol{U}_{\#}$$

are left and right coprime factorizations of $\boldsymbol{K}$. The following are equivalent:

(i) *$\boldsymbol{K}$ stabilizes $\boldsymbol{G}$;*

(ii) *the matrix function* $\begin{pmatrix} \boldsymbol{M} & -\boldsymbol{U} \\ \boldsymbol{N} & \boldsymbol{V} \end{pmatrix}$ *is invertible in $H^{\infty}(\mathbb{C}^{+})$;*

(iii) *the matrix function* $\begin{pmatrix} \boldsymbol{V}_{\#} & \boldsymbol{U}_{\#} \\ -\boldsymbol{N}_{\#} & \boldsymbol{M}_{\#} \end{pmatrix}$ *is invertible in $H^{\infty}(\mathbb{C}^{+})$.*

To prove Lemma 8.2, we need one more lemma. Clearly, our system is equivalent to the system in Fig. 4.

We need the following fact.

Lemma 8.3. *The system in Fig. 4 is internally stable if and only if the transfer function from* $\begin{pmatrix} u \\ v \end{pmatrix}$ *to* $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ *is stable.*

Proof. Suppose that the transfer function from $\begin{pmatrix} u \\ v \end{pmatrix}$ to $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ is stable. We have $y = N\xi + v$ and $w = U\eta + u$, which implies the internal stability of the system in Fig. 3.

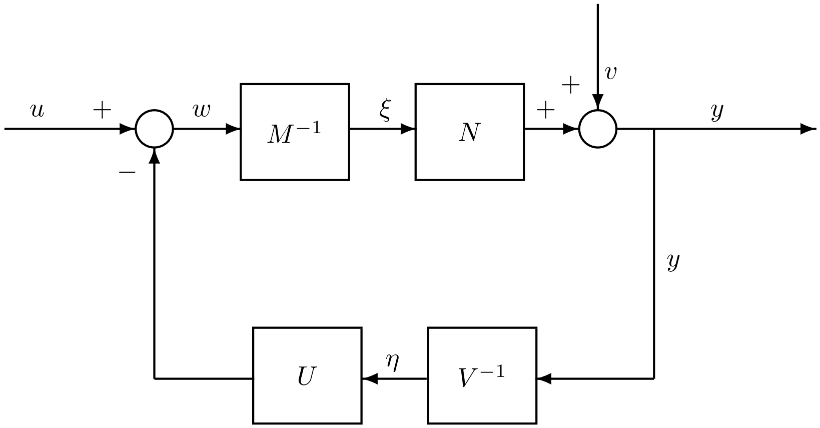

FIGURE 4.

Conversely, suppose that the the transfer function from $\begin{pmatrix} u \\ v \end{pmatrix}$ to $\begin{pmatrix} w \\ y \end{pmatrix}$ is stable. Since by (6.1),

$$\boldsymbol{X}_{\#}\boldsymbol{M} - \boldsymbol{Y}_{\#}\boldsymbol{N} = \boldsymbol{I},$$

we have $\xi = X_{\#}M\xi - Y_{\#}N\xi$. It is clear from Fig. 4 that $M\xi = w$ and $N\xi = y - v$, and so

$$\xi = X_{\#}u - Y_{\#}y + Y_{\#}v.$$

Hence, the transfer function from $\begin{pmatrix} u \\ v \end{pmatrix}$ to ξ is stable. The proof of the fact that the transfer function from $\begin{pmatrix} u \\ v \end{pmatrix}$ to η is stable is the same. ■

Proof of Lemma 8.2. Clearly, it suffices to prove the equivalence of (i) and (ii). The equivalence of (i) and (iii) would follow by passing to transposition.

Obviously, $\begin{pmatrix} \boldsymbol{M} & -\boldsymbol{U} \\ \boldsymbol{N} & \boldsymbol{V} \end{pmatrix}$ is invertible in $H^\infty(\mathbb{C}^+)$ if and only if $\begin{pmatrix} \boldsymbol{M} & \boldsymbol{U} \\ -\boldsymbol{N} & \boldsymbol{V} \end{pmatrix}$ is invertible in $H^\infty(\mathbb{C}^+)$. Let us first show that the matrix function $\begin{pmatrix} \boldsymbol{M} & \boldsymbol{U} \\ -\boldsymbol{N} & \boldsymbol{V} \end{pmatrix}$ is nonsingular, i.e., its inverse exists as a rational matrix function.

We have

$$\begin{pmatrix} \boldsymbol{M} & \boldsymbol{U} \\ -\boldsymbol{N} & \boldsymbol{V} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{K} \\ -\boldsymbol{G} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{M} & \mathbb{O} \\ \mathbb{O} & \boldsymbol{V} \end{pmatrix}.$$

Clearly, $\begin{pmatrix} \boldsymbol{M} & \mathbb{O} \\ \mathbb{O} & \boldsymbol{V} \end{pmatrix}$ is nonsingular. On the other hand, since $\boldsymbol{G}$ is strictly proper, the determinant of $\begin{pmatrix} \boldsymbol{I} & \boldsymbol{K} \\ -\boldsymbol{G} & \boldsymbol{I} \end{pmatrix}$ at ∞ is equal to 1, which implies that $\begin{pmatrix} \boldsymbol{I} & \boldsymbol{K} \\ -\boldsymbol{G} & \boldsymbol{I} \end{pmatrix}$ is also nonsingular.

It is easy to see from Fig. 4 that

$$\begin{pmatrix} M & U \\ -N & V \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Since $\begin{pmatrix} \boldsymbol{M} & \mathbb{O} \\ \mathbb{O} & \boldsymbol{V} \end{pmatrix}$ is nonsingular, we can write

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} M & U \\ -N & V \end{pmatrix}^{-1} \begin{pmatrix} u \\ v \end{pmatrix}.$$

The result now follows from Lemma 8.3. ■

Proof of Theorem 8.1. First of all, it is easy to verify that

$$(\boldsymbol{M}\boldsymbol{Q} - \boldsymbol{Y})(\boldsymbol{X} - \boldsymbol{N}\boldsymbol{Q})^{-1} = (\boldsymbol{X}_\# - \boldsymbol{Q}\boldsymbol{N}_\#)^{-1}(\boldsymbol{Q}\boldsymbol{M}_\# - \boldsymbol{Y}_\#).$$

Suppose that $\boldsymbol{K}$ is given by (8.1) for some stable matrix function $\boldsymbol{Q}$. Let us show that K stabilizes the system. We have

$$\begin{aligned}
(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1} &= \left(\boldsymbol{I} + \boldsymbol{N}\boldsymbol{M}^{-1}(\boldsymbol{M}\boldsymbol{Q} - \boldsymbol{Y})(\boldsymbol{X} - \boldsymbol{N}\boldsymbol{Q})^{-1}\right)^{-1} \\
&= \left(\boldsymbol{X} - \boldsymbol{N}\boldsymbol{Q} + \boldsymbol{N}\boldsymbol{M}^{-1}(\boldsymbol{M}\boldsymbol{Q} - \boldsymbol{Y}))(\boldsymbol{X} - \boldsymbol{N}\boldsymbol{Q})^{-1}\right)^{-1} \\
&= (\boldsymbol{X} - \boldsymbol{N}\boldsymbol{Q})(\boldsymbol{X} - \boldsymbol{N}\boldsymbol{M}^{-1}\boldsymbol{Y})^{-1} \\
&= (\boldsymbol{X} - \boldsymbol{N}\boldsymbol{Q})(\boldsymbol{X} - \boldsymbol{M}_\#^{-1}\boldsymbol{N}_\#\boldsymbol{Y})^{-1}.
\end{aligned}$$

It follows from (6.1) that $\boldsymbol{N}_\#\boldsymbol{Y} = \boldsymbol{M}_\#\boldsymbol{X} - \boldsymbol{I}$. Hence,

$$(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1} = (\boldsymbol{X} - \boldsymbol{N}\boldsymbol{Q})\boldsymbol{M}_\# \in H^\infty(\mathbb{C}^+).$$

It follows that

$$\begin{aligned}
\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1} &= (\boldsymbol{M}\boldsymbol{Q} - \boldsymbol{Y})(\boldsymbol{X} - \boldsymbol{N}\boldsymbol{Q})^{-1}(\boldsymbol{X} - \boldsymbol{N}\boldsymbol{Q})\boldsymbol{M}_\# \\
&= (\boldsymbol{M}\boldsymbol{Q} - \boldsymbol{Y})\boldsymbol{M}_\# \in H^\infty(\mathbb{C}^+) \qquad (8.2)
\end{aligned}$$

and

$$(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1}\boldsymbol{G} = (\boldsymbol{X} - \boldsymbol{N}\boldsymbol{Q})\boldsymbol{M}_\#\boldsymbol{M}_\#^{-1}\boldsymbol{N}_\# = (\boldsymbol{X} - \boldsymbol{N}\boldsymbol{Q})\boldsymbol{N}_\# \in H^\infty(\mathbb{C}^+).$$

Finally,

$$\begin{aligned}
(\boldsymbol{I}+\boldsymbol{K}\boldsymbol{G})^{-1} &= \left(\boldsymbol{I} + (\boldsymbol{X}_\# - \boldsymbol{Q}\boldsymbol{N}_\#)^{-1}(\boldsymbol{Q}\boldsymbol{M}_\# - \boldsymbol{Y}_\#)\boldsymbol{M}_\#^{-1}\boldsymbol{N}_\#\right)^{-1} \\
&= \left(\boldsymbol{X}_\# - \boldsymbol{Q}\boldsymbol{N}_\# + (\boldsymbol{Q}\boldsymbol{M}_\# - \boldsymbol{Y}_\#)\boldsymbol{M}_\#^{-1}\boldsymbol{N}_\#\right)^{-1}(\boldsymbol{X}_\# - \boldsymbol{Q}\boldsymbol{N}_\#) \\
&= (\boldsymbol{X}_\# - \boldsymbol{Y}_\#\boldsymbol{M}_\#^{-1}\boldsymbol{N}_\#)^{-1}(\boldsymbol{X}_\# - \boldsymbol{Q}\boldsymbol{N}_\#) \\
&= (\boldsymbol{X}_\# - \boldsymbol{Y}_\#\boldsymbol{N}\boldsymbol{M}^{-1})^{-1}(\boldsymbol{X}_\# - \boldsymbol{Q}\boldsymbol{N}_\#).
\end{aligned}$$

It follows from (6.1) that $\boldsymbol{Y}_{\#}\boldsymbol{N} = \boldsymbol{X}_{\#}\boldsymbol{M} - \boldsymbol{I}$. Hence,

$$(\boldsymbol{I} + \boldsymbol{K}\boldsymbol{G})^{-1} = \boldsymbol{M}(\boldsymbol{X}_{\#} - \boldsymbol{Q}\boldsymbol{N}_{\#}) \in H^{\infty}(\mathbb{C}^{+}).$$

Thus K makes the system internally stable.

Let us establish the converse. Suppose that K stabilizes internally the system. Let $\boldsymbol{K} = \boldsymbol{U}\boldsymbol{V}^{-1}$ be a right coprime factorization of K. Put $R = N_{\#}U + M_{\#}V$. It follows from (6.1) that

$$\begin{pmatrix} \boldsymbol{X}_{\#} & -\boldsymbol{Y}_{\#} \\ -\boldsymbol{N}_{\#} & \boldsymbol{M}_{\#} \end{pmatrix} \begin{pmatrix} \boldsymbol{M} & -\boldsymbol{U} \\ \boldsymbol{N} & \boldsymbol{V} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & -(\boldsymbol{X}_{\#}\boldsymbol{U} + \boldsymbol{Y}_{\#}\boldsymbol{V}) \\ \mathbb{O} & \boldsymbol{R} \end{pmatrix}. \tag{8.3}$$

By (6.1) and Lemma 8.2, both matrix functions on the left of (8.3) are invertible in $H^{\infty}(\mathbb{C}^{+})$. It follows that $\boldsymbol{R}^{-1} \in H^{\infty}(\mathbb{C}^{+})$. Put $\boldsymbol{Q} = (\boldsymbol{X}_{\#}\boldsymbol{U} + \boldsymbol{Y}_{\#}\boldsymbol{V})\boldsymbol{R}^{-1} \in H^{\infty}(\mathbb{C}^{+})$. Then we have from (8.3)

$$\begin{pmatrix} \boldsymbol{X}_{\#} & -\boldsymbol{Y}_{\#} \\ -\boldsymbol{N}_{\#} & \boldsymbol{M}_{\#} \end{pmatrix} \begin{pmatrix} \boldsymbol{M} & -\boldsymbol{U} \\ \boldsymbol{N} & \boldsymbol{V} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{Q}\boldsymbol{R} \\ \mathbb{O} & \boldsymbol{R} \end{pmatrix}.$$

Multiplying this equality on the left by $\begin{pmatrix} \boldsymbol{M} & \boldsymbol{Y} \\ \boldsymbol{N} & \boldsymbol{X} \end{pmatrix}$, we find from (6.1)

$$\begin{pmatrix} \boldsymbol{M} & -\boldsymbol{U} \\ \boldsymbol{N} & \boldsymbol{V} \end{pmatrix} = \begin{pmatrix} \boldsymbol{M} & \boldsymbol{Y} \\ \boldsymbol{N} & \boldsymbol{X} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{Q}\boldsymbol{R} \\ \mathbb{O} & \boldsymbol{R} \end{pmatrix}.$$

Thus

$$\begin{pmatrix} -\boldsymbol{U} \\ \boldsymbol{V} \end{pmatrix} = \begin{pmatrix} \boldsymbol{M} & \boldsymbol{Y} \\ \boldsymbol{N} & \boldsymbol{X} \end{pmatrix} \begin{pmatrix} -\boldsymbol{Q}\boldsymbol{R} \\ \boldsymbol{R} \end{pmatrix} = \begin{pmatrix} (\boldsymbol{Y} - \boldsymbol{M}\boldsymbol{Q})\boldsymbol{R} \\ (\boldsymbol{X} - \boldsymbol{N}\boldsymbol{Q})\boldsymbol{R} \end{pmatrix},$$

and so

$$\boldsymbol{K} = \boldsymbol{U}\boldsymbol{V}^{-1} = (\boldsymbol{M}\boldsymbol{Q} - \boldsymbol{Y})(\boldsymbol{X} - \boldsymbol{N}\boldsymbol{Q})^{-1},$$

which completes the proof. ■

9. Solution of the Robust Stabilization Problem. The Model Matching Problem

In this section we reduce the robust stabilization problem to the so-called model matching problem. It turns out that in the case $\boldsymbol{G}$ has no poles on the imaginary axis, the model matching problem reduces to the Nehari problem.

Consider the general model matching problem given in Fig. 5. Here T_1, T_2, and T_3 are given stable plants. The problem is to design a stable plant Q that minimizes the energy of the output z in the worst possible case under the assumption that the energy of the input w is at most 1. In other words, the problem is to design a stable plant Q such that the cascade system T_3QT_2 models the performance of T_1 as precisely as possible.

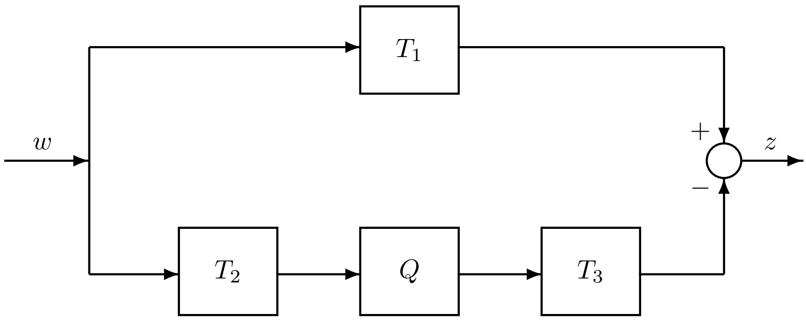

FIGURE 5. Model matching problem

Consider the transfer functions $\boldsymbol{T}_1$, $\boldsymbol{T}_2$, $\boldsymbol{T}_3$, and $\boldsymbol{Q}$ of T_1, T_2, T_3, and Q. Clearly, the model matching problem is to find a stable transfer function $\boldsymbol{Q}$ such that the norm

$$\|\boldsymbol{T}_1 - \boldsymbol{T}_3\boldsymbol{Q}\boldsymbol{T}_2\|_{H^\infty(\mathbb{C}^+)}$$

is small possible. Let $\boldsymbol{T}_3 = \boldsymbol{\Upsilon}\boldsymbol{F}$ and $\boldsymbol{T}_2^{\mathrm{t}} = \boldsymbol{\Theta}\boldsymbol{G}$ be inner–outer factorizations. Consider now the special case when *both* $\boldsymbol{\Upsilon}$ *and* $\boldsymbol{\Theta}$ *are square matrix functions and the operators of multiplication by* $\boldsymbol{F}$ *and* $\boldsymbol{G}$ *map the spaces of bounded analytic vector functions in* $\mathbb{C}^+$ *onto the spaces of bounded analytic vector functions in* $\mathbb{C}^+$ *of the corresponding dimensions.* Then we have

$$\|\boldsymbol{T}_1 - \boldsymbol{T}_3\boldsymbol{Q}\boldsymbol{T}_2\|_{H^\infty(\mathbb{C}^+)} = \|\boldsymbol{\Upsilon}^*\boldsymbol{T}_1\overline{\boldsymbol{\Theta}} - \boldsymbol{F}\boldsymbol{Q}\boldsymbol{G}^{\mathrm{t}}\|_{L^\infty}.$$

It is easy to see that under our assumptions on $\boldsymbol{F}$ and $\boldsymbol{G}$ the set of matrix functions of the form $\boldsymbol{F}\boldsymbol{Q}\boldsymbol{G}^{\mathrm{t}}$ where $\boldsymbol{Q}$ is an arbitrary matrix function in $H^\infty(\mathbb{C}^+)$ coincides with the space of bounded analytic matrix functions in $H^\infty(\mathbb{C}^+)$ that have the same size as $\boldsymbol{\Upsilon}^*\boldsymbol{T}_1\overline{\boldsymbol{\Theta}}$. Thus by Theorem 2.2.2,

$$\inf_{\boldsymbol{Q}\in H^\infty(\mathbb{C}^+)} \|\boldsymbol{T}_1 - \boldsymbol{T}_3\boldsymbol{Q}\boldsymbol{T}_2\|_{H^\infty(\mathbb{C}^+)} = \|H_{\boldsymbol{\Upsilon}^*\boldsymbol{T}_1\overline{\boldsymbol{\Theta}}}\|$$

and in this special case the model matching problem reduces to the Nehari problem. It can be shown that in more general cases the model matching problem reduces to the four block problem considered in the Concluding Remarks to Chapter 2.

Let us return now to the robust stabilization problem. By Theorem 7.1, we have to look for rational transfer functions $\boldsymbol{K}$ that internally stabilize the feedback system with plant G such that $\|\boldsymbol{K}(\boldsymbol{I}+\boldsymbol{G}\boldsymbol{K})^{-1}\|_{L^\infty} < \varepsilon^{-1}$. By (8.2) and Theorem 8.1, this problem reduces to the problem of minimizing the norm

$$\|(\boldsymbol{M}\boldsymbol{Q} - \boldsymbol{Y})\boldsymbol{M}_\#\|_{H^\infty(\mathbb{C}^+)}$$

over rational matrix functions $\boldsymbol{Q}$ in $H^\infty(\mathbb{C}^+)$. This corresponds to the model matching problem with $T_1 = MYM_\#$, $T_2 = M_\#$, and $T_3 = M$. If $\boldsymbol{G}$ has no poles on the imaginary axis, it is easy to see that T_2 and T_3

satisfy the above assumption, and so the problem of robust stabilization reduces to the Nehari problem. Moreover, in this case the inner factors $\boldsymbol{\Upsilon}$ and $\boldsymbol{\Theta}$ of $\boldsymbol{M}$ and $\boldsymbol{M}_{\#}^{\mathrm{t}}$ are finite Blaschke–Potapov products, and so the problem of robust stabilization reduces to the Nehari problem for the *rational* matrix function $\boldsymbol{\Upsilon}^*\boldsymbol{M}\boldsymbol{Y}\boldsymbol{M}_{\#}\overline{\boldsymbol{\Theta}}$. Namely, it reduces to the problem of finding stable rational matrix functions $\boldsymbol{R}$ such that

$$\|\boldsymbol{\Upsilon}^*\boldsymbol{M}\boldsymbol{Y}\boldsymbol{M}_{\#}\overline{\boldsymbol{\Theta}} - \boldsymbol{R}\|_{L^\infty} < \varepsilon^{-1}. \tag{9.1}$$

Clearly, this problem has a solution if and only if $\|H_{\boldsymbol{\Upsilon}^*\boldsymbol{M}\boldsymbol{Y}\boldsymbol{M}_{\#}\overline{\boldsymbol{\Theta}}}\| < \varepsilon^{-1}$.

Recall that in §5.4 and §5.5 for any $\rho \in (\|H_{\boldsymbol{\Upsilon}^*\boldsymbol{M}\boldsymbol{Y}\boldsymbol{M}_{\#}\overline{\boldsymbol{\Theta}}}\|, \varepsilon^{-1})$ a parametrization formula has been given for all bounded analytic matrix functions $\boldsymbol{R}$ satisfying $\|\boldsymbol{\Upsilon}^*\boldsymbol{M}\boldsymbol{Y}\boldsymbol{M}_{\#}\overline{\boldsymbol{\Theta}} - \boldsymbol{R}\|_{L^\infty} \le \rho$. It will be shown in Chapter 14 (Corollary 14.12.4) that there exists a rational matrix function $\boldsymbol{R}$ that minimizes the norm $\|\boldsymbol{\Upsilon}^*\boldsymbol{Y}\boldsymbol{M}_{\#}\overline{\boldsymbol{\Theta}} - \boldsymbol{R}\|_{L^\infty}$. Moreover, in Chapter 14 we give an algorithm how to find such an optimal matrix function $\boldsymbol{R}$.

Concluding Remarks

"H^∞ control theory" was initiated by Zames [1–3]. Let us also mention the papers Helton [1] and Tannenbaum [1]. We recommend Francis [1], Helton [3], Doyle [1], Dahleh and Diaz-Bobillo [1], Chui and Chen [1], Doyle, Francis, and Tannenbaum [1], and Desoer and Vidiasagar [1] for an introduction in this theory.

We refer the reader to Fuhrmann [4] for an introduction in linear systems and their connections with Hankel operators; see also Balakrishnan [1]. Backward shift realizations of symbols of bounded Hankel operators were constructed in Fuhrmann [2] and Helton [1]; see Fuhrmann [4]. Theorem 2.5 was obtained in Young [1]; its original proof is simplified in §2, the simplification is inspired by Megretskii, Peller, and Treil [2]. Theorems 2.8 and 2.9 are due to Young [1]. For realizations of rational functions see Bart, Gohberg, and Kaashoek [1].

The construction of balanced realizations with continuous time was reduced to the construction of balanced realizations with discrete time in Ober and Montgomery-Smith [1]; see also Salamon [1].

We refer the reader to Glover [1,2], Young [3], and Diaz-Bobillo [1] for more information on the model reduction problem.

The problem of robust stabilization is treated in Francis [1], Diaz-Bobillo [1]; see also Young [4]. It is a special case of the so-called standard problem; see Francis [1]. The latter also includes as special cases the tracking problem and the model matching problem; see Francis [1]; see also Foias and Tannenbaum [1] for the weighted sensitivity problem.

Coprime factorizations are discussed in detail in Francis [1]. The idea of coprime factorizations over H^∞ is due to Vidyasagar [1]; see also Vidyasagar [2].

The proof of Theorem 5.1 given in §7 is taken from Ball, Gohberg, and Rodman [1].

Parametrization formula (8.1) was found in Youla, Jabr, and Bonjiorno [1]; see Francis [1].

The reduction of the standard problem to the model matching problem can be found in Francis [1]. Recall that the robust stabilization problem is a special case of the standard problem. More information about the model matching problem can be found in Doyle, Francis, and Tannenbaum [1].

12

The Inverse Spectral Problem for Self-Adjoint Hankel Operators

In §8.5 we have considered a geometric problem in the theory of stationary Gaussian processes and we have reduced this problem to the problem of the description of the bounded linear operators on Hilbert space that are unitarily equivalent to moduli of Hankel operators. In this chapter we are going to solve the latter problem, which in turn will lead to a solution of the above geometric problem in prediction theory.

Moreover, we consider in this chapter a considerably more delicate problem of describing the bounded self-adjoint operators on Hilbert space that are unitarily equivalent to Hankel operators on ℓ^2. In other words, we are going to characterize those self-adjoint operators Γ on a Hilbert space $\mathcal{H}$ for which there exists an orthonormal basis $\{e_j\}_{j\ge 0}$ such that Γ has Hankel matrix with respect to $\{e_j\}_{j\ge 0}$, i.e., $(\Gamma e_j, e_k) = \alpha_{j+k}$, $j, k \ge 0$, for some sequence $\{a_j\}_{j\ge 0}$ of complex numbers. Clearly, Γ is self-adjoint if and only if $\alpha_j \in \mathbb{R}$ for any $j \in \mathbb{Z}_+$.

We have already mentioned in §8.5 that a Hankel operator is always noninvertible and its kernel is either trivial or infinite-dimensional. Thus if Γ is unitarily equivalent to a Hankel operator, then Γ is noninvertible and either $\operatorname{Ker}\Gamma = \{\mathbb{O}\}$ or $\dim \operatorname{Ker}\Gamma = \infty$. It turns out, however, that these two necessary conditions are not sufficient for a self-adjoint operator Γ to be unitarily equivalent to a Hankel operator. We have to add another condition. Roughly speaking it says that the spectral measure of Γ is "nearly symmetric" with respect to the origin. We state the third condition in terms of the spectral multiplicity function of Γ.

Recall (see Appendix 1.4) that each self-adjoint operator Γ on a separable Hilbert space is unitarily equivalent to multiplication by the coordinate function on the von Neumann integral

$$\int \oplus \mathcal{H}(t) d\mu(t),$$

where μ is a scalar spectral measure of Γ. The spectral multiplicity function ν of Γ is defined μ-a.e. by

$$\nu(t) = \dim \mathcal{H}(t).$$

Recall also that two self-adjoint operators are unitarily equivalent if and only if they have mutually absolutely continuous scalar spectral measures and their spectral multiplicity functions coincide almost everywhere with respect to their scalar spectral measures (see Appendix 1.4).

Suppose now that Γ is a self-adjoint operator on Hilbert space with a scalar spectral measure μ and spectral multiplicity function ν. The third necessary condition is that $|\nu(t) - \nu(-t)|$ is at most 2 on the absolutely continuous spectrum of Γ and at most 1 on the singular spectrum of Γ. To be more precise, we have to explain what we mean by $\nu(t) - \nu(-t)$.

We define the Borel measure $\mu_\#$ by

$$\mu_\#(\Delta) = \mu(\Delta) + \mu(-\Delta)$$

for Borel subsets Δ of $\mathbb{R}$. We can assume that the multiplicity function ν is defined $\mu_\#$-almost everywhere and if δ is a Borel set such that $\mu(\delta) = 0$, then $\nu(t) = 0$ for $\mu_\#$-almost all $t \in \delta$. Indeed, we can consider the measure λ defined by $d\lambda = \nu d\mu$. Clearly, λ is absolutely continuous with respect to $\mu_\#$. We can consider now the Radon–Nikodym density of λ with respect to $\mu_\#$ and assume that ν is equal to this density. Clearly, we have changed the function ν only on the set of zero μ-measure.

Now we are in a position to state the main result of the chapter. As usual μ_{a} and μ_{s} are the absolutely continuous and the singular components of μ.

Theorem 0.1. *Let Γ be a bounded self-adjoint operator on Hilbert space with a scalar spectral measure μ and spectral multiplicity function ν. Then Γ is unitarily equivalent to a Hankel operator if and only if the following conditions are satisfied:*

(C1) *either* $\operatorname{Ker}\Gamma = \{\mathbb{O}\}$ *or* $\dim \operatorname{Ker}\Gamma = \infty$*;*

(C2) Γ *is noninvertible;*

(C3) $|\nu(t) - \nu(-t)| \le 2$*,* μ_{a}*-a.e. and* $|\nu(t) - \nu(-t)| \le 1$*,* μ_{s}*-a.e.*

Note that (C3) means in particular that if one of the numbers $\nu(t)$ and $\nu(-t)$ is infinite, then the other one must also be infinite almost everywhere.

Clearly, (C1) is equivalent to the condition that

(C1′) $\nu(0) = 0$ or $\nu(0) = \infty$,

while (C2) is equivalent to the condition

(C2′) $0 \in \operatorname{supp} \nu$.

It follows easily from Theorem 0.1 that if K is a self-adjoint operator on Hilbert space such that $K \ge \mathbb{O}$, K is noninvertible, and $\operatorname{Ker} K = \{\mathbb{O}\}$ or

$\dim \operatorname{Ker} K = \infty$, then K is unitarily equivalent to the modulus of a Hankel operator.

Let us now compare the problem to describe the operators unitarily equivalent to the modulus of a Hankel operator with the problem to describe the operators unitarily equivalent to a self-adjoint Hankel operator. If Γ is a self-adjoint operator, ν_Γ is the spectral multiplicity function of Γ and $\nu_{|\Gamma|}$ is the spectral multiplicity function of $|\Gamma|$. Then $\nu_{|\Gamma|}(t) = \nu_\Gamma(t)+\nu_\Gamma(-t)$, $t > 0$, $\mu_\#$-a.e. Thus if we know that Γ satisfies (C1) and (C2), the problem is how we can distribute $\nu_{|\Gamma|}(t)$ between $\nu_\Gamma(t)$ and $\nu_\Gamma(-t)$ in order that Γ be unitarily equivalent to a Hankel operator. Theorem 0.1 gives a solution to this problem.

The necessity of conditions (C1)–(C3) will be proved in §1. As we have already observed, we have to prove only the necessity of (C3). It is easy to see that (C3) implies the following inequality for the multiplicities of eigenvalues of a self-adjoint Hankel operator Γ: if $\lambda \in \mathbb{R}$, then

$$|\dim \operatorname{Ker}(\Gamma - \lambda I) - \dim \operatorname{Ker}(\Gamma + \lambda I)| \le 1 \tag{0.1}$$

(if one of the dimensions is infinite, then the other one must also be infinite). In §2 we give a simple proof of (0.1) for an arbitrary Hankel operator Γ (not necessarily self-adjoint) and an arbitrary $\lambda \in \mathbb{C}$.

In §§3–6 we use linear systems with continuous time (see Chapter 11) to construct a Hankel operator with prescribed spectral properties. We prove that if Γ is a self-adjoint operator that satisfies (C1), (C2), and the condition

$$|\nu(t) - \nu(-t)| \le 1, \quad \mu\text{-a.e.},$$

then there exists a balanced linear system (see Chapter 11) with continuous time such that the Hankel operator associated with it is unitarily equivalent to Γ. We obtain explicit formulas for the operators A, B, C that determine the linear system.

The above results do not seem to be satisfactory. However, they allow us to solve completely the problem of characterizing the bounded linear operators that are unitarily equivalent to the moduli of Hankel operators.

We show in §7 that if Γ is a positive self-adjoint operator with multiple spectrum, then there exists no proper balanced system with continuous time for which the corresponding Hankel operator is unitarily equivalent to Γ. This shows that in contrast with the problem of describing all possible moduli of Hankel operators, the proper balanced linear systems with continuous time cannot produce enough Hankel operators to solve the problem completely.

Recall that we consider here only proper linear systems, i.e., those linear systems that involve bounded operators A, B, C. It is also possible to consider (generalized) linear systems for which A is the generator of a contractive semigroup, B is an operator from $\mathbb{C}$ to a Hilbert space that is larger than $\mathcal{K}$, and C is a linear functional defined on a dense subset of $\mathcal{K}$ (see

Chapter 11). As we have mentioned in Chapter 11, for any bounded Hankel operator Γ on $L^2(\mathbb{R}_+)$ there exists a generalized balanced linear system such that the Hankel operator associated with it coincides with Γ (see Ober and Montgomery-Smith [1] and Salamon [1]). However, it is not clear how to evaluate the spectra of Hankel operators associated with systems that involve unbounded operators A, B, and C.

Note that positive Hankel operators with multiple spectra do exist. A classical example of such an operator is the Carleman operator K defined on $L^2(\mathbb{R}_+)$ by

$$(Kf)(s) = \int_0^\infty \frac{f(t)}{s+t} dt$$

(see §9.2). Other interesting examples of such operators are given by Howland [2–4].

In §8 we show how one can simplify the construction given in §4 to solve the problem to describe the operators unitarily equivalent to the moduli of Hankel operators. We also solve in §8 the geometrical problem on past and future that has been posed in §8.5.

In §§9–14 we prove that conditions (C1)–(C3) are sufficient for Γ to be unitarily equivalent to a Hankel operator. We use methods of linear systems with discrete time (see Chapter 11). Namely, we prove that if Γ satisfies (C1)–(C3), then there exists a balanced linear system with discrete time such that the corresponding Hankel operator is unitarily equivalent to Γ.

Finally, in §15 we obtain the Aronszain–Donoghue theorem in perturbation theory as a consequence of our construction of linear systems with discrete time and condition (C3) for Hankel operators.

We can also consider the problem of spectral characterization of the self-adjoint operators that are unitarily equivalent to block Hankel operators on $\ell^2(\mathbb{C}^n)$, i.e., operators given by block Hankel matrices $\{\Omega_{j+k}\}_{j,k\ge 0}$, where $\Omega_j \in \mathbb{M}_{n,n}$.

Theorem 0.2. *Let Γ be a bounded self-adjoint operator on Hilbert space with a scalar spectral measure μ and spectral multiplicity function ν. Then Γ is unitarily equivalent to a block Hankel operator on $\ell^2(\mathbb{C}^n)$ if and only if the following conditions are satisfied:*

(C1) *either* $\operatorname{Ker}\Gamma = \{\mathbb{O}\}$ *or* $\dim \operatorname{Ker}\Gamma = \infty$*;*

(C2) Γ *is noninvertible;*

($\mathrm{C3}_n$) $|\nu(t) - \nu(-t)| \le 2n$*,* μ_{a}*-a.e. and* $|\nu(t) - \nu(-t)| \le n$*,* μ_{s}*-a.e.*

The sufficiency of conditions (C1), (C2), and ($\mathrm{C3}_n$) follows easily from Theorem 0.1 and the fact that if Γ satisfies (C1), (C2), and ($\mathrm{C3}_n$), then it can be represented as an orthogonal sum of n operators each of which satisfies (C1)–(C3). Necessity can be proved by the same method as in the case $n = 1$ (see §1).

Finally, we can consider the case of block Hankel operators on $\ell^2(\mathcal{K})$, where $\mathcal{K}$ is a separable infinite-dimensional Hilbert space.

Theorem 0.3. *Let* Γ *be a bounded self-adjoint operator on Hilbert space with a scalar spectral measure* μ *and spectral multiplicity function* ν *and let* $\mathcal{K}$ *be a separable infinite-dimensional Hilbert space. Then* Γ *is unitarily equivalent to a block Hankel operator on* $\ell^2(\mathcal{K})$ *if and only if the following conditions are satisfied:*

(C1) *either* $\operatorname{Ker}\Gamma = \{\mathbb{O}\}$ *or* $\dim \operatorname{Ker}\Gamma = \infty$*;*

(C2) Γ *is noninvertible.*

In view of the above remark we restrict ourselves in this chapter to the proof of Theorem 0.1.

1. Necessary Conditions

The main purpose of this section is to prove that conditions (C1)–(C3) are necessary for a self-adjoint operator Γ to be unitarily equivalent to a Hankel operator on ℓ^2. We have already observed (see §8.5) that (C1) and (C2) are necessary. Thus we prove in this section the following fact.

Theorem 1.1. *Let* Γ *be a bounded self-adjoint Hankel operator on* ℓ^2 *with a scalar spectral measure* μ *and spectral multiplicity function* ν*. Then*

$$|\nu(t) - \nu(-t)| \le 2, \quad \mu_{\mathrm{a}}\text{-a.e.}$$

and

$$|\nu(t) - \nu(-t)| \le 1, \quad \mu_{\mathrm{s}}\text{-a.e.}$$

To prove Theorem 1.1, we need the description of the set of operators that intertwine two given self-adjoint operators. Let A_1 and A_2 be bounded self-adjoint operators on Hilbert spaces and let Q be a bounded linear operator that intertwines A_1 and A_2, i.e.,

$$QA_1 = A_2Q. \tag{1.1}$$

We can realize A_1 and A_2 as multiplications by the independent variables on

$$\int \oplus \mathcal{H}_1(t) d\mu_1(t) \quad \text{and} \quad \int \oplus \mathcal{H}_2(t) d\mu_2(t).$$

Next, we can assume that $\mathcal{H}_1(t)$ and $\mathcal{H}_2(t)$ are defined $(\mu_1 + \mu_2)$-almost everywhere and if $\mu_1(\delta) = 0$ $(\mu_2(\delta) = 0)$ for a Borel set δ, then $\mathcal{H}_1(t) = \{\mathbb{O}\}$ $(\mathcal{H}_2(t) = \{\mathbb{O}\})$ on δ, $(\mu_1 + \mu_2)$-almost everywhere.

Lemma 1.2. *Let* Q *be a bounded linear operator that satisfies* (1.1)*. Then there exists a* $(\mu_1 + \mu_2)$*-measurable bounded operator function* q*,*

$$q(t) : \mathcal{H}_1(t) \to \mathcal{H}_2(t),$$

such that

$$(Qf)(t) = q(t)f(t), \quad f \in \int \oplus \mathcal{H}_1(t) d\mu_1(t).$$

Proof. Without loss of generality we may assume that there exists a Borel measure μ mutually absolutely continuous with $\mu_1 + \mu_2$ and such that μ_j is the restriction of μ to the set

$$\{t \in \mathbb{R} : \ \mathcal{H}_j(t) \neq \{\mathbb{O}\}\}, \quad j = 1, 2.$$

Put

$$\mathcal{H}_j \stackrel{\text{def}}{=} \int \oplus \mathcal{H}_j(t) d\mu_j(t) = \int \oplus \mathcal{H}_j(t) d\mu(t), \quad j = 1, 2,$$

and

$$\mathcal{H} = \int \oplus \mathcal{H}(t) d\mu(t),$$

where

$$\mathcal{H}(t) = \mathcal{H}_1(t) \oplus \mathcal{H}_2(t).$$

Then $\mathcal{H}_1$ and $\mathcal{H}_2$ are naturally imbedded in $\mathcal{H}$. Let A be multiplication by the independent variable on $\mathcal{H}$. Consider the operator $Q_\#$ defined by

$$Q_\# f = Q P_1 f,$$

where P_1 is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_1$. Obviously, $\operatorname{Range} Q_\# \subset \mathcal{H}_2$, $Q = Q_\# \big| \mathcal{H}_1$, and $Q_\# A = A Q_\#$. Then there exists a bounded measurable operator function $q_\#$, $q_\#(t) : \mathcal{H}(t) \to \mathcal{H}(t)$, such that $(Q_\# f)(t) = q_\#(t) f(t)$ (see Appendix 1.4). It is easy to see that $q_\#(t) x = \mathbb{O}$ for $x \in \mathcal{H}_2(t)$ and $\operatorname{Range} q_\#(t) \subset \mathcal{H}_2(t)$. We can now define $q(t)$ by

$$q(t) = q_\#(t) \big| \mathcal{H}_1(t).$$

Clearly, $q(t)$ is an operator from $\mathcal{H}_1(t)$ to $\mathcal{H}_2(t)$ and $(Qf)(t) = q(t) f(t)$. ■

Before we proceed to the proof of Theorem 1.1 we adopt the following terminology. If f is a Borel function, ρ is a Borel measure, and Δ is a Borel set, we say that f *is supported on* Δ (ρ *is supported on* Δ), if f is zero almost everywhere outside Δ (ρ is zero outside Δ).

Proof of Theorem 1.1. Suppose that Γ is unitarily equivalent to a Hankel operator. Then $\mathcal{H}$ has an orthonormal basis $\{e_j\}_{j \ge 0}$ such that Γ has Hankel matrix in this basis, i.e., $(\Gamma e_j, e_k) = \alpha_{j+k}$, $j, k \ge 0$, where $\{\alpha_j\}_{j \ge 0}$ is a sequence of real numbers ($\alpha_j \in \mathbb{R}$, since Γ is self-adjoint).

Consider the shift operator S on $\mathcal{H}$ defined by

$$S \sum_{n \ge 0} x_n e_n = \sum_{n \ge 0} x_n e_{n+1}.$$

Put $\alpha = \sum\limits_{j=0}^{\infty} \alpha_j e_j = \Gamma e_0$ and $\alpha_* = \sum\limits_{j=0}^{\infty} \bar{\alpha}_j e_j = \Gamma^* e_0$. It is easy to see that the following commutation relations hold:

$$S^* \Gamma = \Gamma S, \tag{1.2}$$

$$S \Gamma f - \Gamma S^* f = (f, e_0) S \alpha - (f, S \alpha_*) e_0, \quad f \in \mathcal{H}. \tag{1.3}$$

Let us first prove the inequality $|\nu(t)-\nu(-t)| \le 2$, μ-a.e. Let f be a function in $\mathcal{H}$ such that $f(t) \perp e_0(t)$ and $f(t) \perp (S\alpha_*)(t)$, μ-a.e. Consider the function $(S^*-S)f$. We have

$$\Gamma(S^*-S)f = \Gamma S^* f - \Gamma S f = S\Gamma f - S^*\Gamma f = -(S^*-S)\Gamma f \tag{1.4}$$

by (1.2) and (1.3), since $f \perp e_0$ and $f \perp S\alpha_*$.

Let A_1 be the restriction of Γ to

$$\tilde{\mathcal{H}} = \{f \in \mathcal{H} : \ f(t) \perp e_0(t) \quad \text{and} \quad f(t) \perp (S\alpha_*)(t), \quad \mu\text{-a.e.}\}$$

and let $A_2 = -\Gamma$. Consider the operator $Q : \tilde{\mathcal{H}} \to \mathcal{H}$ defined by $Qf = (S^*-S)f$. By (1.4), $QA_1 = A_2Q$. Thus by Lemma 1.2, there exists a bounded weakly measurable function q, $q(t) : \tilde{\mathcal{H}}(t) \to \mathcal{H}(-t)$, such that

$$(Qf)(t) = q(t)f(-t),$$

where

$$\tilde{\mathcal{H}}(t) = \{x \in \mathcal{H}(t) : \ x \perp e_0(t) \quad \text{and} \quad x \perp (S\alpha_*)(t)\}.$$

Let us show that $\operatorname{Ker} q(t) = \{\mathbb{O}\}$, μ-a.e. Indeed, assume that there exists a Borel set Δ, $\mu(\Delta) > 0$, such that $\operatorname{Ker} q(t) \ne \{\mathbb{O}\}$ for any $t \in \Delta$. Let P_t be the orthogonal projection of $\tilde{\mathcal{H}}(t)$ onto $\operatorname{Ker} q(t)$. Then the function $t \mapsto P_t$ is weakly measurable. This follows from the fact that

$$P_t = \lim_{\varepsilon \to 0} \chi_\varepsilon(q^*(t)q(t)),$$

where $\chi_\varepsilon(s) = 0$ for $s > \varepsilon$ and $\chi_\varepsilon(s) = 1$ for $0 \le s \le \varepsilon$. Therefore (see Appendix 1.4) the spaces $\operatorname{Ker} q(t)$ form a measurable family in a natural way, and we can consider the direct integral

$$\int \oplus \operatorname{Ker} q(t)\, d\mu(t) \ne \{\mathbb{O}\}.$$

It is easy to see that

$$\{\mathbb{O}\} \ne \int \oplus \operatorname{Ker} q(t)\, d\mu(t) \subset \operatorname{Ker} Q.$$

However, it is very easy to verify straightforwardly that $\operatorname{Ker}(S^*-S) = \{\mathbb{O}\}$.

Thus $q(t)$ is an injective map from $\tilde{\mathcal{H}}(t)$ to $\mathcal{H}(-t)$ which implies that

$$\nu(-t) = \dim \mathcal{H}(-t) \ge \dim \tilde{\mathcal{H}}(t) \ge \nu(t) - 2, \quad \mu\text{-a.e.}$$

Interchanging the roles of t and $-t$, we obtain

$$|\nu(t) - \nu(-t)| \le 2.$$

Let us now prove that $|\nu(t)-\nu(-t)| \le 1$ on the singular spectrum. Let $\Delta_{\rm s}$ be a Borel set such that $\mu_{\rm s}$ is supported on $\Delta_{\rm s}$ and $\Delta_{\rm s}$ has zero Lebesgue measure.

Lemma 1.3. *Let f be a function supported on $\Delta_{\rm s}$ such that $f(t) \perp e_0(t)$ and if $e_0(t) = 0$, then $f(t) \perp (S\alpha_*)(t)$, μ-a.e. Then*

$$\Gamma S^* f = S\Gamma f.$$

Let us first complete the proof of Theorem 1.1 and then prove Lemma 1.3. Put

$$\check{\mathcal{H}}(t) = \left\{ x \in \mathcal{H}(t) : \left\{ \begin{array}{ll} x \perp e_0(t), & e_0(t) \neq \mathbb{O}, \\ x \perp (S\alpha_*)(t), & e_0(t) = \mathbb{O} \end{array} \right. \right\}, \quad t \in \Delta_{\mathrm{s}},$$

and

$$\check{\mathcal{H}} = \{ f : \ f \ \text{ is supported on } \ \Delta_{\mathrm{s}} \ \text{ and } \ f(t) \in \check{\mathcal{H}}(t), \quad \mu\text{-a.e.}\}.$$

As above we put $A_1 \stackrel{\text{def}}{=} A|\check{\mathcal{H}}$, $A_2 \stackrel{\text{def}}{=} -A$ and consider the operator $Q : \check{\mathcal{H}} \to \mathcal{H}$ defined by $Qf = (S^* - S)f$. By Lemma 1.3, $QA_1 = A_2 Q$ and by Lemma 1.2,

$$(Qf)(t) = q(t)f(-t),$$

where q is a weakly measurable operator-valued function on Δ_{s} such that $q(t) : \check{\mathcal{H}}(t) \to \mathcal{H}(-t)$ and $\operatorname{Ker} q(t) = \{\mathbb{O}\}$, μ-a.e. As above, it follows that

$$\nu(-t) = \dim \mathcal{H}(-t) \geq \dim \check{\mathcal{H}}(t) \geq \nu(t) - 1,$$

which implies that

$$|\nu(t) - \nu(-t)| \leq 1, \quad \mu_{\mathrm{s}}\text{-a.e.} \quad \blacksquare$$

Proof of Lemma 1.3. Put

$$\Delta_0 = \{ t \in \Delta_{\mathrm{s}} : \ e_0(t) \neq 0 \}$$

(recall that $\{e_j\}_{j\geq 0}$ is an orthonormal basis in which Γ has Hankel matrix). If f is supported on $\Delta_{\mathrm{s}} \backslash \Delta_0$, then $f \perp S\alpha_*$ and $f \perp e_0$, and it follows from (1.3) that $\Gamma S^* f = S\Gamma f$.

Thus we can assume without loss of generality that f is supported on Δ_0. Let k be a positive integer. Put

$$\Delta_k \stackrel{\text{def}}{=} \{ t \in \Delta_{\mathrm{s}} : \ 2^{-k} \leq \|e_0(t)\| \leq 2^k, \ \|f(t)\| \leq 2^k \}.$$

Clearly $\Delta_0 = \bigcup_{k\geq 1} \Delta_k$. So it is sufficient to consider the case when f is supported on Δ_k for some k.

Let $\{I_j\}$ be a cover of Δ_k by disjoint open intervals. Put $\Lambda_j = \Delta_k \cap I_j$ (k is fixed), $f_j = \chi_j f$, $g_j = \chi_j e_0$, where χ_j is the characteristic function of Λ_j.

It follows from (1.2) and (1.3) that

$$\Gamma(S + S^*)f - (S + S^*)\Gamma f = c e_0 \tag{1.5}$$

for some $c \in \mathbb{C}$. Therefore it is sufficient to show that

$$(\Gamma(S + S^*)f - (S + S^*)\Gamma f, e_0) = 0.$$

We have

$$\begin{aligned} (\Gamma(S + S^*)f \ - \ (S + S^*)\Gamma f, e_0) &= \sum_{j\geq 1} (\Gamma(S + S^*)f_j - (S + S^*)\Gamma f_j, e_0) \\ &= \sum_{j\geq 1} \frac{1}{(e_0, g_j)} (\Gamma(S + S^*)f_j - (S + S^*)\Gamma f_j, g_j) \end{aligned}$$

in view of (1.5).

Clearly,

$$(e_0, g_j) = \int_{\Lambda_j} \|e_0(t)\|^2 d\mu(t) \ge 2^{-2k}\mu\Lambda_j.$$

Let $\lambda_j \in \Lambda_j$. Put $g^{(j)} = \Gamma g_j - \lambda_j g_j$, $f^{(j)} = \Gamma f_j - \lambda_j f_j$. Then

$$\|g^{(j)}\| \le |I_j| \cdot \|g_j\|, \quad \|f^{(j)}\| \le |I_j| \cdot \|f_j\|,$$

where $|I_j|$ is the length of I_j. We have

$$\begin{aligned}
&\left|\left((\Gamma(S+S^*)f - (S+S^*)\Gamma f), e_0\right)\right| \\
&\le 2^{2k} \sum_{j\ge1} \frac{1}{\mu\Lambda_j} |((S+S^*)f_j, \Gamma g_j) - (\Gamma f_j, (S+S^*)g_j)| \\
&= 2^{2k} \sum_{j\ge1} \frac{1}{\mu\Lambda_j} \left(\left|((S+S^*)f_j, g^{(j)}) - (f^{(j)}, (S+S^*)g_j)\right|\right) \\
&\le \text{const}\, 2^{2k} \sum_{j\ge1} \frac{|I_j|}{\mu\Lambda_j} \|f_j\| \cdot \|g_j\| \\
&= \text{const}\, 2^{2k} \sum_{j\ge1} \frac{|I_j|}{\mu\Lambda_j} \left(\int_{\Lambda_j} \|f_j(t)\|^2 d\mu(t)\right)^{1/2} \left(\int_{\Lambda_j} \|g_j(t)\|^2 d\mu(t)\right)^{1/2} \\
&\le \text{const}\, 2^{2k} \sum_{j\ge1} \frac{|I_j|}{\mu\Lambda_j} 2^{2k}(\mu\Lambda_j)^{1/2} 2^{2k}(\mu\Lambda_j)^{1/2} = \text{const}\, 2^{6k} \sum_{j\ge1} |I_j|.
\end{aligned}$$

Since $\boldsymbol{m}(\Delta_k) = 0$, we can make $\sum_{j\ge1} |I_j|$ as small as possible. ■

2. Eigenvalues of Hankel Operators

As we have noticed in the Introduction to the chapter, it follows from Theorem 1.1 that for any self-adjoint Hankel operator Γ and for any $\lambda \in \mathbb{R}$

$$|\dim \operatorname{Ker}(\Gamma - \lambda I) - \dim \operatorname{Ker}(\Gamma + \lambda I)| \le 1. \tag{2.1}$$

The main result of this section shows that (2.1) is also valid for an arbitrary bounded Hankel operator and an arbitrary $\lambda \in \mathbb{C}$. The idea of the proof is the same as in Theorem 1.1, but to include the nonself-adjoint case, we have to replace the inner product by the natural bilinear form on ℓ^2.

We assume here that Γ is a bounded Hankel operator on the Hilbert space ℓ^2, $\{e_n\}_{n\ge0}$ is the standard orthonormal basis in ℓ^2. Then $(\Gamma e_j, e_k) = \alpha_{j+k}$, $j, k \ge 0$, where $\{\alpha_j\}_{j\ge0}$ is a sequence of complex numbers.

It is convenient to consider the following bilinear form on ℓ^2:

$$\langle x, y\rangle = \sum_{j\geq 0} x_j y_j,$$

where $x = \sum_{j\geq 0} x_j e_j$ and $y = \sum_{j\geq 0} y_j e_j$.

Clearly, any Hankel operator Γ is symmetric with respect to this form:

$$\langle \Gamma x, y\rangle = \langle x, \Gamma y\rangle, \quad x, y \in \ell^2.$$

Let S be the shift operator on ℓ^2: $Se_n = e_{n+1}$, $n \geq 0$. Clearly, $\langle Sx, y\rangle = \langle x, S^* y\rangle$, $x, y \in \ell^2$.

Theorem 2.1. *Let Γ be a bounded Hankel operator. Then inequality* (2.1) *holds for any $\lambda \in \mathbb{C}$.*

Proof. Given $\varkappa \in \mathbb{C}$, we put $E_\varkappa = \operatorname{Ker}(\Gamma - \varkappa I)$. Let $\alpha = \sum_{j\geq 0} \alpha_j e_j$. We have

$$S^*\Gamma = \Gamma S, \tag{2.2}$$

$$S\Gamma x - \Gamma S^* x = \langle x, e_0\rangle S\alpha - \langle x, S\alpha\rangle e_0, \quad x \in \ell^2, \tag{2.3}$$

(compare with (1.2) and (1.3)).

Let $\lambda \in \mathbb{C}$. To prove the theorem, we have to show that

$$\dim E_{-\lambda} \geq \dim E_\lambda - 1. \tag{2.4}$$

We consider separately two different cases.

Case 1. There exists a vector $a = \sum_{j\geq 0} a_j e_j$ in E_λ with $a_0 \neq 0$. Let us show that if $x \in E_\lambda$ and $\langle x, e_0\rangle = 0$, then

$$S\Gamma x = \Gamma S^* x, \tag{2.5}$$

which is equivalent in view of (2.2) to

$$\Gamma(S + S^*)x = (S + S^*)\Gamma x.$$

It follows from (2.2) and (2.3) that

$$\Gamma(S + S^*)x - (S + S^*)\Gamma x = ce_0$$

for some $c \in \mathbb{C}$. Since $\langle a, e_0\rangle \neq 0$ and

$$\langle \Gamma(S + S^*)x - (S + S^*)\Gamma x, a\rangle = a_0 c,$$

it is sufficient to show that

$$\langle \Gamma(S + S^*)x - (S + S^*)\Gamma x, a\rangle = 0.$$

We have

$$\begin{aligned}\langle \Gamma(S + S^*)x - (S + S^*)\Gamma x, a\rangle &= \langle (S + S^*)x, \Gamma a\rangle - \langle (S + S^*)\Gamma x, a\rangle \\ &= \lambda\langle (S + S^*)x, a\rangle - \lambda\langle (S + S^*)x, a\rangle = 0,\end{aligned}$$

since both x and a belong to E_λ.

Now it is easy to show that if $x \in E_\lambda$ and $\langle x, S\alpha\rangle = 0$, then $(S - S^*)x \in E_{-\lambda}$. Indeed,

$$\Gamma(S - S^*)x = \Gamma Sx - \Gamma S^*x = S^*\Gamma x - S\Gamma x = -(S - S^*)\Gamma x = -\lambda(S - S^*)x$$

by (2.2) and (2.5). Since $\operatorname{Ker}(S - S^*) = \{\mathbb{O}\}$, it follows that $S - S^*$ is a one-to-one map of $\{x \in E_\lambda : \langle x, S\alpha\rangle = 0\}$ into $E_{-\lambda}$, which proves (2.4).

Case 2. For any $x \in E_\lambda$, $\langle x, e_0\rangle = 0$. In this case it follows directly from (2.2) and (2.3) that if $x \in E_\lambda$ and $\langle x, S\alpha\rangle = 0$, then (2.5) holds. As in Case 1, this implies that $S - S^*$ is a one-to-one map of $\{x \in E_\lambda : \langle x, S\alpha\rangle = 0\}$ into $E_{-\lambda}$. ■

3. Linear Systems with Continuous Time and Lyapunov Equations

In this section we make some preparations to construct a Hankel operator associated with a balanced linear system with continuous time that has prescribed spectral properties. Namely, we establish useful facts on the unitary equivalence (modulo the kernel) of the Hankel operator $\boldsymbol{\Gamma}_h$ corresponding to a balanced system $\{A, B, C\}$ and certain operators related to the system, and introduce Lyapunov equations.

We consider here (proper) balanced linear SISO systems $\{A, B, C\}$ (see §11.3), where A is a bounded linear operator on a separable Hilbert space $\mathcal{K}$, $B : \mathbb{C} \to \mathcal{K}$ and $C : \mathcal{K} \to \mathbb{C}$ are bounded linear operators. Then there are vectors b and c in $\mathcal{K}$ such that

$$Bu = ub, \quad u \in \mathbb{C}, \quad Cx = (x, c), \quad x \in \mathcal{K}.$$

In Chapter 11 we have associated with such a linear system $\{A, B, C\}$ the Hankel operator $\boldsymbol{\Gamma}_h$, $h = h_{\{A,B,C\}}$, on $L^2(\mathbb{R}_+)$, where

$$h_{\{A,B,C\}}(t) = Ce^{tA}B = (e^{tA}b, c), \quad t > 0.$$

Recall also that with such a system $\{A, B, C\}$ we associate the observability and controllability Gramians W_{c} and W_{o} defined by

$$W_{\mathrm{c}} = \int_{\mathbb{R}_+} e^{tA}BB^*e^{tA^*}\,dt,$$

and

$$W_{\mathrm{o}} = \int_{\mathbb{R}_+} e^{tA^*}C^*Ce^{tA}dt$$

if the integrals converge in the weak operator topology.

We consider here balanced linear systems (see §11.3), i.e., those minimal systems for which the above integrals converge in the weak operator topology and $W_{\mathrm{o}} = W_{\mathrm{c}}$.

We are going to establish the following result.

Theorem 3.1. *Let Γ be a self-adjoint operator such that*
(C1) *either* $\operatorname{Ker}\Gamma = \{\mathbb{O}\}$ *or* $\dim\operatorname{Ker}\Gamma = \infty$;
(C2) Γ *is noninvertible;*
(C3′) $|\nu(t) - \nu(-t)| \le 1, \quad \mu$*-a.e.*
Then there exists a balanced linear SISO system $\{A, B, C\}$ *such that the corresponding Hankel operator* $\boldsymbol{\Gamma}_h$ *is unitarily equivalent to* Γ.

The proof of Theorem 3.1 will be completed in §5. First we need some auxiliary results on linear systems.

Theorem 3.2. *Let* $\{A, B, C\}$ *be a balanced linear SISO system with continuous time,* $\mathrm{W} = \mathrm{W}_{\mathrm{o}} = \mathrm{W}_{\mathrm{c}}$, *and let* $\boldsymbol{\Gamma}_h$ *be the Hankel operator associated with it. Then* $\boldsymbol{\Gamma}_h$ *is bounded and the restriction of* $|\boldsymbol{\Gamma}_h|$ *to* $(\operatorname{Ker}\boldsymbol{\Gamma}_h)^\perp$ *is unitarily equivalent to* W.

Proof. This is an immediate consequence of Theorem 11.3.1. ■

We consider now linear systems for which the corresponding Hankel operator is self-adjoint.

Theorem 3.3. *Let* $\{A, B, C\}$ *be a balanced linear SISO system such that* $\boldsymbol{\Gamma}_h$ *is self-adjoint,* $W = W_{\mathrm{o}} = W_{\mathrm{c}}$. *Then there exists an operator* $\boldsymbol{J}$ *on* $\mathcal{K}$ *which is self-adjoint and unitary and such that* $\boldsymbol{J}W = W\boldsymbol{J}$, $A^* = \boldsymbol{J}A\boldsymbol{J}$, *and* $c = \boldsymbol{J}b$.

Proof. Recall (see the proof of Theorem 11.3.1) that the operators V_{c} and V_{o} from $\mathcal{K}$ to $L^2(\mathbb{R}_+)$ are defined by

$$(V_{\mathrm{c}}x)(t) = B^* e^{tA^*}x, \quad (V_{\mathrm{o}}x)(t) = Ce^{tA}x, \quad x \in \mathcal{K},$$

and

$$\boldsymbol{\Gamma}_h = V_{\mathrm{o}}V_{\mathrm{c}}^*.$$

Let

$$V_{\mathrm{c}} = U_{\mathrm{c}}W^{1/2}, \quad V_{\mathrm{o}} = U_{\mathrm{o}}W^{1/2}$$

be the polar decompositions of V_{c} and V_{o} ($|V_{\mathrm{c}}| = |V_{\mathrm{o}}| = W^{1/2}$). We have

$$\operatorname{Ker} V_{\mathrm{c}} = \operatorname{Ker} V_{\mathrm{c}}^* V_{\mathrm{c}} = \operatorname{Ker} W_c = \{\mathbb{O}\} = \operatorname{Ker} W_{\mathrm{o}} = \operatorname{Ker} V_{\mathrm{o}}^* V_{\mathrm{c}} = \operatorname{Ker} V_{\mathrm{o}}.$$

Therefore U_{c} and U_{o} are isometries. Clearly, $\operatorname{Range} U_{\mathrm{c}} = \operatorname{clos}\operatorname{Range} V_{\mathrm{c}}$ and $\operatorname{Range} U_{\mathrm{o}} = \operatorname{clos}\operatorname{Range} V_{\mathrm{o}}$.

We have

$$\boldsymbol{\Gamma}_h = U_{\mathrm{o}}WU_{\mathrm{c}}^*.$$

Since $\boldsymbol{\Gamma}_h$ is self-adjoint, it follows

$$U_{\mathrm{o}}WU_{\mathrm{c}}^* = U_{\mathrm{c}}WU_{\mathrm{o}}^*,$$

whence,

$$WU = U^*W, \tag{3.1}$$

where $U = U_{\mathrm{c}}^*U_{\mathrm{o}}$.

Clearly, $\operatorname{Ker} U_{\mathrm{c}}^* = \operatorname{Ker} V_{\mathrm{c}}^* = \operatorname{Ker}\boldsymbol{\Gamma}_h$. Since $\boldsymbol{\Gamma}_h^* = \boldsymbol{\Gamma}_h$, it follows that $\boldsymbol{\Gamma}_h = V_{\mathrm{c}}V_{\mathrm{o}}^*$, which implies that $\operatorname{Ker}\boldsymbol{\Gamma}_h = \operatorname{Ker} V_{\mathrm{o}}^* = \operatorname{Ker} U_{\mathrm{o}}^*$. Therefore U_{c}

and $U_{\rm o}$ are isometries with the same range $(\operatorname{Ker}\boldsymbol{\Gamma}_h)^\perp$, which implies that $U = U_{\rm c}^* U_{\rm o}$ is unitary. Let us show that it is also self-adjoint.

Lemma 3.4. *Let W be a nonnegative operator,* $\operatorname{Ker} W = \{\mathbb{O}\}$ *and let U be a unitary operator such that $WU = U^*W$. Then U is self-adjoint.*

Proof. Multiplying the equality $U^*W = WU$ by U on the left and by U^* on the right, we obtain $UW = WU^*$. Therefore

$$U^*W^2U = WUWU = W^2U^*U = W^2,$$

whence $W^2U = UW^2$. Since W is positive, it follows that $WU = UW$, and so

$$U^*W = WU = UW.$$

Since W has dense range, it follows that $U^* = U$. ■

Let us now complete the proof of Theorem 3.3. We denote $U = U_{\rm c}^* U_{\rm o}$ by $\boldsymbol{J}$. It follows from (3.1) that $W\boldsymbol{J} = \boldsymbol{J}W$. Let us show that $A^* = \boldsymbol{J}A\boldsymbol{J}$. Indeed, by the definition of $\boldsymbol{J}$, $U_{\rm o} = U_{\rm c}\boldsymbol{J}$ and since $\boldsymbol{J}W = W\boldsymbol{J}$, we have $V_{\rm o} = V_{\rm c}\boldsymbol{J}$. This means that

$$Ce^{tA}x = B^*e^{tA^*}\boldsymbol{J}x, \quad x \in \mathcal{K},$$

or, which is the same,

$$(x, e^{tA^*}c) = (x, \boldsymbol{J}e^{tA}b) = (x, \boldsymbol{J}e^{tA}\boldsymbol{J}(\boldsymbol{J}b)) = (x, e^{t\boldsymbol{J}A\boldsymbol{J}}\boldsymbol{J}b).$$

Hence,

$$e^{tA^*}c = e^{t\boldsymbol{J}A\boldsymbol{J}}\boldsymbol{J}b, \quad t \ge 0. \tag{3.2}$$

Substituting $t = 0$, we obtain $c = \boldsymbol{J}b$. Differentiating (3.2), we find that A^* and $\boldsymbol{J}A\boldsymbol{J}$ coincide on the orbit $\{e^{tA^*}c : \ t \ge 0\}$, which is dense since the system is observable. Therefore $A^* = \boldsymbol{J}A\boldsymbol{J}$, which completes the proof. ■

Theorem 3.5. *Let $\{A, B, C\}$ be a balanced linear SISO system, $W_{\rm c} = W_{\rm o} = W$, and let $\boldsymbol{J}$ be an operator that is self-adjoint and unitary and satisfies the equalities $\boldsymbol{J}W = W\boldsymbol{J}$, $A^* = \boldsymbol{J}A\boldsymbol{J}$, $c = \boldsymbol{J}b$. Then the Hankel operator $\boldsymbol{\Gamma}_h$ associated with the system is self-adjoint and $\boldsymbol{\Gamma}_h\big|(\operatorname{Ker}\boldsymbol{\Gamma}_h)^\perp$ is unitarily equivalent to $W\boldsymbol{J}$.*

Proof. Since $A^* = \boldsymbol{J}A\boldsymbol{J}$, we have $e^{tA^*} = \boldsymbol{J}e^{tA}\boldsymbol{J}$. The equality $c = \boldsymbol{J}b$ implies $C = B^*\boldsymbol{J}$, which in turn leads to the equality $V_{\rm c} = V_{\rm o}\boldsymbol{J}$. Therefore

$$\boldsymbol{\Gamma}_h = V_{\rm o}\boldsymbol{J}V_{\rm o}^* = U_{\rm o}W^{1/2}\boldsymbol{J}W^{1/2}U_{\rm o}^* = U_{\rm o}W\boldsymbol{J}U_{\rm o}^*,$$

which proves the result. ■

We are going to use Theorem 3.5 to construct a Hankel operator with given spectral properties modulo the kernel. Namely, let Γ be a self-adjoint operator with a scalar spectral measure μ and spectral multiplicity function ν. Suppose that Γ satisfies conditions (C1), (C2), and (C3′) of Theorem 3.1.

Put $\tilde\Gamma = \Gamma\big|(\operatorname{Ker}\Gamma)^\perp$. Let $\boldsymbol{J}$ be the operator on $(\operatorname{Ker}\Gamma)^\perp$ that is self-adjoint and unitary and satisfies $\tilde\Gamma = \boldsymbol{J}|\tilde\Gamma| = |\tilde\Gamma|\boldsymbol{J}$. We are going to construct a balanced linear system $\{A, B, C\}$ such that $W = |\tilde\Gamma|$,

$A^* = \boldsymbol{J}A\boldsymbol{J}$, $c = \boldsymbol{J}b$, and $W\boldsymbol{J} = \boldsymbol{J}W$. Then by Theorem 3.5, $\boldsymbol{\Gamma}_h\big|(\operatorname{Ker}\boldsymbol{\Gamma}_h)^\perp$ is unitarily equivalent to $\tilde\Gamma$. Later we settle the problem with the kernel.

It is not easy to verify directly that $W = |\tilde\Gamma|$. Fortunately, to prove this equality, we do not have to compute $W_{\rm o}$ and $W_{\rm c}$. We are going to verify instead the corresponding Lyapunov equations.

We say that an operator A generates an *asymptotically stable semigroup* if

$$\lim_{t\to\infty} \|e^{tA}x\| = 0$$

for any x.

Theorem 3.6. *Suppose that A generates an asymptotically stable semigroup on a Hilbert space $\mathcal{K}$ and let K be a bounded operator on $\mathcal{K}$. If the integral*

$$\int_{\mathbb{R}_+} e^{tA^*}Ke^{tA}dt \stackrel{\text{def}}{=} W \tag{3.3}$$

converges in the weak operator topology, then W is a unique solution of the following Lyapunov equation

$$A^*W + WA = -K. \tag{3.4}$$

Proof. Let us show that W satisfies (3.4). We have for $x, y \in \mathcal{K}$

$$\begin{aligned}((A^*W+WA)x,y) &= \int_{\mathbb{R}_+}((A^*e^{tA^*}Ke^{tA}+e^{tA^*}Ke^{tA}A)x,y)dt \\ &= \int_{\mathbb{R}_+}\left(\frac{d}{dt}(e^{tA^*}Ke^{tA}x,y)\right)dt \\ &= \lim_{t\to\infty}(Ke^{tA}x,e^{tA}y)-(Kx,y) = -(Kx,y)\end{aligned}$$

because of asymptotic stability.

Let us now establish the uniqueness of the solution. Suppose

$$A^*X + XA = -K$$

for some operator X. Let $\Delta = W - X$. Then

$$A^*\Delta + \Delta A = \mathbb{O}.$$

Clearly,

$$\begin{aligned}\frac{d}{dt}(\Delta e^{tA}x, e^{tA}y) &= (\Delta Ae^{tA}x,e^{tA}y)+(\Delta e^{tA}x,Ae^{tA}y) \\ &= ((A^*\Delta+\Delta A)e^{tA}x,e^{tA}y) = 0.\end{aligned}$$

Since $\lim_{t\to\infty}\|e^{tA}x\| = 0$ for any $x \in \mathcal{K}$, it follows that $(\Delta x, y) = 0$ for any $x, y \in \mathcal{K}$. Hence, $\Delta = \mathbb{O}$. ■

The following result shows that if W satisfies the Lyapunov equation (3.4), we can obtain the convergence of the integral (3.3) for free.

Theorem 3.7. *Let A be an operator such that $\|e^{tA}\| \le M < \infty$, $t \ge 0$, and let K be a nonnegative operator. If W is a solution of the Lyapunov equation*

$$A^*W + WA = -K,$$

then the integral

$$\int_{\mathbb{R}_+} e^{tA^*} K e^{tA} dt$$

converges in the weak operator topology.

Proof. Let $x, y \in \mathcal{K}$. We have

$$(e^{tA^*} K e^{tA} x, y) = -(e^{tA^*}(A^*W + WA)e^{tA}x, y) = -\frac{d}{dt}(e^{tA^*} W e^{tA} x, y).$$

Therefore

$$\int_0^T (e^{tA^*} K e^{tA} x, x) dt = (Wx, x) - (e^{TA^*} W e^{TA} x, x) \le \|W\| \cdot (M+1).$$

Since $(e^{tA^*} K e^{tA} x, x) \ge 0$ for $x \in \mathcal{K}$, it follows that the integral

$$\int_0^\infty (e^{tA^*} K e^{tA} x, x) dt$$

converges for any $x \in \mathcal{K}$. The result follows from the polarization identity. ■

Theorems 3.6 and 3.7 show that to solve the problem modulo the kernel, it is sufficient to construct a SISO linear system $\{A, B, C\}$ such that

(i) the operators A and A^* generate asymptotically stable semigroups;
(ii) the operator $W = |\tilde{\Gamma}|$ is a solution of the Lyapunov equations

$$A^*W + WA = -C^*C, \quad AW + WA^* = -BB^*;$$

(iii) $A^* = \boldsymbol{J}A\boldsymbol{J}$, $c = \boldsymbol{J}b$.

If $A^* = \boldsymbol{J}A\boldsymbol{J}$, then A^* generates an asymptotically stable semigroup if and only if so does A. It follows easily from (iii) that both Lyapunov equations in (ii) coincide. Therefore it is sufficient to verify the following properties:

(i′) A generates an asymptotically stable semigroup;
(ii′) $A^*W + WA = -C^*C$;
(iii′) $A^* = \boldsymbol{J}A\boldsymbol{J}$, $c = \boldsymbol{J}b$.

4. Construction of a Linear System with Continuous Time

Let Γ be a self-adjoint operator on Hilbert space that satisfies conditions (C1), (C2), and (C3′) of Theorem 3.1. As before we put $\tilde{\Gamma} = \Gamma \big| (\operatorname{Ker}\Gamma)^\perp$.

Then the operator $|\tilde{\Gamma}|$ is unitarily equivalent to the operator W of multiplication by $\boldsymbol{t}$ on the von Neumann integral

$$\mathcal{K} = \int_{\sigma(W)} \oplus E(\boldsymbol{t}) d\rho(\boldsymbol{t}). \tag{4.1}$$

Here ρ is a scalar spectral measure of $|\tilde{\Gamma}|$. Note that to avoid a conflict of notation, we use in §§4–8 the bold symbol $\boldsymbol{t}$ for the coordinate variable in von Neumann integrals while t is used for the time variable.

We can also assume that the spaces $E(\boldsymbol{t})$ are imbedded in an infinite-dimensional space E with an orthonormal basis $\{e_j\}_{j\ge1}$ and

$$E(\boldsymbol{t}) = \operatorname{span}\{e_k : \ 1 \le k < \nu_W(\boldsymbol{t}) + 1\}, \quad E(\boldsymbol{t}) = \{\mathbb{O}\} \quad \text{if} \quad \nu_W(\boldsymbol{t}) = 0,$$

where ν_W is the spectral multiplicity function of W, $\nu_W(\boldsymbol{t}) = \dim E(\boldsymbol{t})$.

Clearly, $\nu_W(\boldsymbol{t}) = \nu(\boldsymbol{t}) + \nu(-\boldsymbol{t})$, ρ-a.e., where ν is the spectral multiplicity function of Γ. Recall that ν satisfies condition (C3$'$) of Theorem 3.1.

Consider the sets

$$\sigma_+ = \{\boldsymbol{t} \in \sigma(W) : \ \nu(\boldsymbol{t}) \ge \nu(-\boldsymbol{t})\},$$

$$\sigma_- = \{\boldsymbol{t} \in \sigma(W) : \ \nu(\boldsymbol{t}) < \nu(-\boldsymbol{t})\}.$$

We define the operator $\boldsymbol{J}$ on $\mathcal{K}$ as multiplication by the operator-valued function J:

$$J(\boldsymbol{t}) = \xi(\boldsymbol{t}) \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & -1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$\xi(\boldsymbol{t}) = \begin{cases} 1, & \boldsymbol{t} \in \sigma_+, \\ -1, & \boldsymbol{t} \in \sigma_-. \end{cases}$$

Clearly, $\tilde{\Gamma}$ is unitarily equivalent to $\boldsymbol{J}W$ and we can assume that $\tilde{\Gamma} = \boldsymbol{J}W$.

Recall that the scalar spectral measure ρ of W is not uniquely defined and we can always replace it with a mutually absolutely continuous measure by multiplying it by a positive weight w in $L^1(\rho)$.

Let $\mathcal{K}_0$ be the subspace of $\mathcal{K}$ that consists of functions f of the form

$$f(\boldsymbol{t}) = \varphi(\boldsymbol{t}) e_1, \quad \varphi \in L^2(\rho).$$

$\mathcal{K}_0$ can be identified naturally with $L^2(\rho)$. Let A_0 be the integral operator on $\mathcal{K}_0 = L^2(\rho)$ defined by

$$(A_0 f)(\boldsymbol{s}) = \int_{\sigma(W)} k(\boldsymbol{s}, \boldsymbol{t}) f(\boldsymbol{t}) d\rho(\boldsymbol{t}),$$

where

$$k(\boldsymbol{s},\boldsymbol{t}) = \begin{cases} -1/(\boldsymbol{s}+\boldsymbol{t}), & \boldsymbol{s},\boldsymbol{t}\in\sigma_+ \quad \text{or} \quad \boldsymbol{s},\boldsymbol{t}\in\sigma_-, \\ -1/(\boldsymbol{s}-\boldsymbol{t}), & \boldsymbol{s}\in\sigma_+,\ \boldsymbol{t}\in\sigma_- \text{ or } \boldsymbol{s}\in\sigma_-,\ \boldsymbol{t}\in\sigma_+. \end{cases} \tag{4.2}$$

The operator A_0 certainly need not be bounded. However, the following lemma allows us to change the measure ρ so that A_0 becomes a Hilbert-Schmidt operator. We extend A_0 to $\mathcal{K}$ by putting $A_0\big|\mathcal{K}_0^\perp = \mathbb{O}$.

Lemma 4.1. *There exists $w \in L^1(\rho)$, $w > 0$, ρ-a.e., such that the integral operator*

$$f \mapsto \int_{\sigma(W)} k(\boldsymbol{s},\boldsymbol{t})f(\boldsymbol{t})d\tilde{\rho}(\boldsymbol{t})$$

is a Hilbert–Schmidt operator on $L^2(\tilde{\rho})$, where $d\tilde{\rho} = wd\rho$.

We postpone the proof of Lemma 4.1 until §6.

Let us define the vectors b and c in $\mathcal{K}$ by

$$c(\boldsymbol{t}) = e_1, \quad b(\boldsymbol{t}) = J(\boldsymbol{t})c(\boldsymbol{t}) = \xi(\boldsymbol{t})c(\boldsymbol{t}), \quad \boldsymbol{t}\in\sigma(W).$$

It is easy to see that

$$A_0^* = \boldsymbol{J}A_0\boldsymbol{J} \quad \text{and} \quad A_0^*W + WA_0 = -C^*C. \tag{4.3}$$

However, $\operatorname{Ker} A_0 = \mathcal{K}_0^\perp$ is nontrivial except for the case when W has simple spectrum. Therefore A_0 does not generate an asymptotically stable semigroup in general. To overcome this obstacle, we perturb A_0 by an operator D such that the perturbed operator still satisfies (4.3) but generates an asymptotically stable semigroup. Of course, to get asymptotic stability, it is not sufficient to kill the kernel.

Let $\{a_k^{(n)}\}_{k=1}^{n-1}$, $n \in \mathbb{N}\cup\{\infty\}$, be positive numbers such that

$$\sum_{k=1}^{n-1}\left(a_k^{(n)}\right)^2 < \frac{1}{2n^2}, \quad n\in\mathbb{N}, \quad \text{and} \quad \sum_{k=1}^{\infty}\left(a_k^{(\infty)}\right)^2 < \infty. \tag{4.4}$$

We define D as multiplication by the operator-valued function d:

$$d(\boldsymbol{t}) = \begin{pmatrix} 0 & a_1^{(n)} & 0 & 0 & \cdots \\ -a_1^{(n)} & 0 & a_2^{(n)} & 0 & \cdots \\ 0 & -a_2^{(n)} & 0 & a_3^{(n)} & \cdots \\ 0 & 0 & -a_3^{(n)} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.5}$$

in the basis $\{e_k\}_{1\le k<n(t)+1}$, where $n = \nu_W(\boldsymbol{t}) = \dim E(\boldsymbol{t})$.

Theorem 4.2. *Let $A = A_0 + D$, where D is defined by* (4.5) *and suppose that* (4.4) *holds. Then*

$$A^*W + WA = -C^*C, \quad A^* = \boldsymbol{J}A\boldsymbol{J},$$

and A generates an asymptotically stable semigroup.

As we have observed in §3, Theorem 4.2 implies the following result, which solves the problem modulo the kernel.

Corollary 4.3. *Let $\boldsymbol{\Gamma}_h$ be the Hankel operator associated with the linear system constructed above. Then $\boldsymbol{\Gamma}_h \big| (\operatorname{Ker} \boldsymbol{\Gamma}_h)$ is unitarily equivalent to $\boldsymbol{J}W = \tilde{\Gamma}$.*

Proof of Theorem 4.2. Clearly, D commutes with W and $D^* = -D$. Thus the operator $A = A_0 + D$ satisfies the Lyapunov equation

$$A^*W + WA = -C^*C. \tag{4.6}$$

It is easy to show that $D^* = \boldsymbol{J}D\boldsymbol{J}$, which implies $A^* = \boldsymbol{J}A\boldsymbol{J}$.

It remains to prove that A generates an asymptotically stable semigroup. Let us show first that $A + A^* \le \mathbb{O}$, which would imply that $\{e^{tA}\}_{t\ge 0}$ is a semigroup of contractions (see Sz.-Nagy and Foias [1], §3.8 and §9.4). We need the following lemma.

Lemma 4.4. *Let W be a nonnegative self-adjoint operator with trivial kernel and let R be a self-adjoint operator such that*

$$RW + WR \le \mathbb{O}.$$

Then $R \le \mathbb{O}$.

Proof. Let $K = -(RW + WR)$. Since $-W \le \mathbb{O}$ and $\operatorname{Ker} W = \{\mathbb{O}\}$, it is easy to see that $-W$ generates an asymptotically stable semigroup. So we can apply Theorems 3.6 and 3.7, with $-W$ playing the role of A and $-R$ playing the role of W. Then we find that R is the unique solution of the equation

$$XW + WX = -K,$$

and the solution is given by

$$R = -\int_{\mathbb{R}_+} e^{-tW} K e^{-tW}\,dt.$$

Hence, $R \le \mathbb{O}$. ■

It follows from (4.6) and from the identity

$$AW + WA^* = -BB^* \tag{4.7}$$

that

$$(A + A^*)W + W(A + A^*) = -C^*C - BB^* \le \mathbb{O},$$

and so $R = A + A^*$ satisfies the hypotheses of Lemma 4.4. Thus $A + A^* \le \mathbb{O}$.

It is well known (see e.g., Sz.-Nagy and Foias [1], §9.4) that the above inequality implies that

$$\sigma(A) \subset \{\zeta \in \mathbb{C} : \ \operatorname{Re} \zeta \le 0\}.$$

Let us show that A has no eigenvalues on the imaginary axis. Let

$$Ax = \mathrm{i}\omega x, \quad x \in \mathcal{K}, \quad x \ne \mathbb{O}, \quad \omega \in \mathbb{R}.$$

Then

$$\begin{aligned} -(C^*Cx, x) &= (A^*Wx, x) + (WAx, x) \\ &= (Wx, Ax) + (Ax, Wx) = -\mathrm{i}\omega(x, Wx) + \mathrm{i}\omega(x, Wx) = 0, \end{aligned}$$

and so $Cx = 0$, i.e., $x \perp c$. Similarly, if

$$A^*\tilde{x} = \mathrm{i}\tilde{\omega}\tilde{x}, \quad \tilde{x} \in \mathcal{K}, \quad \tilde{\omega} \in \mathbb{R},$$

then $B^*\tilde{x} = 0$, i.e., $\tilde{x} \perp b$.

Applying equality (4.6) to the eigenvector x and taking into account that $Cx = 0$, we obtain

$$A^*Wx = -\mathrm{i}\omega Wx,$$

i.e., Wx is an eigenvector of A^*. Now applying equality (4.7) to the eigenvector Wx and taking into account that $B^*Wx = 0$, we obtain

$$AW^2x = \mathrm{i}\omega W^2x.$$

Repeating this procedure, we obtain

$$AW^{2n}x = \mathrm{i}\omega W^{2n}x, \quad n \geq 0.$$

It follows that for any eigenvector x of A with eigenvalue $\mathrm{i}\omega$ and for any bounded measurable function φ

$$A\varphi(W)x = \mathrm{i}\omega\varphi(W)x. \tag{4.8}$$

Consider the representation of x in the direct integral:

$$x(\boldsymbol{t}) = \sum_{k=1}^{n(\boldsymbol{t})} x_k(\boldsymbol{t})e_k.$$

It follows from (4.8) that $\varphi(W)x$ is orthogonal to c, i.e.,

$$\int \varphi(\boldsymbol{t})x_1(\boldsymbol{t})d\rho(\boldsymbol{t}) = 0$$

for any measurable φ. It follows that $x_1(\boldsymbol{t}) = 0$ a.e., i.e., $x \perp \mathcal{K}_0$. Hence, $Ax = Dx$, and so $Dx = \mathrm{i}\omega x$, which means

$$\begin{pmatrix} 0 & a_1^{(n)} & 0 & 0 & \cdots \\ -a_1^{(n)} & 0 & a_2^{(n)} & 0 & \cdots \\ 0 & -a_2^{(n)} & 0 & a_3^{(n)} & \cdots \\ 0 & 0 & -a_3^{(n)} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 0 \\ x_2(\boldsymbol{t}) \\ x_3(\boldsymbol{t}) \\ x_4(\boldsymbol{t}) \\ \vdots \end{pmatrix} = \mathrm{i}\omega \begin{pmatrix} 0 \\ x_2(\boldsymbol{t}) \\ x_3(\boldsymbol{t}) \\ x_4(\boldsymbol{t}) \\ \vdots \end{pmatrix},$$

where $x(\boldsymbol{t}) = \sum_{k=1}^{n} x_k(\boldsymbol{t})e_k$, $n = \nu_W(\boldsymbol{t}) = \dim E(\boldsymbol{t})$.

Performing multiplication, we obtain

$$\begin{pmatrix} a_1^{(n)} x_2(\boldsymbol{t}) \\ a_2^{(n)} x_3(\boldsymbol{t}) \\ -a_2^{(n)} x_2(\boldsymbol{t}) + a_3^{(n)} x_4(\boldsymbol{t}) \\ -a_3^{(n)} x_3(\boldsymbol{t}) + a_4^{(n)} x_5(\boldsymbol{t}) \\ \vdots \end{pmatrix} = \mathrm{i}\omega \begin{pmatrix} 0 \\ x_2(\boldsymbol{t}) \\ x_3(\boldsymbol{t}) \\ x_4(\boldsymbol{t}) \\ \vdots \end{pmatrix}.$$

Comparing the components from top to bottom, we see that $x_k(\boldsymbol{t}) = 0$ for any k, i.e., $x = \mathbb{O}$.

To establish asymptotic stability, we need the following result that follows easily from Proposition 6.7 in Chapter II of Sz.-Nady and Foias [1] by applying Cayley transform (see Sz.-Nady and Foias [1], Ch. IV, §4).

Stability test. *Let $\{e^{tA}\}_{t\ge 0}$ be a strongly continuous semigroup of contractions on a Hilbert space $\mathcal{K}$ such that*

(i) *the spectrum $\sigma(A)$ of its generator is contained in $\{\zeta : \operatorname{Re}\zeta \le 0\}$;*

(ii) *A has no eigenvalues on the imaginary axis* $\mathrm{i}\mathbb{R}$;

(iii) *the set $\sigma(A) \cap \mathrm{i}\mathbb{R}$ is at most countable. Then the semigroup $\{e^{tA}\}_{t\ge 0}$ is asymptotically stable, i.e.*

$$\lim_{t\to\infty} \|e^{tA}x\| = 0 \quad \text{for any} \quad x \in \mathcal{K}.$$

We have already proved that our operator A satisfies (i) and (ii). It remains to show that A satisfies (iii) provided (4.3) holds.

Let D_n be the operator on $\operatorname{span}\{e_k : 1 \le k < n+1\}$, $1 \le n \le \infty$, given by the matrix

$$D_n = \begin{pmatrix} 0 & a_1^{(n)} & 0 & 0 & \cdots \\ -a_1^{(n)} & 0 & a_2^{(n)} & 0 & \cdots \\ 0 & -a_2^{(n)} & 0 & a_3^{(n)} & \cdots \\ 0 & 0 & -a_3^{(n)} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We have $D^* = -D$ (i.e., $\mathrm{i}D$ is self-adjoint), and so

$$\sigma(D) \subset \operatorname{clos}\left(\bigcup_{n=1}^{\infty} \sigma(D_n)\right) \bigcup \sigma(D_\infty).$$

Clearly, D_n is a Hilbert–Schmidt operator for $1 \le n \le \infty$, and its Hilbert–Schmidt norm satisfies $\|D_n\|_{\boldsymbol{S}_2} \le 1/n$ for $n < \infty$.

Therefore $\sigma(D_n) \subset [-\mathrm{i}/n, \mathrm{i}/n]$, $n < \infty$, $\sigma(D_n)$ is finite for any $n < \infty$, $\sigma(D_\infty)$ is countable and can accumulate only at 0. Hence, the only possible accumulation point of the set $\left(\bigcup_{n=1}^{\infty}\sigma(D_n)\right)\bigcup\sigma(D_\infty)$ is 0. Consequently, $\sigma(D)$ is at most countable.

Since A_0 is a Hilbert–Schmidt operator, A is a compact perturbation of D. So the essential spectrum $\sigma_e(A)$ of A is equal to $\sigma_e(D)$, and for any $\lambda \notin \sigma_e(A)$ we have $\operatorname{ind}(A - \lambda I) = 0$ (see Appendix 1.2). So if $\lambda \in \sigma(A)\backslash\sigma_e(A)$, then λ must be an eigenvalue of A. We have already proved that A has no eigenvalues on $\mathrm{i}\mathbb{R}$, which implies that

$$\sigma(A)\cap \mathrm{i}\mathbb{R} = \sigma_e(A)\cap \mathrm{i}\mathbb{R} = \sigma_e(D)\cap \mathrm{i}\mathbb{R} \subset \sigma(D)\cap \mathrm{i}\mathbb{R},$$

and the last set is at most countable. ■

5. The Kernel of $\boldsymbol{\Gamma}_h$

In this section we prove Theorem 3.1. In the previous section we have constructed a linear system $\{A, B, C\}$ such that the corresponding Hankel operator $\boldsymbol{\Gamma}_h$ restricted to $(\operatorname{Ker}\boldsymbol{\Gamma}_h)^\perp$ is unitarily equivalent to $\Gamma\big|(\operatorname{Ker}\Gamma)^\perp$. To prove Theorem 3.1 completely, we have to solve the problem of the description of $\operatorname{Ker}\boldsymbol{\Gamma}_h$. The solution of this problem is given by the following theorem where we consider the Hankel operator $\boldsymbol{\Gamma}_h$ associated with the system $\{A, B, C\}$ that we have constructed in the previous section.

Theorem 5.1. *Let ρ be the scalar spectral measure of W in* (4.1). *Then* $\operatorname{Ker}\boldsymbol{\Gamma}_h = \{\mathbb{O}\}$ *if and only if* $\int_{\sigma(W)}(1/\boldsymbol{s})d\rho(\boldsymbol{s}) = \infty$.

Theorem 3.1 follows now from Theorem 5.1 and the following lemma.

Lemma 5.2. *Let ρ be a finite positive Borel measure on* $[0, a]$, $a > 0$, *such that* $0 \in \operatorname{supp}\rho$ *and*

$$\iint (k(\boldsymbol{s}, \boldsymbol{t}))^2 d\rho(\boldsymbol{s})d\rho(\boldsymbol{t}) < \infty, \tag{5.1}$$

where k is defined by (4.2). *Then we can change ρ by multiplying it by a positive weight in* $L^1(\rho)$ *so that* (5.1) *still holds and*

$$\int \frac{1}{\boldsymbol{s}} d\rho(\boldsymbol{s}) = \infty.$$

Remark. It is obvious that under the hypotheses of Lemma 5.2 one can change a measure ρ (by multiplying it by a positive weight in $L^1(\rho)$) so that (5.1) holds and

$$\int \frac{1}{\boldsymbol{s}} d\rho(\boldsymbol{s}) < \infty.$$

Indeed, it is sufficient to take a weight that is sufficiently small near the origin.

The proof of Lemma 5.2 will be given in §6. Let us first derive Theorem 3.1 and then prove Theorem 5.1.

Proof of Theorem 3.1. Let Γ be a self-adjoint operator that satisfies the assumptions of Theorem 1.1. Put $\tilde{\Gamma} = \Gamma\big|(\operatorname{Ker}\Gamma)^\perp$. Let $W = |\tilde{\Gamma}|$ and consider the representation of W in the form (4.1) where ρ is a scalar spectral measure of W. Let $A, B, C, \boldsymbol{J}$ be as in §4 and let $\boldsymbol{\Gamma}_h$ be the Hankel operator associated with the system $\{A, B, C\}$.

Suppose that W is invertible. Since Γ is noninvertible, the subspace $\operatorname{Ker}\Gamma$ is infinite-dimensional. The operator $\boldsymbol{\Gamma}_h\big|(\operatorname{Ker}\boldsymbol{\Gamma}_h)^\perp$ being unitarily equivalent to $W\boldsymbol{J}$ is also invertible. So $\operatorname{Ker}\boldsymbol{\Gamma}_h$ is infinite-dimensional, which implies that $\boldsymbol{\Gamma}_h$ is unitarily equivalent to Γ.

Suppose now that W is non-invertible. If $\operatorname{Ker}\Gamma = \{\mathbb{O}\}$, we can choose by Lemma 5.2 a scalar spectral measure ρ of W so that $\int(1/\boldsymbol{s})d\rho(\boldsymbol{s}) = \infty$. Then by Theorem 5.1, $\operatorname{Ker}\boldsymbol{\Gamma}_h = \{\mathbb{O}\}$, and so $\boldsymbol{\Gamma}_h$ is unitarily equivalent to Γ.

If Γ has an infinite-dimensional kernel, then by the Remark to Lemma 5.2, we can choose ρ so that $\int(1/\boldsymbol{s})d\rho(\boldsymbol{s}) < \infty$, and by Theorem 5.1, $\boldsymbol{\Gamma}_h$ has an infinite-dimensional kernel. Again, $\boldsymbol{\Gamma}_h$ is unitarily equivalent to Γ. ■

Proof of Theorem 5.1. Let us first prove that $\operatorname{Ker}\boldsymbol{\Gamma}_h$ is nontrivial provided $\int(1/\boldsymbol{s})d\rho(\boldsymbol{s}) < \infty$. Assume that $\operatorname{Ker}\boldsymbol{\Gamma}_h = \{\mathbb{O}\}$. Since $\boldsymbol{\Gamma}_h = V_{\rm o}V_{\rm c}^* = V_{\rm o}\boldsymbol{J}V_{\rm o}^*$, it follows that $\operatorname{Ker}V_{\rm o}^* = \{\mathbb{O}\}$, which is equivalent to the fact that $V_{\rm o}$ has dense range in $L^2(\mathbb{R}_+)$.

Let $\{\Phi_t\}_{t\ge 0}$ be the semigroup of backward translations on $L^2(\mathbb{R}_+)$,

$$(\Phi_t f)(s) = f(s+t), \quad s, t \ge 0.$$

It is easy to see that

$$V_{\rm o}e^{tA} = \Phi_t V_{\rm o}. \tag{5.2}$$

Clearly, the condition $\int(1/\boldsymbol{s})d\rho(\boldsymbol{s}) < \infty$ means that $c \in \operatorname{Range}W^{1/2}$.

Let $V_{\rm o} = U_{\rm o}W^{1/2}$ be the polar decomposition of $V_{\rm o}$ (see the proof of Theorem 3.2). Since $\operatorname{Ker}\boldsymbol{\Gamma}_h = \{\mathbb{O}\}$, it follows that $U_{\rm o}$ is unitary. Therefore $c \in \operatorname{Range}V_{\rm o}^*$. Let $c = V_{\rm o}^* f,\ f \in L^2(\mathbb{R})$. Define the operator $F : L^2(\mathbb{R}) \to \mathbb{C}$ by

$$F\varphi = (\varphi, f).$$

Then obviously, $FV_{\rm o} = C$, where as above $Cx = (x, c)$. Therefore by (5.2),

$$Ce^{tA} = F\Phi_t V_{\rm o},$$

and so

$$e^{tA^*}C^*Ce^{tA} = V_{\rm o}^*\Phi_t^*F^*F\Phi_t V_{\rm o},$$

which implies

$$\int_{\mathbb{R}_+} e^{tA^*}C^*Ce^{tA}dt = W = V_{\rm o}^*\left(\int_{\mathbb{R}_+}\Phi_t^*F^*F\Phi_t\,dt\right)V_{\rm o}.$$

Since $V_{\rm o} = U_{\rm o}W^{1/2}$ and $V_{\rm o}^* = W^{1/2}U_{\rm o}^*$, we have

$$U_{\rm o}^*\left(\int_{\mathbb{R}} \Phi_t^* F^* F \Phi_t \, dt\right) U_{\rm o} = I$$

and bearing in mind that $U_{\rm o}$ is unitary, we obtain

$$\int_{\mathbb{R}_+} \Phi_t^* F^* F \Phi_t \, dt = I. \tag{5.3}$$

Consider the function f_τ, $\tau \ge 0$, defined by

$$f_\tau(t) = \begin{cases} f(t-\tau), & t \ge \tau, \\ 0, & t < \tau, \end{cases}$$

and the operators $F_\tau : L^2(\mathbb{R}) \to \mathbb{C}$ defined by

$$F_\tau x = (x, f_\tau).$$

Since $F\Phi_\tau = F_\tau$, (5.3) can be rewritten as

$$\int_{\mathbb{R}_+} F_\tau^* F_\tau d\tau = I. \tag{5.4}$$

Clearly, $F_\tau^* F_\tau$ is the integral operator with kernel function $\varkappa_\tau(s,t) = f_\tau(s)\overline{f_\tau(t)}$. We have

$$\varkappa(s,t) \stackrel{\text{def}}{=} \int_0^\infty \varkappa_\tau(s,t) d\tau = \int_0^\infty f_\tau(s)\overline{f_\tau(t)} d\tau = \int_0^{\min\{s,t\}} f(s-\tau)\overline{f(t-\tau)} d\tau,$$

whence

$$|\varkappa(s,t)| \le \|f\|_2^2.$$

Therefore (5.4) implies that

$$(\varphi, \psi) = \int_0^\infty \int_0^\infty \varkappa(s,t)\varphi(s)\overline{\psi(t)} ds dt \tag{5.5}$$

at least for compactly supported φ and ψ in L^2.

Now let

$$\varphi(s) = \psi(s) = \begin{cases} 1/\sqrt{\varepsilon}, & 0 \le s \le \varepsilon, \\ 0, & s > \varepsilon. \end{cases}$$

Then (5.5) implies that

$$1 = \|\varphi\|_2^2 = (\varphi, \varphi) \le \varepsilon \|\varkappa\|_\infty \le \varepsilon \|f\|_2^2 < 1$$

for a sufficiently small ε. The contradiction obtained proves that $\operatorname{Ker} \boldsymbol{\Gamma}_h \ne \{\mathbb{O}\}$.

Let us show that if $\operatorname{Ker} \boldsymbol{\Gamma}_h \ne \{\mathbb{O}\}$, then $\int \frac{1}{\boldsymbol{s}} d\rho(\boldsymbol{s}) < \infty$. It is easy to see that the subspace $\boldsymbol{K} = (\operatorname{Ker} \boldsymbol{\Gamma}_h)^\perp$ is invariant under the semigroup of backward translations $\{\Phi_s\}_{s \ge 0}$. Let $\Psi_s \stackrel{\text{def}}{=} \Phi_s \big| \boldsymbol{K}$. Since $\operatorname{Ker} \boldsymbol{\Gamma}_h = \operatorname{Ker} V_{\rm o}^*$, it follows that $\boldsymbol{K} = \operatorname{clos} \operatorname{Range} V_0^*$ and so by (5.2),

$$V_{\rm o} e^{tA} = \Psi_t V_{\rm o}. \tag{5.6}$$

We need the following lemma.

Lemma 5.3. *The semigroup* $\{\Psi_t\}_{t\ge 0}$ *has bounded generator, i.e.,* $\Psi_t = e^{tG}$, $t \ge 0$, *for a bounded linear operator* G *on* $\boldsymbol{K}$.

Let us first complete the proof of Theorem 5.1 and then prove Lemma 5.3. By Lemma 5.3, $\Psi_t = e^{tG}$, $t \ge 0$. It follows from (5.6) that

$$V_{\rm o}A = GV_{\rm o},$$

and so

$$G = V_{\rm o}AV_{\rm o}^{-1}$$

($V_{\rm o}AV_{\rm o}^{-1}$ is defined on a dense subset of $\boldsymbol{K}$ and extends by continuity to a bounded operator).

Put

$$R \stackrel{\rm def}{=} G + G^* = V_{\rm o}AV_{\rm o}^{-1} + (V_{\rm o}^{-1})^*A^*V_{\rm o}^*.$$

Multiplying this equality by $V_{\rm o}$ on the right and $V_{\rm o}^*$ on the left and bearing in mind that $V_{\rm o}^*V_{\rm o} = W$, we obtain

$$WA + A^*W = V_{\rm o}^*RV_{\rm o}.$$

On the other hand, we have from (4.6)

$$WA + A^*W = -C^*C = -(\cdot, c)c.$$

So $V_{\rm o}^*RV_{\rm o} = -C^*C$. Hence, $c \in \operatorname{Range} V_{\rm o}^*$ or, which is equivalent, $c \in \operatorname{Range} W^{1/2}$ (see the proof of Theorem 3.2). Clearly, the last condition exactly means that $\int s^{-1}d\rho(s) < \infty$. ■

Proof of Lemma 5.3. It is slightly more convenient to consider here the Fourier transform $\boldsymbol{\mathcal{F}}$ defined by

$$(\boldsymbol{\mathcal{F}}h)(y) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} h(x)e^{-\mathrm{i}xy}dx, \quad h \in L^1(\mathbb{R}).$$

To show that our contractive semigroup has bounded generator, we apply the inverse Fourier transform $\boldsymbol{\mathcal{F}}^{-1}$, which maps $L^2(\mathbb{R}_+)$ onto the Hardy class $H^2(\mathbb{C}_+)$. Put $\check{\Phi}_s = \boldsymbol{\mathcal{F}}^{-1}\Phi_s\boldsymbol{\mathcal{F}}$ and $\check{\Psi}_s = \boldsymbol{\mathcal{F}}^{-1}\Psi_s\boldsymbol{\mathcal{F}}$. Then

$$\check{\Psi}_s f = \mathbb{P}_+ e_{-s} f,$$

where $e_{-s}(t) = e^{-\mathrm{i}st}$ and $\mathbb{P}_+$ is the orthogonal projection from $L^2(\mathbb{R})$ onto $H^2(\mathbb{C}_+)$. Clearly,

$$\check{\Psi}_s = \check{\Phi}_s|\check{\boldsymbol{K}},$$

where $\check{\boldsymbol{K}} = \mathcal{F}^{-1}\boldsymbol{K}$.

Obviously, $\check{\boldsymbol{K}}$ is an invariant subspace of the semigroup $\{\check{\Phi}_s\}_{s\ge 0}$, and so by Beurling's theorem (see Appendix 2.2), $\check{\boldsymbol{K}}$ has the form $\boldsymbol{K}_\vartheta = H^2(\mathbb{C}_+) \ominus \vartheta H^2(\mathbb{C}_+)$, where ϑ is an inner function in $\mathbb{C}_+$.

It is sufficient to show that the semigroup $\{\check{\Psi}_s\}_{s\ge 0}$ has bounded generator. Since $\{\check{\Psi}_s\}_{s\ge 0}$ is a contractive semigroup, we can consider its *cogenerator* T defined by

$$T = \lim_{s\to 0+} \varphi_s(\check{\Psi}_s)\,,$$

where

$$\varphi_s(\lambda) = \frac{\lambda - 1 + s}{\lambda - 1 - s}\,.$$

Then $\|T\| \le 1$ (see Sz.-Nagy and Foias [1], §3.8). Moreover, if we show that $1 \notin \sigma(T)$, then the bounded operator $\breve{G} \stackrel{\text{def}}{=} (T+I)(T-I)^{-1}$ is the generator of the semigroup $\{\breve{\Psi}_s\}_{s\ge 0}$, i.e., $\breve{\Psi}_s = e^{s\breve{G}}$, $s > 0$ (see Sz.-Nagy, Foias [1], §3.8).

Let us first evaluate the cogenerator T. Taking into account that for $\psi \in H^\infty(\mathbb{C}_+)$

$$\psi(\breve{\Psi}_s)f = \mathbb{P}_+\psi(e_{-s})f, \quad f \in \boldsymbol{K}_\vartheta,$$

we can conclude that the cogenerator T of the semigroup $\breve{\Psi}_s$ is given by the formula

$$Tf = \mathbb{P}_+\varphi f, \quad f \in \boldsymbol{K}_\vartheta,$$

where $\varphi(t) = \lim\limits_{s\to 0+} \varphi_s(e^{-\mathrm{i}st}) = (s+\mathrm{i})/(s-\mathrm{i})$.

Let ω be the conformal map of the unit disc $\mathbb{D}$ onto the half-plane $\mathbb{C}_+$ defined by

$$\omega(z) = \mathrm{i}\frac{1+z}{1-z}.$$

Put $\breve{\vartheta} \stackrel{\text{def}}{=} \vartheta \circ \omega$. Then $\breve{\vartheta}$ is an inner function in $\mathbb{D}$. The operator $\mathcal{U}$,

$$(\mathcal{U}f)(t) = \pi^{1/2}\frac{1}{t+\mathrm{i}}(f\circ\omega^{-1})(t) = \pi^{1/2}\frac{1}{t+\mathrm{i}} f\left(\frac{t-\mathrm{i}}{t+\mathrm{i}}\right), \quad t \in \mathbb{R},$$

maps unitarily H^2 onto $H^2(\mathbb{C}_+)$ and $K_{\breve{\vartheta}}$ onto $\boldsymbol{K}_\vartheta$. Moreover, $\mathcal{U}^{-1}T\mathcal{U}f = S^*\big|K_{\breve{\vartheta}}$, where S^* is backward shift on H^2 (see Appendix 2.1). Thus T is unitarily equivalent to $S^*\big|K_{\breve{\vartheta}}$.

To complete the proof, we have to show that the spectrum $\sigma(\breve{\vartheta})$ (see Appendix 2.1) of the inner function $\breve{\vartheta}$ does not contain the point 1, since the spectrum of $S^*\big|(H^2 \ominus \breve{\vartheta}H^2)$ is equal to $\overline{\sigma(\breve{\vartheta})}$, i.e., the complex conjugate of $\sigma(\breve{\vartheta})$ (see Appendix 2.1).

If ϑ is an inner function in $\mathbb{C}_+$, we put by definition $\sigma(\vartheta) = \omega(\sigma(\breve{\vartheta}))$, the spectrum of ϑ. Note that $\sigma(\vartheta)$ can contain ∞ (this happens if and only if $1 \in \sigma(\breve{\vartheta})$).

Let us show that $\sigma(\vartheta) \subset -\mathrm{i}\overline{\sigma(A)}$, which would imply that $\sigma(\vartheta)$ is bounded and since

$$\sigma(T) = \overline{\sigma(\breve{\vartheta})} = \omega^{-1}(\sigma(\vartheta)),$$

it would follow that $1 \notin \sigma(T)$, which would complete the proof.

We are going to use the notion of pseudocontinuation. As in the case of the unit disk a function $f \in H^2(\mathbb{C}_+)$ is said to have a *pseudocontinuation* if there exists a meromorphic function g in the Nevanlinna class

$$N(\mathbb{C}_-) \stackrel{\text{def}}{=} \left\{\frac{g_1}{g_2}:\ g_1, g_2 \in H^\infty(\mathbb{C}_-)\right\}$$

such that the boundary value of g coincides with the boundary values of f almost everywhere on $\mathbb{R}$ (see Appendix 2.2). In this case g is called a pseudocontinuation of f. It is well known that if $f \in \boldsymbol{K}_\vartheta$, then f has a pseudocontinuation (see Appendix 2.2, the case of functions in $\mathbb{C}_+$ reduces very easily to the case of functions in $\mathbb{D}$).

We are going to use the following facts (see Appendix 2.1 and Appendix 2.2):

(i) if a function $f \in H^2(\mathbb{C}_+)$ has a pseudocontinuation and f extends analytically across an interval $I \subset \mathbb{R}$, then its analytic extension coincides with its pseudocontinuation;

(ii) if ϑ is an inner function such that $\mathbb{R} \not\subset \sigma(\vartheta)$, then ϑ extends analytically to $\mathbb{C} \setminus \overline{\sigma(\vartheta)}$;

(iii) if $\mathbb{R} \not\subset \sigma(\vartheta)$ and $f \in \boldsymbol{K}_\vartheta$, then f extends analytically to $\mathbb{C} \setminus \overline{\sigma(\vartheta)}$.

Again, the above results reduce very easily to the case of functions in $\mathbb{D}$.

Now we are ready to prove that $\sigma(\vartheta) \subset \mathrm{i}\overline{\sigma(A)}$. Put $\widehat{\mathbb{C}} \stackrel{\text{def}}{=} \mathbb{C} \cup \{\infty\}$. For a function $f \in \boldsymbol{K}_\vartheta$ we consider its pseudocontinuation and denote this pseudocontinuation also by f. We need the following lemma.

Lemma 5.4. *Let ϑ be an inner function in $\mathbb{C}_+$ and let σ be a closed subset of its spectrum $\sigma(\vartheta)$ such that $\sigma \neq \sigma(\vartheta)$. Then there exists a nontrivial inner divisor ϑ_1 of ϑ (i.e., $\vartheta/\vartheta_1 \neq \text{const}$) such that every function $f \in \boldsymbol{K}_\vartheta$ that extends analytically to $\widehat{\mathbb{C}} \setminus \overline{\sigma}$ belongs to $\boldsymbol{K}_{\vartheta_1}$.*

Let us first complete the proof of Lemma 5.3. Suppose that $\sigma(\vartheta) \not\subset -\mathrm{i}\overline{\sigma(A)}$. Put $\sigma = \sigma(\vartheta) \cap -\mathrm{i}\overline{\sigma(A)}$. Let ϑ_1 be the inner function satisfying the conclusion of Lemma 5.4. Consider an arbitrary $f = \boldsymbol{\mathcal{F}}^{-1} V_{\mathrm{o}} x$, $x \in \mathcal{K}$. Since

$$
\begin{aligned}
(\boldsymbol{\mathcal{F}}^{-1} V_{\mathrm{o}} x)(s) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} C e^{tA} x e^{2\pi \mathrm{i} s t} dt \\
&= \frac{1}{\sqrt{2\pi}} ((A + \mathrm{i} s I)^{-1} x, c), \quad \operatorname{Im} s > 0, \qquad (5.7)
\end{aligned}
$$

the function f extends analytically outside the set $\mathrm{i}\sigma(A)$. On the other hand, $f \in \boldsymbol{K}_\vartheta$ and by (iii), it extends analytically to $\mathbb{C} \setminus \overline{\sigma(\vartheta)}$. Hence, f extends analytically to $\widehat{\mathbb{C}} \setminus \overline{\sigma}$. By Lemma 5.4, $f \in \boldsymbol{K}_{\vartheta_1}$ and since $\{\boldsymbol{\mathcal{F}}^{-1} V_{\mathrm{o}} x : x \in \mathcal{K}\}$ is dense in $\boldsymbol{K}_\vartheta$, it follows that $\boldsymbol{K}_\vartheta \subset \boldsymbol{K}_{\vartheta_1}$ which contradicts the fact that ϑ_1 is a nontrivial inner divisor of ϑ. Thus $\sigma(\vartheta) \subset -\mathrm{i}\overline{\sigma(A)}$, and as we have already observed, this implies the boundedness of G. ■

Remark. It can be shown that $\sigma(T) = \overline{\sigma(\tilde{\vartheta})} = \omega^{-1}(-\mathrm{i}\overline{\sigma(A)})$ and $\sigma(T)$ must be symmetric about the real line: $\sigma(T) = \overline{\sigma(T)}$. Thus $\sigma(A)$ must be symmetric too: $\sigma(A) = \sigma(A^*)$.

Proof of Lemma 5.4. Recall that $\boldsymbol{K}_\vartheta = H^2(\mathbb{C}_+) \cap \vartheta H^2(\mathbb{C}_-)$. Consider first the simplest case. Suppose that there exists a point $\lambda \in \mathbb{C}_+$ such that $\lambda \in \sigma(\vartheta) \setminus \sigma$. Then we can put $\vartheta_1 \stackrel{\text{def}}{=} \vartheta / b_\lambda$, where b_λ is the Blaschke factor with zero at λ. Let f be a function in $\boldsymbol{K}_\vartheta$ that extends analytically to $\bar{\lambda}$. We have $f = (f/\vartheta)\vartheta$, where $f/\vartheta \in H^2(\mathbb{C}_-)$. The pseudocontinuation of ϑ is defined naturally by

$$\vartheta(\zeta) = 1/\overline{\vartheta(\bar{\zeta})}, \quad \zeta \in \mathbb{C}_-.$$

Thus the pseudocontinuation of ϑ has a pole at λ, and so $(f/\vartheta)(\lambda) = 0$. The function $1/b_\lambda$ can be considered as a Blaschke factor in $\mathbb{C}_-$ with zero at $\overline{\lambda}$. Since the function f/ϑ belongs to $H^2(\mathbb{C}_-)$ and vanishes at $\overline{\lambda}$, it follows that $1/b_\lambda$ is a divisor of f/ϑ, and so

$$\frac{f}{\vartheta_1} = \frac{b_\lambda f}{\vartheta} \in H^2(\mathbb{C}_-),$$

which means that $f \in \boldsymbol{K}_{\vartheta_1}$.

Suppose now that $\mathbb{C}_+ \cup (\sigma(\vartheta) \setminus \sigma) = \varnothing$. Put $\widehat{\mathbb{R}} \stackrel{\text{def}}{=} \mathbb{R} \cup \{\infty\}$. Then there exists an open connected subset $\mathcal{I}$ of $\widehat{\mathbb{R}}$ such that $\mathcal{I} \cap \sigma(\vartheta) \neq \varnothing$ and $\mathcal{I}$ is separated from σ.

Let τ be a nontrivial inner divisor of ϑ whose spectrum is contained in $\mathcal{I}$ and let $\vartheta_1 = \vartheta/\tau$. Let us show that ϑ_1 satisfies the conclusion of the lemma. Let f be a function in $\boldsymbol{K}_\vartheta$ that extends analytically to $\mathbb{C} \setminus \sigma$, and so f is analytic in a neighborhood of $\operatorname{clos}\mathcal{I}$. Since $f \in \mathcal{N}(\mathbb{C}_-)$, we have $f = g_1/g_2$, where g_1, $g_2 \in H^\infty(\mathbb{C}_-)$, and g_1 and g_2 have no nontrivial common inner factor. We need the following fact.

Lemma 5.5. *Let φ_1 and φ_2 be bounded analytic functions in $\mathbb{D}$ that have no nonconstant common inner factor. If φ_1/φ_2 extends analytically to a neighborhood of a closed arc J of $\mathbb{T}$, then the spectra of the inner components of φ_1 and φ_2 do not intersect J.*

Let us first complete the proof of Lemma 5.4. Applying the conformal map from $\mathbb{C}_-$ onto $\mathbb{D}$, we find from Lemma 5.5 that the spectra of the inner components of g_1 and g_2 are separated from $\mathcal{I}$.

Let $\vartheta^\#$ and $\tau^\#$ be the inner functions in $\mathbb{C}_-$ defined by

$$\vartheta^\#(\zeta) = \overline{\vartheta(\bar{\zeta})}, \quad \tau^\#(\zeta) = \overline{\tau(\bar{\zeta})}, \quad \zeta \in \mathbb{C}_-.$$

Then $f/\vartheta \in H^2(\mathbb{C}_-)$ and $(f/\vartheta)(\zeta) = f(\zeta)\vartheta^\#(\zeta)$, $\zeta \in \mathbb{C}_-$. Since the spectrum of the inner factor of g_2 is separated from $\mathcal{I}$, it is easy to see that $\tau^\#$ is a divisor of g_1. It follows that the function $f\vartheta^\#/\tau^\#$ in $\mathbb{C}_-$ belongs to $H^2(\mathbb{C}_-)$. Clearly, the boundary values of this function coincide with the boundary values of f/ϑ_1, and so $f \in \boldsymbol{K}_{\vartheta_1}$. ■

Proof of Lemma 5.5. It is easy to see that the zeros of φ_1 and φ_2 in $\mathbb{D}$ are separated from J. Thus we can divide φ_1 and φ_2 by the corresponding Blaschke product and reduce the situation to the case when φ_1 and φ_2 have no zeros in $\mathbb{D}$. Then $\log|\varphi_1/\varphi_2|$ is the Poisson integral of a real measure υ on $\mathbb{T}$. Clearly,

$$\upsilon = \lim_{r \to 1} \upsilon_r$$

in the weak-$*$ topology, where

$$dv_r(\zeta) = \log\left|\frac{\varphi_1(r\zeta)}{\varphi_2(r\zeta)}\right| d\boldsymbol{m}(\zeta).$$

Since $\log|\varphi_1/\varphi_2|$ is smooth in a neighborhood of J in $\mathbb{D}$, it follows that the restriction of v_r to J converges in the norm to the restriction of v to J, and so v is absolutely continuous on J, which means that the singular measures of the inner components of φ_1 and φ_2 are supported outside J. ■

6. Proofs of Lemmas 4.1 and 5.2

The aim of this section is to prove Lemmas 4.1 and 5.2. It is easy to see that they are easy consequences of the following fact.

Lemma 6.1. *Let ρ be a finite positive Borel measure on $[0,a]$, $a>0$, which has no mass at 0 and let σ_+ and σ_- be disjoint sets such that $\sigma_+ \cup \sigma_- = \operatorname{supp}\rho$. Then there exists a weight $w \in L^1(\rho)$ that is positive ρ-a.e. and such that the measure $\tilde\rho$, $d\tilde\rho = wd\rho$, satisfies the following conditions:*

(a) *the integral operator A_0 on $L^2(\tilde\rho)$ with kernel function* (4.2) *belongs to the Hilbert–Schmidt class $\boldsymbol{S}_2$;*

(b) *if $0 \in$ supp ρ, then we can find a weight w satisfying* (a) *and such that*

$$\int \frac{1}{\boldsymbol{s}} d\tilde\rho(\boldsymbol{s}) = \infty.$$

Proof. We begin with (a). In the case when one of the sets σ_+, σ_- is empty the result is trivial. Assume that both σ_+ and σ_- have positive measure. Let $\sigma_+^{(n)}$ and $\sigma_-^{(n)}$ be compact subsets of σ_+ and σ_- such that $\sigma_+^{(n)} \subset \sigma_+^{(n+1)}$, $\sigma_-^{(n)} \subset \sigma_-^{(n+1)}$, $n \ge n+1$, and

$$\lim_{n\to\infty} \rho\big(\sigma_+ \setminus \sigma_+^{(n)}\big) = \lim_{n\to\infty} \rho\big(\sigma_- \setminus \sigma_-^{(n)}\big) = 0.$$

Clearly, the restriction of k to $(\sigma_+^{(n)} \cup \sigma_-^{(n)}) \times (\sigma_+^{(n)} \cup \sigma_-^{(n)})$ is bounded.

Suppose we have already defined w on $\sigma_+^{(n)} \cup \sigma_-^{(n)}$. Let us define it on $\sigma_+^{(n+1)} \cup \sigma_-^{(n+1)}$. We can easily do it so that

$$\iint\limits_{\Delta_n} (k(\boldsymbol{s},\boldsymbol{t}))^2 d\tilde\rho(\boldsymbol{s}) d\tilde\rho(\boldsymbol{t}) < \frac{1}{2^n},$$

where

$$\Delta_n = \big(\sigma_+^{(n+1)} \cup \sigma_-^{(n+1)}\big) \times \big(\sigma_+^{(n+1)} \cup \sigma_-^{(n+1)}\big) \backslash \big(\sigma_+^{(n)} \cup \sigma_-^{(n)}\big) \times \big(\sigma_+^{(n)} \cup \sigma_-^{(n)}\big),$$

$d\tilde{\rho} = w d\rho$, and w is positive a.e. on $\left(\sigma_+^{(n+1)} \cup \sigma_-^{(n+1)}\right) \setminus \left(\sigma_+^{(n)} \cup \sigma_-^{(n)}\right)$. Doing in this way, define w on $\operatorname{supp} \rho$. Clearly, the measure $\tilde{\rho}$, $d\tilde{\rho} = w d\mu$, satisfies (a).

To prove (b) we consider first the simplest case when one of the sets σ_+, σ_- is empty and so $k(\boldsymbol{s}, \boldsymbol{t}) = -1/(\boldsymbol{s}+\boldsymbol{t})$. Without loss of generality we can assume that supp $\rho \subset [0,1]$.

Let $\delta_n = (2^{-n}, 2^{-n+1}]$, $n \geq 1$. Consider the increasing sequence $\{n_j\}_{j\geq 1}$ of integers such that $\rho(\delta_{n_j}) > 0$ and $\rho(\delta_n) = 0$ if $n \neq n_j$ for any j. Since $0 \in$ supp ρ, the sequence $\{n_j\}_{j\geq 1}$ is infinite. We define the weight w by

$$w(\boldsymbol{s}) = \frac{2^{-n_j}}{\rho(\delta_{n_j}) j \log(j+1)}, \quad \boldsymbol{s} \in \delta_{n_j}. \tag{6.1}$$

Then

$$\begin{aligned}
\int_0^1 \frac{w(\boldsymbol{s}) d\rho(\boldsymbol{s})}{\boldsymbol{s}} &= \sum_{j\geq 1} \int_{\delta_{n_j}} \frac{w(\boldsymbol{s}) d\rho(\boldsymbol{s})}{\boldsymbol{s}} \\
&= \sum_{j\geq 1} \frac{2^{-n_j}}{\rho(\delta_{n_j}) j \log(j+1)} \int_{\delta_{n_j}} \frac{d\rho(\boldsymbol{s})}{\boldsymbol{s}} \\
&\geq \sum_{j\geq 1} \frac{2^{-n_j}}{\rho(\delta_{n_j}) j \log(j+1)} \rho(\delta_{n_j}) \cdot 2^{n_j - 1} \\
&= \frac{1}{2} \sum_{j\geq 1} \frac{1}{j \log(j+1)} = \infty.
\end{aligned}$$

Thus

$$\int_0^1 \frac{d\tilde{\rho}(\boldsymbol{s})}{\boldsymbol{s}} = \infty.$$

On the other hand,

$$\begin{aligned}
&\iint (k(\boldsymbol{s}, \boldsymbol{t}))^2 d\tilde{\rho}(\boldsymbol{s}) d\tilde{\rho}(\boldsymbol{t}) = \sum_{j,r\geq 1} \iint_{\delta_{n_j} \times \delta_{n_r}} \frac{w(\boldsymbol{s}) w(\boldsymbol{t})}{(\boldsymbol{s}+\boldsymbol{t})^2} d\rho(\boldsymbol{s}) d\rho(\boldsymbol{t}) \\
&\leq \sum_{1\leq j<\infty,\, r\leq j} \frac{1}{\rho(\delta_{n_j})\rho(\delta_{n_r}) j \log(j+1) r \log(r+1)} \iint_{\delta_{n_j} \times \delta_{n_r}} \frac{2^{-n_j - n_r} d\rho(\boldsymbol{s}) d\rho(\boldsymbol{t})}{(2^{-n_j} + 2^{-n_r})^2} \\
&\leq 2 \sum_{1\leq j<\infty,\, r\leq j} \frac{1}{\rho(\delta_{n_j})\rho(\delta_{n_r}) j \log(j+1) r \log(r+1)} 2^{n_r} 2^{-n_j} \rho(\delta_{n_j}) \rho(\delta_{n_r}) \\
&= 2 \sum_{1\leq j<\infty} \sum_{1\leq r\leq j} \frac{2^{n_r} 2^{-n_j}}{j \log(j+1) r \log(r+1)}.
\end{aligned}$$

Since
$$\sum_{r=1}^{j}\frac{2^{n_r}}{r\log(r+1)}\le\frac{2^{n_j}}{\log(j+1)},$$
we have
$$\iint(k(\boldsymbol{s},\boldsymbol{t}))^2d\tilde\rho(\boldsymbol{s})d\tilde\rho(\boldsymbol{t})\le 2\sum_{j=1}^{\infty}\frac{1}{j\log^2(j+1)}<\infty.$$

Consider now the general case. Since $0\in\operatorname{supp}\rho$, 0 is an accumulation point of either σ_+ or σ_-. To be definite, assume that 0 is an accumulation point of σ_+. Then we can consider the sets $\delta_n=(2^{-n},2^{-n+1}]\cap\sigma_+$, the increasing sequence $\{n_j\}_{j\ge1}$ such that $\rho(\delta_{n_j})>0$ and $\rho(\delta_n)=0$ if $n\ne n_j$ for any j. We can define w on δ_{n_j} as in (6.1). It follows from the above reasonings that
$$\int_{\sigma_+}\frac{w(\boldsymbol{s})d\rho(\boldsymbol{s})}{\boldsymbol{s}}=\infty$$
and
$$\int_{\sigma_+}\int_{\sigma_+}(k(\boldsymbol{s},\boldsymbol{t}))^2d\tilde\rho(\boldsymbol{s})d\tilde\rho(\boldsymbol{t})<\infty.$$
We can now define w inductively on $(2^{-n_j},2^{-n_{j-1}}]\cap\sigma_-$ to be so small that
$$\int_{2^{-n_j}}^{1}\int_{2^{-n_j}}^{1}(k(\boldsymbol{s},\boldsymbol{t}))^2w(\boldsymbol{s})w(\boldsymbol{t})d\rho(\boldsymbol{s})d\rho(\boldsymbol{t})$$
$$\le 2(1-2^{-j})\int_{\Delta_j}\int_{\Delta_j}(k(\boldsymbol{s},\boldsymbol{t}))^2w(\boldsymbol{s})w(\boldsymbol{t})d\rho(\boldsymbol{s})d\rho(\boldsymbol{t}),$$
where $\Delta_j=\sigma_+\cap(2^{-n_j},1]$. Obviously, this can easily be done. ■

7. Positive Hankel Operators with Multiple Spectrum

We show in this section that the method of constructing Hankel operators with the help of proper linear systems with continuous time has limitations. Namely, we show that with the help of this method it is impossible to construct a positive Hankel operator with multiple spectrum (i.e., the spectral multiplicity function ν takes values greater than 1 on a set of nonzero spectral measure). Recall that such Hankel operators do exist, in particular, the Carleman operator on $L^2(\mathbb{R}_+)$ has spectral multiplicity 2 (see §1.8 and §9.2).

Theorem 7.1. *Let Γ be a nonnegative self-adjoint operator such that $W=\Gamma|(\operatorname{Ker}\Gamma)^\perp$ has multiple spectrum. Then there exists no balanced linear SISO system $\{A,B,C\}$ with continuous time such that $\boldsymbol{\Gamma}_h\big|(\operatorname{Ker}\boldsymbol{\Gamma}_h)^\perp$ is unitarily equivalent to W.*

Recall that $\boldsymbol{\Gamma}_h = \boldsymbol{\Gamma}_{h_{A,B,C}}$ is the Hankel operator associated with the system $\{A, B, C\}$.

Theorem 7.1 shows that not all self-adjoint Hankel operators (even up to unitary equivalence) can be realized by proper balanced linear systems with continuous time.

Proof. Suppose that W is unitarily equivalent to $\boldsymbol{\Gamma}_h \big| (\operatorname{Ker} \boldsymbol{\Gamma}_h)^\perp$ for some balanced linear system $\{A, B, C\}$. Then by Theorems 3.3 and 3.5, $\boldsymbol{J} = I$, $A = A^*$, and $b = c$. We have

$$W = \int_0^\infty e^{tA} C^* C e^{tA} dt,$$

the integral being convergent in the weak operator topology.

Let us show that $\operatorname{Ker} A = \{\mathbb{O}\}$. Indeed if $x \in \operatorname{Ker} A$, then

$$\int_0^\infty (e^{tA} C^* C e^{tA} x, x) dt = \int_0^\infty |(e^{tA} x, c)|^2 dt = \int_0^\infty |(x, c)|^2 dt < \infty,$$

so $x \perp e^{tA} c$ and since c is a cyclic vector of A, we have $x = 0$.

As we have already proved

$$AW + WA = -C^* C \tag{7.1}$$

(see §12.3). Therefore it follows from Lemma 4.4 that $A \le \mathbb{O}$. Since Ker $A = \{\mathbb{O}\}$, A generates an asymptotically stable semigroup.

We can now interchange the roles of A and W in (7.1). Since $-W$ is self-adjoint and asymptotically stable, it follows from Theorems 3.1 and 3.2 that $-A$ is a unique solution of the equation

$$X(-W) + (-W)X = -C^* C$$

and this solution is given by

$$-A = \int_0^\infty e^{-tW} C^* C e^{-tW} dt, \tag{7.2}$$

the integral being convergent in the weak operator topology.

However, it is easy to see from (7.2) that $\operatorname{Ker} A = (\operatorname{span}\{W^k c : \ k \ge 0\})^\perp$ and this subspace is nontrivial since W has multiple spectrum. ■

Remark. In the definition of balanced systems we require that $\operatorname{Ker} W = \{\mathbb{O}\}$. One could think that if we change the definition to admit a nontrivial kernel of W, then Theorem 7.1 would not be true anymore. However, this is not the case. If we admit $\operatorname{Ker} W \ne \{\mathbb{O}\}$ we can consider the space $\mathcal{K}_1 = (\operatorname{Ker} W)^\perp$. Since

$$W = \int_0^\infty e^{tA^*} C^* C e^{tA} dt = \int_0^\infty e^{tA} B B^* e^{tA^*} dt\,,$$

it follows that

$$\mathcal{K}_1 = (\operatorname{Ker} W)^\perp = \operatorname{span}\{A^n b : \ n \ge 0\} = \operatorname{span}\{(A^*)^n c : \ n \ge 0\}\,,$$

so $\mathcal{K}_1$ is a reducing subspace for A and $b, c \in \mathcal{K}_1$. Therefore if we consider the operators $A_1 = P_{\mathcal{K}_1} A \big| \mathcal{K}_1$ and $W_1 = W \big| \mathcal{K}_1$, then $(e^{tA} b, c) = (e^{tA_1} b, c)$, so the Hankel operators corresponding to the systems $\{A, b, c\}$ and $\{A_1, b, c\}$ coincide. But $\operatorname{Ker} W_1 = \{\mathbb{O}\}$ and we arrive at our original definition.

8. Moduli of Hankel Operators, Past and Future, and the Inverse Problem for Rational Approximation

In this section we solve the problem on the geometry of past and future. This problem has been posed in §8.6. To solve this problem, we describe the moduli of Hankel operators up to unitary equivalence. As a consequence of this description we solve the inverse problem for rational approximation in $BMOA$ that has been posed in §6.6.

Theorem 8.1. *Let K be an operator on Hilbert space such that $A \geq \mathbb{O}$. The following are equivalent:*

(i) *K is unitarily equivalent to the modulus of a Hankel operator;*

(ii) *K is unitarily equivalent to the modulus of a self-adjoint Hankel operator;*

(iii) *K is noninvertible, and* $\operatorname{Ker} K$ *is either trivial or infinite-dimensional.*

Proof. We have mentioned in §8.6 that (i) implies (iii). It is trivial that (ii) implies (i). Finally, the implication (iii)$\Longrightarrow$(ii) is an immediate consequence of Theorem 3.1. ■

Note that the implication (iii)$\Longrightarrow$(ii) can be proved much easier that it has been done in the proof of Theorem 3.1. Let us outline how the construction given in the proof of Theorem 3.1 can be simplified to construct a Hankel operator whose modulus is unitarily equivalent to a given operator K.

We are looking for a linear SISO system $\{A, B, C\}$, where $A : \mathcal{K} \to \mathcal{K}$, $B : \mathbb{C} \to \mathcal{K}$, and $C : \mathcal{K} \to \mathbb{C}$,

$$Bu = ub, \quad u \in \mathbb{C}, \quad Cx = (x, c), \quad x \in \mathcal{K},$$

where b, $c \in \mathcal{K}$.

As in the proof of Theorem 3.1 first we solve the problem modulo the kernel. We put $W = K \big| (\operatorname{Ker} K)^{\perp}$ and we construct $\{A, B, C\}$ such that $|\Gamma_h| \big| (\operatorname{Ker} \Gamma_h)^{\perp}$ is unitarily equivalent to W, where $h = h_{A,B,C}$. We do not need the operator $\boldsymbol{J}$. We can make A self-adjoint and we can look for b and c in the form

$$b(\boldsymbol{t}) = c(\boldsymbol{t}) = e_1$$

(we use the same notation as in §12.4). To solve the problem modulo the kernel, we have to find an asymptotically stable self-adjoint operator A

satisfying the Lyapunov equation

$$AW + WA = -C^*C. \tag{8.1}$$

We can define the operator A_0 on the space $\mathcal{K}_0$, which we have identified with $L^2(\rho)$ (see §12.4) by

$$(A_0 f)(\boldsymbol{s}) = \int_{\sigma(W)} \frac{-f(\boldsymbol{t})}{\boldsymbol{s}+\boldsymbol{t}} d\rho(\boldsymbol{t}).$$

It is almost obvious that we can always replace ρ (if necessary) with a measure mutually absolutely continuous with ρ so that A_0 becomes a Hilbert–Schmidt operator. We can now define D as in §12.4 and put $A = A_0 + D$. It is obvious that A satisfies (8.1). The proof of the fact that A generates an asymptotically stable semigroup is the same as in §12.4.

This solves the problem modulo the kernel. To settle the problem with the kernel, we have to repeat what has been done in §12.5.

We proceed now to the solution of the geometrical problem on past and future that has been posed in §8.6.

Theorem 8.2. *Let $\boldsymbol{G}$ be an infinite-dimensional Gaussian space, and let $\mathcal{K}$ and $\mathcal{L}$ be subspaces of $\boldsymbol{G}$ that are under nonzero angle and such that* $\operatorname{clos}(\mathcal{K}+\mathcal{L}) = \boldsymbol{G}$. *Suppose that the triple $t = (\mathcal{K}, \mathcal{L}, \boldsymbol{G})$ satisfies the following properties:*

(i) $n_+(\boldsymbol{t}) = n_-(\boldsymbol{t})$;

(ii) *the operator $\mathcal{P}_\mathcal{K}\mathcal{P}_\mathcal{L}\mathcal{P}_\mathcal{K}\big|\mathcal{K}$ is noninvertible;*

(iii) *either $n_+(\boldsymbol{t}) = 0$ or $n_+(\boldsymbol{t}) = \infty$.*

Then there exists a stationary sequence $\{X_n\}_{n\in\mathbb{Z}}$ in $\boldsymbol{G}$ such that its past coincides with $\mathcal{K}$ and its future coincides with $\mathcal{L}$.

Recall that the numbers $n_\pm(t)$ have been introduced in §8.6.

Proof. The result follows immediately from Theorems 8.6.6 and 8.1. ■

Theorem 8.3. *Let $\boldsymbol{G}$ be an infinite-dimensional Gaussian space, and let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Gaussian subspaces of $\boldsymbol{G}$ that are under nonzero angle and such that* $\operatorname{clos}(\boldsymbol{A}+\boldsymbol{B}) = \boldsymbol{G}$. *Suppose that the triple $t = (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{G})$ satisfies the following properties:*

(i) $n_+(\boldsymbol{t}) = n_-(\boldsymbol{t})$;

(ii) *the operator $\mathcal{P}_{\boldsymbol{A}}\mathcal{P}_{\boldsymbol{B}}\mathcal{P}_{\boldsymbol{A}}\big|\boldsymbol{A}$ is noninvertible;*

(iii) *either $n_+(\boldsymbol{t}) = 0$ or $n_+(\boldsymbol{t}) = \infty$.*

Then there exists a stationary Gaussian process $\{X_n\}_{n\in\mathbb{Z}}$ in $\boldsymbol{G}$ such that its past coincides with $\boldsymbol{A}$ and its future coincides with $\boldsymbol{B}$.

Proof. The result follows immediately from Theorems 8.6.7 and 8.1. ■

Note that it has been shown in §8.6 that conditions (i), (ii), and (iii) in Theorems 8.2 and 8.3 are also necessary.

Another consequence of Theorem 8.1 solves the inverse problem for best approximation in $BMOA$ that has been posed in §6.6 (see §6.6 for the notation).

Theorem 8.4. *Let $\{c_n\}_{n\ge0}$ be a nonincreasing sequence of nonnegative numbers. Then there exists $\varphi \in BMOA$ such that*

$$\rho_n^+(\varphi) = c_n. \quad n \in \mathbb{Z}_+.$$

Proof. By the Kronecker theorem and by the Adamyan–Arov–Krein theorem we have to find a Hankel operator Γ such that

$$s_n(\Gamma) = c_n. \quad n \in \mathbb{Z}_+. \tag{8.2}$$

Consider the diagonal operator D on ℓ^2 defined by $De_n = c_n e_n$, where $\{e_n\}_{n\ge0}$ is the standard orthonormaal basis in ℓ^2 and let $T = D \oplus \mathbb{O}$, where $\mathbb{O}$ is the zero operator on an infinite-dimensional Hilbert space. Clearly, by Theorem 8.1, there exists a Hankel operator Γ such that $|\Gamma|$ is unitarily equivalent to T, which implies (8.2). ■

9. Linear Systems with Discrete Time

We have seen in §7 that proper linear systems with continuous time cannot help to construct a positive Hankel operator with multiple spectrum. On the other hand, we have seen in §9.2 that such operators do exist. From now on we are going to use linear systems with discrete time to solve the problem completely.

The idea of the method can be described briefly as follows. Let Γ be a Hankel operator and let S be the shift operator. Then the operator $\Lambda = S^*\Gamma$ satisfies the equality

$$\Gamma^2 - \Lambda^2 = (\cdot, p)p,$$

where $p = \Gamma e_0$ and e_0 is the first basis vector of the basis in which Γ has Hankel matrix.

Suppose now that Γ is a self-adjoint operator on Hilbert space that satisfies conditions (C1)–(C3). Assume here for simplicity that $\operatorname{Ker}\Gamma = \{\mathbb{O}\}$. The problem is to find a vector p in $\operatorname{Range}\Gamma$ such that $\Gamma^2 - (\cdot, p)p \ge \mathbb{O}$ and to find a self-adjoint operator Λ such that $\Lambda^2 = \Gamma^2 - (\cdot, p)p$. Then we can define a contraction T by $\Lambda = T\Gamma$. It is easy to choose a vector p so that T^* is an isometry. It can also be seen that $\operatorname{Ker} T$ is one-dimensional. If we could prove that T^* is unitarily equivalent to the shift operator, then it would follow from the equality $T\Gamma = \Gamma T^*$ that Γ is unitarily equivalent to a Hankel operator. Clearly, if T is an isometry and $\dim \operatorname{Ker} T = 1$, then T is unitarily equivalent to S^* if and only if

$$\|T^n x\| \to 0$$

for any vector x. However, the verification of the above property (the *asymptotic stability* of T) is the most difficult problem. We reduce it to the verification of the asymptotic stability of another auxiliary operator. The last problem is much simpler since the auxiliary operator has a sparse spectrum on the unit circle.

For a given operator Γ we are going to construct in this section a linear system and we show that if the state space operator T of that system is asymptotically stable (T^* is an isometry if $\operatorname{Ker}\Gamma = \{\mathbb{O}\}$), then the Hankel operator associated with the system is unitarily equivalent to Γ. Such systems are output normal systems (see §11.2). As we have already mentioned, it is very difficult to verify the asymptotic stability of T. In §10 we construct another linear system and show that if T is asymptotically stable then it gives another (balanced) realization of the same Hankel operator and we reduce the verification of the asymptotic stability of T to the verification of asymptotic stability of the adjoint to the state-space operator of the new balanced system. Later we choose parameters and prove asymptotic stability by using the Sz.-Nagy–Foias stability test.

We consider here SISO linear systems with discrete time of the form $\{A, B, C\}$ (see §11.2), where A is a bounded linear operator (state-space operator) on a Hilbert $\mathcal{K}$, $B : \mathbb{C} \to \mathcal{K}$ and $C : \mathcal{K} \to \mathbb{C}$ are bounded linear operators,

$$Bu = ub, \quad u \in \mathbb{C}, \qquad Cx = (x, c), \quad x \in \mathcal{K},$$

where $b, c \in \mathcal{K}$. The system $\{A, B, C\}$ is the system of equations:

$$\begin{cases} x_{n+1} = Ax_n + Bu_n, \\ y_n = Cx_n. \end{cases} \tag{9.1}$$

In what follows we also use the notation $\{A, b, c\}$ for the system (9.1).

We associate with the system (9.1) the formal Hankel operator Γ_α, $\alpha = \alpha_{\{A,B,C\}}$, with the matrix

$$\{\alpha_{j+k}\}_{j,k\ge 0}, \quad \alpha_j = (A^j b, c), \quad j \ge 0.$$

Recall (see §11.2) that in general Γ_α need not be bounded on ℓ^2, but in important cases it is bounded.

Clearly, the system (9.1) is controllable if $\operatorname{span}\{A^n b : n \ge 0\} = \mathcal{K}$ and observable if $\operatorname{span}\{A^{*n} c : n \ge 0\} = \mathcal{K}$ (see §11.2).

Recall that the controllability and observability Gramians are defined by

$$W_{\mathrm{c}} = \sum_{j=0}^{\infty} A^j BB^*(A^*)^j, \tag{9.2}$$

$$W_{\mathrm{o}} = \sum_{j=0}^{\infty} (A^*)^j C^* C A^j \tag{9.3}$$

if the series converge in the weak operator topology. In this case the system is controllable (observable) if and only if $\operatorname{Ker} W_{\mathrm{c}} = \{\mathbb{O}\}$ ($\operatorname{Ker} W_{\mathrm{o}} = \{\mathbb{O}\}$).

Recall that a minimal system is called balanced if the series (9.2) and (9.3) converge and $W_{\mathrm{c}} = W_{\mathrm{o}}$. We also consider here output normal systems, i.e., the systems for which the series (9.2) and (9.3) converge and $W_{\mathrm{o}} = I$ (see §11.2).

Our aim is to prove the following theorem, which together with Theorem 1.1 gives a complete solution of the problem to describe the self-adjoint operators that are unitarily equivalent to a Hankel operator and proves Theorem 0.1 in the Introduction to this chapter.

Theorem 9.1. *Let* Γ *be a self-adjoint operator on Hilbert space that satisfies conditions* (C1), (C2), *and* (C3) *in the introduction to this chapter. Then there exists a SISO balanced linear system* $\{A, b, c\}$ *such that the Hankel operator* Γ_α *associated with the system is unitarily equivalent to* Γ.

The proof of Theorem 1.1 will be given in §13.

Now for a given self-adjoint operator we are going to construct a linear system with discrete time and prove that if the state space operator is asymptotically stable, then the Hankel operator associated with the system is unitarily equivalent to the operator in question. Moreover, we construct an output normal system. The main difficulty, however, consists in the verification of asymptotic stability.

Let R be a self-adjoint operator on a Hilbert space $\mathcal{K}$ with trivial kernel and let q be a vector in $\mathcal{K}$ such that $\|q\| \le 1$. Then $R^2 - (\cdot, p)p \ge \mathbb{O}$, where $p = Rq$. Let Λ be a self-adjoint operator such that

$$R^2 = \Lambda^2 + (\cdot, p)p. \tag{9.4}$$

Clearly, $\Lambda^2 \le R^2$, and so by Lemma 2.1.2, there exists a contraction T (which is unique because of the fact that $\operatorname{Ker} R = \{\mathbb{O}\}$) such that $TR = \Lambda$.

Consider now the linear dynamical system $\{T, p, q\}$. As above $\alpha_j = (T^j p, q)$, $j \ge 0$, and Γ_α is the Hankel operator with matrix $\{\alpha_{j+k}\}_{j,k\ge 0}$.

We say that T is *asymptotically stable* if $\|T^n x\| \to 0$ as $n \to \infty$ for any $x \in \mathcal{K}$.

Theorem 9.2. *Suppose that* T *is asymptotically stable. Then*

(i) *the system* $\{T, p, q\}$ *is output normal;*
(ii) R *is unitarily equivalent to* $\Gamma_\alpha \big| (\operatorname{Ker} \Gamma_\alpha)^\perp$;
(iii) $\operatorname{Ker} \Gamma_\alpha = \{\mathbb{O}\}$ *if and only if* $\|q\| = 1$ *and* $q \notin \operatorname{Range} R$.

Proof. It follows from (9.4) and the definition of T that

$$R^2 = RT^*TR + (\cdot, Rq)Rq,$$

and since $\operatorname{Ker} R = \{\mathbb{O}\}$, we obtain

$$I = T^*T + (\cdot, q)q. \tag{9.5}$$

Therefore

$$\|x\|^2 = \|Tx\|^2 + |(x, q)|^2, \quad x \in \mathcal{K}. \tag{9.6}$$

If we apply (9.6) to Tx and use the fact that T is asymptotically stable, we obtain

$$\|x\|^2 = \sum_{j=0}^{\infty} \|(T^j x, q)|^2, \quad x \in \mathcal{K},$$

which means that the operator $V : \mathcal{K} \to \ell^2$ defined by

$$Vx = ((x,q),(Tx,q),(T^2x,q),...) \tag{9.7}$$

is an isometry. It is easy to see that the Hankel operator Γ_α associated with the system satisfies the equality $\Gamma_\alpha = VRV^*$. Indeed,

$$V^*\{y_j\}_{j\ge0} = \sum_{j\ge0} y_j(T^*)^j q, \quad \{y_j\}_{j\ge0} \in \ell^2. \tag{9.8}$$

So if $\{e_j\}_{j\ge0}$ is the standard orthonormal basis of ℓ^2, then

$$(VRV^*e_j, e_k) = (R(T^*)^j q, (T^*)^k q)$$

and since $RT^* = TR$ (this follows from the definition of T), we have

$$(VRV^*e_j, e_k) = (T^jRq, (T^*)^k q) = (T^{j+k}p, q) = \alpha_{j+k}.$$

Therefore (ii) holds and $\operatorname{Ker}\Gamma_\alpha = \operatorname{Ker} V^*$.

It is clear from (9.7) and (9.8) that the series (9.3) converges in the weak operator topology and $W_{\rm o} = V^*V = I$. On the other hand, since $TR = RT^* = \Lambda$, we have

$$\begin{aligned} W_{\rm c} &= \sum_{j\ge0} T^j((\cdot, Rq)Rq)(T^*)^j = \sum_{j\ge0} T^jR((\cdot,q)q)R(T^*)^j \\ &= \sum_{j\ge0} R(T^*)^j((\cdot,q)q))T^jR = RW_{\rm o}R = R^2, \end{aligned}$$

and so the series (9.2) converges in the weak operator topology.

Note that $\operatorname{Ker}\Gamma_\alpha = \{\mathbb{O}\}$ if and only if V is onto, which is equivalent to the fact that $\{(T^*)^j q\}_{j\ge0}$ is an orthonormal basis in $\mathcal{K}$.

Suppose now that $\|q\| = 1$ and $q \notin \operatorname{Range} R$. We have by the definition of Λ,

$$\Lambda^2 = R(I - (\cdot,q)q)R. \tag{9.9}$$

Since $\operatorname{Ker} R = \{\mathbb{O}\}$, it is clear that $\operatorname{Ker}\Lambda = \{\mathbb{O}\}$. It follows now from the equality $RT^* = \Lambda$ that $\operatorname{Ker} T^* = \{\mathbb{O}\}$.

Applying (9.5) to the vector q, we obtain

$$T^*Tq + q = q,$$

and since $\operatorname{Ker} T^* = \{\mathbb{O}\}$, it follows that $Tq = \mathbb{O}$.

Multiplying (9.5) by T on the left and by T^* on the right, we obtain

$$TT^*TT^* + T((\cdot,q)q)T^* = TT^*, \tag{9.10}$$

and since $Tq = \mathbb{O}$, we have $(TT^*)^2 = TT^*$. Clearly, in view of the fact that $\operatorname{Ker} T^* = \{\mathbb{O}\}$ this means that $TT^* = I$, i.e., T^* is an isometry. Since T is asymptotically stable, T has no unitary part. Clearly, (9.5) means that $\dim \operatorname{Ker} T = 1$, and so T^* is unitarily equivalent to the shift operator S. Therefore the conditions $Tq = \mathbb{O}$ and $\|q\| = 1$ imply that the system $\{(T^*)^j q\}_{j\ge0}$ forms an orthonormal basis in $\mathcal{K}$.

Suppose now that $\{(T^*)^j q\}_{j\ge0}$ is an orthonormal basis in $\mathcal{K}$. Then $\|q\| = 1$ and T^* is an isometry, i.e., $TT^* = I$. Therefore we have from (9.10)

$$T((\cdot,q)q)T^* = (\cdot,Tq)Tq = \mathbb{O},$$

and so $Tq = \mathbb{O}$. Since T^* and R have trivial kernels, the operator $\Lambda = RT^*$ also has trivial kernel. But it is easy to see that (9.9) and the fact that $\|q\| = 1$ imply that $\operatorname{Ker}\Lambda = \{\mathbb{O}\}$ if and only if $q \notin \operatorname{Range} R$. ■

Remark. As we have shown in the proof, in the case $\operatorname{Ker}\Gamma = \{\mathbb{O}\}$ the operator T is unitarily equivalent to backward shift S^*. It is easy to see that otherwise T is unitarily equivalent to the restriction of S^* to its invariant subspace $(\operatorname{Ker}\Gamma_\alpha)^\perp$.

Theorem 9.1 gives us a recipe how to construct a Hankel operator that is unitarily equivalent to our operator Γ. We can put $R = \Gamma\big|(\operatorname{Ker}\Gamma)^\perp$. Then we have to choose a vector q such that $R^2 - (\cdot,Rq)Rq \ge \mathbb{O}$. If $\operatorname{Ker}\Gamma = \{\mathbb{O}\}$, then q must satisfy the conditions $\|q\| = 1$ and $q \notin \operatorname{Range} R$. If $\operatorname{Ker}\Gamma \ne \{\mathbb{O}\}$, at least one of the conditions must be violated. Next, we have to choose a self-adjoint operator Λ such that $\Lambda^2 = R^2 - (\cdot,Rq)Rq$. The problem will be solved if we manage to prove that the operator T uniquely defined by the equality $TR = \Lambda$ is asymptotically stable. However, this is the most difficult point.

10. Passing to Balanced Linear Systems

In the previous section we have constructed an output normal system $\{T,p,q\}$ and we have reduced the problem to the verification of the asymptotic stability of T. In this section we construct a balanced system $\{A,b,c\}$ that produces the same Hankel matrix and we show that the verification of the asymptotic stability of T can be reduced to the verification of the asymptotic stability of A^*. It turns out that the last problem is considerably simpler.

Lemma 10.1. *Let T be a contraction and let K, X be bounded operators on Hilbert space such that X has dense range and*

$$TX = XK.$$

If K is asymptotically stable, then so is T.

Proof. The result follows immediately from the formula $T^jX = XK^j$, $j \ge 1$, and from the facts that $\|T\| \le 1$ and X has dense range. ■

Let T, Λ, R be the operators introduced in §9. Put $W = |R| = (R^*R)^{1/2}$.

Lemma 10.2. *There exists a unique contraction A such that*

$$AW^{1/2} = W^{1/2}T^*. \tag{10.1}$$

If A^* is asymptotically stable, then so is T.

Note that Lemma 10.2 is practically the same as Lemma 11.2.6. We prove it here for convenience.

Proof. The inequality $T^*T \le I$ implies

$$W^2 = R^2 \ge RT^*TR = \Lambda^2 = TR^2T^* = TW^2T^* \ge TWT^*TWT^*.$$

By the Heinz inequality (see Appendix 1.7), this implies that $TWT^* \le W$. Now applying Lemma 2.1.2, we find that there exists a unique contraction A satisfying (10.1).

It follows from (10.1) that

$$TW^{1/2} = W^{1/2}A^*$$

and so by Lemma 10.1, the asymptotic stability of T follows from the asymptotic stability of A^*. ■

Let $\boldsymbol{J}$ be the "argument" of R, i.e.,

$$R = \boldsymbol{J}|R| = \boldsymbol{J}W = W\boldsymbol{J}, \quad \boldsymbol{J} = \boldsymbol{J}^* = \boldsymbol{J}^{-1}.$$

Consider now the linear system $\{A, b, c\}$, where the operator A is defined in Lemma 10.2, $b = W^{1/2}q,\ c = \boldsymbol{J}W^{1/2}q$.

Theorem 10.3. *Let T be asymptotically stable. Then the system $\{A, b, c\}$ is a balanced realization of the Hankel operator associated with the output normal system $\{T, p, q\}$.*

In other words, the system $\{A, b, c\}$ is balanced and $(A^j b, c) = (T^j p, q)$, $j \ge 0$, and so the Hankel operators associated with these two systems coincide. To prove Theorem 10.3, we need the following lemma.

Lemma 10.4. $A^*\boldsymbol{J} = \boldsymbol{J}A$.

Proof. Since $\operatorname{Ker} W = \{\mathbb{O}\}$ and W has dense range, it is sufficient to show that $W^{1/2}A^*\boldsymbol{J}W^{1/2} = W^{1/2}\boldsymbol{J}AW^{1/2}$. We have by (10.1)

$$W^{1/2}A^*\boldsymbol{J}W^{1/2} = TW^{1/2}\boldsymbol{J}W^{1/2} = T\boldsymbol{J}W = TR = \Lambda$$

by the definition of T. On the other hand by (10.1),

$$W^{1/2}\boldsymbol{J}AW^{1/2} = W^{1/2}\boldsymbol{J}W^{1/2}T^* = \boldsymbol{J}WT^* = RT^* = \Lambda,$$

which proves the result. ■

Proof of Theorem 10.3. Let us first show that $\{A, b, c\}$ is a balanced system. We have

$$\begin{aligned} W_{\rm c} &= \sum_{j\ge 0} A^j((\cdot, W^{1/2}q)W^{1/2}q)(A^*)^j \\ &= \sum_{j\ge 0} A^j W^{1/2}((\cdot, q)q)W^{1/2}(A^*)^j \\ &= \sum_{j\ge 0} W^{1/2}(T^*)^j((\cdot, q), q))T^j W^{1/2} \end{aligned}$$

by (10.1). Hence,

$$W_c = W^{1/2}\left(\sum_{j\ge 0}(\cdot,(T^*)^j q)(T^*)^j q\right)W^{1/2} = W,$$

since the operator V defined by (9.7) is an isometry. Clearly, the series (9.2) converges in the weak operator topology. Next,

$$\begin{aligned} W_o &= \sum_{j\ge 0}(A^*)^j((\cdot,\boldsymbol{J}W^{1/2}q)\boldsymbol{J}W^{1/2}q)A^j \\ &= \sum_{j\ge 0}(A^*)^j\boldsymbol{J}W^{1/2}((\cdot,q)q)W^{1/2}\boldsymbol{J}A^j \\ &= \sum_{j\ge 0}\boldsymbol{J}A^j((\cdot,W^{1/2}q)W^{1/2}q)(A^*)^j\boldsymbol{J} \end{aligned}$$

by Lemma 10.4. Hence,

$$W_o = \boldsymbol{J}W_c\boldsymbol{J} = \boldsymbol{J}W\boldsymbol{J} = W$$

and the series (9.3) converges in the weak operator topology.

Let us now prove the equality $(A^j b,c) = (T^j p,f)$, $j\ge 0$. Bearing in mind (10.1) and the equality $RT^* = TR$, we have

$$\begin{aligned} (A^j b,c) &= (A^j W^{1/2}q,\boldsymbol{J}W^{1/2}q) = (W^{1/2}\boldsymbol{J}A^jW^{1/2}q,q) \\ &= (W^{1/2}\boldsymbol{J}W^{1/2}(T^*)^j q,q) = (R(T^*)^j q,q) \\ &= (T^j Rq,q) = (T^j p,q). \quad \blacksquare \end{aligned}$$

Let $\boldsymbol{\mathcal{J}}$ be the "argument" of Λ, i.e.,

$$\Lambda = \boldsymbol{\mathcal{J}}|\Lambda| = |\Lambda|\boldsymbol{\mathcal{J}}, \quad \boldsymbol{\mathcal{J}} = \boldsymbol{\mathcal{J}}^* = \boldsymbol{\mathcal{J}}^{-1}. \tag{10.2}$$

If $\operatorname{Ker}\Lambda \ne \{\mathbb{O}\}$, then $\boldsymbol{\mathcal{J}}$ is not uniquely defined, in which case $\boldsymbol{\mathcal{J}}$ is an arbitrary operator satisfying (10.2).

Note that $W^2 = R^2 \ge \Lambda^2$, and so by the Heinz inequality (see Birman and Solomyak [1], Sec. 10.4), $W \ge |\Lambda|$. Hence, by Lemma 2.1.2, there exists a unique contraction Q such that

$$QW^{1/2} = |\Lambda|^{1/2}. \tag{10.3}$$

Lemma 10.5. $A^* = Q^*\boldsymbol{\mathcal{J}}Q\boldsymbol{J}$.

Proof. We have by (10.1)

$$W^{1/2}A^*\boldsymbol{J}W^{1/2} = TW^{1/2}\boldsymbol{J}W^{1/2} = TR = \Lambda.$$

On the other hand,

$$\Lambda = |\Lambda|^{1/2}\boldsymbol{\mathcal{J}}|\Lambda|^{1/2} = W^{1/2}Q^*\boldsymbol{\mathcal{J}}QW^{1/2}.$$

Thus

$$W^{1/2}A^*\boldsymbol{J}W^{1/2} = W^{1/2}Q^*\boldsymbol{\mathcal{J}}QW^{1/2}.$$

Since $\operatorname{Ker} W^{1/2} = \{\mathbb{O}\}$, we have $A^* \boldsymbol{J} = Q^* \boldsymbol{\mathcal{J}} Q$, which completes the proof. ■

The following lemma describes the structure of Q.

Lemma 10.6. *Let $\mathcal{K}_0$ be the smallest invariant subspace of W that contains $\boldsymbol{J}q$. Then Q has the following structure in the decomposition $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_0^\perp$:*

$$\begin{pmatrix} Q_0 & \mathbb{O} \\ \mathbb{O} & I \end{pmatrix}, \tag{10.4}$$

where Q_0 is a pure contraction (i.e., $\|Q_0 h\| < \|h\|$ for $h \neq 0$).

Proof. Clearly, W^2 and Λ^2 coincide on $\mathcal{K}_0^\perp$ and $\mathcal{K}_0^\perp$ is a reducing subspace for both of them. Therefore the same is true for the operators $W^{1/2}$ and $|\Lambda|^{1/2}$, which together with (10.3) proves (10.4). It remains to show that Q_0 is a pure contraction.

By (10.3),

$$\begin{aligned} W^{1/2} Q^* Q W Q^* Q W^{1/2} &= \Lambda^2 = W^2 - (\cdot, p)p \\ &= W^{1/2}(W - (\cdot, W^{1/2}\boldsymbol{J}q) W^{1/2}\boldsymbol{J}q) W^{1/2}, \end{aligned}$$

and so

$$Q^* Q W Q^* Q = W - (\cdot, W^{1/2}\boldsymbol{J}q) W^{1/2}\boldsymbol{J}q. \tag{10.5}$$

It follows from (10.5) that

$$(W Q^* Q x, Q^* Q x) = (Wx, x) + |(x, W^{1/2}\boldsymbol{J}q)|^2, \quad x \in \mathcal{H}. \tag{10.6}$$

Since Q is a contraction, $\|Qh\| = \|h\|$ if and only if $Q^*Qh = h$ (i.e., $h \in \operatorname{Ker}(I - Q^*Q)$). It follows from (10.6) that if $h \in \operatorname{Ker}(I - Q^*Q)$, then $h \perp W^{1/2}\boldsymbol{J}q$ and by (10.5), we obtain $Q^*QWh = Wh$. So $\operatorname{Ker}(I - Q^*Q)$ is an invariant subspace of W that is orthogonal to $W^{1/2}\boldsymbol{J}q$. Therefore $\operatorname{Ker}(I - Q^*Q)$ is orthogonal to $\mathcal{K}_0$, which proves that Q_0 is a pure contraction. ■

11. Asymptotic Stability

The material of the previous section suggests to us how to construct a Hankel operator that is unitarily equivalent to a given operator Γ. Let $\mathcal{K} = (\operatorname{Ker}\Gamma)^\perp$, $\tilde{\Gamma} = \Gamma\big|\mathcal{K}$, and $W = |\tilde{\Gamma}|$. We have to choose a vector q, operators $\boldsymbol{J}$ and $\boldsymbol{\mathcal{J}}$ such that the following properties hold:

(i) $\boldsymbol{J} = \boldsymbol{J}^* = \boldsymbol{J}^{-1}$, $\boldsymbol{J}W = W\boldsymbol{J}$, and $R \stackrel{\text{def}}{=} \boldsymbol{J}W$ is unitarily equivalent to $\tilde{\Gamma}$;

(ii) $R^2 - (\cdot, Rq)Rq \geq \mathbb{O}$;

(iii) $\boldsymbol{\mathcal{J}} = \boldsymbol{\mathcal{J}}^* = \boldsymbol{\mathcal{J}}^{-1}$, $\boldsymbol{\mathcal{J}}(R^2 - (\cdot, Rq)Rq)^{1/2} = (R^2 - (\cdot, Rq)Rq)^{1/2}\boldsymbol{\mathcal{J}}$;

(iv) the operator $A^* = Q^*\boldsymbol{\mathcal{J}}Q\boldsymbol{J}$ is asymptotically stable, where the contraction Q is uniquely defined by (10.3) and

$$\Lambda = \boldsymbol{\mathcal{J}}(R^2 - (\cdot, Rq)Rq)^{1/2}.$$

If $\operatorname{Ker}\Gamma = \{\mathbb{O}\}$, we have to impose the following condition:

(v) $\|q\| = 1$ and $q \notin \operatorname{Range} R$.

If $\operatorname{Ker}\Gamma \neq \{\mathbb{O}\}$, we have to make the assumption:

(v′) either $\|q\| < 1$ or $q \in \operatorname{Range} R$.

The most difficult problem here is to verify the asymptotic stability of A^*. To this end we shall use the following stability test, which is an immediate consequence of Proposition 6.7 in Chapter II of Sz.-Nagy and Foias [1] (in §4 we have used an analog of this test for semigroups of contractions).

Stability test. *Let $\boldsymbol{T}$ be a contraction on Hilbert space. If $\boldsymbol{T}$ has no eigenvalues on the unit circle $\mathbb{T}$ and the set $\sigma(\boldsymbol{T}) \cap \mathbb{T}$ is at most countable, then $\boldsymbol{T}$ is asymptotically stable.*

We define the *Fredholm spectrum* $\sigma_{\mathrm{F}}(\boldsymbol{T})$ of a bounded linear operator $\boldsymbol{T}$ as the set of points λ in $\mathbb{C}$ for which there exists a sequence $\{x_n\}_{n\geq 1}$ such that $\|x_n\| = 1$, $\operatorname{w-}\lim\limits_{n\to\infty} x_n = \mathbb{O}$ (i.e., $\lim\limits_{n\to\infty} x_n = \mathbb{O}$ in the weak topology), and

$$\lim_{n\to\infty} \|\boldsymbol{T}x_n - \lambda x_n\| = 0.$$

Clearly, it is sufficient to prove that $\sigma_{\mathrm{F}}(A^*) \cap \mathbb{T}$ is at most countable and both A and A^* do not have eigenvalues on $\mathbb{T}$. It follows easily from Lemma 10.5 that $A^* = \boldsymbol{J}A\boldsymbol{J}$, and so the operators A and A^* are unitarily equivalent. Therefore it is sufficient to verify that A^* has no eigenvalues on $\mathbb{T}$ (and certainly that $\sigma_{\mathrm{F}}(A^*) \cap \mathbb{T}$ is at most countable). The following lemma will help us to get rid of eigenvalues on $\mathbb{T}$. Recall that $\mathcal{K}_0$ is the smallest W-invariant subspace of $\mathcal{K}$ that contains $\boldsymbol{J}q$.

Lemma 11.1. *Let $|\zeta| = 1$ and $h \in \mathcal{H}$. Then $A^*h = \zeta h$ if and only if $h \in \mathcal{K}_0^\perp$, $\boldsymbol{J}h \in \mathcal{K}_0^\perp$, and $\boldsymbol{\mathcal{J}}\boldsymbol{J}h = \zeta h$.*

Proof. Suppose that $A^*h = \zeta h$, $|\zeta| = 1$. Since $A^* = Q^*\boldsymbol{\mathcal{J}}Q\boldsymbol{J}$ and Q is a contraction, it follows that $\|Q\boldsymbol{J}h\| = \|\boldsymbol{J}h\| = \|h\|$.

By Lemma 10.6, $\boldsymbol{J}h \in \mathcal{K}_0^\perp$ and $Q\boldsymbol{J}h = \boldsymbol{J}h$. Next, the equality $\|A^*h\| = \|h\|$ implies $\|Q^*\boldsymbol{\mathcal{J}}Q\boldsymbol{J}h\| = \|Q^*\boldsymbol{\mathcal{J}}\boldsymbol{J}h\| = \|\boldsymbol{\mathcal{J}}\boldsymbol{J}h\|$. So by Lemma 10.6,

$$Q^*\boldsymbol{\mathcal{J}}Q\boldsymbol{J}h = Q^*\boldsymbol{\mathcal{J}}\boldsymbol{J}h = \boldsymbol{\mathcal{J}}\boldsymbol{J}h.$$

Hence, $\boldsymbol{\mathcal{J}}\boldsymbol{J}h = \zeta h$ and $h \in \mathcal{K}_0^\perp$.

The converse is trivial. If $\boldsymbol{J}h \in \mathcal{K}_0^\perp$, by Lemma 10.6,

$$\boldsymbol{\mathcal{J}}Q\boldsymbol{J}h = \boldsymbol{\mathcal{J}}\boldsymbol{J}h = \zeta h.$$

Since $h \in \mathcal{K}_0^\perp$, it follows from Lemma 10.6 that $A^*h = Q^*\boldsymbol{\mathcal{J}}Q\boldsymbol{J}h = \zeta h$. ■

The following lemma will be used to evaluate $\sigma_{\mathrm{F}}(A^*) \cap \mathbb{T}$.

Lemma 11.2. *Suppose that*

$$\lim_{n\to\infty} \|(Q^* \boldsymbol{\mathcal{J}} Q - \boldsymbol{\mathcal{J}})g_n\| = 0$$

for any sequence $\{g_n\}_{n\ge1}$ *such that* $\|g_n\| = 1$, $\lim_{n\to\infty} \|Qg_n\| = 1$, *and* $\text{w-}\lim_{n\to\infty} g_n = \mathbb{O}$. *Then* $\sigma_{\rm F}(A^*) \cap \mathbb{T} \subset \sigma_{\rm F}(\boldsymbol{\mathcal{J}}\boldsymbol{J})$.

Proof. Let $\zeta \in \sigma_{\rm F}(A^*)$. Then there is a sequence $\{x_n\}_{n\ge1}$ such that $\|x_n\| = 1$, $\text{w-}\lim_{n\to\infty} x_n = \mathbb{O}$, and $\lim_{n\to\infty} \|A^*x_n - \eta x_n\| = 0$. Let $g_n = \boldsymbol{J}x_n$. Clearly, $\|g_n\| = 1$ and $\text{w-}\lim_{n\to\infty} g_n = \mathbb{O}$. We have

$$\|A^*x_n - \zeta x_n\| = \|Q^*\boldsymbol{\mathcal{J}}Qg_n - \zeta x_n\| \to 0. \tag{11.1}$$

Since $|\zeta| = 1$, it follows that $\|Qg_n\| \to 1$. Therefore by the hypotheses

$$\|A^*x_n - \boldsymbol{\mathcal{J}}\boldsymbol{J}x_n\| = \|Q^*\boldsymbol{\mathcal{J}}Qg_n - \boldsymbol{\mathcal{J}}g_n\| \to 0,$$

which together with (11.1) yields $\|\boldsymbol{\mathcal{J}}\boldsymbol{J}x_n - \zeta x_n\| \to 0$. So $\zeta \in \sigma_{\rm F}(\boldsymbol{\mathcal{J}}\boldsymbol{J})$. ■

12. The Main Construction

The operator W admits a representation as multiplication by t on the direct integral

$$\mathcal{K} = \int_{\sigma(W)} \oplus E(t)d\rho(t), \tag{12.1}$$

where ρ is a spectral measure of W. Let $\nu_W(t) \stackrel{\text{def}}{=} \dim E(t)$ be the spectral multiplicity function of W. Without loss of generality we can assume that all spaces $E(t)$ are imbedded in a Hilbert space E with an orthonormal basis $\{e_j\}_{j\ge1}$ and

$$E(t) = \operatorname{span}\{e_j : 1 \le j < \nu_W(t) + 1\}$$

(see Appendix 1.4).

Recall that ν is the spectral multiplicity function of Γ. Clearly, $\nu(t) = \nu_R(t)$ for $t \ne 0$, where ν_R is the multiplicity function of R. It is also clear that $\nu_W(t) = \nu_R(t) + \nu_R(-t)$, $t > 0$. Note that we have not yet defined the operator R. But since R must be unitarily equivalent to $\tilde{\Gamma} = \Gamma\big|(\operatorname{Ker}\Gamma)^\perp$, the above equalities must be satisfied.

We split the set $\sigma(W)$ into five disjoint subsets:

$$\sigma_0 = \{t \in \sigma(W) : \nu_R(t) = \nu_R(-t)\},$$

$$\sigma_1^{(+)} = \{t \in \sigma(W) : \nu_R(t) = \nu_R(-t) + 1\},$$

$$\sigma_1^{(-)} = \{t \in \sigma(W) : \nu_R(t) = \nu_R(-t) - 1\},$$

$$\sigma_2^{(+)} = \{t \in \sigma(W) : \nu_R(t) = \nu_R(-t) + 2\},$$

$$\sigma_2^{(-)} = \{t \in \sigma(W) : \nu_R(t) = \nu_R(-t) - 2\}.$$

Since Γ satisfies condition (C3), it follows that

$$\sigma_0 \cup \sigma_1^{(+)} \cup \sigma_1^{(-)} \cup \sigma_2^{(+)} \cup \sigma_2^{(-)} = \sigma(W).$$

We choose the operator $\boldsymbol{J}$ to be multiplication by the operator-valued function J that is defined as follows. For $t \in \sigma_0$ we put

$$J(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For $t \in \sigma_1^{(\pm)}$ we put

$$J(t) = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \pm 1 \end{pmatrix}$$

(we assume $J(t) = \pm 1$ if $\nu_W(t) = 1$). Finally, for $t \in \sigma_2^{(\pm)}$ we put

$$J(t) = \begin{pmatrix} \pm 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \pm 1 \end{pmatrix}$$

(we assume

$$J(t) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

if $\nu_W(t) = 2$).

Now we can define R by $R = \boldsymbol{J}W$.

Denote by $\mathcal{K}_0$ the "first level" of $\mathcal{K}$, i.e.,

$$\mathcal{K}_0 = \{f \in \mathcal{K} : f(t) = \varphi(t)e_1, \ \varphi \in L^2(\rho)\}.$$

Clearly, $\mathcal{K}_0$ can naturally be identified with $L^2(\rho)$.

Now let q_0 be a vector in $\mathcal{K}_0$ that satisfies the following properties:

(1) $\|q_0\| = 1$;

(2) q_0 is a cyclic vector of $W\big|\mathcal{K}_0$ (in other words, $q_0(t) \neq 0$, ρ-a.e.);

(3) $q_0 \in \operatorname{Range} W$ if $\operatorname{Ker}\Gamma \neq \{\mathbb{O}\}$ and $q_0 \notin \operatorname{Range} W$ if $\operatorname{Ker}\Gamma = \{\mathbb{O}\}$ (in the last case W is noninvertible and so such a q_0 exists).

We define the vector q by $q = \boldsymbol{J}q_0$, so $q_0 = \boldsymbol{J}q$ and the definition of $\mathcal{K}_0$ is consistent with that given in Lemma 10.6. Let $p = Rq = R\boldsymbol{J}q_0 = Wq_0$ and $\Lambda^2 = W^2 - (\cdot, p)p$. (Note that we do not define the operator Λ; we define only the operator Λ^2, and to define Λ, we have to choose a reasonable square root of Λ^2, which will be done later.) It is easy to see that

$$(W^2x, x) - (x, p)(p, x) = \|Wx\|^2 - |(x, Rq)|^2 \geq \|Wx\|^2 - \|Rx\|^2\|q\|^2 = 0,$$

since $\|Rx\| = \|Wx\|$ and $\|q\| = 1$. So $W^2 - (\cdot, p)p \geq \mathbb{O}$. Obviously, $\mathcal{K}_0$ is a reducing subspace for both W^2 and Λ^2, and W and $|\Lambda| \stackrel{\text{def}}{=} (\Lambda^2)^{1/2}$ coincide on $\mathcal{K}_0^\perp$.

We are going to use the Kato–Rosenblum theorem (see Kato [1], Ch. 10), which says that *if B is a self-adjoint operator and Δ is a self-adjoint operator of class $\boldsymbol{S}_1$, then the absolutely continuous parts of B and $B + \Delta$ are unitarily equivalent.*

Thus by the Kato–Rosenblum theorem, the absolutely continuous parts of W^2 and Λ^2 are unitarily equivalent. So if λ is a scalar spectral measure of $|\Lambda|\Big|\mathcal{K}_0$, then the absolutely continuous components ρ_a and λ_a of the measures ρ and λ are mutually absolutely continuous. Therefore there are disjoint Borel sets Δ, δ_ρ, and δ_λ, such that ρ and λ are mutually absolutely continuous on Δ, ρ is supported on $\delta_\rho \cup \Delta$, λ is supported on $\delta_\lambda \cup \Delta$, and the sets δ_ρ and δ_λ have zero Lebesgue measure. Clearly, we can assume that ρ and λ coincide on Δ. Put $\tilde{\rho} = \rho + \lambda|\delta_\lambda$. Consider the von Neumann integral

$$\tilde{\mathcal{K}} \stackrel{\text{def}}{=} \int_{\Delta \cup \delta_\rho \cup \delta_\lambda} \oplus \tilde{E}(t)\, d\tilde{\rho}(t), \tag{12.2}$$

where

$$\tilde{E}(t) = \begin{cases} \operatorname{span}\{e_k : 1 \leq k < \nu_W(t) + 1\}, & t \in \Delta, \\ \operatorname{span}\{e_k : 2 \leq k < \nu_W(t) + 1\}, & t \in \delta_\rho, \\ \{\zeta e_1 : \zeta \in \mathbf{C}\}, & t \in \delta_\lambda. \end{cases}$$

It is easy to see that p is a cyclic vector of $\Lambda^2\Big|\mathcal{K}_0$, and so $|\Lambda|\Big|\mathcal{K}_0$ is a cyclic operator. Therefore there exists a unitary operator U_0 from $L^2(\rho)$

onto $L^2(\lambda)$ such that $U_0\left(|\Lambda|\Big|\mathcal{K}_0\right)U_0^*$ is multiplication by the independent variable on $L^2(\lambda)$.

The operator $|\Lambda|$ can be realized as multiplication on $\tilde{\mathcal{K}}$. Namely, if we define the unitary operator $U : \mathcal{K} = L^2(\rho) \oplus \mathcal{K}_0^\perp \to L^2(\lambda) \oplus \mathcal{K}_0^\perp$ by the operator matrix

$$U = \begin{pmatrix} U_0 & \mathbb{O} \\ \mathbb{O} & I \end{pmatrix}, \tag{12.3}$$

then $|\Lambda| = U^* M_t U$, where M_t is multiplication by t on $\tilde{\mathcal{K}}$. (Note that $|\Lambda|$ coincides with W on $\mathcal{K}_0^\perp$.)

We define now the operator $\boldsymbol{\mathcal{J}}$ (the argument of Λ) as $\boldsymbol{\mathcal{J}} = U^*\tilde{\boldsymbol{\mathcal{J}}}U$, where $\tilde{\boldsymbol{\mathcal{J}}}$ is multiplication by an operator-valued function $\tilde{\mathcal{J}}$ of the following form.

If $t \in \delta_\lambda$, we put

$$\tilde{\mathcal{J}}(t) = 1$$

(in this case $\dim \tilde{E}(t) = 1$). If $t \in \sigma_2^{(\pm)}$ (in this case $t \in \Delta$ by (C3)), we put

$$\tilde{\mathcal{J}}(t) = \begin{pmatrix} \cos\omega_1(t) & \sin\omega_1(t) & 0 & 0 & \cdots \\ \sin\omega_1(t) & -\cos\omega_1(t) & 0 & 0 & \cdots \\ 0 & 0 & \cos\omega_2(t) & \sin\omega_2(t) & \cdots \\ 0 & 0 & \sin\omega_2(t) & -\cos\omega_2(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For $t \in \sigma_1^{(\pm)} \cap \Delta$ we put

$$\tilde{\mathcal{J}}(t) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cos\omega_1(t) & \sin\omega_1(t) & \cdots & 0 & 0 \\ 0 & \sin\omega_1(t) & -\cos\omega_1(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cos\omega_k(t) & \sin\omega_k(t) \\ 0 & 0 & 0 & \cdots & \sin\omega_k(t) & -\cos\omega_k(t) \end{pmatrix},$$

while for $t \in \sigma_1^{(\pm)} \cap \delta_\rho$ we put

$$\tilde{\mathcal{J}}(t) = \begin{pmatrix} \cos\omega_1(t) & \sin\omega_1(t) & \cdots & 0 & 0 \\ \sin\omega_1(t) & -\cos\omega_1(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cos\omega_k(t) & \sin\omega_k(t) \\ 0 & 0 & \cdots & \sin\omega_k(t) & -\cos\omega_k(t) \end{pmatrix}.$$

($\tilde{\mathcal{J}}(t)$ is defined on $\tilde{E}(t) = \operatorname{span}\{e_k : 2 \le k < \nu_W(t) + 1\}$, which has even dimension for $t \in \sigma_1^{(\pm)} \cap \delta_\rho$; note that in this case we simply delete the first column and the first row from the matrix defined for the previous case

$t \in \sigma_1^{(\pm)} \cap \Delta$.) If $t \in \sigma_0 \cap \Delta$ and $\dim E(t) < \infty$, we put

$$\tilde{\mathcal{J}}(t) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cos\omega_1(t) & \sin\omega_1(t) & \cdots & 0 & 0 & 0 \\ 0 & \sin\omega_1(t) & -\cos\omega_1(t) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cos\omega_k(t) & \sin\omega_k(t) & 0 \\ 0 & 0 & 0 & \cdots & \sin\omega_k(t) & -\cos\omega_k(t) & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \tag{12.4}$$

(note that

$$\tilde{\mathcal{J}}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

if $\dim E(t) = 2$). In the case $t \in \sigma_0 \cap \Delta$ and $\dim E(t) = \infty$, we define $\tilde{\mathcal{J}}(t)$ as

$$\tilde{\mathcal{J}}(t) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & \cos\omega_1(t) & \sin\omega_1(t) & \cdots & 0 & 0 & \cdots \\ 0 & \sin\omega_1(t) & -\cos\omega_1(t) & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & \cos\omega_k(t) & \sin\omega_k(t) & \cdots \\ 0 & 0 & 0 & \cdots & \sin\omega_k(t) & -\cos\omega_k(t) & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{12.5}$$

Finally, in the case $t \in \sigma_0 \cap \delta_\rho$ in the definition of $\tilde{\mathcal{J}}$ we delete the first row and the first column from the matrix (12.4) if $\dim E(t) < \infty$, and we delete the first column and the first row from the matrix (12.5) if $\dim E(t) = \infty$.

We impose the following assumptions on the functions ω_j:

(1′) the operator-valued function $\tilde{\mathcal{J}}$ takes at most a countable set of values $\{\tilde{\mathcal{J}}\}_{n\geq 1}$ (the order of the values is insignificant);
(2′) if $\tilde{\mathcal{J}}$ contains an entry $\cos\omega_j(t)$, then $\omega_j(t) < 1/3n$;
(3′) $\omega_j(t) \neq 0$;
(4′) $\omega_j(t) \to 0$ as $j \to \infty$.

If in the definition of $\tilde{\mathcal{J}}_n$ we replace $\omega_j(t)$ by zero, we obtain a diagonal operator D_n whose entries are equal to 1 or -1. It follows from (4′) that $D_n - \tilde{\mathcal{J}}_n$ is compact. Note that it follows from (2′) that

$$\|\tilde{\mathcal{J}}_n - D_n\| \leq 1/n. \tag{12.6}$$

We can now define Λ by $\Lambda = \boldsymbol{\mathcal{J}}|\Lambda|$. The operators A and Q are defined by (10.1) and (10.3). This completes the construction. Now to prove Theorem

9.1, we have to choose a vector q_0 satisfying the above conditions (1)–(3) and functions ω_j satisfying the above conditions (1′)–(4′) so that the operator A^* is asymptotically stable.

13. Proof of Theorem 9.1

In this section we are going to prove Theorem 9.1. As we have already explained, this will also prove Theorem 0.1, the main result of this chapter. To this end we first show that A^* has no point spectrum on $\mathbb{T}$. Then we reduce the problem to the evaluation of the Fredholm spectrum of $\boldsymbol{\mathcal{J}}\boldsymbol{J}$. Finally, we show how to make $\sigma_{\mathrm{F}}(\boldsymbol{\mathcal{J}}\boldsymbol{J})$ at most countable, which gives us the proof of Theorem 9.1.

Lemma 13.1. *For any choice of* q_0 *and* ω_j *satisfying conditions* (1)–(3) *and* (1′)–(4′) *of* §*12, the operator* A^* *has no eigenvalues on* $\mathbb{T}$.

Proof. The proof is based on the following trivial observation. Let a_{11}, a_{12}, a_{21}, a_{22}, d_1, d_2 be complex numbers such that $a_{12} \neq 0$ and

$$\begin{pmatrix} 0 \\ d_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 0 \\ d_1 \end{pmatrix}.$$

Then $d_1 = d_2 = 0$.

By Lemma 11.1, $A^*h = \zeta h$, $|\zeta| = 1$, if and only if $h \in \mathcal{K}_0^\perp$, $\boldsymbol{J}h \in \mathcal{K}_0^\perp$, and $\boldsymbol{\mathcal{J}}\boldsymbol{J}h = \zeta h$. Put $g = \boldsymbol{J}h = \zeta\boldsymbol{\mathcal{J}}h$. Consider the representations of h and g in the von Neumann integral (12.1):

$$h(t) = \sum_{k\geq1} h_k(t)e_k, \quad g(t) = \sum_{k\geq1} g_k(t)e_k, \tag{13.1}$$

where h_k and g_k are scalar functions. Since $h \perp \mathcal{K}_0$ and $g = \boldsymbol{J}h \perp \mathcal{K}_0$, it follows that $h_1(t) = 0$ and $g_1(t) = 0$ for all t. Let us show that h_k and g_k are identically zero for $k \geq 2$. Suppose that we have already proved that for $1 \leq k \leq n$.

Note that since $h_1(t) = g_1(t) = 0$, the representations of h and g in the von Neumann integral (12.2) coincide with the representations (13.1).

If n is even and $t \in \sigma_2^{(\pm)}$, the equality $g = \boldsymbol{J}h$ implies

$$\begin{pmatrix} 0 \\ g_{n+1}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ h_{n+1}(t) \end{pmatrix}. \tag{13.2}$$

If n is even and $t \notin \sigma_2^{(\pm)}$, the equality $q = \zeta\boldsymbol{\mathcal{J}}h$ implies

$$\begin{pmatrix} 0 \\ g_{n+1}(t) \end{pmatrix} = \begin{pmatrix} \cos\omega_j(t) & \sin\omega_j(t) \\ \sin\omega_j(t) & -\cos\omega_j(t) \end{pmatrix} \begin{pmatrix} 0 \\ h_{n+1}(t) \end{pmatrix}. \tag{13.3}$$

In view of the above observation in both cases we get $g_{n+1}(t) = h_{n+1}(t) = 0$ (recall that $\sin\omega_j(t) \neq 0$).

If n is odd and $t \in \sigma_2^{(\pm)}$, then (13.3) holds, and if n is odd and $t \notin \sigma_2^{(\pm)}$, then (13.2) holds, which again proves that $g_{n+1}(t) = h_{n+1}(t) = 0$. ■

Now we are going to reduce the estimation of the Fredholm spectrum of A^* on the unit circle to that of $\boldsymbol{\mathcal{J}}\boldsymbol{J}$. This would be very easy if we could prove that $I-Q$ is compact. Indeed in this case $A^* - \boldsymbol{\mathcal{J}}\boldsymbol{J} = Q^*\boldsymbol{\mathcal{J}}Q\boldsymbol{J} - \boldsymbol{\mathcal{J}}\boldsymbol{J}$ is compact and so if $\lambda \in \sigma_{\mathrm{F}}(A^*)$, then $\lambda \in \sigma_{\mathrm{F}}(\boldsymbol{\mathcal{J}}\boldsymbol{J})$. It is easy to see that $I-Q$ is compact if W is invertible. Indeed, in this case $Q = (W^{1/2} - |\Lambda|^{1/2})W^{-1/2}$ and since $W^2 - |\Lambda|^2$ is rank one, it follows that $W^{1/2} - |\Lambda|^{1/2}$ is compact. We shall see in §14 that $I - Q$ is compact (and even Hilbert–Schmidt) if $q \in \operatorname{Range} R$ (which corresponds to the case when Γ has nontrivial kernel). However, it is unknown if it is possible to choose $q \notin \operatorname{Range} R$ so that $I-Q$ is compact. It is also unclear whether it is possible to find q so that $I-Q$ is noncompact. It can be shown that $I-Q$ is not a Hilbert–Schmidt operator in general.

Nevertheless the following lemmas will allow us to get rid of Q even in the case $\operatorname{Ker}\Gamma = \{\mathbb{O}\}$.

Lemma 13.2. *There exists a function $\varkappa \in L^\infty(\lambda)$ such that $\varkappa(t) > 0$, λ-a.e., and the operator $M_\psi U_0(I - Q)$ is a compact operator from $L^2(\rho)$ to $L^2(\lambda)$ for any ψ satisfying $|\psi| \le \varkappa$, where M_ψ is multiplication by ψ on $L^2(\lambda)$.*

As before we identify $L^2(\rho)$ with $\mathcal{K}_0$. The proof of Lemma 13.2 will be given in §14.

Lemma 13.3. *Suppose that the function ω_1 in the definition of $\boldsymbol{\mathcal{J}}$ satisfies the inequality $|\sin\omega_1(t)| \le \varkappa(t)$ for $t \in \sigma_2^{(+)} \cup \sigma_2^{(-)}$, where $\varkappa$ is the function from Lemma 13.2. Then $\sigma_{\mathrm{F}}(A^*) \cap \mathbb{T} \subset \sigma_{\mathrm{F}}(\boldsymbol{\mathcal{J}}\boldsymbol{J})$.*

Proof. Let us verify the hypotheses of Lemma 11.2. Let

$$\mathcal{K}_j = \{f \in \mathcal{K} : f(t) = \varphi(t)e_{j+1},\ \varphi \in L^2(\rho)\}, \quad j \ge 0.$$

Clearly, $\mathcal{K}_j$ can naturally be identified with a subspace of $L^2(\rho)$. Then $\mathcal{K} = \bigoplus\limits_{j\ge 0} \mathcal{K}_j$. Let $\{\boldsymbol{\mathcal{J}}_{jk}\}_{j,k\ge 0}$ be the block matrix representation of the operator $\boldsymbol{\mathcal{J}} = U^*\tilde{\boldsymbol{\mathcal{J}}}U$ with respect to the orthogonal decomposition $\mathcal{K} = \bigoplus\limits_{j\ge 0} \mathcal{K}_j$. It follows from the definition of $\boldsymbol{\mathcal{J}}$ that

$$\boldsymbol{\mathcal{J}}_{00} = U_0^*(I - M_u)U_0, \qquad \boldsymbol{\mathcal{J}}_{01} = U_0^* M_v \big| \mathcal{K}_1, \qquad \boldsymbol{\mathcal{J}}_{10} = M_v U_0,$$

where M_u and M_v are multiplications on $L^2(\lambda)$ by the functions u and v defined by

$$u(t) = \begin{cases} 1 - \cos\omega_1(t), & t \in \sigma_2^{(+)} \cup \sigma_2^{(-)}, \\ 0, & t \notin \sigma_2^{(+)} \cup \sigma_2^{(-)}, \end{cases}$$

$$v(t) = \begin{cases} \sin\omega_1(t), & t \in \sigma_2^{(+)} \cup \sigma_2^{(-)}, \\ 0, & t \notin \sigma_2^{(+)} \cup \sigma_2^{(-)}. \end{cases}$$

(In the case $\nu_W(t) = 1$, we have $v(t) = 0$, and so Range$\mathcal{J}_{10} \subset \mathcal{K}_1$.)

Since $|v(t)| \le \varkappa(t)$, it is easy to see that $|u(t)| \le \varkappa(t)$.

Consider now the block matrix representation of the operator $Q^*\mathcal{J}Q$ with respect to the same decomposition of $\mathcal{K}$. It is easy to see that

$$Q^*\mathcal{J}Q = \begin{pmatrix} Q_0^*U_0^*(I-M_u)U_0Q_0 & Q_0^*U_0^*M_v\big|\mathcal{K}_1 & \mathcal{J}_{02} & \cdots \\ M_vU_0Q_0 & \mathcal{J}_{11} & \mathcal{J}_{12} & \cdots \\ \mathcal{J}_{20} & \mathcal{J}_{21} & \mathcal{J}_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

To verify the hypotheses of Lemma 11.2, it is sufficient to prove that for any $g_n, h_n \in \mathcal{K}_0$ such that $\|g_n\| = \|h_n\| = 1$, $\lim\limits_{n\to\infty} \|Q_0 g_n\| = 1$, and

$$\text{w-}\lim_{n\to\infty} g_n = \text{w-}\lim_{n\to\infty} h_n = \mathbb{O},$$

the following equalities hold:

$$\lim_{n\to\infty} \|(Q_0^*U_0^*(I-M_u)U_0Q_0 - U_0^*(I-M_u)U_0)g_n\| = 0, \tag{13.4}$$

$$\lim_{n\to\infty} \|(M_vU_0Q_0 - M_vU_0)g_n\| = 0, \tag{13.5}$$

$$\lim_{n\to\infty} \|(Q_0^*U_0^*M_v - U_0^*M_v)h_n\| = 0. \tag{13.6}$$

Since $|v(t)| \le \varkappa(t)$, it follows from Lemma 13.2 that the operator

$$M_vU_0Q_0 - M_vU_0 = M_vU_0(Q_0 - I)$$

is compact which implies (13.5). Similarly, $Q_0^*U_0^*M_v - U_0^*M_v$ is compact, and so (13.6) holds. Let us prove (13.4). We have

$$\begin{aligned} Q_0^*U_0^*(I-M_u)U_0Q_0 \quad - \quad & U_0^*(I-M_u)U_0 \\ = \quad & Q_0^*Q_0 - I - (Q_0^* - I)U_0^*M_uU_0 \\ - \quad & U_0^*M_uU_0(Q_0 - I) - (Q_0^* - I)U_0^*M_uU_0(Q_0 - I). \end{aligned}$$

By Lemma 13.2, the operators

$$(Q_0^* - I)U_0^*M_uU_0, \quad U_0^*M_uU_0(Q_0 - I), \quad \text{and} \quad (Q_0^* - I)U_0^*M_uU_0(Q_0 - I)$$

are compact. Hence, it remains to show that $\lim\limits_{n\to\infty} \|Q_0^*Q_0 g_n - g_n\| = 0$. Since $\|Q_0\| \le 1$ and

$$\lim_{n\to\infty} (Q_0^*Q_0 g_n, g_n) = \lim_{n\to\infty} \|Q_0 g_n\|^2 = 1,$$

we have $\lim\limits_{n\to\infty} \|Q_0^*Q_0 g_n\| = 1$. It now follows from the spectral theorem for the self-adjoint contraction $Q_0^*Q_0$ that $\lim\limits_{n\to\infty} \|Q_0^*Q_0 g_n - g_n\| = 0$, which proves (13.4) and completes the proof of the lemma. ■

Now it remains to show that we can choose the functions q_0 and ω_j so that $\sigma_{\mathrm{F}}(\mathcal{J}\boldsymbol{J})$ is at most countable.

Let us briefly explain the idea how to make $\sigma_{\mathrm{F}}(\boldsymbol{\mathcal{J}}\boldsymbol{J})$ at most countable. Suppose for a while that $\rho = \lambda$ and $\boldsymbol{\mathcal{J}}$ is multiplication by $\tilde{\mathcal{J}}$ on $\mathcal{K}$. Since both functions J and $\tilde{\mathcal{J}}$ take at most countably many values, so does $J\tilde{\mathcal{J}}$. It can easily be shown that if the functions ω_j satisfy the conditions (1′)–(4′) of §12, then for each t the operator $J(t)\tilde{\mathcal{J}}(t)$ is a small perturbation of a block diagonal operator whose diagonal blocks are of the form

$$\pm 1, \qquad \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The spectrum of each such block is contained in the set $\{1, -1, \mathrm{i}, -\mathrm{i}\}$, and it is easy to see from (12.6) that the set $\bigcup\limits_t \sigma\left(J(t)\tilde{\mathcal{J}}(t)\right)$ is at most countable and can have accumulation points only in the set $\{1, -1, \mathrm{i}, -\mathrm{i}\}$. Since the operator $J(t)\tilde{\mathcal{J}}(t)$ is unitary for each t, it follows that

$$\sigma(\boldsymbol{\mathcal{J}}\boldsymbol{J}) \subset \operatorname{clos}\left(\bigcup_t \sigma\left(J(t)\tilde{\mathcal{J}}(t)\right)\right),$$

and so $\sigma(\boldsymbol{\mathcal{J}}\boldsymbol{J})$ is at most countable.

However, the operators $\boldsymbol{J}$ and $\boldsymbol{\mathcal{J}}$ are multiplications by J and $\tilde{\mathcal{J}}$ in different systems of coordinates. Nevertheless, if these two systems of coordinates are "sufficiently close" to each other, we can use similar considerations to prove that $\sigma(\boldsymbol{\mathcal{J}}\boldsymbol{J})$ is at most countable.

Theorem 13.4. *There exist a function q_0 in $L^2(\rho)$ that satisfies conditions* (1)–(3) *of* §*12 and a function η in $L^\infty(\rho)$, $\eta(t) > 0$, ρ-a.e., such that if the functions ω_j satisfy conditions* (1′)–(4′) *of* §*12 and $|\omega_1(t)| \le \eta(t)$ for almost all $t \in \sigma_2^{(+)} \cup \sigma_2^{(-)}$, then $\sigma_{\mathrm{F}}(\boldsymbol{\mathcal{J}}\boldsymbol{J})$ is at most countable.*

To prove the theorem we represent the operator $\boldsymbol{\mathcal{J}}$ as a product of three operators. We define the operator-valued functions $\tilde{\mathcal{J}}_2^{(+)}, \tilde{\mathcal{J}}_2^{(-)}$, and $\tilde{\mathcal{J}}_1$ on $\Delta \cup \delta_\lambda \cup \delta_\rho$ by

$$\tilde{\mathcal{J}}_2^{(\pm)}(t) = \begin{cases} \tilde{\mathcal{J}}(t), & t \in \sigma_2^{(\pm)}, \\ I, & t \notin \sigma_2^{(\pm)}, \end{cases}$$

$$\tilde{\mathcal{J}}_1(t) = \begin{cases} \tilde{\mathcal{J}}(t), & t \notin \sigma_2^{(+)} \cup \sigma_2^{(-)}, \\ I, & t \in \sigma_2^{(+)} \cup \sigma_2^{(-)}. \end{cases}$$

Let $\tilde{\boldsymbol{\mathcal{J}}}_2^{(+)}, \tilde{\boldsymbol{\mathcal{J}}}_2^{(-)}$, and $\tilde{\boldsymbol{\mathcal{J}}}_1$ be multiplications by $\tilde{\mathcal{J}}_2^{(+)}, \tilde{\mathcal{J}}_2^{(-)}$, and $\tilde{\mathcal{J}}_1$ on the von Neumann integral (12.2). Put $\boldsymbol{\mathcal{J}}_2^{(\pm)} = U^*\tilde{\boldsymbol{\mathcal{J}}}_2^{(\pm)}U$ and $\boldsymbol{\mathcal{J}}_1 = U^*\tilde{\boldsymbol{\mathcal{J}}}_1 U$. Clearly, $\boldsymbol{\mathcal{J}} = \boldsymbol{\mathcal{J}}_2^{(+)}\boldsymbol{\mathcal{J}}_2^{(-)}\boldsymbol{\mathcal{J}}_1$. It is easy to see that $\mathcal{K}_0$ is a reducing subspace for $\boldsymbol{\mathcal{J}}_1$ and $\boldsymbol{\mathcal{J}}_1 \big| \mathcal{K}_0 = I$.

For a Borel subset γ of $\sigma(W)$ we denote by $\mathcal{K}(\gamma)$ the subspace of $\mathcal{K}$ that consists of functions supported on γ and by $\mathcal{E}(\gamma)$ the orthogonal projection onto $\mathcal{K}(\gamma)$ (in other words, $\mathcal{E}$ is the spectral measure of W). Put $\sigma_1 = \sigma(W)\backslash\left(\sigma_2^{(+)} \cup \sigma_2^{(-)}\right)$.

We call an operator on $\mathcal{K}$ *essentially block diagonal* (with respect to the decomposition $\mathcal{K} = \mathcal{K}(\sigma_1) \oplus \mathcal{K}\left(\sigma_2^{(+)}\right) \oplus \mathcal{K}\left(\sigma_2^{(-)}\right)$ if its block matrix has compact off-diagonal entries.

It is easy to see from the definition of $\boldsymbol{\mathcal{J}}$ that the operator $\boldsymbol{\mathcal{J}}_1$ is block diagonal. It is also easy to show that the operators $\boldsymbol{\mathcal{J}}_2^{(+)}$ and $\boldsymbol{\mathcal{J}}_2^{(-)}$ are essentially block diagonal if the operators

$$(\boldsymbol{\mathcal{J}}_2^{(\pm)} - I)\mathcal{E}\left(\sigma(W)\backslash\sigma_2^{(\pm)}\right) \tag{13.7}$$

are compact. Therefore in this case $\boldsymbol{\mathcal{J}}$ is also essentially block diagonal. It follows easily from the compactness of the operators (13.7) that the operators

$$\mathcal{E}\left(\sigma_2^{(\pm)}\right)\boldsymbol{\mathcal{J}}\mathcal{E}\left(\sigma_2^{(\pm)}\right) - \mathcal{E}\left(\sigma_2^{(\pm)}\right)\boldsymbol{\mathcal{J}}_2^{(\pm)}\mathcal{E}\left(\sigma_2^{(\pm)}\right) \tag{13.8}$$

and

$$\mathcal{E}(\sigma_1)\boldsymbol{\mathcal{J}}\mathcal{E}(\sigma_1) - \mathcal{E}(\sigma_1)\boldsymbol{\mathcal{J}}_1\mathcal{E}(\sigma_1) \tag{13.9}$$

are also compact.

The following lemma proves the compactness of the operators (13.7).

Lemma 13.5. *There exist a function q_0 in $L^2(\rho)$ that satisfies conditions* (1)–(3) *of* §*12 and a function η in $L^\infty(\rho)$, $\eta(t) > 0$, ρ-a.e., such that if the functions ω_j satisfy conditions* (1′)–(4′) *of* §*12 and $|\omega_1(t)| \le \eta(t)$ for almost all $t \in \sigma_2^{(+)} \cup \sigma_2^{(-)}$, then the operators* (13.8) *and* (13.9) *are compact.*

The proof of Lemma 13.5 will be given in §14.

Proof of Theorem 13.4. Let us choose q_0, η, and ω_j satisfying the hypotheses of Lemma 13.5. Then the operator $\boldsymbol{\mathcal{J}}$ is essentially block diagonal with respect to the decomposition $\mathcal{K} = \mathcal{K}(\sigma_1) \oplus \mathcal{K}\left(\sigma_2^{(+)}\right) \oplus \mathcal{K}\left(\sigma_2^{(-)}\right)$. Obviously, the operator $\boldsymbol{J}$ is block diagonal. Hence, $\boldsymbol{\mathcal{J}}\boldsymbol{J}$ is essentially block diagonal. Therefore, to prove that $\sigma_{\mathrm{F}}(\boldsymbol{\mathcal{J}}\boldsymbol{J})$ is at most countable, it is sufficient to prove that the Fredholm spectrum of each diagonal block is at most countable on the unit circle.

Consider the block

$$\mathcal{E}(\sigma_1)\boldsymbol{\mathcal{J}}\boldsymbol{J}\Big|\mathcal{K}(\sigma_1) = \mathcal{E}(\sigma_1)\boldsymbol{\mathcal{J}}\mathcal{E}(\sigma_1)\boldsymbol{J}\Big|\mathcal{K}(\sigma_1).$$

It follows from the compactness of the operators (13.8) and (13.8) that

$$\sigma_{\mathrm{F}}\left(\mathcal{E}(\sigma_1)\boldsymbol{\mathcal{J}}\boldsymbol{J}\Big|\mathcal{K}(\sigma_1)\right) = \sigma_{\mathrm{F}}\left(\mathcal{E}(\sigma_1)\boldsymbol{\mathcal{J}}_1\boldsymbol{J}\Big|\mathcal{K}(\sigma_1)\right).$$

But both operators $\mathcal{E}(\sigma_1)\boldsymbol{\mathcal{J}}_1\mathcal{E}(\sigma_1)$ and $\mathcal{E}(\sigma_1)\boldsymbol{J}\mathcal{E}(\sigma_1)$ are multiplications on the von Neumann integral (12.1). Thus the reasoning given before the statement of Theorem 13.4 works in this case, and so $\sigma\left(\mathcal{E}(\sigma_1)\boldsymbol{\mathcal{J}}\boldsymbol{J}\,\middle|\,\mathcal{K}(\sigma_1)\right)$ is at most countable.

The situation with two other blocks is a bit more complicated. Consider for example the block

$$\mathcal{E}\left(\sigma_2^{(+)}\right)\boldsymbol{\mathcal{J}}\boldsymbol{J}\,\Big|\,\mathcal{K}\left(\sigma_2^{(+)}\right)=\mathcal{E}\left(\sigma_2^{(+)}\right)\boldsymbol{\mathcal{J}}\mathcal{E}\left(\sigma_2^{(+)}\right)\boldsymbol{J}\,\Big|\,\mathcal{K}\left(\sigma_2^{(+)}\right).$$

Again by the compactness of the operators (13.8) and (13.8), we have

$$\sigma_{\mathrm{F}}\left(\mathcal{E}\left(\sigma_2^{(+)}\right)\boldsymbol{\mathcal{J}}\boldsymbol{J}\,\Big|\,\mathcal{K}(\sigma_2)\right)=\sigma_{\mathrm{F}}\left(\mathcal{E}\left(\sigma_2^{(+)}\right)\boldsymbol{\mathcal{J}}_2\boldsymbol{J}\,\Big|\,\mathcal{K}(\sigma_2)\right).$$

Let $J_2^{(+)}$ be the operator-valued function defined by

$$J_2^{(+)}(t)=\begin{cases} J(t), & t\in\sigma_2^{(+)},\\ I, & t\in\sigma(W)\backslash\sigma_2^{(+)},\end{cases}$$

and let $\boldsymbol{J}_2^{(+)}$ be multiplication by $J_2^{(+)}$ on the von Neumann integral (12.1). Obviously, $\mathcal{E}\left(\sigma_2^{(+)}\right)\boldsymbol{J}\mathcal{E}\left(\sigma_2^{(+)}\right)=\mathcal{E}\left(\sigma_2^{(+)}\right)\boldsymbol{J}_2^{(+)}\mathcal{E}\left(\sigma_2^{(+)}\right)$, and so

$$\begin{aligned}\mathcal{E}\left(\sigma_2^{(+)}\right)\boldsymbol{\mathcal{J}}_2^{(+)}\boldsymbol{J}\mathcal{E}\left(\sigma_2^{(+)}\right) &= \mathcal{E}\left(\sigma_2^{(+)}\right)\boldsymbol{\mathcal{J}}_2^{(+)}\mathcal{E}\left(\sigma_2^{(+)}\right)\boldsymbol{J}\mathcal{E}\left(\sigma_2^{(+)}\right)\\ &= \mathcal{E}\left(\sigma_2^{(+)}\right)\boldsymbol{\mathcal{J}}_2^{(+)}\mathcal{E}\left(\sigma_2^{(+)}\right)\boldsymbol{J}_2^{(+)}\mathcal{E}\left(\sigma_2^{(+)}\right)\\ &= \mathcal{E}\left(\sigma_2^{(+)}\right)\boldsymbol{\mathcal{J}}_2^{(+)}\boldsymbol{J}_2^{(+)}\mathcal{E}\left(\sigma_2^{(+)}\right).\end{aligned}$$

We have

$$\boldsymbol{\mathcal{J}}_2^{(+)}\boldsymbol{J}_2^{(+)}=U^*\tilde{\boldsymbol{\mathcal{J}}}_2^{(+)}U\boldsymbol{J}_2^{(+)}=U^*\tilde{\boldsymbol{\mathcal{J}}}_2^{(+)}(U\boldsymbol{J}_2^{(+)}U^*)U.$$

Since $\mathcal{K}_0$ is a reducing subspace of $\boldsymbol{J}_2^{(+)}$, it follows from (12.3) that the operator $\tilde{\boldsymbol{J}}_2^{(+)}=U\boldsymbol{J}_2^{(+)}U^*$ is multiplication on the von Neumann integral (12.2) by the operator-valued function $\tilde{J}_2^{(+)}$ defined by

$$\tilde{J}_2^{(+)}(t)e_j=\begin{cases} J_2^{(+)}(t)e_j, & 2\le j<\nu_W+1, & t\in\delta_\rho\cup\Delta,\\ e_1, & j=1, & t\in\delta_\rho\cup\Delta\cup\delta_\lambda.\end{cases}$$

Now the operators $\tilde{\boldsymbol{\mathcal{J}}}_2^{(+)}$ and $U\boldsymbol{J}_2^{(+)}U^*$ are both multiplications on (12.2), and so the reasoning given before the statement of Theorem 13.4 works, which shows that

$$\sigma\left(\tilde{\boldsymbol{\mathcal{J}}}_2^{(+)}U\boldsymbol{J}_2^{(+)}U^*\right)=\sigma\left(\boldsymbol{\mathcal{J}}_2^{(+)}\boldsymbol{J}'\right)$$

is at most countable.

To show that $\sigma\left(\mathcal{E}\left(\sigma_2^{(+)}\right)\boldsymbol{\mathcal{J}}_2^{(+)}\boldsymbol{J}_2^{(+)}\,\Big|\,\mathcal{K}\left(\sigma_2^{(+)}\right)\right)\bigcap\mathbb{T}$ is at most countable, we need the following lemma.

Lemma 13.6. *Let Z be a contraction and V a unitary operator on Hilbert space. Then*

$$\sigma(Z^*VZ)\cap\mathbb{T}\subset\sigma(V).$$

Let us first complete the proof of Theorem 13.4. It follows from Lemma 13.6 that

$$\sigma\left(\mathcal{E}\left(\sigma_2^{(+)}\right)\boldsymbol{\mathcal{J}}_2^{(+)}\boldsymbol{J}_2^{(+)}\mathcal{E}\left(\sigma_2^{(+)}\right)\right)\bigcap\mathbb{T}\subset\sigma\left(\boldsymbol{\mathcal{J}}_2^{(+)}\boldsymbol{J}_2^{(+)}\right),$$

which implies that $\sigma\left(\mathcal{E}\left(\sigma_2^{(+)}\right)\boldsymbol{\mathcal{J}}_2^{(+)}\boldsymbol{J}_2^{(+)}\Big|\mathcal{K}\left(\sigma_2^{(+)}\right)\right)\bigcap\mathbb{T}$ is at most countable. In a similar way it can be shown that $\sigma\left(\mathcal{E}\left(\sigma_2^{(-)}\right)\boldsymbol{\mathcal{J}}\boldsymbol{J}\mathcal{E}\left(\sigma_2^{(-)}\right)\right)\bigcap\mathbb{T}$ is at most countable, which completes the proof. ■

Proof of Lemma 13.6. Suppose that $\zeta\in\sigma(Z^*VZ)\cap\mathbb{T}$ and $\zeta\notin\sigma(V)$. Since Z^*VZ is a contraction and $|\zeta|=1$, it follows that there exists a sequence of vectors $\{f_n\}_{n\ge0}$ such that $\|f_n\|=1$ and

$$((\bar\zeta Z^*VZ-I)f_n,f_n)=\bar\zeta(Z^*VZf_n,f_n)-\|f_n\|^2\to0. \tag{13.10}$$

Since $\zeta\notin\sigma(V)$, we have $1\notin\sigma(\bar\zeta V)$ and it follows from the spectral theorem for unitary operators that there exists a positive ε such that

$$\operatorname{Re}(\bar\zeta Vg,g)\le(1-\varepsilon)\|g\|^2$$

for any g. Hence,

$$\begin{aligned}\operatorname{Re}((\bar\zeta Z^*VZ-I)f_n,f_n) &= \operatorname{Re}\bar\zeta(Z^*VZf_n,f_n)-\|f_n\|\\ &\le (1-\varepsilon)\|Zf_n\|^2-1\le-\varepsilon,\end{aligned}$$

which contradicts (13.10). ■

Now we are in a position to prove Theorem 9.1.

Proof of Theorem 9.1. We can choose the functions q_0 and ω_j that satisfy the hypotheses of Theorem 13.4 and Lemma 13.3. It follows from Lemma 13.1 that A^* has no eigenvalues on $\mathbb{T}$. Next, it follows from Lemma 13.3 that

$$\sigma_{\mathrm F}(A^*)\cap\mathbb{T}\subset\sigma_{\mathrm F}(\boldsymbol{\mathcal{J}}\boldsymbol{J}),$$

and so by Theorem 13.4, the set $\sigma_{\mathrm F}(A^*)$ is at most countable. Consequently, by the stability test (see §11), the operator A^* is asymptotically stable, which by Lemmas 10.1 and 10.2 imply that T is asymptotically stable. The result now follows from Theorem 9.2. ■

14. Proofs of Lemmas 13.2 and 13.5

In this section we prove Lemmas 13.2 and 13.5. The following lemma will help us to represent the operator $U_0(I-Q_0)$ as an integral operator.

Recall that the space $\mathcal{K}_0$ is reducing for both W^2 and Λ^2. As usual we identify $\mathcal{K}_0$ with $L^2(\rho)$. Clearly, $M\stackrel{\text{def}}{=}W\big|\mathcal{K}_0$ is multiplication by the independent variable on $L^2(\rho)$. Let $\tilde{M}$ be multiplication by the independent variable on $L^2(\lambda)$. Then $|\Lambda| \Big| \mathcal{K}_0 = U^* \tilde{M} U$ and

$$(W^2 - \Lambda^2) \Big| \mathcal{K}_0 = M^2 - U_0^* \tilde{M}^2 U_0 = (\cdot, p)p. \tag{14.1}$$

Lemma 14.1. *Let $r = U_0 p \in L^2(\lambda)$. Then for any continuously differentiable function φ on* $\operatorname{supp}\rho \cup \operatorname{supp}\lambda$ *the following equality holds:*

$$\left(\left(U_0\varphi\left(M^2\right) - \varphi\left(\tilde{M}^2\right)U_0\right)f\right)(t) = \int \frac{\varphi(t^2) - \varphi(s^2)}{t^2 - s^2} r(t)\overline{p(s)}f(s)d\rho(s). \tag{14.2}$$

Note that Lemma 14.1 is a consequence of the theory of double operator integrals (see Birman and Solomyak [2]).

Proof of Lemma 14.1. It follows from (14.1) that

$$U_0 M^2 - \tilde{M}^2 U_0 = (\cdot, p)r,$$

which is equivalent to the equality

$$\left(\left(UM_0^2 - \tilde{M}_0^2 U\right)f\right)(t) = \int r(t)\overline{p(s)}f(s)d\rho(s).$$

Clearly,

$$U_0 M^4 - \tilde{M}^4 U_0 = \left(U_0 M^2 - \tilde{M}^2 U_0\right) M^2 + \tilde{M}^2 \left(U_0 M^2 - \tilde{M}^2 U_0\right),$$

which is equivalent to

$$\left(\left(U_0 M^4 - \tilde{M}^4 U_0\right)f\right)(t) = \int (s^2 + t^2) r(t)\overline{p(s)}f(s)d\rho(s).$$

Similarly,

$$\left(\left(U_0 M^{2n} - \tilde{M}^{2n} U_0\right)f\right)(t) = \int \frac{t^{2n} - s^{2n}}{t^2 - s^2} r(t)\overline{p(s)}f(s)d\rho(s),$$

which proves (14.2) for $\varphi(t) = t^{2n}$. The result now follows from the fact that the set of polynomials of t^2 is dense in $C^1[0, \|W\|^2]$ and the fact that the Hilbert–Schmidt norm of the integral operator on the right-hand side of (14.2) is at most $\mathrm{const} \cdot \|\varphi\|_{C^1}$. ■

Proof of Lemma 13.2. Let $\varphi(s) = s^{1/4}$, $s > 0$. It follows from Lemma 14.1 that

$$((U_0 M^{1/2} - \tilde{M}^{1/2} U_0)f)(t) = \int \frac{t^{1/2} - s^{1/2}}{t^2 - s^2} r(t)\overline{p(s)}f(s)d\rho(s) \tag{14.3}$$

at least in the case when $0 \notin \operatorname{supp} f$. Indeed, in this case we can change φ in a small neighborhood of 0 to make it continuously differentiable.

On the other hand,

$$I - Q = I - |\Lambda|^{1/2} W^{-1/2} = \left(W^{1/2} - |\Lambda|^{1/2}\right) W^{-1/2}$$

on the dense subset Range $W^{1/2}$ of $\mathcal{K}$. Therefore

$$I - Q_0 = \left(M^{1/2} - U_0^* \tilde{M}^{1/2} U_0\right) M^{-1/2}$$

on the range of $M^{1/2}$. It follows from (14.3) that

$$\begin{aligned}(U_0(I-Q_0)f)(t) &= \left(\left(U_0 M^{1/2} - \tilde{M}^{1/2} U_0\right) M^{-1/2} f\right)(t) \\ &= \int \frac{t^{1/2} - s^{1/2}}{t^2 - s^2} r(t)\overline{p(s)} s^{-1/2} f(s) d\rho(s)\end{aligned}$$

whenever $0 \notin \operatorname{supp} f$. Since $p = W q_0$, we have

$$(U_0(I-Q_0)f)(t) = \int \frac{t^{1/2} - s^{1/2}}{t^2 - s^2} r(t)\overline{q_0(s)} s^{1/2} f(s) d\rho(s) \tag{14.4}$$

whenever $0 \notin \operatorname{supp} f$.

Consider first the case $q_0 \notin$ Range W (i.e., $\operatorname{Ker}\Gamma = \{\mathbb{O}\}$). In this case $p \notin$ Range W^2, and so $\operatorname{Ker}\Lambda = \{\mathbb{O}\}$. Therefore $\lambda(\{0\}) = 0$. Consequently, for λ-almost all t the function

$$s \mapsto \frac{t^{1/2} - s^{1/2}}{t^2 - s^2} s^{1/2} \overline{q_0(s)}$$

belongs to $L^2(\rho)$. Therefore there exists a function $\varkappa$ in $L^\infty(\lambda)$ such that $\varkappa(t) > 0$, λ-a.e., and

$$\iint \left| \frac{t^{1/2} - s^{1/2}}{t^2 - s^2} s^{1/2} r(t) \overline{q_0(s)} \varkappa(t) \right|^2 d\rho(s) d\lambda(t) < \infty.$$

It follows that the operator $M_\varphi U(I - Q_0)$ is Hilbert–Schmidt whenever $|\varphi(t)| \le \varkappa(t)$, λ-a.e.

Consider now the case $q_0 \in$ Range W. Let us show that in this case $(I - Q_0)$ is a Hilbert–Schmidt operator. Let $q_0 = Wh$, $h \in L^2(\rho)$. It follows from (14.4) that

$$(U_0(I-Q_0)f)(t) = \int \frac{t^{1/2} s^{3/2} - s^2}{t^2 - s^2} \overline{h(s)} r(t) f(s) d\rho(s) \tag{14.5}$$

whenever $0 \in \operatorname{supp} f$. However, it is easy to see that

$$\sup_{t,s>0} \left| \frac{t^{1/2} s^{3/2} - s^2}{t^2 - s^2} \right| < \infty,$$

and so the integral operator on the right-hand side of (14.5) is Hilbert–Schmidt. ■

To prove Lemma 13.5, we need one more lemma. We denote by $\mathcal{E}_0(\gamma)$ the orthogonal projection on $L^2(\rho)$ onto the subspace of functions supported on γ and by $\mathcal{E}_1(\gamma)$ the corresponding projection on $L^2(\lambda)$. Note that $\mathcal{E}_0$ is the spectral measure of $W \big| \mathcal{K}_0$ and $\mathcal{E}_1$ is the spectral measure of $U(|\Lambda| \big| \mathcal{K}_0)U^*$. As usual, M_ψ denotes multiplication by ψ.

Lemma 14.2. *There exist a function q_0 satisfying conditions* (1)–(3) *of §12 and a function η that is positive ρ-a.e. on $\sigma_2^{(+)} \cup \sigma_2^{(-)}$ such that*

$$M_\psi \mathcal{E}_1 \left(\sigma_2^{(\pm)}\right) U_0 \mathcal{E}_0 \left(\sigma(W) \backslash \sigma_2^{(\pm)}\right) \tag{14.6}$$

is a compact operator from $L^2(\rho)$ to $L^2(\lambda)$ for any function ψ satisfying $|\psi(t)| \le \eta(t)$ a.e.

Proof. Let us first show that for any disjoint compact sets γ_1 and γ_2,

$$\left(\mathcal{E}_1(\gamma_2) U \mathcal{E}_0(\gamma_1) f\right)(t) = \int\limits_{\gamma_1} \frac{f(s)}{s^2 - t^2} r(t) \overline{p(s)} d\rho(s), \quad t \in \gamma_2. \tag{14.7}$$

Let φ be a continuously differentiable function such that

$$\varphi(s) = \begin{cases} 1, & s^{1/2} \in \gamma_1, \\ 0, & s^{1/2} \in \gamma_2\,. \end{cases}$$

Then by Lemma 14.1,

$$\left(\left(U_0 \varphi(M^2) - \varphi(\tilde{M}^2) U_0\right) f\right)(t) = \int \frac{\varphi(s^2) - \varphi(t^2)}{s^2 - t^2} r(t) \overline{p(s)} d\rho(s).$$

Clearly, $\left(\varphi\left(\tilde{M}^2\right) U_0 f\right)(t) = 0$ for $t \in \gamma_2$. Hence, $\mathcal{E}_1(\gamma_2)\varphi\left(\tilde{M}^2\right) U_0 = 0$. It is also clear that if f is supported on γ_1, then $\varphi(M^2) f = f$. It follows that for f supported on γ_1,

$$(U_0 f)(t) = \int \frac{1}{s^2 - t^2} r(t) \overline{p(s)} f(s) d\rho(s), \quad t \in \gamma_2,$$

which implies (14.7).

Note that if γ_1 and γ_2 are arbitrary disjoint Borel subsets, then we can approximate them by compact subsets and pass to the limit, which proves that (14.7) holds for f in a dense subset of $L^2(\rho)$.

Therefore to prove the lemma, it is sufficient to find a function q_0 that satisfies conditions (1)–(3) of §12 and such that for almost all $t \in \sigma_2^{(\pm)}$,

$$\int\limits_{\sigma(W) \backslash \sigma_2^{(\pm)}} \frac{|p(s)|^2}{|s^2 - t^2|} d\rho(s) < \infty. \tag{14.8}$$

(Recall that $p = W q_0$, and so $p(s) = s q_0(s)$.) Indeed, in this case it follows from (14.7) with $\gamma_1 = \sigma_2^{(\pm)}$ and $\gamma_2 = \sigma(W) \backslash \sigma_2^{(\pm)}$ that if η is a positive function such that

$$\int\limits_{\sigma_2^{(\pm)}} |r(t)|^2 (\eta(t))^2 \left(\int\limits_{\sigma(W) \backslash \sigma_2^{(\pm)}} \frac{|p(s)|^2}{|s^2 - t^2|} d\rho(s) \right) d\lambda(t) < \infty,$$

then the operators (14.6) are Hilbert–Schmidt, provided $|\psi| \le \eta$.

Let $\left\{\beta_n^{(+)}\right\}_{n\ge 1}$, $\left\{\beta_n^{(-)}\right\}_{n\ge 1}$, and $\{\gamma_n\}_{n\ge 1}$ be increasing sequences of compact sets such that

$$\beta_n^{(\pm)} \subset \sigma_2^{(\pm)}, \quad \lambda\left(\sigma_2^{(\pm)} \backslash \beta_n^{(\pm)}\right) \to 0, \quad \gamma_n \subset \sigma_1, \quad \rho(\sigma_1 \backslash \gamma_n) \to 0, \tag{14.9}$$

where $\sigma_1 = \sigma(W) \backslash \left(\sigma_2^{(+)} \cup \sigma_2^{(-)}\right)$. Note that since ρ and λ are mutually absolutely continuous on Δ, we have $\rho\left(\sigma_2^{(\pm)} \backslash \beta_n^{(\pm)}\right) \to 0$.

Consider first the case $\operatorname{Ker}\Gamma \neq \{\mathbb{O}\}$. In this case we have to choose q_0 in Range W such that $\|q_0\|_{L^2(\rho)} = 1$ and $q_0(s) \neq 0$, ρ-a.e.

We can easily define a positive function q_0 on $\beta_1^{(+)} \cup \beta_1^{(-)} \cup \gamma_1$ such that

$$\int\limits_{\beta_1^{(+)} \cup \beta_1^{(-)} \cup \gamma_1} \frac{|q_0(s)|^2}{s^2} d\rho(s) \le 1,$$

$$\int\limits_{\beta_1^{(-)} \cup \gamma_1} \frac{|p(s)|^2}{|s^2 - t^2|^2} d\rho(s) \le 1, \quad t \in \beta_1^{(+)},$$

and

$$\int\limits_{\beta_1^{(+)} \cup \gamma_1} \frac{|p(s)|^2}{|s^2 - t^2|^2} d\rho(s) \le 1, \quad t \in \beta_1^{(-)}.$$

Then we can proceed by induction. At the nth step we define q_0 on

$$\left(\beta_n^{(+)} \cup \beta_n^{(-)} \cup \gamma_n\right) \backslash \left(\beta_{n-1}^{(+)} \cup \beta_{n-1}^{(-)} \cup \gamma_{n-1}\right)$$

so that

$$\int\limits_{\beta_n^{(+)} \cup \beta_n^{(-)} \cup \gamma^{(n)}} \frac{|q_0(s)|^2}{s^2} d\rho(s) \le 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}},$$

$$\int\limits_{\beta_n^{(-)} \cup \gamma_n} \frac{|p(s)|^2}{|s^2 - t^2|^2} d\rho(s) \le \frac{1}{2^{k-1}} + \dots + \frac{1}{2^{n-1}}, \tag{14.10}$$

$$t \in \beta_k^{(+)} \backslash \beta_{k-1}^{(+)}, \quad 1 \le k \le n,$$

and

$$\int\limits_{\beta_n^{(+)} \cup \gamma_n} \frac{|p(s)|^2}{|s^2 - t^2|^2} d\rho(s) \le \frac{1}{2^{k-1}} + \cdots + \frac{1}{2^{n-1}}, \tag{14.11}$$

$$t \in \beta_k^{(-)} \backslash \beta_{k-1}^{(-)}, \quad 1 \le k \le n,$$

(here $\beta_0^{(+)} = \beta_0^{(-)} = \gamma_0 = \varnothing$).

Passing to the limit, we obtain a function q_0 such that $q_0(s) \neq 0$, ρ-a.e., $q_0 \in \operatorname{Range} W$ and (14.8) holds with $p(s) = sq_0(s)$. The only condition that can be violated is $\|q_0\|_{L^2(\rho)} = 1$ but obviously we can multiply q_0 by a suitable constant to achieve this condition.

Consider now the case $\operatorname{Ker}\Gamma = \{\mathbb{O}\}$. We have to construct a function q_0 such that $\|q_0\|_{L^2(\rho)} = 1$, $q_0(s) \neq 0$, ρ-a.e., $q_0 \notin \operatorname{Range} W$, and (14.8) holds. In this case the operator W is noninvertible. Hence, $\rho\big((0,\varepsilon)\big) > 0$ for any $\varepsilon > 0$. Therefore the same is true for the restriction of ρ to at least one of the sets $\sigma_2^{(+)}$, $\sigma_2^{(-)}$, σ_1. To be definite, suppose that

$$\rho\left((0,\varepsilon) \cap \sigma_2^{(+)}\right) > 0$$

for any $\varepsilon > 0$.

Clearly, we can assume that $0 \in \sigma_2^{(+)}$. It is easy to see that there exists a compact subset $\beta_1^{(+)}$ of $\sigma_2^{(+)}$ such that

$$\rho\Big((0,\varepsilon) \cap \beta_1^{(+)}\Big) > 0$$

for any $\varepsilon > 0$. We can now choose a positive function q_0 on $\beta_1^{(+)}$ such that

$$\int\limits_{\beta_1^{(+)}} \frac{|q_0(s)|^2}{s^2}\, d\rho(s) = \infty.$$

Then we can choose increasing sequences $\left\{\beta_n^{(\pm)}\right\}_{n\ge 1}$ and $\{\gamma_n\}_{n\ge 1}$ of compact sets satisfying (14.9) and proceed by induction as in the first case. At the nth step we can define q_0 on $\beta_n^{(+)} \cup \beta_n^{(-)} \cup \gamma_n$ so that (14.10) and (14.11) hold. Passing to the limit and multiplying, if necessary, by a suitable constant, we obtain a function q_0 with desired properties. ■

Proof of Lemma 13.5. Let p and η satisfy the assumptions of Lemma 13.5. Suppose that $|\sin \omega_1(t)| \le \eta(t)$ for $t \in \sigma_2^{(+)} \cup \sigma_2^{(-)}$. It is easy to see that $1 - \cos \omega_1(t) \le \eta(t)$, $t \in \sigma_2^{(+)} \cup \sigma_2^{(-)}$.

We have to show that the operators

$$\left(\boldsymbol{\mathcal{J}}_2^{(\pm)} - I\right) \mathcal{E}\left(\sigma(W)\backslash\sigma_2^{(\pm)}\right)$$

are compact. Since

$$\left(\boldsymbol{\mathcal{J}}_2^{(\pm)} - I\right) \mathcal{E}\left(\sigma(W)\backslash\sigma_2^{(\pm)}\right) f = \mathbb{O}$$

for $f \in \mathcal{K}_0^{\perp}$, it is sufficient to show that the operators

$$\left(\boldsymbol{\mathcal{J}}_2^{(\pm)} - I\right) \mathcal{E}_0\left(\sigma(W)\backslash\sigma_2^{(\pm)}\right) P_0$$

are compact, where P_0 is the orthonormal projection onto $\mathcal{K}_0$. Let us prove that the operator $\left(\boldsymbol{\mathcal{J}}_2^{(+)} - I\right) \mathcal{E}\left(\sigma(W)\backslash\sigma_2^{(+)}\right) P_0$ is compact. The proof for the other operator is the same. It is easy to see from the definition of

$\boldsymbol{\mathcal{J}}_2^{(+)}$ that for $f \in \mathcal{K}_0$ the vector $\left(\boldsymbol{\mathcal{J}}_2^{(+)} - I\right) f$ can be represented in the decomposition $\mathcal{K} = \bigoplus_{j \ge 0} \mathcal{K}_j$ as

$$\left(\boldsymbol{\mathcal{J}}_2^{(+)} - I\right) f = \begin{pmatrix} U_0^* M_u U_0 f \\ M_v U_0 f \\ \mathbb{O} \\ \mathbb{O} \\ \vdots \end{pmatrix},$$

where M_u and M_v are multiplications on $L^2(\lambda)$ by the functions u and v defined by

$$u(t) = \begin{cases} \cos \omega_1(t) - 1, & t \in \sigma_2^{(+)}, \\ 0, & t \notin \sigma_2^{(+)}, \end{cases}$$

$$v(t) = \begin{cases} \sin \omega_1(t), & t \in \sigma_2^{(+)}, \\ 0, & t \notin \sigma_2^{(+)}. \end{cases}$$

By Lemma 14.2, $M_u U_0 \mathcal{E}_0 \left(\sigma(W) \backslash \sigma_2^{(+)}\right)$ and $M_v U_0 \mathcal{E}_0 \left(\sigma(W) \backslash \sigma_2^{(+)}\right)$ are compact operators, which proves that $\left(\boldsymbol{\mathcal{J}}_2^{(+)} - I\right) \mathcal{E}_0 \left(\sigma(W) \setminus \sigma_2^{(+)}\right) P_0$ is also compact. ■

15. A Theorem in Perturbation Theory

In this section we find a connection between results on Hankel operators obtained in previous sections and a theorem in perturbation theory.

In §§9-14 for an operator Γ satisfying conditions (C1)–(C3) we have constructed an output normal system $\{T, p, q\}$ such that the Hankel operator associated with this system is unitarily equivalent to Γ. If $\operatorname{Ker} \Gamma = \{\mathbb{O}\}$, then $R = \Gamma$ and T is unitarily equivalent to backward shift S^*. In this case the vectors $\{(T^*)^j q\}_{j \ge 0}$ form an orthonormal basis in $\mathcal{K}$ and R has Hankel matrix $\{\alpha_{j+k}\}_{j,k \ge 0}$ in this basis. It is easy to see that the operator Λ also has Hankel matrix (namely, $\{\alpha_{j+k+1}\}_{j,k \ge 0}$) in the same basis. So Λ must satisfy the same conditions (C1)–(C3).

If $\operatorname{Ker} \Gamma \ne \{\mathbb{O}\}$, then it follows from the Remark following Theorem 9.2 that T is unitarily equivalent to the restriction of S^* to $(\operatorname{Ker} \Gamma)^\perp$. It is easy to see that in this case $\Lambda = S^* \Gamma \big| (\operatorname{Ker} \Gamma)^\perp$ and the operator $S^* \Gamma$ is a Hankel operator whose kernel is contained in $\operatorname{Ker} \Gamma$.

Therefore in both cases the spectral multiplicity function ν_Λ of the operator Λ must satisfy (C3).

A natural question can arise if we look at our definition of Λ. The operator Λ was defined with the help of multiplication by the function $\tilde{\mathcal{J}}$. It is easy to see from (12.4) that for $t \in \sigma_0 \cap \Delta$

$$\nu_\Lambda(t) - \nu_\Lambda(-t) = 2. \tag{15.1}$$

Hence, if the restriction of the measure λ to the set $\sigma_0 \cap \Delta$ contains a nonzero singular component, then (15.1) contradicts condition (C3). However, this can never happen: the singular components of the measures ρ and λ are mutually singular and so the restriction of λ to $\sigma_0 \cap \Delta$ is absolutely continuous. This is a consequencce of the following Aronszajn–Donoghue theorem (see Reed and Simon [1], Sec. XIII.6).

Theorem 15.1. *Let K be a self-adjoint operator on Hilbert space and let*

$$L = K + (\cdot, r)r,$$

where r is a cyclic vector of K. Then the singular components of the spectral measures of L and R are mutually singular.

Theorem 15.1 has an elementary proof. However, it is possible to deduce it from the results on Hankel operators obtained in this chapter.

Proof of Theorem 15.1. Without loss of generality we can assume that both L and K are positive and invertible. Since K is cyclic, it is unitarily equivalent to multiplication by the independent variable on $L^2(\rho)$, where ρ is a positive Borel measure with compact support in $(0, \infty)$. Let Γ be the orthogonal sum of $K^{1/2}$, $-K^{1/2}$ and the zero operator on an infinite-dimensional space. Clearly, Γ satisfies (C1)–(C3). Let $R = \Gamma \big| (\operatorname{Ker} \Gamma)^\perp$ and $W = |R|$. Then W is unitarily equivalent to $K^{1/2} \oplus K^{1/2}$. We can now put $p = r$, $q_0 = W^{-1}x$ (in this case W is invertible). Without loss of generality we can assume that $\|q_0\|_{L^2(\rho)} = 1$. We can now define the operators $\boldsymbol{J}$, $\boldsymbol{\mathcal{J}}$, Λ as above. In our case $\sigma(W) = \sigma_0$, and so the operator A^* defined by (10.1) is asymptotically stable for any choice of a cyclic vector q_0 of W. (Note that in this special case the proof of the asymptotic stability of A^* is considerably simpler than in the general case.)

Assume that the singular components of the measures ρ and λ are not mutually singular. In this case the restriction of λ to Δ (see §12) has a nonzero singular component. It follows from (12.4) that

$$\nu_\Lambda(t) = 2, \quad \nu_\Lambda(-t) = 0, \quad \lambda_s\text{-a.e. on } \Delta. \tag{15.2}$$

However, as we have already observed, ν_Λ must satisfy (C3) and so

$$|\nu_\Lambda(t) - \nu_\Lambda(-t)| \le 1, \quad \lambda_s\text{-a.e.},$$

which contradicts (15.2). ∎

Concluding Remarks

The results of this chapter were obtained in Megretskii, Peller, and Treil [2] (see also the announcement in Megretskii, Peller, and Treil [1]).

The problem to identify nonnegative operators on Hilbert space that are unitarily equivalent to the moduli of Hankel operators was posed in Khrushchëv and Peller [1]. It was shown in Khrushchëv and Peller [1] that if $\{c_n\}_{n\ge 0}$ is a nonincreasing sequence of nonnegative numbers and $\varepsilon > 0$, then there exists a Hankel operator Γ such that

$$(1-\varepsilon)c_n \le s_n(\Gamma) \le (1+\varepsilon)c_n, \quad n \ge 0.$$

The next step toward the solution of the problem posed in Khrushchëv and Peller [1] was made in Treil [1] and [2]. He showed that if a nonnegative operator satisfies (C1) and (C2) and has simple discrete spectrum, then it is unitarily equivalent to the modulus of a Hankel operator. Later Vasyunin and Treil [1] proved that the same is true for nonnegative operators with discrete spectrum satisfying (C1) and (C2).

In Ober [1] and [2] a new approach was offered. Namely, Ober obtained the results of Treil and Vasyunin [1] "modulo the kernel" using linear systems with continuous time. In Treil [7] Ober's approach was developed to solve the problem on the moduli of Hankel operators completely.

It turned out, however, that the problem of characterizing the self-adjoint operators unitarily equivalent to Hankel operators is considerably more complicated. This problem was solved in Megretskii, Peller, and Treil [2] and this solution is given in this chapter.

Theorem 2.1 was proved in Megretskii, Peller, and Treil [2]. For compact self-adjoint Hankel operators inequality (2.1) was proved earlier in Peller [19] by another method. Note that a weaker inequality was obtained in Clark [1]. Another proof of (2.1) for compact self-adjoint Hankel operators was given in the preprint Helton and Woerdeman [1]. Adamyan informed the author that he had found another proof of (2.1) in the compact case but had not published the proof.

In §8 solutions of the problem on the geometry of past and future and the inverse problem for rational approximation in $BMOA$ are given. Both problems were posed in Khrushchëv and Peller [1].

One can pose the problem to characterize the normal operators that are unitarily equivalent to Hankel operators. However, this problem trivially reduces to the problem of characterizing the self-adjoint operators unitarily equivalent to Hankel operators. Indeed, if Γ is a normal Hankel operator, then there exists a complex number τ such that $|\tau| = 1$ and the operator $\tau\Gamma$ is self-adjoint (see Nikol'skii [4], 5.6.5 (b)).

Finally, we mention here the following result of Abakumov [1]. He showed that for an arbitrary finite set $\lambda_1, \cdots, \lambda_n$ of distinct nonzero complex numbers and for an arbitrary set $k_1, \cdots, k_n$ of positive integers there exists a finite rank Hankel operator Γ such that its nonzero eigenvalues are precisely

$\lambda_1, \cdots, \lambda_n$ and the algebraic multiplicity of λ_j is k_j. Recall that the algebraic multiplicity of λ is, by definition, the dimension of $\bigcup\limits_{j\geq 1} \mathrm{Ker}(\lambda I - \Gamma)^j$.

13

Wiener–Hopf Factorizations and the Recovery Problem

In Chapter 7 we considered the recovery problem for unimodular functions. It is very important in applications to be able to solve the same problem for unitary-valued matrix functions. Namely, for a unitary-valued function U and a space X of functions on $\mathbb{T}$ we consider in this chapter the problem of under which natural assumptions we can conclude that

$$\mathbb{P}_- U \in X \quad \Longrightarrow \quad U \in X.$$

Recall that for a matrix (or vector) function Φ we use the notation $\Phi \in X$ if all entries of Φ belong to X and $\mathbb{P}_- \Phi$ is obtained from Φ by applying $\mathbb{P}_-$ to all entries.

In this chapter we find a solution to this recovery problem which is similar to the one given in Chapter 7 for scalar functions. As in the scalar case we consider separately the case of $\mathcal{R}$-spaces introduced in §7.1 and the the case of decent function spaces (i.e., spaces satisfying the axioms (A1)–(A4) stated in §7.2). The case of $\mathcal{R}$-spaces is considered in §1.

It turns out that the recovery problem for decent function spaces is closely related to the so-called heredity problem for Wiener–Hopf factorizations. Recall that in §3.5 it has been proved that if a matrix function Φ is the symbol of a Fredholm Toeplitz operator, then Φ admits a Wiener–Hopf factorization. We are going to prove in this section that if X is a decent function space and $\Phi \in X$, then the factors in the Wiener–Hopf factorization of Φ belong to the same space X. In fact, we are going to solve the recovery problem and the heredity problem for Wiener–Hopf factorizations simultaneously. The solution will be based on properties of maximizing vectors of vectorial Hankel operators that will be discussed in §2.

1. The Recovery Problem in $\mathcal{R}$-spaces

In this section we find a solution to the recovery problem for $\mathcal{R}$-spaces that is similar to the one found in §7.1. Suppose that X is a Banach (quasi-Banach) $\mathcal{R}$-space and Φ is an $m \times n$ matrix function in X (recall that this means that all entries of Φ are in X). Then unless otherwise stated, by its norm (or quasinorm) $\|\Phi\|_X$ we mean

$$\|\Phi\|_X \stackrel{\text{def}}{=} \sup\{\|(\Phi x, y)\|_X : \ x \in \mathbb{C}^n, \ y \in \mathbb{C}^m, \ \|x\| = \|y\| = 1\}.$$

Theorem 1.1. *Let X be a linear $\mathcal{R}$-space and let U be an $n \times n$ unitary-valued function such that $\mathbb{P}_- U \in X$. If the Toeplitz operator $T_U : H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^n)$ has dense range, then $\mathbb{P}_+ U \in X$. If in addition to that X is a (quasi-)Banach $\mathcal{R}$-space, then*

$$\|\mathbb{P}_+ U\|_X \le \text{const}\, \|\mathbb{P}_- U\|_X.$$

Proof. The result follows immediately from Theorem 4.4.11. ■

Theorem 1.2. *Let X be a linear $\mathcal{R}$-space and let U be an $n \times n$ unitary-valued matrix function such that $\mathbb{P}_- U \in X$. Suppose that the Toeplitz operator $T_U : H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^n)$ is Fredholm. Then $\mathbb{P}_+ U \in X$.*

Proof. Since $\dim \operatorname{Ker} T_{U^*} < \infty$, it follows that there exists $m \in \mathbb{Z}_+$ such that the restriction of T_{U^*} to $z^m H^2(\mathbb{C}^n)$ has trivial kernel. Hence, $\operatorname{Ker} T_{z^m U^*} = \{\mathbb{O}\}$. Consequently, the Toeplitz operator $T_{\bar{z}^m U}$ has dense range in $H^2(\mathbb{C}^n)$. It follows easily from Lemma 7.1.1 that $\mathbb{P}_- \bar{z}^m U \in X$. By Theorem 1.1, we may conclude that $\bar{z}^m U \in X$. It follows now from Lemma 7.1.1 that $U \in X$. ■

Corollary 1.3. *Let U be an $n \times n$ unitary-valued function. Suppose that either T_U has dense range in $H^2(\mathbb{C}^n)$ or T_U is a Fredholm operator on $H^2(\mathbb{C}^n)$. Then the following assertions hold:*

(1) *if $\mathbb{P}_- U \in B_p^{1/p}$, $0 < p < \infty$, then $U \in B_p^{1/p}$;*

(2) *if $\mathbb{P}_- U \in VMO$, then $U \in VMO$.*

The result follows from the fact that VMO and $B_p^{1/p}$ are $\mathcal{R}$-spaces; see §7.1.

Theorem 1.4. *Let U be an $n \times n$ unitary-valued function such that T_U has dense range in $H^2(\mathbb{C}^n)$. If $\mathbb{P}_- U$ is a rational matrix function, then so is $\mathbb{P}_- U$ and*

$$\deg \mathbb{P}_+ U \le \deg \mathbb{P}_- U.$$

Recall that the McMillan degree $\deg \Phi$ of an operator function Φ is defined in §2.5.

Proof. The result follows immediately from Theorems 2.5.3 and 4.4.11. ■

2. Maximizing Vectors of Vectorial Hankel Operators

In this section we consider vectorial Hankel operators whose symbols belong to a decent function space (see §7.2). We study properties of their maximizing vectors and obtain results similar to the results of §7.2 in the scalar case. The proofs in the vector case are also similar to scalar proofs given in §7.2.

The following theorems are the main results of this section.

Theorem 2.1. *Suppose that X is a function space satisfying the axioms* (A1)–(A3). *Let $\Phi \in X(\mathbb{M}_{m,n})$ and let $\varphi \in H^2(\mathbb{C}^n)$ be a maximizing vector of the Hankel operator $H_\Phi : H^2(\mathbb{C}^n) \to H^2_-(\mathbb{C}^m)$. Then $\varphi \in X_+(\mathbb{C}^n)$.*

Theorem 2.2. *Suppose that X, Φ, and φ satisfy the hypotheses of Theorem 2.1. If $\varphi(\tau) = 0$ for some $\tau \in \mathbb{T}$, then $(1-\bar{\tau}z)^{-1}\varphi \in X_+(\mathbb{C}^n)$ and $(1-\bar{\tau}z)^{-1}\varphi$ is also a maximizing vector of H_Φ.*

Recall that

$$X_+ = \{f \in X : \hat{f}(j) = 0,\ j < 0\} \quad \text{and} \quad X_- = \{f \in X : \hat{f}(j) = 0,\ j \ge 0\}.$$

Proof of Theorem 2.1. Without loss of generality we may assume that the norm of the Hankel operator $H_\Phi : H^2(\mathbb{C}^n) \to H^2_-(\mathbb{C}^m)$ is equal to 1.

Consider the self-adjoint operator $H^*_\Phi H_\Phi$ on $H^2(\mathbb{C}^n)$. It follows from (A1) and (A2) that it maps $X_+(\mathbb{C}^n)$ into itself. Let R be the restriction of $H^*_\Phi H_\Phi$ to $X_+(\mathbb{C}^n)$. By (A3), R is a compact operator on $X_+(\mathbb{C}^n)$.

Clearly, $X_+(\mathbb{C}^n) \subset H^2(\mathbb{C}^n)$. We can imbed naturally the space $H^2(\mathbb{C}^n)$ in the dual space $X^*_+(\mathbb{C}^n)$ as follows. Let $g \in H^2(\mathbb{C}^n)$. We associate with it the linear functional $\mathcal{J}(g)$ on $X_+(\mathbb{C}^n)$ defined by

$$f \mapsto (f, g) = \int_{\mathbb{T}} (f(\zeta), g(\zeta))_{\mathbb{C}^n} d\boldsymbol{m}(\zeta).$$

Note that $\mathcal{J}(\lambda_1 g_1 + \lambda_2 g_2) = \overline{\lambda_1}\mathcal{J}(g_1) + \overline{\lambda_2}\mathcal{J}(g_2)$, g_1, $g_2 \in H^2$, λ_1, $\lambda_2 \in \mathbb{C}$.

The imbedding $\mathcal{J}$ allows us to identify $H^2(\mathbb{C}^n)$ with a linear subset of $X^*_+(\mathbb{C}^n)$. Since $H^*_\Phi H_\Phi$ is self-adjoint, it is easy to see that $R^* g = H^*_\Phi H_\Phi g$ for $g \in H^2(\mathbb{C}^n)$. Hence,

$$\operatorname{Ker}(I - R) \subset \operatorname{Ker}(I - H^*_\Phi H_\Phi) \subset \operatorname{Ker}(I - R^*).$$

Since R is a compact operator, it follows from the Riesz–Schauder theorem (see Yosida [1], Ch. X, §5) that $\dim \operatorname{Ker}(I-R) = \dim \operatorname{Ker}(I-R^*)$. Therefore

$$\operatorname{Ker}(I - R) = \operatorname{Ker}(I - H^*_\Phi H_\Phi) = \operatorname{Ker}(I - R^*). \tag{2.1}$$

Clearly, the subspace $\operatorname{Ker}(I - H^*_\Phi H_\Phi)$ is the space of maximizing vectors of H_Φ, and it follows from (2.1) that each maximizing vector of H_Φ belongs to X_+. ■

Proof of Theorem 2.2. Suppose now that φ is a maximizing vector of H_Φ and $\varphi(\tau) = 0$. Consider the following continuous linear functional ω on

$X_+(\mathbb{C}^n)$:

$$\omega(f) = (\mathbb{P}_+(\varphi^* f))(\tau). \tag{2.2}$$

Let us show that $R^*\omega = \omega$. Let F be a matrix function in $H^\infty(\mathbb{M}_{m,n})$ such that $\|\Phi - F\|_\infty = \|H_\Phi\| = 1$. Put $U \stackrel{\text{def}}{=} \Phi - F$. Then by Theorem 2.2.3,

$$H_\Phi\varphi = U\varphi, \quad \|(H_\Phi\varphi)(\zeta)\|_{\mathbb{C}^m} = \|\varphi(\zeta)\|_{\mathbb{C}^n},\ \zeta \in \mathbb{T}, \quad \text{and} \quad U^*U\varphi = \varphi. \tag{2.3}$$

We have

$$(R^*\omega)(f) = \omega(Rf) = \omega(H_U^* H_U f) = (\mathbb{P}_+(\varphi^* H_U^* H_U f))(\tau)$$

by (2.2). Therefore

$$\begin{aligned}(R^*\omega)(f) &= (\mathbb{P}_+(\varphi^*\mathbb{P}_+(U^*H_U f)))(\tau)\\ &= (\mathbb{P}_+(\varphi^* U^* H_U f))(\tau) - (\mathbb{P}_+(\varphi^*\mathbb{P}_-(U^*H_U f)))(\tau).\end{aligned}$$

Since $\varphi^*\mathbb{P}_-(U^*H_U f) \in H^2_-$, it follows that $\mathbb{P}_+(\varphi^*\mathbb{P}_-(U^*H_U f)) = 0$. Hence,

$$\begin{aligned}(R^*\omega)(f) &= (\mathbb{P}_+(\varphi^* U^* H_U f))(\tau)\\ &= (\mathbb{P}_+((\mathbb{P}_-U\varphi)^* H_U f))(\tau) + (\mathbb{P}_+((\mathbb{P}_+U\varphi)^* H_U f))(\tau).\end{aligned}$$

Clearly, $\mathbb{P}_+((\mathbb{P}_+U\varphi)^* H_U f) = 0$, and so

$$\begin{aligned}(R^*\omega)(f) &= (\mathbb{P}_+((H_U\varphi)^* H_U f))(\tau) = (\mathbb{P}_+((H_U\varphi)^*\mathbb{P}_-(Uf)))(\tau)\\ &= (\mathbb{P}_+((H_U\varphi)^* Uf))(\tau) - (\mathbb{P}_+((H_U\varphi)^*\mathbb{P}_+(Uf)))(\tau).\end{aligned}$$

Since $(H_U\varphi)^*\mathbb{P}_+(Uf) \in H^2$, we have

$$(\mathbb{P}_+((H_U\varphi)^*\mathbb{P}_+(Uf)))(\tau) = (H_U\varphi)^*(\tau)(\mathbb{P}_+(Uf))(\tau).$$

By the hypotheses $\varphi(\tau) = 0$, and so by (2.3), $(H_U\varphi)(\tau) = 0$. Hence,

$$(R^*\omega)(f) = (\mathbb{P}_+((H_U\varphi)^* Uf))(\tau).$$

By (2.3),

$$\begin{aligned}(R^*\omega)(f) &= (\mathbb{P}_+((U\varphi)^* Uf))(\tau) = (\mathbb{P}_+((U\varphi)^* Uf))(\tau)\\ &= (\mathbb{P}_+(\varphi^* U^* Uf))(\tau) = (\mathbb{P}_+(\varphi^* f))(\tau) = \omega(f),\end{aligned}$$

which proves (2.2).

It follows from (2.2) that $\omega \in \operatorname{Ker}(I - R^*)$, and by Theorem 2.1, $\omega \in \operatorname{Ker}(I - R)$. Hence, there must be a function $\psi \in X_+(\mathbb{C}^n)$ such that ψ is a maximizing vector of H_Φ and

$$\omega(f) = \int_{\mathbb{T}} (f(\zeta), \psi(\zeta))_{\mathbb{C}^n}\, d\boldsymbol{m}(\zeta), \quad f \in X_+(\mathbb{C}^n).$$

To prove that $\psi = (1 - \bar{\tau}z)^{-1}\varphi$, we show that these two functions have the same Taylor coefficients. Let $k \in \mathbb{Z}_+$, $\gamma \in \mathbb{C}^n$. We have

$$(\hat{\psi}(k), \gamma)_{\mathbb{C}^n} = \overline{\omega(z^k\gamma)} = \overline{(\mathbb{P}_+ z^k\varphi^*\gamma)(\tau)} = \overline{\sum_{j=0}^{k} \bar{\tau}^{k-j}(\hat{\varphi}(j))^*\gamma},$$

and so

$$\hat{\psi}(k) = \sum_{j=0}^{k} \bar{\tau}^{k-j} \hat{\varphi}(j).$$

On the other hand,

$$(1 - \bar{\tau}z)^{-1}\varphi = \left(\sum_{k\ge 0} \bar{\tau}^k z^k\right)\varphi = \sum_{k\ge 0} z^k \left(\sum_{j=0}^{k} \bar{\tau}^{k-j}\hat{\varphi}(j)\right)$$

which completes the proof. ■

3. Wiener–Masani Factorizations

We study in this section hereditary properties of Wiener–Masani factorizations of positive-definite matrix functions. Let W be a positive-definite matrix function in $L^1(\mathbb{M}_{n,n})$ such that $\log\det W \in L^1$. By the Wiener–Masani theorem, W admits a factorization of the form

$$W = F^*F, \tag{3.1}$$

where F is an outer matrix function in $H^2(\mathbb{M}_{n,n})$. Such factorizations are called *Wiener–Masani factorizations.* We do not need this result in full generality and we refer the reader to Rozanov [1] for the proof. We prove the following special case of the Wiener–Masani theorem.

Theorem 3.1. *Let W be a positive definite function in $L^\infty(\mathbb{M}_{n,n})$ such that $W^{-1} \in L^\infty(\mathbb{M}_{n,n})$. Then there exists an invertible matrix function F in $H^\infty(\mathbb{M}_{n,n})$ such that* (3.1) *holds.*

Proof. Consider the subspace $W^{1/2}H^2(\mathbb{C}^n)$. It is easy to see that it is a completely nonreducing invariant invariant subspace of multiplication by z on $L^2(\mathbb{C}^n)$ (see Appendix 2.3). Then $W^{1/2}$ admits a factorization in the form $W^{1/2} = UF$, where F is an outer matrix function and U is an isometric-valued matrix function (see Appendix 2.3). Since $W^{1/2}$ is invertible in $L^\infty(\mathbb{M}_{n,n})$, it follows that U is unitary-valued and F is invertible in $H^\infty(\mathbb{M}_{n,n})$. We have

$$W = (UF)^*UF = F^*F. \quad \blacksquare$$

We consider the case when W is a positive definite matrix function in X such that $W(\zeta)$ is invertible for all ζ in $\mathbb{T}$, where X is a decent space of functions on $\mathbb{T}$. We prove that such a matrix function W admits a Wiener–Masani factorization of the form (3.1) such that F belongs to the same space X.

First we establish the uniqueness of a Wiener–Masani factorization modulo a constant unitary factor.

Theorem 3.2. *Let W be a matrix function in $L^1(\mathbb{M}_{n,n})$ such that*

$$W = F^*F = G^*G,$$

where F and G are outer functions in $H^2(\mathbb{M}_{n,n})$. Then there exists a constant unitary matrix $\mathfrak{U}$ such that $G = \mathfrak{U}F$.

Proof. Put $U = W^{1/2}F^{-1}$. Then U is unitary-valued. Indeed,

$$U^*U = (F^*)^{-1}WF^{-1} = (F^*)^{-1}F^*FF^{-1} = \boldsymbol{I}_n,$$

where $\boldsymbol{I}_n$ is the constant matrix function identically equal to the $n \times n$ identity matrix I_n.

Similarly, if $V = W^{1/2}G^{-1}$, then V is unitary-valued. We have

$$W^{1/2} = UF = VG.$$

Since F is outer, it follows that

$$\operatorname{clos}\{W^{1/2}q : \ q \text{ is a polynomial in } H^2(\mathbb{C}^n)\} = UH^2(\mathbb{C}^n).$$

Similarly, since G is outer, we have

$$\operatorname{clos}\{W^{1/2}q : \ q \text{ is a polynomial in } H^2(\mathbb{C}^n)\} = VH^2(\mathbb{C}^n).$$

Hence, the invariant subspaces $UH^2(\mathbb{C}^n)$ and $VH^2(\mathbb{C}^n)$ of multiplication by z on $L^2(\mathbb{C}^n)$ coincide. Therefore there exists a unitary matrix Q such that $V = UQ$ (see Appendix 2.3). We have

$$G = V^*W^{1/2} = Q^*U^*W^{1/2} = Q^*F,$$

which proves the result. ■

Theorem 3.3. *Let X be a decent function space and let W be a positive definite $n \times n$ matrix function in X such that $W(\zeta)$ is invertible for any $\zeta \in \mathbb{T}$. Then W admits a Wiener–Masani factorization of the form* (3.1) *with F invertible in $X_+(\mathbb{M}_{n,n})$.*

It is easy to see that if F is a function invertible in $X_+(\mathbb{M}_{n,n})$, then F is outer.

To prove Theorem 3.3, we need a lemma. Note that if Φ is an $n \times n$ matrix function in X, then the Toeplitz operator T_Φ maps $X_+(\mathbb{C}^n)$ into itself and can be considered as a bounded operator on $X_+(\mathbb{C}^n)$.

Lemma 3.4. *Let W be a positive-definite $n \times n$ matrix function in X such that $W(\zeta)$ is invertible for any $\zeta \in \mathbb{T}$. Then the Toeplitz operator T_W is an invertible operator on $X_+(\mathbb{C}^n)$.*

Proof of Lemma 3.4. We have

$$T_W T_{W^{-1}} = I - H_W^* H_{W^{-1}}, \tag{3.2}$$

where $H_{W^{-1}} : H^2(\mathbb{C}^n) \to H^2_-(\mathbb{C}^n)$ is a Hankel operator while $H_W^* : H^2_-(\mathbb{C}^n) \to H^2(\mathbb{C}^n)$ is the adjoint to the Hankel operator H_W. It is easy to see that the operator $H_W^* H_{W^{-1}}$ maps $X_+(\mathbb{C}^n)$ into itself and we can consider it as a bounded operator on $X_+(\mathbb{C}^n)$. In fact, it is a compact operator on $X_+(\mathbb{C}^n)$ since by the axiom (A3), the Hankel operator $H_{W^{-1}}$ is a compact operator from $X_+(\mathbb{C}^n)$ to $X_-(\mathbb{C}^n)$.

Thus in (3.2) we may assume that both sides of the equation are operators on $X_+(\mathbb{C}^n)$. Let us show that $T_W T_{W^{-1}}$ is an invertible operator on $X_+(\mathbb{C}^n)$. Since $H_W^* H_{W^{-1}}$ is compact, it suffices to show that $\operatorname{Ker} T_W T_{W^{-1}} = \{\mathbb{O}\}$. This will be proved if we show that $\operatorname{Ker} T_W = \operatorname{Ker} T_{W^{-1}} = \{\mathbb{O}\}$.

Let us show that $\operatorname{Ker} T_W = \{\mathbb{O}\}$, the proof of the fact that $\operatorname{Ker} T_{W^{-1}} = \{\mathbb{O}\}$ is the same. Suppose that $f \in \operatorname{Ker} T_W$. We have

$$\begin{aligned} 0 &= (T_W f, f)_{L^2(\mathbb{C}^n)} = (\mathbb{P}_+ W f, f)_{L^2(\mathbb{C}^n)} = (W f, \mathbb{P}_+ f)_{L^2(\mathbb{C}^n)} \\ &= (W f, f)_{L^2(\mathbb{C}^n)} = \int_{\mathbb{T}} (W(\zeta) f(\zeta), f(\zeta))_{\mathbb{C}^n}\, d\boldsymbol{m}(\zeta). \end{aligned}$$

Since $W(\zeta)$ is positive definite and invertible, it follows that $f = \mathbb{O}$.

It follows now from the invertibility of $T_W T_{W^{-1}}$ that T_W maps $X_+(\mathbb{C}^n)$ onto itself. A similar argument shows that

$$T_{W^{-1}} T_W = I - H_{W^{-1}}^* H_W$$

is an invertible operator on $X_A(\mathbb{C}^n)$, which implies that the operator T_W on $X_+(\mathbb{C}^n)$ has trivial kernel. ■

Proof of Theorem 3.3. Since T_W is invertible as an operator on $X_+(\mathbb{C}^n)$, there exist functions $g_j \in X_+(\mathbb{C}^n)$, $1 \le j \le n$, such that

$$T_W g_j = \begin{pmatrix} \mathbb{O} \\ \vdots \\ \mathbf{1} \\ \vdots \\ \mathbb{O} \end{pmatrix}$$

($\mathbf{1}$ is in the jth row). Consider the $n \times n$ matrix function

$$G = \begin{pmatrix} g_1 & \cdots & g_n \end{pmatrix}.$$

Clearly,

$$(\mathbb{P}_+ W G)(\zeta) = I, \quad \text{a.e. on } \mathbb{T}.$$

It follows that $\overline{WG} \in H^2$.

Put $\Xi = G^* W G$. Then $\Xi \in \overline{H^1}$ since both G^* and WG are in $\overline{H^2}$. Clearly, $\Xi^* = \Xi$, and so Ξ is a constant function.

Let us prove that $\det \Xi \ne 0$. We have $\det \Xi = |\det G(\zeta)|^2 \det W(\zeta)$ for $\zeta \in \mathbb{T}$. Since the columns $g_1(\zeta), \cdots, g_n(\zeta)$ of $G(\zeta)$ are linearly independent on $\mathbb{T}$, it follows that $\det G(\zeta) \ne 0$ on $\mathbb{T}$. Clearly, $\det W(z) \ne 0$, $\zeta \in \mathbb{T}$. Hence, $\det \Xi \ne 0$, and so Ξ is invertible.

Put $F \stackrel{\text{def}}{=} \Xi^{1/2} G^{-1}$. Clearly, F, $F^{-1} \in X_+$. We have

$$F^* F = G^{*-1} \Xi^{1/2} \Xi^{1/2} G^{-1} = W.$$

The last equality is an immediate consequence of the definition of Ξ. ■

Corollary 3.5. *Suppose that W satisfies the hypotheses of Theorem 3.3. If G is an outer $n \times n$ matrix function in H^2 such that $G^*G = W$, then G, $G^{-1} \in X$.*

Proof. The result follows immediately from Theorems 3.3 and 3.2. ■

Corollary 3.6. *Suppose that W satisfies the hypotheses of Theorem 3.3. If Ω is an outer $n \times n$ matrix function in H^2 such that $\Omega\Omega^* = W$, then Ω, $\Omega^{-1} \in X$.*

Proof. Consider the transposed matrices: $W^{\mathrm{t}} = (\Omega^{\mathrm{t}})^*\Omega^{\mathrm{t}}$. Then Ω^{t} is also an outer function since a square matrix function in H^2 is outer if and only if its determinant is a scalar outer function (Appendix 2.3). The result follows now from Corollary 3.5. ■

4. Isometric-Outer Factorizations

In this section we consider representations of matrix functions as a product of an isometric-valued function and an outer function. Suppose that Φ is an $n \times m$ matrix function in L^2 such that $\det \Phi^*\Phi \in L^1$. Then it follows from the Wiener–Masani theorem mentioned in the previous section that Φ admits a factorization $\Phi = UF$, where F is an $m \times m$ outer matrix function in H^2 and U is an $n \times m$ *isometric-valued* function, i.e., for almost all $\zeta \in \mathbb{T}$

$$\|U(\zeta)x\|_{\mathbb{C}^m} = \|x\|_{\mathbb{C}^n}$$

for any $x \in \mathbb{C}^n$. Indeed, if $\Phi^*\Phi = F^*F$ is a Wiener–Masani factorization of $\Phi^*\Phi$, then it is easy to see that the matrix function $U = \Phi F^{-1}$ is isometric-valued and $\Phi = UF$. Such an isometric-outer factorization is unique modulo a multiplicative unitary constant, which follows from the corresponding uniqueness result for Wiener–Masani factorizations.

We consider here matrix functions whose entries belong to a decent function space X and we prove that in this case the factors U and F also belong to X.

Theorem 4.1. *Let X be a decent function space and let Φ be an $n \times m$ matrix function in X such that* $\operatorname{rank} \Phi(\zeta) = m$ *for all $\zeta \in \mathbb{T}$. Then Φ admits a factorization*

$$\Phi = UF,$$

where U is an $n \times m$ isometric-valued function in X and F is an $m \times m$ outer function such that F, $F^{-1} \in X$.

Proof. Let $W \stackrel{\mathrm{def}}{=} \Phi^*\Phi$. Then W is positive definite and W is invertible in X. By Theorem 3.3, W admits a factorization

$$W = F^*F,$$

where F is an invertible function in X. Put $U = \Phi F^{-1}$. Clearly, $U \in X$. We have

$$U^*U = F^{*-1}\Phi^*\Phi F^{-1} = F^{*-1}F^*FF^{-1} \equiv I,$$

and so U is isometric-valued. ■

5. The Recovery Problem and Wiener–Hopf Factorizations of Unitary-Valued Functions

In this section we consider matrix functions that belong to a decent function space X. We use the results of §2 to solve two problems: the Wiener–Hopf factorization problem for unitary-valued functions and the recovery problem for unitary-valued functions.

As in §1 we solve the recovery problem for a unitary-valued unction U under the assumption that the Toeplitz operator $T_U : H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^n)$ is Fredholm or under the assumption that T_U has a dense range.

Theorem 5.1. *Suppose that X is a decent function space. Let U be a unitary-valued matrix function such that the Toeplitz operator T_U has dense range in $H^2(\mathbb{C}^n)$. If $\mathbb{P}_-U \in X$, then $U \in X$.*

Theorem 5.2. *Suppose that X is a decent function space. Let U be a unitary-valued matrix function such that the Toeplitz operator T_U is Fredholm on $H^2(\mathbb{C}^n)$. If $\mathbb{P}_-U \in X$, then $U \in X$.*

If T_U is Fredholm, then $T_{\bar{z}^N U}$ maps $H^2(\mathbb{C}^n))$ onto itself for sufficiently large N. Therefore it is sufficient to prove Theorem 5.1. Let us also show that under the hypotheses of Theorem 5.1 the Toeplitz operator $T_U : H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^n)$ is Fredholm. Since $X \subset C(\mathbb{T})$, it follows that $\mathbb{P}_-U \in VMO$. By Corollary 1.3, $U \in VMO$. Therefore the Hankel operators H_U and H_{U^*} are compact, and so the fact that T_U is Fredholm follows from the formulas

$$I - T_{U^*}T_U = H_U^* H_U, \quad I - T_U T_{U^*} = H_{U^*}^* H_{U^*}$$

Now let U be a unitary-valued function such that T_U is Fredholm. Then by Theorem 3.5.2, U admits a factorization

$$U = \Psi_2^* \Lambda \Psi_1, \tag{5.1}$$

where $\Psi_1^{\pm 1}$, $\Psi_2^{\pm 1} \in H^2(\mathbb{C}^n)$ and Λ is a diagonal matrix of the form

$$\Lambda = \begin{pmatrix} z^{d_1} & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & z^{d_2} & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & z^{d_n} \end{pmatrix}, \quad d_1, \cdots, d_n \in \mathbb{Z}.$$

It is easy to see that Theorems 5.1 and 5.2 follow immediately from the following theorem, which solves the Wiener–Hopf factorization problem in X for unitary-valued functions. In fact, we obtain an even stronger result since instead of assuming that $U \in X$ we assume only that $\mathbb{P}_-U \in X$.

Theorem 5.3. *Suppose that X is a decent function space. Let U be a unitary-valued function such that $\mathbb{P}_-U \in X$ and the Toeplitz operator T_U on $H^2(\mathbb{C}^n)$ is Fredholm. Then the functions Ψ_1, Ψ_1^{-1}, Ψ_2, and Ψ_2^{-1} in* (5.1) *belong to X.*

Proof. Let us first prove that $\Psi_1 \in X$ and $\Psi_1^{-1} \in X$. Without loss of generality we may assume that all the indices of the factorization (5.1) are negative. Otherwise, we can multiply U by $\bar{z}^N$ for a sufficiently large integer N. Let ξ be an arbitrary nonzero vector in $\mathbb{C}^n$. It is easy to see that $\Psi_1^{-1}\xi$ is a maximizing vector for H_U. It follows now from Theorem 2.1 that $\Psi_1^{-1}\xi \in X$, and so $\Psi_1^{-1} \in X$.

Let us show that $\Psi_1 \in X$. Put $G = \Psi^{-1} \in X$. It is sufficient to show that the matrix $G(\zeta)$ is invertible for all $\zeta \in \mathbb{T}$.

Consider $\operatorname{Ker} T_U$. It admits the following description:

$$\operatorname{Ker} T_U = \left\{ G \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} : \ q_j \text{ is a polynomial in } H^2, \ \deg q_j < |d_j| \right\}. \tag{5.2}$$

Indeed, it is easy to see that the right-hand side of (5.2) is contained in the left-hand side and by Theorem 3.5.7, the dimensions of both sides are equal.

Assume that $G(\zeta_0)$ is noninvertible for some $\zeta_0 \in \mathbb{T}$. Then $G(\zeta_0)\xi = \mathbb{O}$ for some nonzero $\xi \in \mathbb{C}^n$. It is easy to see from (5.2) that $G\xi$ is a nonzero function in $\operatorname{Ker} T_U$. Clearly, a nonzero vector in $H^2(\mathbb{C}^n)$ is maximizing for H_U if and only if it belongs to $\operatorname{Ker} T_U$.

By Theorem 2.2, the function $(1-\overline{\zeta_0}z)^{-1}Ge$ is a maximizing vector of H_U, and so it belongs to $\operatorname{Ker} T_U$. However, it is easy to see that $(1-\overline{\zeta_0}z)^{-1}Ge$ does not belong to the right-hand side of (5.2).

Let us show that Ψ_2, $\Psi_2^{-1} \in X$. Since U is unitary-valued, it follows that

$$(U^*U)(\zeta) = (\Psi_1^*\Lambda^*\Psi_2\Psi_2^*\Lambda\Psi_1)(\zeta) = I, \quad \zeta \in \mathbb{T},$$

and so

$$\Psi_2\Psi_2^* = \Lambda\Psi_1^{*-1}\Psi_1^{-1}\Lambda^*. \tag{5.3}$$

Since Ψ_2 is an outer function and the right-hand side of (5.3) is invertible in X, the result follows from Corollary 3.6. ■

6. Wiener–Hopf Factorizations. The General Case

We consider in this section Wiener–Hopf factorizations of matrix functions in decent function spaces X. If Φ is an $n \times n$ matrix function in X such that $\det \Phi(\zeta) \neq 0$ for any ζ in $\mathbb{T}$, then the Toeplitz operator T_Φ on

$H^2(\mathbb{C}^n)$ is Fredholm, and so Φ admits a Wiener–Hopf factorization

$$\Phi = \Psi_2^* \Lambda \Psi_1, \tag{6.1}$$

where Ψ_1, Ψ_1^{-1}, Ψ_2, $\Psi_2^{-1} \in H^2$ and Λ is a diagonal matrix function of the form

$$\Lambda = \begin{pmatrix} z^{d_1} & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & z^{d_2} & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & z^{d_n} \end{pmatrix}, \quad d_1, \cdots, d_n \in \mathbb{Z}.$$

We are going to prove in this section that for $\Phi \in X$ the matrix functions Ψ_1 and Ψ_2 are invertible functions in X.

Theorem 6.1. *Let X be a decent function space and let Φ be an $n \times n$ matrix function in X such that $\det \Phi(\zeta) \neq 0$ for any ζ in $\mathbb{T}$. Suppose that* (6.1) *is a Wiener–Hopf factorization of Φ. Then Ψ_1, Ψ_1^{-1}, Ψ_2, $\Psi_2^{-1} \in X$.*

Proof. By Theorem 4.1, Φ admits a representation $\Phi = U\Psi$, where U is a unitary-valued function in X and Ψ is an outer function that is invertible in X.

Suppose that Φ admits a factorization in the form (6.1). We have

$$U = \Psi_2^* \Lambda \Psi_1 \Psi^{-1}.$$

It is easy to see that $\Psi_1 \Psi^{-1} \in H^2$ and $(\Psi_1 \Psi^{-1})^{-1} \in H^2$. By Theorem 5.3, Ψ_2, $\Psi_2^{-1} \in X$ and $\Psi_1 \Psi^{-1}$, $(\Psi_1 \Psi^{-1})^{-1} \in X$, and so Ψ_1, $\Psi_1^{-1} \in X$. ■

Concluding Remarks

The results of §1 were obtained in Peller [24]. Theorems 2.1 and 2.2 were proved in Peller and Young [5] under the assumption that the trigonometric polynomials are dense in X. In Alexeev and Peller [2] it was shown how to generalize the proof to the case of arbitrary decent spaces. The results of §§3–4 are due to Peller [24]. Theorems 5.1 and 5.2 were also found in Peller [24].

The heredity problem for Wiener–Hopf factorizatiions was studied by many authors. Classical results by Plemelj, Muskhelishvili, and Vekua solved the Wiener–Hopf factorization problem in Hölder classes Λ_α, $0 < \alpha < 1$ (see Muskhelishvili [1] and Vekua [1]). In Gohberg and Krein [1] the Wiener–Hopf factorization problem was solved in the class $\mathcal{F}\ell^1$. In Gohberg [2], and Budyanu and Gohberg [1–2] heredity results for Wiener–Hopf factorizations were obtained for a class of function spaces that is close to the class of decent spaces (see also Clancey and Gohberg [1]).

In Peller [24] a new method to obtain heredity results for Wiener–Hopf factorizations was found. It is based on properties of maximizing vectors of Hankel operators. Theorem 5.3 was proved in Peller [24].

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Analytic Approximation of Matrix Functions

We study in this chapter the problem of approximating an essentially bounded matrix function on $\mathbb{T}$ by bounded analytic matrix functions in $\mathbb{D}$. For such a matrix function $\Phi \in L^\infty(\mathbb{M}_{m,n})$ its L^∞ norm $\|\Phi\|_{L^\infty}$ is, by definition,

$$\|\Phi\|_{L^\infty} = \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} \|\Phi(\zeta)\|_{\mathbb{M}_{m,n}}.$$

As usual we denote by $H^\infty(\mathbb{M}_{m,n})$ the subspace of $L^\infty(\mathbb{M}_{m,n})$ that consists of bounded analytic matrix functions in $\mathbb{D}$. Recall that the space $\mathbb{M}_{m,n}$ of $m \times n$ matrices is equipped with the operator norm of the space of linear operators from $\mathbb{C}^n$ to $\mathbb{C}^m$.

We have seen in §1.1 that for a scalar continuous function φ on $\mathbb{T}$ there exists a unique best approximation f by bounded analytic functions. In this chapter we study the problem finding best approximations for matrix functions. As we have seen in Chapter 11, this problem is very important in applications.

In this chapter we call best approximations of Φ by functions F in $H^\infty(\mathbb{M}_{m,n})$ *optimal solutions* of the Nehari problem. Note that this terminology slightly differs from the terminology used in Chapter 5, where $\Phi - F$ has been called an optimal solution of the Nehari problem.

It is easy to see that unlike the scalar case the continuity (and even any smoothness) of a matrix function Φ does not guarantee the uniqueness of an optimal solution of the Nehari problem. Indeed, if $\Phi = \begin{pmatrix} \bar{z} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}$, then $\operatorname{dist}_{L^\infty}(\Phi, H^\infty(\mathbb{M}_{2,2})) = 1$, since $\operatorname{dist}_{L^\infty}(\bar{z}, H^\infty) = 1$. Clearly, for any scalar function $f \in H^\infty$ such that $\|f\|_\infty \le 1$ we have

$$\left\| \begin{pmatrix} \bar{z} & \mathbb{O} \\ \mathbb{O} & -f \end{pmatrix} \right\|_{L^\infty} = 1,$$

and so $\begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & f \end{pmatrix}$ is a best approximation of Φ. However, if we were asked to find among the above best approximations the "very best" one, we would probably say that it is the zero matrix function, since it does a better job than just minimizing the L^∞-norm. It seems natural to impose additional constraints on a best approximation and choose among best approximations the "very best". To do this, we minimize lexicographically the essential suprema of the singular values $s_j\big((\Phi - F)(\zeta)\big)$, $0 \le j \le \min\{m,n\} - 1$, of $(\Phi - F)(\zeta)$, $\zeta \in \mathbb{T}$,

Given a matrix function $\Phi \in L^\infty(\mathbb{M}_{m,n})$ we define inductively the sets $\boldsymbol{\Omega}_j$, $0 \le j \le \min\{m,n\} - 1$, by

$$\boldsymbol{\Omega}_0 = \left\{ F \in H^\infty(\mathbb{M}_{m,n}) : \ F \text{ minimizes } \ \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} \|\Phi(\zeta) - F(\zeta)\|_{\mathbb{M}_{m,n}} \right\},$$

$$\boldsymbol{\Omega}_j = \left\{ F \in \boldsymbol{\Omega}_{j-1} : \ F \text{ minimizes } \ \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} s_j\big(\Phi(\zeta) - F(\zeta)\big) \right\}.$$

We put

$$t_j \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} s_j\big(\Phi(\zeta) - F(\zeta)\big) \quad \text{for} \quad F \in \boldsymbol{\Omega}_j, \quad 0 \le j \le \min\{m,n\} - 1.$$

Functions in $\boldsymbol{\Omega}_{\min\{m,n\}-1}$ are called *superoptimal approximations* of Φ by bounded analytic matrix functions. The numbers $t_j = t_j(\Phi)$ are called the *superoptimal singular values* of Φ. Note that the functions in $\boldsymbol{\Omega}_0$ are just the best approximations by analytic matrix functions. Clearly, $t_0(\Phi) = \|H_\Phi\|$. It is easy to see that in the above example the zero function is the unique superoptimal approximation.

The notion of a superoptimal approximation is important in control theory. In the robust stabilization problem one seeks a controller that stabilizes all functions in an L^∞-norm ball of maximal radius about a nominal transfer function, and this reduces mathematically to a Nehari problem (see Chapter 11). If we choose the *superoptimal* solution, then we optimize robustness with respect not only to the worst direction of uncertainty (in the space of transfer functions), but also to other directions. We refer the reader to the papers Postlethwaite, Tsai, and Gu [1], Limebeer, Halikias, and Glover [1], Kwakernaak [1], and Foo and Postlethwaite [1] written by engineers for further discussions of superoptimal approximation in connection with control theory.

It can be shown easily by a compactness argument that for an arbitrary $\Phi \in L^\infty(\mathbb{M}_{m,n})$ there exists a superoptimal approximation by bounded analytic matrix functions. As in the scalar case a superoptimal approximation is not unique in general. However, we will see in this chapter that under the condition $\Phi \in H^\infty + C$ (or even under a certain weaker condition) the superoptimal approximation is unique.

It is easy to see that if Φ is a continuous diagonal $n \times n$ matrix function of the form

$$\Phi = \begin{pmatrix} \varphi_1 & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & \varphi_2 & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & \varphi_n \end{pmatrix},$$

and $f_j \in H^\infty$ is the best approximation of φ_j, then

$$F = \begin{pmatrix} f_1 & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & f_2 & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & f_n \end{pmatrix}$$

is the unique superoptimal approximation of Φ by bounded analytic matrix functions.

In the general case the situation is much more complicated, but we are going to perform a sort of diagonalization too.

In §1 we introduce a remarkable class of so-called balanced unitary-valued matrix functions and its subclass of thematic matrix functions, and we study their properties. In §2 we begin the diagonalization procedure and parametrize the optimal solutions of the Nehari problem under the condition $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$. We prove the uniqueness of a superoptimal approximation in §3 under the condition $\Phi \in H^\infty + C$. In §4 we apply another method that allows us to prove uniqueness under the weaker condition on Φ: $\|H_\Phi\|_{\mathrm{e}}$ is less than any nonzero superoptimal singular value of Φ. Note that both methods give a construction of the unique superoptimal approximation.

Section 5 is devoted to the construction of so-called thematic and partially thematic factorizations. We define thematic indices associated with such factorizations. We introduce the notion of a very badly approximable matrix function and under suitable conditions on the essential norm of the Hankel operator we characterize the badly approximable matrix functions and the very badly approximable matrix functions in terms of such factorizations.

In §6 we study so-called admissible and superoptimal matrix weights. They are used in §7 to study thematic indices, which may depend on the choice of a (partial) thematic factorization. However, we show in §7 that the sum of the thematic indices that correspond to the superoptimal singular values equal to a specific number depends only on the matrix function Φ itself rather than on the choice of a factorization. We continue the study of invariance properties of (partial) thematic factorization in Sections 9 and 10. In §9 we prove the invariance of so-called residual entries, while in §10 we introduce monotone (partial) thematic factorizations, and prove that among (partial) thematic factorizations it is always possible to choose a monotone one and the indices of a monotone factorization are uniquely determined by the function Φ itself.

Section 8 concerns inequalities that involve superoptimal singular values. In particular, we prove an important inequality between the superoptimal singular values $t_j(\Phi)$ and the Hankel singular values $s_j(H_\Phi)$.

In §11 we suggest a more constructive approach to find the unique superoptimal approximation. It is given in terms of solutions of corona problems. It is used in §12 to study the hereditary properties of the nonlinear operator of superoptimal approximation as well as in §13 to study continuity properties of this operator.

In §14 we study the problem of finding so-called unitary interpolants U of Φ. We are interested in those unitary interpolants U for which the Toeplitz operator T_U is Fredholm, in which case we study the indices of Wiener–Hopf factorizations of U. We give a solution to this problem in terms of the superoptimal singular values of Φ and the indices of monotone partial thematic factorizations.

Section 15 gives an alternative approach to the construction of superoptimal approximation which is based on so-called canonical factorizations. Unlike thematic factorizations, all factors in canonical factorizations are uniquely determined by Φ modulo constant unitary factors. A special role in that section is played by the very badly approximable unitary-valued functions U such that $\|H_U\|_{\mathrm{e}} < 1$. In §16 we give a characterization of such unitary-valued functions.

In §17 we study the problem of superoptimal approximation by meromorphic matrix functions whose McMillan degree is at most k. Unlike the case of analytic approximation the condition $\Phi \in H^\infty + C$ does not guarantee that a superoptimal approximation is unique. We prove uniqueness under the additional assumption $s_{k-1}(H_\Phi) > s_k(H_\Phi)$.

In §18 we study the problem of superoptimal analytic approximation of infinite matrix functions. We establish uniqueness under the condition that H_Φ is compact. We also prove analogs of many results obtained in previous sections for finite matrix functions.

Finally, in §19 we revisit the problem of parametrization of all optimal solutions of the Nehari problem under the condition $\|H_\Psi\|_{\mathrm{e}} < \|H_\Psi\|$ and we obtain the Adamyan–Arov–Krein parametrization formula using the method developed in §2.

1. Balanced Matrix Functions

We consider here a very important class of matrix functions that will play an essential role in studying approximation problem for matrix functions.

Definition. Let n be a positive integer and let r be an integer such that $0 < r < n$. Suppose that Υ is an $n \times r$ inner and co-outer matrix function and Θ is an $n \times (n-r)$ inner and co-outer matrix function (see Appendix 2.3). If the matrix function

$$\mathcal{V} = \begin{pmatrix} \Upsilon & \overline{\Theta} \end{pmatrix}$$

is unitary-valued, it is called an *r-balanced matrix function*. If $r = 0$ or $r = n$, it is natural to say that an r-balanced matrix function is a constant unitary matrix function. An $n \times n$ matrix function $\mathcal{V}$ is called *balanced* if it is r-balanced for some r, $0 \le r \le n$. A special role will be played by 1-balanced matrix functions. We also call them *thematic matrix functions*.

We will see that balanced matrix functions have many nice properties that can justify the term "balanced". First we show that any inner and co-outer matrix function can be completed to a balanced matrix function.

Theorem 1.1. *Let n be a positive integer, $0 < r < n$, and let Υ be an $n \times r$ inner and co-outer matrix function. Then the subspace $\mathcal{L} \stackrel{\text{def}}{=} \operatorname{Ker} T_{\Upsilon^{\mathrm{t}}}$ has the form*

$$\mathcal{L} = \Theta H^2(\mathbb{C}^{n-r}),$$

where Θ is an inner and co-outer $n \times (n-r)$ matrix function such that $\begin{pmatrix} \Upsilon & \overline{\Theta} \end{pmatrix}$ is balanced.

Proof. Clearly, the subspace $\mathcal{L}$ of $H^2(\mathbb{C}^n)$ is invariant under multiplication by z. By the Beurling–Lax–Halmos theorem (see Appendix 2.3), $\mathcal{L} = \Theta H^2(\mathbb{C}^l)$ for some $l \le n$ and an $n \times l$ inner matrix function Θ.

Let us first prove that Θ is co-outer. Suppose that $\Theta^{\mathrm{t}} = \mathcal{O}F$, where $\mathcal{O}$ is an inner matrix function and F is an outer matrix function. Since Θ is inner, it is easy to see that $\mathcal{O}$ has size $l \times l$ while F has size $l \times n$. It follows that $\mathcal{O}^*\Theta^{\mathrm{t}} = F$, and so $F^{\mathrm{t}} = \Theta\overline{\mathcal{O}}$. Since both Θ and $\overline{\mathcal{O}}$ take isometric values almost everywhere on $\mathbb{T}$, the matrix function F^{t} is inner.

Let us show that

$$F^{\mathrm{t}} H^2(\mathbb{C}^l) \subset \mathcal{L}. \tag{1.1}$$

First of all, it is easy to see that $\Upsilon^{\mathrm{t}}\Theta$ is the zero matrix function. Now let $f \in H^2(\mathbb{C}^l)$. We have

$$\Upsilon^{\mathrm{t}} F^{\mathrm{t}} f = \Upsilon^{\mathrm{t}}\Theta\overline{\mathcal{O}} f = \mathbb{O},$$

which proves (1.1).

It follows from (1.1) that

$$F^{\mathrm{t}} H^2(\mathbb{C}^l) = \Theta\overline{\mathcal{O}} H^2(\mathbb{C}^l) \subset \Theta H^2(\mathbb{C}^l).$$

Multiplying the last inclusion by Θ^*, we have

$$\overline{\mathcal{O}} H^2(\mathbb{C}^l) \subset H^2(\mathbb{C}^l),$$

which implies that $\mathcal{O}$ is a constant unitary matrix function.

Let us now prove that $l = n - r$. First of all, it is evident that the columns of $\Upsilon(\zeta)$ are orthogonal to the columns of $\overline{\Theta}(\zeta)$ almost everywhere on $\mathbb{T}$. Hence, the matrix function $\begin{pmatrix} \Upsilon & \overline{\Theta} \end{pmatrix}$ takes isometric values almost everywhere, and so $l \le n - r$.

To show that $l \ge n - r$, consider the functions $P_{\mathcal{L}}C$, where $P_{\mathcal{L}}$ is the orthogonal projection onto $\mathcal{L}$ and C is a constant function that we identify with a vector in $\mathbb{C}^n$. Note that $C \perp \mathcal{L}$ if and only if the vectors $f(0)$ and C are orthogonal in $\mathbb{C}^n$ for any $f \in \mathcal{L}$. Let us prove that

$$\dim\{f(0):\ f \in \mathcal{L}\} \ge n - r. \tag{1.2}$$

Since Υ is co-outer, it is easy to see that $\operatorname{rank}\Upsilon(0) = r$. Without loss of generality we may assume that $\Upsilon = \begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \end{pmatrix}$, where Υ_1 has size $(n-r)\times r$, Υ_2 has size $r \times r$, and the matrix $\Upsilon_2(0)$ is invertible. Now let K be an arbitrary vector in $\mathbb{C}^{n-r}$. Put

$$f = (\det \Upsilon_2(0))^{-1} \det \Upsilon_2 \begin{pmatrix} K \\ -(\Upsilon_2^{\mathrm{t}})^{-1}\Upsilon_1^{\mathrm{t}} K \end{pmatrix}.$$

Clearly, $f \in H^2(\mathbb{C}^n)$. We have

$$\Upsilon^{\mathrm{t}} f = \begin{pmatrix} \Upsilon_1^{\mathrm{t}} & \Upsilon_2^{\mathrm{t}} \end{pmatrix} f = (\det \Upsilon_2(0))^{-1} \det \Upsilon_2 (\Upsilon_1^{\mathrm{t}} K - \Upsilon_2^{\mathrm{t}}(\Upsilon_2^{\mathrm{t}})^{-1}\Upsilon_1^{\mathrm{t}} K) = 0,$$

and so $f \in \mathcal{L}$. On the other hand, it is easy to see that

$$f(0) = \begin{pmatrix} K \\ -(\Upsilon_2^{\mathrm{t}}(0))^{-1}\Upsilon_1^{\mathrm{t}}(0) K \end{pmatrix}$$

and since K is an arbitrary vector in $\mathbb{C}^{n-r}$, this proves (1.2).

We have already observed that

$$\{f(0):\ f \in \mathcal{L}\} = \mathbb{C}^n \ominus \{C \in \mathbb{C}^n:\ P_{\mathcal{L}}C = \mathbb{O}\},$$

and so it follows from (1.2) that

$$\dim\{P_{\mathcal{L}}C:\ C \in \mathbb{C}^n\} \ge n - r.$$

It is easy to see that for $C \in \mathbb{C}^n$ we have

$$P_{\mathcal{L}}C = \Theta \mathbb{P}_+ \Theta^* C = \Theta(\Theta^*(0))C.$$

Clearly, $\Theta(\Theta^*(0))C$ belongs to the linear span of the columns of Θ. This completes the proof of the fact that $l = n - r$ and proves that $\begin{pmatrix} \Upsilon & \overline{\Theta} \end{pmatrix}$ is a balanced matrix function. ■

Let us now show that a balanced completion is unique modulo a constant unitary factor.

Theorem 1.2. *Let $\begin{pmatrix} \Upsilon & \overline{\Theta}_1 \end{pmatrix}$ and $\begin{pmatrix} \Upsilon & \overline{\Theta}_2 \end{pmatrix}$ be r-balanced $n \times n$ matrix functions. Then there exists an $(n-r)\times(n-r)$ unitary matrix $\mathcal{O}$ such that $\Theta_2 = \Theta_1 \mathcal{O}$.*

Proof. Let $\mathcal{L} = \operatorname{Ker} T_{\Upsilon^{\mathrm{t}}}$. It is easy to see that

$$\Theta_2 H^2(\mathbb{C}^{n-r}) \subset \mathcal{L} = \Theta_1 H^2(\mathbb{C}^{n-r}).$$

It follows (see Appendix 2.3) that $\Theta_2 = \Theta_1 \mathcal{O}$ for an inner matrix function $\mathcal{O}$. Clearly, $\mathcal{O}$ has size $(n-r)\times(n-r)$. Hence, $\Theta_2^{\mathrm{t}} = \mathcal{O}^{\mathrm{t}}\Theta_1^{\mathrm{t}}$, $\mathcal{O}^{\mathrm{t}}$ is inner, and since Θ_1 is co-outer, it follows that $\mathcal{O}$ is a unitary constant. ■

The following theorem establishes a very important property of balanced matrix functions.

Theorem 1.3. *Let V be an $n \times n$ balanced matrix function. Then the Toeplitz operator T_V on $H^2(\mathbb{C}^n)$ has dense range and trivial kernel.*

We need the following lemma.

Lemma 1.4. *Let G be a co-outer matrix function in $H^\infty(\mathbb{M}_{m,n})$ and let f be a function in $L^2(\mathbb{C}^n)$ such that $Gf \in H^2(\mathbb{C}^m)$. Then $f \in H^2(\mathbb{C}^n)$.*

Proof of Lemma 1.4. Since G^{t} is an outer function, there exists a sequence of analytic polynomial matrix functions $\{Q_j\}_{j\ge 0}$ of size $m \times n$ such that $\{G^{\mathrm{t}} Q_j\}_{j\ge 0}$ converges in $H^2(\mathbb{M}_n)$ to $\boldsymbol{I}_n$, where $\boldsymbol{I}_n$ is the function identically equal to the identity matrix of size $n \times n$. Then the sequence $\{Q_j^{\mathrm{t}} G f\}_{j\ge 0}$ of $H^2(\mathbb{C}^n)$ functions converges to f in $L^2(\mathbb{C}^n)$, and so $f \in H^2(\mathbb{C}^n)$. ■

Proof of Theorem 1.3. Let $V = \begin{pmatrix} \Upsilon & \overline{\Theta} \end{pmatrix}$, where Υ is an $n \times r$ inner and co-outer matrix function and Θ is an $n \times (n-r)$ inner and co-outer matrix function. Let us first show that $\operatorname{Ker} T_V = \{\mathbb{O}\}$. Suppose that $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \operatorname{Ker} T_V$, where $f_1 \in H^2(\mathbb{C}^r)$ and $f_2 \in H^2(\mathbb{C}^{n-r})$. We have

$$Vf = \begin{pmatrix} \Upsilon & \overline{\Theta} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \Upsilon f_1 + \overline{\Theta} f_2 \in H^2_-(\mathbb{C}^n). \tag{1.3}$$

Since V is unitary-valued, it follows from (1.3) that

$$\Upsilon^*(\Upsilon f_1 + \overline{\Theta} f_2) = f_1 \in H^2_-(\mathbb{C}^r),$$

which implies that $f_1 = \mathbb{O}$ and so $\overline{\Theta} f_2 \in H^2_-(\mathbb{C}^n)$. Hence, $\Theta \bar{z} \bar{f_2} \in H^2(\mathbb{C}^n)$. By Lemma 1.4, $\bar{z}\bar{f_2} \in H^2(\mathbb{C}^{n-r})$, which means that $f_2 \in H^2_-(\mathbb{C}^{n-r})$, and so $f_2 = \mathbb{O}$.

Let us now prove that $\operatorname{Ker} T_{V^*} = \{\mathbb{O}\}$. Let $f \in \operatorname{Ker} T_{V^*}$. We have

$$V^* f = \begin{pmatrix} \Upsilon^* \\ \Theta^{\mathrm{t}} \end{pmatrix} f = \begin{pmatrix} \Upsilon^* f \\ \Theta^{\mathrm{t}} f \end{pmatrix} \in H^2_-(\mathbb{C}^n). \tag{1.4}$$

Hence, $\Theta^{\mathrm{t}} f \in H^2_-(\mathbb{C}^{n-r})$. Since both Θ^{t} and f are analytic, it follows that $\Theta^{\mathrm{t}} f = \mathbb{O}$. Therefore the vector $f(\zeta)$ is orthogonal in $\mathbb{C}^n$ to the columns of $\overline{\Theta}(\zeta)$ for almost all $\zeta \in \mathbb{T}$. Since V is unitary-valued, we have $f = \Upsilon \chi$, where $\chi = \Upsilon^* f \in L^2(\mathbb{C}^r)$. By Lemma 1.4, $\chi \in H^2(\mathbb{C}^r)$. On the other hand, it follows from (1.4) that $\Upsilon^* \Upsilon \chi = \chi \in H^2_-(\mathbb{C}^r)$, and so $\chi = \mathbb{O}$. ■

Corollary 1.5. *Let V be an $n \times n$ balanced matrix function. Then the Toeplitz operator $T_{\overline{V}}$ on $H^2(\mathbb{C}^n)$ has dense range and trivial kernel.*

Proof. To deduce Corollary 1.5 from Theorem 1.3, it is sufficient to rearrange the columns of $\overline{V}$. ■

We proceed now to the study of the property of analyticity of minors of balanced matrix functions. Let $V = \begin{pmatrix} \Upsilon & \overline{\Theta} \end{pmatrix}$ be an r-balanced $n \times n$ matrix function. We are going to study its minors $V_{\imath_1 \cdots \imath_k, \jmath_1 \cdots \jmath_k}$ of order k, i.e., the determinants of the submatrix of V with rows $\imath_1, \cdots, \imath_k$ and columns $\jmath_1, \cdots, \jmath_k$. Here $1 \le \imath_1 < \cdots < \imath_k \le n$ and $1 \le \jmath_1 < \cdots < \jmath_k \le n$. By a *minor of V on the first r columns* we mean a minor $V_{\imath_1\cdots\imath_k,\jmath_1\cdots\jmath_k}$ with $k \ge r$ and $\jmath_1 = 1, \cdots, \jmath_r = r$. Similarly, by a *minor of V on the last $n-r$ columns* we mean a minor $V_{\imath_1\cdots\imath_k,\jmath_1\cdots\jmath_k}$ with $k \ge n-r$ and $\jmath_{k-n+r+1} = r+1, \cdots, \jmath_k = n$.

Theorem 1.6. *Let V be an r-balanced matrix function of size $n \times n$. Then all minors of V on the first r columns are in H^∞ while all minors of V on the last $n-r$ columns are in $\overline{H^\infty}$.*

To prove Theorem 1.6, we are going to use the language of wedge products. If $0 \le k \le n$, we consider the vector space $\underbrace{\mathbb{C}^n \wedge \mathbb{C}^n \wedge \cdots \wedge \mathbb{C}^n}_{k}$ that consists of wedge products of the form $C_1 \wedge \cdots \wedge C_k$ with $C_1, \cdots, C_k \in \mathbb{C}^n$. In fact, one can identify the space of such wedge products with the space of $\begin{pmatrix} n \\ k \end{pmatrix}$ minors of order k of $n \times k$ matrices $\begin{pmatrix} C_1 & C_2 & \cdots & C_k \end{pmatrix}$. If $\psi_1, \cdots, \psi_k$ are vector functions in $L^\infty(\mathbb{C}^n)$, we can consider the wedge product $\psi_1 \wedge \cdots \wedge \psi_k$ as a function with values in $\underbrace{\mathbb{C}^n \wedge \mathbb{C}^n \wedge \cdots \wedge \mathbb{C}^n}_{k}$. Then $\psi_1 \wedge \cdots \wedge \psi_k$ is in $H^\infty(\underbrace{\mathbb{C}^n \wedge \mathbb{C}^n \wedge \cdots \wedge \mathbb{C}^n}_{k})$ if and only if all minors of order k of the matrix function $\begin{pmatrix} \psi_1 & \cdots & \psi_k \end{pmatrix}$ are in H^∞. We refer the reader to Gantmakher [1] for more information on wedge products.

Proof. It is easy to see that it is sufficient to prove the theorem for the minors on the first r columns of V. To deduce the second assertion of the theorem, we can consider the matrix function $\overline{V}$ and rearrange its columns to make it $(n-r)$-balanced.

Denote by $\Upsilon_1, \cdots, \Upsilon_r$ and $\Theta_1, \cdots, \Theta_{n-r}$ the columns of Υ and Θ. In the proof of Theorem 1.1 we have observed that for any constant $C \in \mathbb{C}^n$ we have

$$P_{\mathcal{L}}C = \Theta\Theta^*(0)C$$

and

$$\dim\{P_{\mathcal{L}}C : \ C \in \mathbb{C}^n\} = \dim\{\Theta^*(0)C : \ C \in \mathbb{C}^n\} = n - r.$$

It follows that there exist $C_1, \cdots, C_{n-r} \in \mathbb{C}^n$ such that

$$\Theta_j = P_{\mathcal{L}}C_j, \quad 1 \le j \le n-r. \tag{1.5}$$

If $1 \le d \le n-r$ and $1 \le j_1 < j_2 < \cdots < j_d \le n-r$, we consider the vector function

$$\Upsilon_1 \wedge \cdots \wedge \Upsilon_r \wedge \overline{\Theta}_{j_1} \wedge \cdots \wedge \overline{\Theta}_{j_d}$$

whose $\begin{pmatrix} n \\ r+d \end{pmatrix}$ components are the minors of order $r+d$ of the matrix function $\begin{pmatrix} \Upsilon_1 & \cdots & \Upsilon_r & \overline{\Theta}_{j_1} & \cdots & \overline{\Theta}_{j_d} \end{pmatrix}$.

It follows from (1.5) that

$$\Theta_j = C_j - P_{\mathcal{L}^\perp}C_j,$$

where $P_{\mathcal{L}^\perp}$ is the orthogonal projection onto $\mathcal{L}^\perp = \operatorname{clos}\operatorname{Range} T_{\overline{\Upsilon}}$. We have

$$\begin{aligned}&\Upsilon_1 \wedge \cdots \wedge \Upsilon_r \wedge \overline{\Theta}_{j_1} \wedge \cdots \wedge \overline{\Theta}_{j_d}\\ &= \Upsilon_1 \wedge \cdots \wedge \Upsilon_r \wedge (\overline{C}_{j_1} - \overline{P_{\mathcal{L}^\perp} C_{j_1}}) \wedge \cdots \wedge (\overline{C}_{j_d} - \overline{P_{\mathcal{L}^\perp} C_{j_d}}).\end{aligned}$$

The components of this vector belong to L^∞ and can be approximated in $L^{2/d}$ by vector functions of the form

$$\Upsilon_1 \wedge \cdots \wedge \Upsilon_r \wedge (\overline{C}_{j_1} - \overline{g}_{j_1}) \wedge \cdots \wedge (\overline{C}_{j_d} - \overline{g}_{j_d}), \tag{1.6}$$

where $g_{j_1}, \cdots, g_{j_d} \in \operatorname{Range} T_{\overline{\Upsilon}}$. Hence, it is sufficient to prove that the components of (1.6) belong to $H^{2/d}$. Let $g_{j_l} = \mathbb{P}_+ \overline{\Upsilon} f_l$ for $f_l \in H^2(\mathbb{C}^r)$, $1 \le l \le d$. We have

$$\begin{aligned}&\Upsilon_1 \wedge \cdots \wedge \Upsilon_r \wedge (\overline{C}_{j_1} - \overline{g}_{j_1}) \wedge \cdots \wedge (\overline{C}_{j_d} - \overline{g}_{j_d})\\ =\ &\Upsilon_1 \wedge \cdots \wedge \Upsilon_r \wedge (\overline{C}_{j_1} - \overline{\mathbb{P}_+ \overline{\Upsilon} f_1}) \wedge \cdots \wedge (\overline{C}_{j_d} - \overline{\mathbb{P}_+ \overline{\Upsilon} f_d})\\ =\ &\Upsilon_1 \wedge \cdots \wedge \Upsilon_r \wedge (\overline{C}_{j_1} - \Upsilon \bar{f}_1 + \overline{\mathbb{P}_- \overline{\Upsilon} f_1}) \wedge \cdots \wedge (\overline{C}_{j_d} - \Upsilon \bar{f}_d + \overline{\mathbb{P}_- \overline{\Upsilon} f_d}).\end{aligned}$$

Clearly, almost everywhere on $\mathbb{T}$ the vectors $\Upsilon(\zeta)\overline{f_l(\zeta)}$ are linear combinations of $\Upsilon_1(\zeta), \cdots, \Upsilon_r(\zeta)$. Therefore if we expand the above wedge product using the multilinearity of $\wedge$, all terms containing $\Upsilon \bar{f}_l$ give zero contribution. Thus we have

$$\begin{aligned}&\Upsilon_1 \wedge \cdots \wedge \Upsilon_r \wedge (\overline{C}_{j_1} - \Upsilon \bar{f}_1 + \overline{\mathbb{P}_- \overline{\Upsilon} f_1}) \wedge \cdots \wedge (\overline{C}_{j_d} - \Upsilon \bar{f}_d + \overline{\mathbb{P}_- \overline{\Upsilon} f_d})\\ &= \Upsilon_1 \wedge \cdots \wedge \Upsilon_r \wedge (\overline{C}_{j_1} + \overline{\mathbb{P}_- \overline{\Upsilon} f_1}) \wedge \cdots \wedge (\overline{C}_{j_d} + \overline{\mathbb{P}_- \overline{\Upsilon} f_d}) \in H^{2/d}. \qquad \blacksquare\end{aligned}$$

The following fact is an immediate consequence of Theorem 1.6.

Corollary 1.7. *Let V be a balanced matrix function. Then* $\det V$ *is a constant function of modulus 1.*

Finally, we obtain in this section the following result, which will play an important role in this chapter.

Theorem 1.8. *Let $0 < r \le \min\{m, n\}$ and let V and W^{t} be r-balanced matrix functions of sizes $n \times n$ and $m \times m$, respectively. Then*

$$WH^\infty(\mathbb{M}_{m,n})V \bigcap \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & L^\infty(\mathbb{M}_{m-r,n-r}) \end{pmatrix} = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & H^\infty(\mathbb{M}_{m-r,n-r}) \end{pmatrix}. \tag{1.7}$$

Proof. Let

$$V = \begin{pmatrix} \Upsilon & \overline{\Theta} \end{pmatrix}, \quad W = \begin{pmatrix} \Omega & \Xi \end{pmatrix}^{\mathrm{t}},$$

where Υ has size $n \times r$ and Ω has size $m \times r$. Clearly, (1.7) is equivalent to the following equality:

$$\begin{aligned}&H^\infty(\mathbb{M}_{m,n}) \bigcap W^* \begin{pmatrix} \mathbb{O} & 0 \\ \mathbb{O} & L^\infty(\mathbb{M}_{m-r,n-r}) \end{pmatrix} V^*\\ &= W^* \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & H^\infty(\mathbb{M}_{m-r,n-r}) \end{pmatrix} V^*.\end{aligned}$$

Let us show first that the right-hand side of this equality is contained in the left-hand side. Let $F \in H^\infty(\mathbb{M}_{m-r,n-r})$. We have

$$\begin{aligned} W^* \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & F \end{pmatrix} V^* &= \begin{pmatrix} \overline{\Omega} & \Xi \end{pmatrix} \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & F \end{pmatrix} \begin{pmatrix} \Upsilon^* \\ \Theta^{\mathrm{t}} \end{pmatrix} \\ &= \Xi F \Theta^{\mathrm{t}} \in H^\infty(\mathbb{M}_{m,n}). \end{aligned}$$

Let us now prove the opposite inclusion. Suppose that $G \in L^\infty(\mathbb{M}_{m-r,n-r})$ and

$$W^* \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & G \end{pmatrix} V^* \in H^\infty(\mathbb{M}_{m,n}).$$

We have to prove that $G \in H^\infty(\mathbb{M}_{m-r,n-r})$. As above one can easily see that

$$W^* \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & G \end{pmatrix} V^* = \Xi G \Theta^{\mathrm{t}} \in H^\infty(\mathbb{M}_{m,n}).$$

Since Ξ is co-outer, it follows from Lemma 1.4 that $G\Theta^{\mathrm{t}} \in H^\infty(\mathbb{M}_{m-r,n})$ (one should apply Lemma 1.4 to each column of $G\Theta^{\mathrm{t}}$). Again, by Lemma 1.4, since Θ^{t} is outer, it follows that $G \in H^\infty(\mathbb{M}_{m-r,n-r})$. ■

2. Parametrization of Best Approximations

In this section we consider matrix functions $\Phi \in L^\infty(\mathbb{M}_{m,n})$ such that the Hankel operator $H_\Phi : H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^m)$ has a maximizing vector. For such matrix functions we obtain a parametrization of the set of best approximations of Φ by bounded analytic matrix functions. Among such matrix functions Φ we characterize the *badly approximable* matrix functions, i.e., such that

$$\|\Phi\|_{L^\infty} = \operatorname{dist}_{L^\infty}(\Phi, H^\infty).$$

Let us first make a couple of obvious observations. Suppose that $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and $F \in H^\infty(\mathbb{M}_{m,n})$ is a superoptimal approximation of Φ by bounded analytic matrix functions. Then the transposed matrix function F^{t} is a superoptimal approximation of Φ^{t}. Moreover, the matrix functions Φ and Φ^{t} have the same superoptimal singular values.

Next, suppose that $\mathbb{P}_-\Phi \neq \mathbb{O}$, $f \in H^2(\mathbb{C}^n)$, and $g = t_0^{-1}\bar{z}\overline{H_\Phi f} \in H^2(\mathbb{C}^m)$. (Recall that the largest superoptimal singular value $t_0 = t_0(\Phi)$ is equal to $\|H_\Phi\|$.) Then f is a maximizing vector of H_Φ if and only if g is a maximizing vector of $H_{\Phi^{\mathrm{t}}}$. Indeed, f is a maximizing vector of H_Φ if and only if $t_0^{-1}H_\Phi f = \bar{z}\bar{g}$ is a maximizing vector of H_Φ^*, i.e., g is nonzero and

$$\|H_\Phi^* \bar{z}\bar{g}\|_2 = \|H_\Phi^*\| \cdot \|\mathbb{P}_+\Phi^*\bar{z}\bar{g}\|_2 = \|H_\Phi^*\| \cdot \|g\|_2.$$

It is easy to see that the last equality is equivalent to the following one:

$$\|\mathbb{P}_-\Phi^{\mathrm{t}} g\|_2 = \|H_{\Phi^{\mathrm{t}}}\| \cdot \|g\|_2,$$

which in turn is equivalent to the fact that g is a maximizing vector of $H_{\Phi^{\mathrm{t}}}$.

Let us first consider the simplest case $\min\{m,n\} = 1$ and prove the uniqueness of a best approximation.

Theorem 2.1. *Suppose that Φ is a matrix function in $L^\infty(\mathbb{M}_{m,n})$ such that H_Φ has a maximizing vector. If $\min\{m,n\} = 1$, then Φ has a unique best approximation by bounded analytic matrix functions.*

Proof. In view of the above observations it is sufficient to assume that $n = 1$. Let $F \in H^\infty(\mathbb{M}_{m,n})$ be a best approximation of Φ. Suppose that $\boldsymbol{f} \in H^2$ is a maximizing vector of H_Φ. By Theorem 2.2.3,

$$H_\Phi \boldsymbol{f} = \mathbb{P}_-(\Phi - F)\boldsymbol{f} = (\Phi - F)\boldsymbol{f},$$

and so

$$\Phi - F = \frac{H_\Phi \boldsymbol{f}}{\boldsymbol{f}}.$$

Clearly, the last equality uniquely determines F. ■

Suppose now that $\min\{m,n\} > 1$, $\boldsymbol{f}$ is a maximizing vector of H_Φ, and $\boldsymbol{g} = t_0^{-1}\bar{z}\overline{H_\Phi \boldsymbol{f}}$. Let h be a scalar outer function in H^2 such that $|h(\zeta)| = \|\boldsymbol{f}(\zeta)\|_{\mathbb{C}^n}$ for almost all $\zeta \in \mathbb{T}$. Let ϑ_1 be a greatest common inner divisor of the entries of $\boldsymbol{f}$, i.e., ϑ_1 is a scalar inner function such that all entries of f are divisible by ϑ_1 and any other common inner divisor of all entries of $\boldsymbol{f}$ is also a divisor of ϑ_1. Clearly, $\boldsymbol{f}$ admits a factorization of the form

$$\boldsymbol{f} = \vartheta_1 h \boldsymbol{v}, \tag{2.1}$$

where $\boldsymbol{v}$ is an inner and co-outer column function. By Theorem 2.2.3, $\|\boldsymbol{g}(\zeta)\|_{\mathbb{C}^m} = \|\boldsymbol{f}(\zeta)\|_{\mathbb{C}^n}$ for almost all $\zeta \in \mathbb{T}$. Hence, $\boldsymbol{g}$ admits the following factorization:

$$\boldsymbol{g} = \vartheta_2 h \boldsymbol{w}, \tag{2.2}$$

where ϑ_2 is a scalar inner function and $\boldsymbol{w}$ is an inner and co-outer column function.

By Theorem 1.1, the column functions $\boldsymbol{v}$ and $\boldsymbol{w}$ admits thematic (in other words, 1-balanced) completions, i.e., there exist inner and co-outer matrix functions Θ and Ξ such that the matrix functions

$$V \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix} \quad \text{and} \quad W^{\mathrm{t}} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{w} & \overline{\Xi} \end{pmatrix} \tag{2.3}$$

are unitary-valued.

Note that if $n = 1$, then $\boldsymbol{f}$ is a scalar function, it admits a factorization of the form $\boldsymbol{f} = \vartheta_1 h$, where h is an outer function in H^2 and ϑ is an inner function. In this case $\boldsymbol{v}$ is identically equal to 1 and V is a scalar function identically equal to 1 as well. The same can be said in the case $m = 1$.

Theorem 2.2. *Let Φ be a matrix function in $L^\infty(\mathbb{M}_{m,n})$ such that the Hankel operator $H_\Phi : H^2(\mathbb{C}^n) \to H^2_-(\mathbb{C}^m)$ is nonzero and has a maximizing vector. Suppose that V and W are as above and F is a best approximation of Φ by bounded analytic matrix functions. Then there exists*

$\Psi \in L^\infty(\mathbb{M}_{m-1,n-1})$ *such that* $\|\Psi\|_{L^\infty} \le t_0$ *and*

$$\Phi - F = W^* \begin{pmatrix} t_0 u & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^*, \tag{2.4}$$

where $u = \bar{z}\bar{\vartheta}_1\bar{\vartheta}_2\bar{h}/h$, *and* h, ϑ_1, *and* ϑ_2 *are given by* (2.1) *and* (2.2).

If $n = 1$, by $\begin{pmatrix} t_0 u & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix}$ we mean the matrix function

$$\begin{pmatrix} t_0 u \\ \mathbb{O} \end{pmatrix}$$

of size $m \times 1$. If $m = 1$, then this is the $1 \times n$ matrix function

$$\begin{pmatrix} t_0 u & \mathbb{O} \end{pmatrix}.$$

We will always use this way of writing, i.e., if a submatrix has size 0, this means that it does not exist.

Proof. By Theorem 2.2.3, $\bar{z}\bar{\boldsymbol{g}} = t_0^{-1} H_\Phi \boldsymbol{f} = t_0^{-1}(\Phi - F)\boldsymbol{f}$. It is easy to see that the upper left entry of the matrix function $W(\Phi - F)V$ is

$$\begin{aligned} \boldsymbol{w}^{\mathrm{t}}(\Phi - F)\boldsymbol{v} &= \frac{\bar{\vartheta}_2}{h}\boldsymbol{g}^{\mathrm{t}}(\Phi - F)\frac{\bar{\vartheta}_1}{h}\boldsymbol{f} = \frac{\bar{\vartheta}_2}{h}\boldsymbol{g}^{\mathrm{t}}\frac{\bar{\vartheta}_1}{h}H_\Phi \boldsymbol{f} \\ &= t_0 \bar{z}\frac{\bar{\vartheta}_2}{h}\frac{\bar{\vartheta}_1}{h}\boldsymbol{g}^{\mathrm{t}}\bar{\boldsymbol{g}} = t_0\bar{z}\bar{\vartheta}_1\bar{\vartheta}_2\frac{|h|^2}{h^2} = t_0 u. \end{aligned}$$

Clearly, $\|(W(\Phi - F)V)(\zeta)\|_{\mathbb{M}_{m,n}} \le t_0$ for almost all $\zeta \in \mathbb{T}$. Since u is a unimodular function, $|t_0 u(\zeta)| = t_0$ almost everywhere, and so the upper right and the lower left entries of $(W(\Phi - F)V)(\zeta)$ must be the zero. Hence, $W(\Phi - F)V$ has the form

$$W^* \begin{pmatrix} t_0 u & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^*,$$

where $\Psi \in L^\infty(\mathbb{M}_{m-1,n-1})$. Since V and W are unitary-valued functions and $\|\Phi - F\|_{L^\infty} \le t_0$, it follows that $\|\Phi\|_{L^\infty} \le t_0$. ■

We can parametrize now all best approximations of Φ by bounded analytic matrix functions. We fix a best approximation F and consider the factorization (2.4).

Theorem 2.3. *Let* $\min\{m, n\} > 1$ *and let* Φ *be a matrix function in* $L^\infty(\mathbb{M}_{m,n})$ *such that* H_Φ *is nonzero and has a maximizing vector. Suppose that* F *is a best approximation of* Φ *and* $\Phi - F$ *satisfies* (2.4). *Let* $Q \in H^\infty(\mathbb{M}_{m,n})$. *Then* Q *is a best approximation of* Φ *if and only if there exists* $G \in H^\infty(\mathbb{M}_{m-1,n-1})$ *such that* $\|\Psi - G\|_{L^\infty} \le t_0$ *and*

$$\Phi - Q = W^* \begin{pmatrix} t_0 u & \mathbb{O} \\ \mathbb{O} & \Psi - G \end{pmatrix} V^*. \tag{2.5}$$

Proof. Suppose that Q is a best approximation of Φ by bounded analytic matrix functions. Then by Theorem 2.2, $\Phi - Q$ admits a factorization

$$\Phi - Q = W^* \begin{pmatrix} t_0 u & \mathbb{O} \\ \mathbb{O} & \Psi^\circ \end{pmatrix} V^*,$$

where u is the same as in (2.4) and Ψ° is a function in $L^\infty(\mathbb{M}_{m-1,n-1})$ such that $\|\Psi^\circ\|_{L^\infty} \le t_0$. Then

$$Q - F = W^* \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \Psi - \Psi^\circ \end{pmatrix} V^*$$

and by Theorem 1.8, $G \stackrel{\text{def}}{=} \Psi - \Psi^\circ \in H^\infty(\mathbb{M}_{m-1,n-1})$.

Now let G be a matrix function in $H^\infty(\mathbb{M}_{m-1,n-1})$ such that $\|\Psi - G\|_{L^\infty} \le t_0$. Again, by Theorem 1.8, there exists a matrix function $Q \in H^\infty(\mathbb{M}_{m,n})$ such that

$$Q - F = W^* \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & G \end{pmatrix} V^*.$$

Then (2.5) holds and clearly, $\|\Phi - Q\|_{L^\infty} = t_0$, i.e., Q is a best approximation of Φ. ■

Theorems 2.2 and 2.3 allow one to reduce the problem of finding a superoptimal approximation of Φ to the same problem for the matrix function Ψ, which has size $(m-1) \times (n-1)$. Indeed, the following result follows immediately from Theorems 2.2 and 2.3.

Theorem 2.4. *Suppose that Φ satisfies the hypotheses of Theorem 2.2 and* (2.4) *holds. Then $t_j(\Phi) = t_{j-1}(\Psi)$ for $j \ge 1$. A function $Q \in H^\infty(\mathbb{M}_{m,n})$ is a superoptimal approximation of Φ if and only if $\Phi - Q$ admits a factorization of the form* (2.5) *for a superoptimal approximation G of Ψ. Moreover,*

$$s_0\big((\Phi - Q)(\zeta)\big) = t_0, \quad \textit{for almost all} \quad \zeta \in \mathbb{T}, \tag{2.6}$$

and

$$s_j\big((\Phi - Q)(\zeta)\big) = s_{j-1}\big((\Psi - G)(\zeta)\big), \quad j \ge 1, \quad \textit{for almost all} \quad \zeta \in \mathbb{T}. \tag{2.7}$$

Proof. Suppose that Q is a best approximation of Φ and (2.5) holds. Equalities (2.6) and (2.7) follow immediately from the facts that u is unimodular and $\|\Psi\|_{L^\infty} \le t_0$. This in turn implies that $t_j(\Phi) = t_{j-1}(\Psi)$ for $j \ge 1$ as well as the rest of the theorem. ■

We have now arrived at the problem of finding a superoptimal approximation of Ψ that has size $(m-1) \times (n-1)$. If we knew that H_Ψ also possesses a maximizing vector, we could iterate the above procedure and eventually reduce the problem to the case $\min\{m,n\} = 1$. In the general case it is not true that under the hypotheses of Theorem 2.2 the Hankel operator H_Ψ has a maximizing vector. Indeed, we can consider the diagonal matrix function

$$\Phi = \begin{pmatrix} \varphi_1 & \mathbb{O} \\ \mathbb{O} & \varphi_2 \end{pmatrix},$$

such that φ_1 and φ_2 are scalar functions in L^∞, $\|H_{\varphi_1}\| > \|H_{\varphi_2}\|$, H_{φ_1} has a maximizing vector but H_{φ_2} has no maximizing vector. It is easy to see that in this case $\Psi = \varphi_2$.

However, in certain important cases we can make the conclusion that H_Ψ must have a maximizing vector that will allow us to construct a superoptimal approximation and prove uniqueness. This will be done in §3 in the case $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$ and in §4 in the case when the essential norm of H_Φ is less than the smallest nonzero superoptimal singular value of Φ.

We can describe now the badly approximable matrix functions Φ for which H_Φ has a maximizing vector. As in the scalar case a matrix function $\Phi \in L^\infty(\mathbb{M}_{m,n})$ is called *badly approximable* if

$$\operatorname{dist}_{L^\infty}(\Phi, H^\infty(\mathbb{M}_{m,n})) = \|\Phi\|_{L^\infty}.$$

Theorem 2.5. *Suppose that $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and $H_\Phi \neq \mathbb{O}$. The following are equivalent:*

(i) Φ *is badly approximable and H_Φ has a maximizing vector;*

(ii) Φ *admits a factorization of the form*

$$\Phi = W^* \begin{pmatrix} tu & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^*,$$

where $t > 0$, V and W^{t} are thematic matrix functions, $\|\Psi\|_{L^\infty} \le t$, and u is a scalar unimodular function of the form

$$u = \bar{z}\bar{\vartheta}\bar{h}/h$$

for an inner function ϑ and an outer function h in H^2.

If (ii) *holds, then $\|H_\Phi\| = t$.*

Proof. If (i) holds, the existence of a desired factorization follows from Theorem 2.2.

Conversely, suppose (ii) holds with V and W of the form (2.3). Clearly, $\|\Phi\|_{L^\infty} = t$, and so $\|H_\Phi\| \le t$. Put $f = h\boldsymbol{v} \in H^2(\mathbb{C}^n)$. We have

$$\begin{aligned} \Phi f &= W^* \begin{pmatrix} tu & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} \begin{pmatrix} \boldsymbol{v}^* \\ \Theta^{\mathrm{t}} \end{pmatrix} h\boldsymbol{v} \\ &= W^* \begin{pmatrix} tu & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} \begin{pmatrix} h \\ \mathbb{O} \end{pmatrix} \\ &= \begin{pmatrix} \bar{\boldsymbol{w}} & \Xi \end{pmatrix} \begin{pmatrix} t\bar{z}\bar{\vartheta}\bar{h} \\ \mathbb{O} \end{pmatrix} = t\bar{z}\bar{\vartheta}\bar{h}\bar{\boldsymbol{w}} \in H^2_-(\mathbb{C}^m). \end{aligned}$$

Hence, $H_\Phi f = \Phi f$, $\|H_\Phi f\|_{L^2(\mathbb{C}^m)} = \|h\|_{H^2} = \|f\|_{H^2(\mathbb{C}^n)}$. It follows that $\|H_\Phi\| = t$ and f is a maximizing vector of H_Φ. ■

3. Superoptimal Approximation of $H^\infty + C$ Matrix Functions

In this section we consider the case when $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$. By the Hartman theorem, the Hankel operator $H_\Phi : H^2(\mathbb{C}^n) \to H^2_-(\mathbb{C}^m)$ is compact, and so it has a maximizing vector. Therefore by Theorem 2.2, if

$H_\Phi \neq \mathbb{O}$ and F is a best approximation of Φ by bounded analytic matrix functions, then $\Phi - F$ admits a factorization of the form (2.4). We show that in that factorization Ψ must belong to $(H^\infty + C)(\mathbb{M}_{m-1,n-1})$, and so we can iterate this procedure and reduce the problem to the case $\min\{m,n\} = 1$. Uniqueness will follow from Theorem 2.1.

To prove that $\Psi \in (H^\infty + C)(\mathbb{M}_{m-1,n-1})$, we use Theorem 1.6 on the analyticity of minors. Later in this chapter we consider other methods to prove the uniqueness of a superoptimal approximation.

Recall that the maximizing vectors $\boldsymbol{f}$ and $\boldsymbol{g}$ of H_Φ and $H_{\Phi^{\mathrm{t}}}$ admit factorizations (2.1) and (2.2), and the unitary-valued matrix function V and W are of the form (2.3). First we prove that the unimodular function u in (2.4) belongs to QC.

Theorem 3.1. *Let $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$ and $H_\Phi \neq \mathbb{O}$. If F is a best approximation of Φ by bounded analytic matrix functions and*

$$\Phi - F = W^* \begin{pmatrix} t_0 u & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^*, \tag{3.1}$$

then $u \in QC$.

Proof. It can be easily seen from (3.1) that

$$\boldsymbol{w}^{\mathrm{t}}(\Phi - F)\boldsymbol{v} = t_0 u,$$

and so $u \in H^\infty + C$. Hence, $\mathbb{P}_- u \in VMO$.

Since $u = \bar{z}\bar{\vartheta}_1\bar{\vartheta}_2\bar{h}/h$, we have $uh = \bar{z}\bar{\vartheta}_1\bar{\vartheta}_2\bar{h} \in H^2_-$, and so $h \in \operatorname{Ker} T_u$. By Theorem 3.1.4, $\operatorname{Ker} T_u^* = \{\mathbb{O}\}$, and so T_u has dense range in H^2. By Theorem 7.1.4 applied to $X = VMO$, $u \in VMO$, and so $u \in QC$. ■

Now we are in a position to show that the entry Ψ in (3.1) belongs to $H^\infty + C$.

Theorem 3.2. *Suppose that $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$, $H_\Phi \neq \mathbb{O}$, F is a best approximation of Φ by bounded analytic matrix functions, and* (3.1) *holds. Then $\Psi \in (H^\infty + C)(\mathbb{M}_{m-1,n-1})$.*

Proof. By (3.1),

$$W(\Phi - F)V = \begin{pmatrix} t_0 u & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix}. \tag{3.2}$$

For $2 \le j \le m$ and $2 \le k \le n$ we consider the 2×2 minor of the right-hand side of (3.2) with indices $1j, 1k$. Clearly, it is equal to $t_0 u_0 \psi_{j-1,k-1}$, where $\Psi = \{\psi_{st}\}_{1 \le s \le m-1, 1 \le t \le n-1}$. On the other hand, the same minor of the left-hand side of (3.2) equals

$$\sum_{s<t,\iota<\varkappa} W_{1j,st}(\Phi - F)_{st,\iota\varkappa} V_{\iota\varkappa,1k},$$

and so

$$\psi_{j-1,k-1} = t_0^{-1}\bar{u} \sum_{s<t,\iota<\varkappa} W_{1j,st}(\Phi - F)_{st,\iota\varkappa} V_{\iota\varkappa,1k}.$$

By Theorem 1.6, $V_{\iota\varkappa,1k} \in H^\infty + C$ and $W_{1j,st} \in H^\infty + C$. By Theorem 3.1, $\bar{u} \in H^\infty + C$. Therefore $\psi_{j-1,k-1} \in H^\infty + C$, which proves that $\Psi \in (H^\infty + C)(\mathbb{M}_{m-1,n-1})$. ■

Theorem 3.3. *Let $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$. Then Φ has a unique superoptimal approximation by bounded analytic matrix functions.*

Proof. If $\min\{m,n\} = 1$, the result follows from Theorem 2.1. Otherwise, let F be a best approximation of Φ by bounded analytic matrix functions. Then by Theorem 2.2, $\Phi - F$ admits a factorization of the form (3.1). Theorem 2.4 reduces the result to the fact that Ψ has a unique superoptimal approximation. Since by Theorem 3.2, $\Psi \in (H^\infty + C)(\mathbb{M}_{m-1,n-1})$ we can iterate this procedure and the result follows by induction on $\min\{m,n\}$. ■

Theorem 3.4. *Let $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$ and let Q be the unique superoptimal approximation of Φ by bounded analytic matrix functions. Then*

$$s_j\big((\Phi - Q)(\zeta)\big) = t_j(\Phi), \quad 0 \le j \le \min\{m,n\} - 1, \quad \zeta \in \mathbb{T}.$$

Proof. The result follows by induction from Theorem 2.4. ■

4. Superoptimal Approximation of Matrix Functions Φ with Small Essential Norm of H_Φ

In this section we consider the case when the essential norm of the Hankel operator H_Φ is less than the largest nonzero superoptimal singular value of Φ and prove that Φ has a unique superoptimal approximation by bounded analytic matrix functions. The method used in §3 does not work in this case. We use another method, which is considerably more complicated.

First we assume that $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\| = t_0$. This condition guarantees that H_Φ has a maximizing vector $\boldsymbol{f} \in H^2(\mathbb{C}^n)$. As in §2 we put $\boldsymbol{g} = t_0^{-1}\bar{z}\overline{H_\Phi \boldsymbol{f}} \in H^2(\mathbb{C}^m)$. The vector functions $\boldsymbol{f}$ and $\boldsymbol{g}$ admit admit factorizations (2.1) and (2.2). Consider the unitary-valued functions V and W as in (2.3), i.e.,

$$\boldsymbol{f} = \vartheta_1 h \boldsymbol{v}, \quad \boldsymbol{g} = \vartheta_2 h \boldsymbol{w}, \quad \text{and} \quad V = \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix}, \quad W = \begin{pmatrix} \boldsymbol{w} & \overline{\Xi} \end{pmatrix}^{\mathrm{t}}.$$

By Theorem 2.2, if F is a best approximation of Φ by bounded analytic matrix functions, then $\Phi - F$ admits a factorization of the form

$$\Phi - F = W^* \begin{pmatrix} t_0 u & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^*, \tag{4.1}$$

where $u = \bar{z}\bar{\vartheta}_1\bar{\vartheta}_2\bar{h}/h$, and h, ϑ_1 and ϑ_2 are given by (2.1) and (2.2).

We would like to continue this process and apply the above procedure to Ψ. By Theorem 2.4, $\|H_\Psi\| = t_1(\Phi)$. If $t_1(\Phi) = 0$, then Ψ has a unique best approximation equal to Ψ and F is the unique superoptimal approximation of Φ. Suppose now that $t_1(\Phi) \neq 0$. We would be able to apply the above procedure to Ψ if we knew that $\|H_\Psi\|_{\mathrm{e}} < t_1(\Phi)$. By our assumption, $\|H_\Phi\|_{\mathrm{e}} < t_1$. Everything will be fine if we prove that

$$\|H_\Psi\|_{\mathrm{e}} \le \|H_\Phi\|_{\mathrm{e}}. \tag{4.2}$$

This is the key result of this section.

Theorem 4.1. *Suppose that $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$, F is a best approximation of Φ by bounded analytic matrix functions, and $\Phi - F$ admits a factorization* (4.1)*. Then* (4.2) *holds.*

We are going to use the following formula for the essential norm of a Hilbert space operator. Let T be a bounded linear operator from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$. Then

$$\|T\|_{\mathrm{e}} = \inf\{\limsup_{j\to\infty} \|Tx_j\|_{\mathcal{H}_2}\}, \tag{4.3}$$

where the infimum is taken over all sequences $\{x_j\}_{j\ge 0}$ in $\mathcal{H}_1$ such that $\|x_j\|_{\mathcal{H}_1} = 1$ and $x_j \to \mathbb{O}$ in the weak topology of $\mathcal{H}_1$.

To prove Theorem 4.1, we need several other results.

Theorem 4.2. *Under the hypotheses of Theorem 4.1 the Toeplitz operator T_u is Fredholm,* $\operatorname{ind} T_u > 0$*, and ϑ_1 and ϑ_2 are finite Blaschke products.*

Proof. It is easy to see from (4.1) that

$$u = t_0^{-1}\boldsymbol{w}^{\mathrm{t}}(\Phi - F)\boldsymbol{v}.$$

Since

$$\operatorname{dist}_{L^\infty}\big(\Phi, (H^\infty + C)(\mathbb{M}_{m,n})\big) = \|H_\Phi\|_{\mathrm{e}} < t_0$$

and $\|\boldsymbol{v}\|_{H^\infty} = \|\boldsymbol{w}^{\mathrm{t}}\|_{H^\infty} = 1$, it follows that

$$\|H_u\|_{\mathrm{e}} = \operatorname{dist}_{L^\infty}(u, H^\infty) < 1. \tag{4.4}$$

Clearly, $T_u h = \mathbb{P}_+ \bar z \bar\vartheta_1 \bar\vartheta_2 \bar h = \mathbb{O}$, and so $\operatorname{Ker} T_u \ne \{\mathbb{O}\}$. By Theorem 3.1.4, T_u has dense range in H^2.

It follows from (4.4) that $\operatorname{Ker} T_u$ is finite-dimensional, and so by Theorem 4.4.7, $\|H_{\bar u}\|_{\mathrm{e}} = \|H_u\|_{\mathrm{e}} < 1$. The Fredholmness of T_u follows now from Corollary 3.1.16. Since $\operatorname{Ker} T_u \ne \{\mathbb{O}\}$, we have $\operatorname{ind} T_u > 0$.

It remains to show that both ϑ_1 and ϑ_2 are finite Blaschke products. Let $\vartheta = \vartheta_1\vartheta_2$. If ϑ is not a finite Blaschke product, then for any $N \in \mathbb{Z}_+$ there exist inner divisors $\tau_1, \cdots, \tau_N$ of ϑ such that τ_j is divisible by τ_{j+1} for $j \le N-1$. It is easy to see that the H^2 functions $\tau_1 h, \cdots, \tau_N h$ are linear independent and belong to $\operatorname{Ker} T_u$, which contradicts the fact that $\dim \operatorname{Ker} T_u < \infty$. ■

Theorem 4.3. *Under the hypotheses of Theorem 4.1 the Toeplitz operators $T_{\overline{\boldsymbol{v}}} : H^2 \to H^2(\mathbb{C}^n)$ and $T_{\overline{\boldsymbol{w}}} : H^2 \to H^2(\mathbb{C}^m)$ are left invertible.*

Proof. Clearly, $\|H_\Phi\| = \|H_{\Phi^{\mathrm{t}}}\|$ and $\|H_\Phi\|_{\mathrm{e}} = \|H_{\Phi^{\mathrm{t}}}\|_{\mathrm{e}}$, and so it is sufficient to prove that $T_{\overline{\boldsymbol{w}}}$ is left invertible and apply this result to Φ^{t}.

Let us first show that $\operatorname{Ker} T_{\overline{\boldsymbol{w}}} = \{\mathbb{O}\}$. Indeed, suppose that $\varphi \in \operatorname{Ker} T_{\overline{\boldsymbol{w}}}$. Then $\overline{\boldsymbol{w}}\varphi \in H^2_-(\mathbb{C}^m)$, i.e., $\boldsymbol{w}\bar{z}\bar{\varphi} \in H^2(\mathbb{C}^m)$. Since $\boldsymbol{w}$ is co-outer, by Lemma 1.4, $\bar{z}\bar{\varphi} \in H^2(\mathbb{C}^m)$, which means that $f \in H^2_-(\mathbb{C}^m)$, and so $f = \mathbb{O}$.

If $T_{\overline{\boldsymbol{w}}}$ is not left invertible, then there exists a sequence $\{\varphi_j\}_{j\ge 0}$ in H^2 such that $\|\varphi_j\|_2 = 1$, $\varphi_j \to \mathbb{O}$ in the weak topology of H^2 and $\|T_{\overline{\boldsymbol{w}}}\varphi_j\|_2 \to 0$. By Theorem 4.2, the Toeplitz operator T_u is onto. Hence, there exists a sequence $\{\omega_j\}_{j\ge 0}$ in $(\operatorname{Ker} T_u)^\perp$ such that $T_u\omega_j = \varphi_j$ and $\|\omega_j\|_2 \le \text{const}$. Since T_u is Fredholm, it follows that $\omega_j \to \mathbb{O}$ in the weak topology of H^2.

Put $\rho_j = \omega_j\boldsymbol{v}$. By (4.1), we have

$$(\Phi - F)\rho_j = W^* \begin{pmatrix} t_0 u\omega_j \\ \mathbb{O} \\ \vdots \\ \mathbb{O} \end{pmatrix} = t_0 u\omega_j\overline{\boldsymbol{w}} = t_0(\varphi_j + \psi_j)\overline{\boldsymbol{w}},$$

where $\psi_j = H_u\omega_j \in H^2_-$. We have

$$\begin{aligned} \|H_\Phi\rho_j\|_2^2 &= \|\mathbb{P}_-(\Phi - F)\rho_j\|_2^2 = \|t_0\mathbb{P}_-((\varphi_j + \psi_j)\overline{\boldsymbol{w}})\|_2^2 \\ &= \|t_0(\varphi_j + \psi_j)\overline{\boldsymbol{w}}\|_2^2 - \|t_0\mathbb{P}_+((\varphi_j + \psi_j)\overline{\boldsymbol{w}})\|_2^2 \\ &= \|t_0 u\omega_j\overline{\boldsymbol{w}}\|_2^2 - \|t_0\mathbb{P}_+\varphi_j\overline{\boldsymbol{w}}\|_2^2 \\ &= \|t_0 u\omega_j\overline{\boldsymbol{v}}\|_2^2 - \|t_0\mathbb{P}_+\varphi_j\overline{\boldsymbol{w}}\|_2^2, \end{aligned}$$

since $\|\boldsymbol{v}(\zeta)\|_{\mathbb{C}^n} = \|\boldsymbol{w}(\zeta)\|_{\mathbb{C}^m}$, $\zeta \in \mathbb{T}$. Hence,

$$\|H_\Phi\rho_j\|_2^2 = t_0^2(\|\rho_j\|^2 - \|T_{\overline{\boldsymbol{w}}}\varphi_j\|_2^2). \tag{4.5}$$

Clearly, $\|\rho_j\|_2 \le \text{const}$, $\rho_j \to \mathbb{O}$ weakly. Since $\|T_{\overline{\boldsymbol{w}}}\varphi_j\|_2 \to 0$, it follows from (4.3) and (4.5) that $\|H_\Phi\|_{\mathrm{e}} = t_0 = \|H_\Phi\|$, which contradicts the assumptions. ■

Recall that a matrix function $\Upsilon \in H^\infty(\mathbb{M}_{m,n})$ is called *left invertible in* H^∞ if there exists a matrix function $\Omega \in H^\infty(\mathbb{M}_{n,m})$ such that $\Omega(\zeta)\Upsilon(\zeta) = I_n$ for all $\zeta \in \mathbb{D}$ (I_n is the $n \times n$ identity matrix).

Corollary 4.4. *Under the hypotheses of Theorem 4.1 the column functions $\boldsymbol{v}$ and $\boldsymbol{w}$ are left invertible in H^∞.*

Proof. The result follows immediately from Theorems 4.3 and 3.6.1. ■

Theorem 4.5. *Under the hypotheses of Theorem 4.1 the matrix functions Θ and Ξ are left invertible in H^∞.*

Proof. As in the proof of Theorem 4.3 it is sufficient to prove that Θ is left invertible in H^∞. By Theorem 4.3, $T_{\overline{\boldsymbol{v}}}$ is left invertible. Since $\overline{\boldsymbol{v}}$ takes isometric values, by Theorem 3.4.4, $\|H_{\overline{\boldsymbol{v}}}\| < 1$. Clearly,

$$\|H_{\overline{\boldsymbol{v}}}\| = \|H_{\overline{V}}\| < 1.$$

By Corollary 1.5, $T_{\overline{V}}$ has dense range and trivial kernel. By Theorem 4.4.11,

$$\|H_{\overline{V}}\| = \|H_{V^{\mathrm{t}}}\|.$$

Clearly,

$$\|H_{V^{\mathrm{t}}}\| = \|H_{\overline{\Theta}}\| < 1.$$

Since $\overline{\Theta}$ takes isometric values, again by Theorem 3.4.4, the Toeplitz operator $T_{\overline{\Theta}}$ is left invertible, and so by Theorem 3.6.1, Θ is left invertible in H^∞. ■

Remark. It is easy to see that if $V = \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix}$ is a thematic matrix function such that the Toeplitz operator T_V is invertible, then Θ is left invertible in H^∞. Indeed, in this case $T_{\overline{\Theta}}$ is left invertible and by Theorem 3.6.1, Θ is left invertible in H^∞.

Corollary 4.6. *Under the hypotheses of Theorem 4.1 the Toeplitz operators T_V and $T_{V^{\mathrm{t}}}$ on $H^2(\mathbb{C}^n)$ and the Toeplitz operators T_W and $T_{W^{\mathrm{t}}}$ on $H^2(\mathbb{C}^m)$ are invertible.*

Proof. As usual it is sufficient to prove the invertibility of T_V and $T_{V^{\mathrm{t}}}$. Since V is unitary-valued, it follows from Corollary 3.4.5 that T_V is invertible if and only if $\|H_V\| < 1$ and $\|H_{V^*}\| < 1$. Clearly, $\|H_V\| = \|H_{\overline{\Theta}}\|$ and $\|H_{V^*}\| = \|H_{\overline{\boldsymbol{v}}}\|$. It has been shown in the proof of Theorem 4.5 that

$$\|H_{\overline{\boldsymbol{v}}}\| = \|H_{\overline{\Theta}}\| < 1.$$

This proves the invertibility of T_V. The function V^{t} is also unitary-valued, and so by Corollary 3.4.5, it is sufficient to show that $\|H_{V^{\mathrm{t}}}\| < 1$ and $\|H_{\overline{V}}\| < 1$. The now result follows from the obvious equalities

$$\|H_{V^{\mathrm{t}}}\| = \|H_V\|, \quad \|H_{\overline{V}}\| = \|H_{V^*}\|. \quad ■$$

Now we are in a position to prove Theorem 4.1.

Proof of Theorem 4.1. We are going to use formula (4.3) for the essential norm of an operator. Let $\{\xi_j\}_{j\ge 0}$ be a sequence of functions in $H^2(\mathbb{C}^{n-1})$ such that $\|\xi_j\|_2 = 1$ and $\xi_j \to \mathbb{O}$ weakly. Put $\eta_j = H_\Psi \xi_j$. We are going to construct a sequence $\{\xi_j^\#\}_{j\ge 0}$ in $H^2(\mathbb{C}^n)$ such that the $\xi_j^\#$ and $\eta_j^\# \stackrel{\text{def}}{=} H_\Phi \xi_j^\#$ satisfy

$$\frac{\xi_j^\#}{\|\xi_j^\#\|_2} \to \mathbb{O} \quad \text{weakly} \quad \text{and} \quad \frac{\|\eta_j^\#\|_2}{\|\xi_j^\#\|_2} \ge \|\eta_j\|_2, \quad j \in \mathbb{Z}_+. \tag{4.6}$$

Then if we put $x_j = \dfrac{\xi_j^\#}{\|\xi_j^\#\|_2}$ in (4.3), it would follow from (4.3) that $\|H_\Psi\|_{\mathrm{e}} \le \|H_\Phi\|_{\mathrm{e}}$.

By Theorem 4.5, there exist matrix functions $A \in H^\infty(\mathbb{M}_{n-1,n})$ and $B \in H^\infty(\mathbb{M}_{m-1,m})$ such that $A\Theta = \boldsymbol{I}_{n-1}$ and $B\Xi = \boldsymbol{I}_{m-1}$, where $\boldsymbol{I}_k$ denotes the constant matrix function identically equal to I_k.

We need the following fact.

Lemma 4.7. *Let $\eta \in H^2_-(\mathbb{C}^{m-1})$ and let*

$$\chi = -\mathbb{P}_+ \boldsymbol{w}^{\mathrm{t}} B^* \eta.$$

Then

$$W^* \begin{pmatrix} \chi \\ \eta \end{pmatrix} \in H^2_-(\mathbb{C}^m).$$

Proof. Since W is unitary-valued, we have

$$\boldsymbol{I}_m = W^*W = \overline{\boldsymbol{w}}\boldsymbol{w}^{\mathrm{t}} + \Xi\Xi^*,$$

and so

$$\Xi = \Xi(B\Xi)^* = \Xi\Xi^*B^* = (\boldsymbol{I}_m - \overline{\boldsymbol{w}}\boldsymbol{w}^{\mathrm{t}})B^*.$$

Hence,

$$\begin{aligned} W^*\begin{pmatrix} \chi \\ \eta \end{pmatrix} &= \begin{pmatrix} \boldsymbol{w} & \Xi \end{pmatrix}\begin{pmatrix} \chi \\ \eta \end{pmatrix} \\ &= \overline{\boldsymbol{w}}\chi + (\boldsymbol{I}_m - \overline{\boldsymbol{w}}\boldsymbol{w}^{\mathrm{t}})B^*\eta = B^*\eta + \overline{\boldsymbol{w}}(\chi + \boldsymbol{w}^{\mathrm{t}}B^*\eta). \end{aligned}$$

Clearly, $B^*\eta \in H^2_-(\mathbb{C}^m)$ and $\overline{\boldsymbol{w}}(\chi + \boldsymbol{w}^{\mathrm{t}}B^*\eta) = \overline{\boldsymbol{w}}\mathbb{P}_-\boldsymbol{w}^{\mathrm{t}}B^*\eta \in H^2_-(\mathbb{C}^m)$, which proves the result. ■

Let us complete the proof of Theorem 4.1. We apply Lemma 4.7 to the functions η_j, $j \ge 0$. We obtain a sequence of scalar functions $\{\chi_j\}_{j\ge 0}$ in H^2 such that $\chi_j \to \mathbb{O}$ weakly and

$$W^*\begin{pmatrix} \chi_j \\ \eta_j \end{pmatrix} \in H^2_-(\mathbb{C}^m). \tag{4.7}$$

Put

$$\xi_j^{\#} = A^{\mathrm{t}}\xi_j + q_j\boldsymbol{v}.$$

The scalar functions $q_j \in H^2$ will be chosen later.

We have

$$\begin{pmatrix} t_0u & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^*\xi_j^{\#} = \begin{pmatrix} t_0uq_j + t_0u\boldsymbol{v}^*A^{\mathrm{t}}\xi_j \\ \Psi\xi_j \end{pmatrix}. \tag{4.8}$$

By Theorem 4.2, the Toeplitz operator T_u is onto, and so we can pick q_j as follows:

$$q_j = t_0^{-1}\big(T_u\big|(\operatorname{Ker} T_u)^{\perp}\big)^{-1}(\chi_j - t_0\mathbb{P}_+u\boldsymbol{v}^*A^{\mathrm{t}}\xi_j).$$

Then $q_j \to \mathbb{O}$ weakly and

$$\mathbb{P}_+(t_0uq_j + t_0u\boldsymbol{v}^*A^{\mathrm{t}}\xi_j) = \chi_j.$$

It follows that $\xi_j^{\#} \to \mathbb{O}$ weakly.

Let us show that the $\xi_j^{\#}$ and $\eta_j^{\#}$ satisfy (4.6). Since that $\xi_j^{\#} \to \mathbb{O}$ weakly, to prove that $\dfrac{\xi_j^{\#}}{\|\xi_j^{\#}\|} \to \mathbb{O}$ weakly, it is sufficient to show that the norms $\|\xi_j^{\#}\|$ are separated away from 0. We have

$$\|\xi_j^{\#}\|_2^2 = \|V^*\xi_j^{\#}\|_2^2 = \left\|\begin{pmatrix} q_j + \boldsymbol{v}^*A^{\mathrm{t}}\xi_j \\ \xi_j \end{pmatrix}\right\|_2^2 = \|q_j + \boldsymbol{v}^*A^{\mathrm{t}}\xi_j\|_2^2 + \|\xi_j\|_2^2 \ge 1. \tag{4.9}$$

Since $\eta_j = H_\Psi\xi_j$, we have $\Psi\xi_j - \eta_j \in H^2(\mathbb{C}^{m-1})$. It is easy to see from the definition of W that

$$W^*\begin{pmatrix} \mathbb{O} \\ \Psi\xi_j - \eta_j \end{pmatrix} \in H^2(\mathbb{C}^m).$$

It follows now from (4.8) that

$$\begin{aligned}\eta_j^{\#} &= \mathbb{P}_-\Phi\xi_j^{\#} = \mathbb{P}_- W^*\begin{pmatrix} t_0 u & \mathbb{O} \\ \mathbb{O} & \Psi\end{pmatrix} V^*\xi_j^{\#} \\ &= \mathbb{P}_- W^*\begin{pmatrix} t_0 u q_j + t_0 u \boldsymbol{v}^* A^{\mathrm{t}}\xi_j \\ \Psi\xi_j\end{pmatrix} \\ &= \mathbb{P}_- W^*\begin{pmatrix} \chi_j + \omega_j \\ \eta_j\end{pmatrix},\end{aligned}$$

where

$$\omega_j \stackrel{\mathrm{def}}{=} \mathbb{P}_-(t_0 u q_j + t_0 u \boldsymbol{v}^* A^{\mathrm{t}}\xi_j).$$

It is easy to see that

$$W^*\begin{pmatrix} \omega_j \\ \mathbb{O}\end{pmatrix} \in H^2_-(\mathbb{C}^m)$$

and it follows from (4.7) that

$$\mathbb{P}_- W^*\begin{pmatrix} \chi_j + \omega_j \\ \eta_j\end{pmatrix} = W^*\begin{pmatrix} \chi_j + \omega_j \\ \eta_j\end{pmatrix}.$$

Hence,

$$\|\eta_j^{\#}\|_2^2 = \|\chi_j + \omega_j\|_2^2 + \|\eta_j\|_2^2 = t_0^2\|q_j + \boldsymbol{v}^* A^{\mathrm{t}}\xi_j\|_2^2 + \|H_\Psi\xi_j\|_2^2.$$

Since $\|H_\Psi\| \le t_0$, this, together with (4.9), yields

$$\frac{\|\eta_j^{\#}\|_2}{\|\xi_j^{\#}\|_2} = \left(\frac{t_0^2\|q_j + \boldsymbol{v}^* A^{\mathrm{t}}\xi_j\|_2^2 + \|H_\Psi\xi_j\|_2^2}{\|q_j + \boldsymbol{v}^* A^{\mathrm{t}}\xi_j\|_2^2 + \|\xi_j\|_2^2}\right)^{1/2} \ge \frac{\|\eta_j\|_2}{\|\xi_j\|_2} = \|\eta_j\|_2.$$

This completes the proof. ■

We can deduce now from Theorem 4.1 the uniqueness of a superoptimal approximation in the case when $\|H_\Phi\|_{\mathrm{e}}$ is less than the smallest nonzero superoptimal singular value of Φ.

Theorem 4.8. *Suppose that $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and $\|H_\Phi\|_{\mathrm{e}}$ is less than the smallest nonzero superoptimal singular value of Φ. Then Φ has a unique superoptimal approximation by bounded analytic matrix functions.*

Proof. If $\min\{m,n\} = 1$, the result follows from Theorem 2.1. Otherwise, let F be a best approximation of Φ by bounded analytic matrix functions. Then by Theorem 2.2, $\Phi - F$ admits a factorization of the form (4.1). Theorem 2.4 reduces the result to the fact that Ψ has a unique superoptimal approximation. Since by Theorem 4.1, $\|H_\Psi\|_{\mathrm{e}}$ is less than the smallest nonzero superoptimal singular value of Ψ, we can iterate this procedure and the result follows by induction on $\min\{m,n\}$. ■

As in the previous section we can prove that if Q is the superoptimal approximation of Φ, then all singular values of $\Phi - Q$ are constant on $\mathbb{T}$.

Theorem 4.9. *Suppose that Φ satisfies the hypotheses of Theorem 4.8. Let Q be the unique superoptimal approximation of Φ by bounded analytic matrix functions. Then*

$$s_j\big((\Phi - Q)(\zeta)\big) = t_j(\Phi), \quad 0 \le j \le \min\{m,n\} - 1, \quad \zeta \in \mathbb{T}.$$

Proof. The result follows by induction from Theorem 2.4. ■

5. Thematic Factorizations and Very Badly Approximable Functions

For a matrix function Φ satisfying the assumptions imposed in the previous section we analyze the algorithm of finding the superoptimal approximation Q described in the previous sections and obtain so-called thematic factorizations of the error function $\Phi - F$. We obtain a characterization of the very badly approximable functions in terms of thematic factorizations. Then we obtain certain sufficient conditions for a matrix function to be very badly approximable that are similar to those obtained in §7.5 in the scalar case. However, unlike the scalar case they are not necessary.

Definition. A matrix function $\Phi \in L^\infty(\mathbb{M}_{m,n})$ is called *very badly approximable* if the zero matrix function is a superoptimal approximation of Φ by bounded analytic matrix functions.

It is easy to see that for a matrix function $\Phi \in L^\infty(\mathbb{M}_{m,n})$ a function $Q \in H^\infty(\mathbb{M}_{m,n})$ is a superoptimal approximation of Φ if and only if $\Phi - Q$ is a very badly approximable matrix function.

Consider first the case when for $r \le \min\{m,n\}$ we have $t_{r-1} > \|H_\Phi\|_{\mathrm{e}}$ and $t_{r-1} > t_r$, and apply successively the procedure described in §4. Recall that the sets $\boldsymbol{\Omega}_j$ are defined in the introduction to Chapter 14. As usual $\boldsymbol{I}_j$ denotes the constant matrix function identically equal to the $j \times j$ identity matrix I_j and $t_j = 0$ for $j \ge \min\{m,n\}$.

Theorem 5.1. *Let $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and $1 \le r \le \min\{m,n\}$. Suppose that $\|H_\Phi\|_{\mathrm{e}} < t_{r-1}$ and $t_r < t_{r-1}$. Suppose that F is a bounded analytic matrix function in $\boldsymbol{\Omega}_{r-1}$. Then $\Phi - F$ admits a factorization*

$$\Phi - F = W_0^* \cdots W_{r-1}^* \begin{pmatrix} t_0 u_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_1 u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & t_{r-1} u_{r-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \Psi \end{pmatrix} V_{r-1}^* \cdots V_0^*, \tag{5.1}$$

in which the V_j and W_j for $j \ge 1$ have the form

$$W_j = \begin{pmatrix} \boldsymbol{I}_j & \mathbb{O} \\ \mathbb{O} & \breve{W}_j \end{pmatrix}, \quad V_j = \begin{pmatrix} \boldsymbol{I}_j & \mathbb{O} \\ \mathbb{O} & \breve{V}_j \end{pmatrix}, \quad 1 \le j \le r - 1, \tag{5.2}$$

the $W_0^t, \breve{W}_j^t, V_0, \breve{V}_j$ *are thematic matrix functions, the* u_j *are unimodular functions such that* T_{u_j} *is Fredholm, and* $\operatorname{ind} T_{u_j} > 0$*,*

$$\Psi \in L^\infty(\mathbb{M}_{m-r,n-r}), \quad \|\Psi\|_{L^\infty} \le t_{r-1}, \quad \text{and} \quad \|H_\Psi\| < t_{r-1}. \tag{5.3}$$

Proof. The result follows by successive application of Theorem 2.2 and Theorem 4.1. ■

Factorizations of the form (5.1) with Ψ satisfying (5.3) are called *partial thematic factorizations.* With such a factorization we associate the *factorization indices* (or *thematic indices*) k_j defined by

$$k_j = \operatorname{ind} T_{u_j}. \tag{5.4}$$

The matrix function Ψ is called the *residual entry* of the partial thematic factorization (5.1).

Note that by Theorems 3.1 and 3.2, if $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$, then the unimodular functions u_j in (5.1) belong to QC, and so in this case we can write

$$k_j = -\operatorname{wind} u_j$$

(see §3.3, where it is explained how to define the winding number of an invertible function in $H^\infty + C$).

Suppose now that $\|H_\Phi\|_e$ is less than the smallest nonzero superoptimal singular value of Φ. In this case we can apply the algorithm described in the previous section to find the unique superoptimal approximation Q and obtain a thematic factorization of $\Phi - Q$. To be more precise, we mean the following.

Theorem 5.2. *Suppose that* $\Phi \in L^\infty(\mathbb{M}_{m,n})$ *and* $t_0, t_1, \cdots, t_{r-1}$ *are all nonzero superoptimal singular values of* Φ*. Let* $Q \in H^\infty(\mathbb{M}_{m,n})$ *be the superoptimal approximation of* Φ *by bounded analytic matrix functions. Then* $\Phi - Q$ *admits a factorization of the form*

$$\Phi - F = W_0^* \cdots W_{r-1}^* \begin{pmatrix} t_0u_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_1u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & t_{r-1}u_{r-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \end{pmatrix} V_{r-1}^* \cdots V_0^*, \tag{5.5}$$

in which the V_j *and* W_j*,* $0 \le j \le r-1$*, have the form* (5.2)*, the* $W_0^t, \breve{W}_j^t, V_0, \breve{V}_j$ *are thematic matrix functions, the* u_j *are unimodular functions such that* T_{u_j} *is Fredholm, and* $\operatorname{ind} T_{u_j} > 0$.

Note that the lower right entry of the diagonal matrix function on the right-hand side of (5.5) has size $(m-r) \times (n-r)$. As usual if $m = r$ or $n = r$, this means that the corresponding zero row or column does not exist.

Proof. We proceed in exactly the same way as in the proof of Theorem 5.1. We can arrive at the situation when the corresponding lower right entry

Ψ of the diagonal matrix function is zero. In this case the best approximation of Ψ is the zero function and the process terminates. Otherwise, we arrive at the situation when $m = r$ or $n = r$, in which case the process also terminates. In both cases we obtain a factorization of the form (5.5). ■

Factorizations of the form (5.5) are called *thematic factorizations.* The corresponding thematic indices are defined by (5.4).

Theorem 5.2 shows that a very badly approximable function satisfying the assumptions of Theorem 5.2 admits a thematic factorization. The following result says that the converse is also true.

Theorem 5.3. *Let Φ be a matrix function of the form*

$$\Phi = W_0^* \cdots W_{r-1}^* \begin{pmatrix} s_0u_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & s_1u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & s_{r-1}u_{r-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \end{pmatrix} V_{r-1}^* \cdots V_0^*,$$

where the V_j and W_j, $0 \le j \le r-1$, have the form (5.2)*, the $W_0^t, \breve{W}_j^t, V_0, \breve{V}_j$ are thematic matrix functions,*

$$s_0 \ge s_1 \ge \cdots \ge s_{r-1},$$

the u_j are unimodular functions of the form $u_j = \bar{z}\bar{\vartheta}_j\bar{h}_j/h_j$, ϑ_j being an inner function and h_j being an outer function in H^2. Then Φ is very badly approximable and $t_j(\Phi) = s_j$, $0 \le j \le r-1$.

Proof. By Theorem 2.5, Φ is badly approximable and $s_0 = t_0(\Phi) = \|H_\Phi\|$. The rest of the theorem follows by induction from Theorem 2.4 ■.

In the scalar case there is a geometric description of the badly approximable functions φ under the condition $\|H_\varphi\|_e < \|H_\varphi\|$ (see Theorem 7.5.5). Namely such a function φ is badly approximable if and only if $\|f\| = \text{const}$ and T_φ is Fredholm with positive index. For $\varphi \in H^\infty + C$ the last condition is equivalent to the fact that $\operatorname{wind} \varphi < 0$. Note that $\operatorname{ind} T_\varphi > 0$ if and only if $T_{z\varphi}$ has dense range in H^2.

It follows from Theorem 4.9 that if $\Phi \in L^\infty(\mathbb{M}_{m,n})$ satisfies the condition that $\|H_\Phi\|_e$ is less than the smallest nonzero superoptimal singular value of Φ, then all singular values of $\Phi(\zeta)$ are constant. Moreover,

$$s_j(\Phi(\zeta)) = t_j(\Phi), \quad \text{for almost all} \quad \zeta \in \mathbb{T}.$$

It turns out that if in addition to that $m \le n$ and $t_{m-1}(\Phi) \ne 0$, then the Toeplitz operator $T_{z\Phi}$ has dense range in $H^2(\mathbb{C}^m)$.

Theorem 5.4. *Let Φ be a matrix function in $L^\infty(\mathbb{M}_{m,n})$ such that $\|H_\Phi\|_e$ is less than the smallest nonzero superoptimal singular value of Φ. Suppose that Φ is very badly approximable, $m \le n$, and $t_{m-1}(\Phi) > 0$. Then the Toeplitz operator $T_{z\Phi} : H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^m)$ has dense range.*

Proof. We argue by induction on m. Suppose that $m = 1$. Without loss of generality we may assume that $\|H_\Phi\| = 1$. Let $\boldsymbol{f} \in H^2(\mathbb{C}^n)$ be a maximizing vector of H_Φ and let $\boldsymbol{g} = \bar{z}\overline{H_\Phi \boldsymbol{f}}$. By Theorem 2.2.3, $\Phi\boldsymbol{f} = \bar{z}\bar{\boldsymbol{g}}$ and

$$|\boldsymbol{g}(\zeta)| = \|\Phi(\zeta)\|_{\mathbb{M}_{1,n}}\|\boldsymbol{f}(\zeta)\|_{\mathbb{C}^n} = \|\Phi^*(\zeta)\|_{\mathbb{C}^n}\|\boldsymbol{f}(\zeta)\|_{\mathbb{C}^n} = \|\boldsymbol{f}(\zeta)\|_{\mathbb{C}^n}.$$

We are going to use the following elementary fact. Suppose that $x, y \in \mathbb{C}^n$ and $c \in \mathbb{C}$ satisfy

$$x^* y = c, \quad |c| = \|x\| \cdot \|y\|.$$

Then $x^* = c\|y\|^{-2}y^*$.

We apply this fact for $x = \Phi(\zeta)^*$, $y = \boldsymbol{f}(\zeta)$, and $c = \overline{\zeta\boldsymbol{g}(\zeta)}$ and obtain

$$\Phi = (\boldsymbol{f}^*\boldsymbol{f})^{-1}\bar{z}\bar{\boldsymbol{g}}\boldsymbol{f}^*. \tag{5.6}$$

Consider the following factorizations:

$$\boldsymbol{f} = \vartheta_1 h \boldsymbol{v}, \quad \boldsymbol{g} = \vartheta_2 h,$$

where ϑ_1 and ϑ_2 are scalar inner functions, h is a scalar outer function in H^2, and $\boldsymbol{v}$ is an inner and co-outer function in $H^\infty(\mathbb{C}^n)$.

It follows from (5.6) that

$$\Phi = \bar{z}\bar{\vartheta}_1\bar{\vartheta}_2\frac{\bar{h}}{h}\boldsymbol{v}^*. \tag{5.7}$$

We have to prove that $\operatorname{Ker} T^*_{z\Phi} = \{\mathbb{O}\}$. Suppose that $\chi \in \operatorname{Ker} T^*_{z\Phi}$. Then $\bar{z}\Phi^*\chi \in H^2_-$, and it follows from (5.7) that

$$\vartheta_1\vartheta_2(h/\bar{h})\chi\boldsymbol{v} \in H^2_-,$$

and so all entries of the column function $\chi\boldsymbol{v}$ belong to $\operatorname{Ker} T_{\vartheta_1\vartheta_2(h/\bar{h})}$. By Theorem 4.4.10, $\operatorname{Ker} T_{\vartheta_1\vartheta_2(h/\bar{h})} = \mathbb{O}$. Since $\boldsymbol{v} \neq \mathbb{O}$, it follows that $\chi = \mathbb{O}$. This proves the theorem for $m = 1$.

Suppose now that $m > 1$. Then, by Theorem 2.2, Φ admits a factorization of the form

$$\Phi = W^* \begin{pmatrix} t_0 u & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^*,$$

where V and W^{t} are thematic matrix functions, u is a unimodular function of the form $u = \bar{z}\bar{\vartheta}\bar{h}/h$ with inner ϑ and outer $h \in H^2$, and Ψ is a very badly approximable function of size $(m-1)\times(n-1)$. By Theorems 2.4 and 4.1, $\|H_\Psi\|_{\mathrm{e}}$ is less than the smallest nonzero superoptimal singular value of Ψ. Thus by the inductive hypotheses, $T_{z\Psi}$ has dense range in $H^2(\mathbb{C}^{m-1})$.

Let

$$V = \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix}, \quad W = \begin{pmatrix} \boldsymbol{w} & \Xi \end{pmatrix}^{\mathrm{t}}.$$

We want to show that $\operatorname{Ker} T^*_{z\Phi} = \{\mathbb{O}\}$. Suppose that $\xi \in \operatorname{Ker} T^*_{z\Phi}$. Then $\bar{z}\Phi^*\xi \in H^2_-(\mathbb{C}^n)$, and so

$$\boldsymbol{v}^*\bar{z}\Phi^*\xi = \boldsymbol{v}^*\bar{z}V\begin{pmatrix} t_0\bar{u} & \mathbb{O} \\ \mathbb{O} & \Psi^* \end{pmatrix}W\xi = t_0\bar{z}\begin{pmatrix} \bar{u} & \mathbb{O} \end{pmatrix}W\xi = t_0\bar{z}\bar{u}\boldsymbol{w}^{\mathrm{t}}\xi \in H^2_-.$$

Hence, $\boldsymbol{w}^{\mathrm{t}}\xi \in \operatorname{Ker} T_{\bar z\bar u}$. Again by Theorem 4.4.10, $\operatorname{Ker} T_{\bar z\bar u} = \{\mathbb{O}\}$, and so $\boldsymbol{w}^{\mathrm{t}}\xi = \mathbb{O}$. We have

$$\xi = W^*W\xi = W^*\begin{pmatrix} \boldsymbol{w}^{\mathrm{t}} \\ \Xi^* \end{pmatrix}\xi = \begin{pmatrix} \boldsymbol{w} & \Xi \end{pmatrix}\begin{pmatrix} \mathbb{O} \\ \Xi^*\xi \end{pmatrix} = \Xi\Xi^*\xi.$$

Put $\eta = \Xi^*\xi \in L^2(\mathbb{C}^{n-1})$. Then $\xi = \Xi\eta$. By Lemma 1.4, $\eta \in H^2(\mathbb{C}^{n-1})$.

We have

$$\begin{aligned} \bar z\Phi^*\xi &= \bar z V\begin{pmatrix} t_0\bar u & \mathbb{O} \\ \mathbb{O} & \Psi^* \end{pmatrix} W\Xi\eta \\ &= \bar z V\begin{pmatrix} t_0\bar u & \mathbb{O} \\ \mathbb{O} & \Psi^* \end{pmatrix}\begin{pmatrix} \mathbb{O} \\ \eta \end{pmatrix} \\ &= \bar z V\begin{pmatrix} \mathbb{O} \\ \Psi^*\eta \end{pmatrix} = \bar z\overline{\Theta}\Psi^*\eta. \end{aligned}$$

By the assumption, $\bar z\overline{\Theta}\Psi^*\eta \in H^2_-(\mathbb{C}^m)$. Since Θ is co-outer, it follows easily from Lemma 1.4 that $\bar z\Psi^*\eta \in H^2_-(\mathbb{C}^{m-1})$, i.e., $\eta \in \operatorname{Ker} T^*_{z\Psi}$. By the inductive hypothesis, $\eta = \mathbb{O}$, and so $\xi = \mathbb{O}$. ■

It is easy to see that the same reasoning proves the following more general fact.

Theorem 5.5. *Let Φ be a very badly approximable matrix function in $L^\infty(\mathbb{M}_{m,n})$ such that $\|H_\Phi\|_{\mathrm{e}}$ is less than the smallest nonzero superoptimal singular value of Φ. If $f \in \operatorname{Ker} T_{\bar z\Phi^*}$, then $\Phi^* f = \mathbb{O}$.*

Remark. It will follow from the results of §12 that if $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$ in the hypotheses of Theorem 5.4, then $T_{z\Phi} : H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^m)$ is onto.

However, unlike the scalar case the hypotheses of Theorem 5.4 together with the condition that all $s_j(\Phi)$ are constant on $\mathbb{T}$ do not guarantee that Φ is very badly approximable.

Example. Let $\boldsymbol{e} : \mathbb{T} \to \mathbb{C}^2$ be a continuous function such that $\|\boldsymbol{e}(\zeta)\|_{\mathbb{C}^2} = 1$, $\zeta \in \mathbb{T}$. Let $0 < \alpha < 1$ and let

$$\Phi(\zeta) = \bar\zeta(\alpha I_2 + (1-\alpha)\boldsymbol{e}(\zeta)\boldsymbol{e}^*(\zeta)), \quad \zeta \in \mathbb{T}.$$

Then it is easy to see that Φ is continuous,

$$s_0(\Phi(\zeta)) = \|\Phi(\zeta)\|_{\mathbb{M}_{2,2}} = 1, \quad \text{and} \quad s_1(\Phi(\zeta)) = \alpha.$$

Let us show that $\operatorname{Ker} T_{z\Phi} = \{\mathbb{O}\}$. Suppose that $\varphi \in \operatorname{Ker} T^*_{z\Phi}$. Then

$$0 = (T^*_{z\Phi}\varphi, \varphi) = (\varphi, z\Phi\varphi) = \alpha\|\varphi\|_2^2 + (1-\alpha)\|\boldsymbol{e}^*\varphi\|_2^2.$$

Thus $\varphi = \mathbb{O}$, and so $\operatorname{Ker} T_{z\Phi} = \{\mathbb{O}\}$. However, for suitable $\boldsymbol{e}$, Φ is not even badly approximable. Indeed, if Φ is badly approximable and f is a maximizing vector of H_Φ, then by Theorem 2.2.3, $f(\zeta)$ is a maximizing vector of $\Phi(\zeta)$ for almost all $\zeta \in \mathbb{T}$, and so $f(\zeta) = \chi(\zeta)\boldsymbol{e}(\zeta)$ almost everywhere on

$\mathbb{T}$ for some scalar function χ. But we can choose $\boldsymbol{e}$ so that $\chi\boldsymbol{e} \notin H^2(\mathbb{C}^2)$ for any scalar function χ. For example, we can take

$$\boldsymbol{e} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} \\ \xi \end{pmatrix},$$

where ξ is a continuous unimodular function on $\mathbb{T}$ such that $\operatorname{Ker} H_\xi = \{\mathbb{O}\}$.

Such a function ξ can easily be constructed. For example, if ξ is a continuous nonconstant unimodular function such that $\xi(\zeta) = 1$ for $\operatorname{Re}\zeta \ge 0$ but ξ is not constant. Let us show that $\operatorname{Ker} H_\xi = \{\mathbb{O}\}$. Indeed, if $\psi \in \operatorname{Ker} H_\xi$ and $\psi \ne \mathbb{O}$, then $\xi\psi \in H^2$ and since ξ is unimodular, the H^2 functions ψ and $\xi\psi$ have the same moduli on $\mathbb{T}$. Thus they can be factorized as follows:

$$\psi = \vartheta_1 h, \quad \xi\psi = \vartheta_2 h,$$

for some outer function $h \in H^2$ and inner functions ϑ_1 and ϑ_2. Then $\vartheta_1(\zeta) = \vartheta_2(\zeta)$ for $\operatorname{Re}\zeta \ge 0$ but $\vartheta_1 \ne \vartheta_2$, which is impossible for H^∞ functions.

We can associate with a matrix function $\Phi \in L^\infty(\mathbb{M}_{m,n})$ another numerical sequence $\tau_j = \tau_j(\Phi)$, $0 \le j < \min\{m, n\}$:

$$\tau_j = \inf\{\|H_{\Phi - G}\| : \ G \in L^\infty(\mathbb{M}_{m,n}), \ \operatorname{rank} g(\zeta) \le j \text{ a.e. on } \mathbb{T}\}.$$

Clearly, $\tau_0 = t_0$. Suppose that $\|H_\Phi\|_{\mathrm{e}}$ is less than the smallest nonzero superoptimal singular value of Φ. It follows from Theorem 5.2 that $\tau_j(\Phi) \le t_j(\Phi)$. Indeed, without loss of generality we may assume that Φ is very badly approximable (otherwise we can subtract from Φ its superoptimal approximation). Then Φ admits a factorization of the form (5.5). Put

$$G = W_0^* \cdots W_{r-1}^* \begin{pmatrix} t_0 u_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & t_1 u_1 & \cdots & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & t_{j-1}u_{j-1} & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \end{pmatrix} V_{r-1}^* \cdots V_0^*.$$

Clearly, $\operatorname{rank} G(\zeta) = j$ for almost all $\zeta \in \mathbb{T}$. It is also easy to see that $\|\Phi - G\|_{L^\infty} = t_j$, and so $\tau_j \le t_j$.

However, the following example shows that it is not true in general that $\tau_j = t_j$.

Example. Let

$$\Phi = \begin{pmatrix} \bar{z} & \mathbb{O} \\ \mathbb{O} & \alpha\bar{z} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\bar{z}}{\sqrt{2}} \\ \frac{-z}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},$$

where $0 < \alpha < 1$. Clearly, the Φ is represented in the form of a thematic factorization. Hence, $t_0(\Phi) = 1$ and $t_1(\Phi) = \alpha$. Now put

$$G = \begin{pmatrix} \bar{z} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\bar{z}}{\sqrt{2}} \\ \frac{-z}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Clearly, $\operatorname{rank} G(\zeta) = 1$ for $\zeta \in \mathbb{T}$. It is easy to see that

$$\Phi - G = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \frac{-\alpha}{\sqrt{2}} & \frac{\alpha\bar{z}}{\sqrt{2}} \end{pmatrix}.$$

Let

$$F = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \frac{-\alpha}{\sqrt{2}} & \mathbb{O} \end{pmatrix} \in H^\infty(\mathbb{M}_{2,2}).$$

Then

$$\|H_{\Phi - G}\| \le \|\Phi - G - F\|_{L^\infty} = \frac{\alpha}{\sqrt{2}} < \alpha.$$

6. Admissible and Superoptimal Weights

Let $\mathfrak{W} \in L^\infty(\mathbb{M}_{n,n})$ be a *matrix weight*, i.e., a bounded matrix function whose values are nonnegative self-adjoint matrices. Given $\Phi \in L^\infty(\mathbb{M}_{m,n})$, we say that $\mathfrak{W}$ is *admissible* for the Hankel operator H_Φ if

$$\|H_\Phi \xi\|^2 \le (\mathfrak{W}\xi, \xi) \stackrel{\text{def}}{=} \int_{\mathbb{T}} (\mathfrak{W}(\zeta)\xi(\zeta), \xi(\zeta)) d\boldsymbol{m}(\zeta), \quad \xi \in H^2(\mathbb{C}^n).$$

If $\mathfrak{W} = c\boldsymbol{I}_n$ for $c > 0$, then it is easy to see that $\mathfrak{W}$ is admissible if and only if $\|H_\Phi\| \le \sqrt{c}$. The following theorem generalizes the matrix version of the Nehari theorem.

Theorem 6.1. *Suppose that $\mathfrak{W} \in L^\infty(\mathbb{M}_{n,n})$ is an admissible weight for H_Φ, $\Phi \in L^\infty(\mathbb{M}_{m,n})$. Then there exists $F \in H^\infty(\mathbb{M}_{m,n})$ such that $(\Phi - F)^*(\Phi - F) \le \mathfrak{W}$.*

In other words Theorem 6.1 says that the Hankel operator H_Φ has a symbol $\Phi_\#$ satisfying $\Phi_\#^*\Phi_\# \le \mathfrak{W}$. In this case we say that $\Phi_\#$ is *dominated by* the admissible weight $\mathfrak{W}$.

Proof. Put $\mathfrak{W}_\varepsilon = \mathfrak{W} + \varepsilon\boldsymbol{I}_n$. By Theorem 13.3.1, there exists an invertible matrix function G_ε in $H^\infty(\mathbb{M}_{n,n})$ such that $G_\varepsilon^* G_\varepsilon = \mathfrak{W}_\varepsilon$. Clearly, the weight $\mathfrak{W}_\varepsilon$ is admissible, and so

$$\|H_\Phi \xi\|_2 \le \|G_\varepsilon \xi\|_2, \quad \xi \in H^2(\mathbb{C}^n),$$

which is equivalent to the fact that

$$\|H_\Phi G_\varepsilon^{-1} \xi\|_2 \le \|\xi\|_2, \quad \xi \in H^2(\mathbb{C}^n).$$

Since G_ε is invertible in $H^\infty(\mathbb{M}_{n,n})$, $H_\Phi G_\varepsilon^{-1}$ is a bounded Hankel operator. By Theorem 2.2.2, this Hankel operator has symbol $\Psi_\varepsilon \in L^\infty(\mathbb{M}_{m,n})$ such that $\|\Psi\|_{L^\infty} \le 1$. Then $\Phi_\varepsilon = \Psi G_\varepsilon$ is a symbol of H_Φ and

$$\Phi_\varepsilon^* \Phi_\varepsilon = G_\varepsilon^* \Psi_\varepsilon^* \Psi_\varepsilon G_\varepsilon \le G_\varepsilon^* G_\varepsilon = \mathfrak{W}_\varepsilon = \mathfrak{W} + \varepsilon \boldsymbol{I}_n.$$

It remains to choose a sequence $\{\varepsilon_j\}_{j\ge 0}$ that converges to 0 and such that the sequence $\{\Phi_{\varepsilon_j}\}_{j\ge 0}$ converges in the weak-$*$ topology to a matrix function, say $\Phi_\#$. Clearly, $\Phi_\#$ is a symbol of H_Φ dominated by $\mathfrak{W}$. ■

Definition. Let $\mathfrak{W}$ be an admissible weight for a Hankel operator H_Φ. Consider the numbers

$$s_j^\infty(\mathfrak{W}) \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} s_j\big(\mathfrak{W}(\zeta)\big), \quad 0 \le j \le n-1.$$

$\mathfrak{W}$ is called a *superoptimal weight for* H_Φ if it lexicographically minimizes the numbers $s_0^\infty(\mathfrak{W}), s_1^\infty(\mathfrak{W}), \cdots, s_{n-1}^\infty(\mathfrak{W})$ among all admissible weights, i.e.,

$$s_0^\infty(\mathfrak{W}) = \min\{s_0^\infty(\mathfrak{V}):\ \mathfrak{V} \text{ is admissible}\},$$

$$s_1^\infty(\mathfrak{W}) = \min\{s_1^\infty(\mathfrak{V}):\ \mathfrak{V} \text{ is admissible}, s_0^\infty(\mathfrak{V}) \text{ is minimal possible}\}, \text{ etc.}$$

The following result shows that a superoptimal weight always exists.

Theorem 6.2. *Let* $\Phi \in L^\infty(\mathbb{M}_{m,n})$ *and let* $F \in H^\infty(\mathbb{M}_{m,n})$ *be a superoptimal approximation of* Φ*. Then* $(\Phi - F)^*(\Phi - F)$ *is a superoptimal weight for* H_Φ*.*

Recall that a superoptimal approximation always exists (see the Introduction to this chapter).

Proof. Put $\mathfrak{W} = (\Phi - F)^*(\Phi - F)$. Clearly,

$$s_j\big((\mathfrak{W})(\zeta)\big) = 0, \quad j \le \min\{m,n\}.$$

Suppose $\mathfrak{W}$ is not superoptimal. Then there exists an admissible weight $\mathfrak{V}$ such that for some j_0, $0 \le j_0 \le \min\{m,n\} - 1$,

$$s_j^\infty(\mathfrak{V}) = s_j^\infty(\mathfrak{W}), \quad 0 \le j \le j_0, \quad \text{and} \quad s_{j_0}^\infty(\mathfrak{V}) < s_{j_0}^\infty(\mathfrak{W}). \tag{6.1}$$

By Theorem 6.1, there exists $G \in H^\infty(\mathbb{M}_{m,n})$ such that $\Phi - G$ is dominated by $\mathfrak{V}$, i.e., $(\Phi - G)^*(\Phi - G) \le (\Phi - F)^*(\Phi - F)$. It follows now from (6.1) that

$$s_j\big((\Phi - G)(\zeta)\big) \le \sup_{\zeta\in\mathbb{T}} s_j\big((\Phi - F)(\zeta)\big), \quad 0 \le j \le j_0,$$

while

$$\sup_{\zeta\in\mathbb{T}} s_{j_0}\big((\Phi - G)(\zeta)\big) < \sup_{\zeta\in\mathbb{T}} s_{j_0}\big((\Phi - F)(\zeta)\big),$$

which contradicts the fact that F is a superoptimal approximation of Φ. ■

The following theorem says that if Φ satisfies the hypotheses of Theorem 4.8, then there is a unique superoptimal weight for H_Φ.

Theorem 6.3. *Let Φ be a matrix function in $L^\infty(\mathbb{M}_{m,n})$ such that $\|H_\Phi\|_{\mathrm{e}}$ is less than the smallest nonzero superoptimal singular value of Φ. Then H_Φ has a unique superoptimal weight.*

Proof. Let F be the unique superoptimal approximation of Φ by bounded analytic matrix functions. By Theorem 6.2, $\mathfrak{W} \stackrel{\text{def}}{=} (\Phi - F)^*(\Phi - F)$ is a superoptimal weight for H_Φ. By Theorem 4.9,

$$s_j\big(\mathfrak{W}(\zeta)\big) = s_j^\infty(\mathfrak{W}) = t_j^2(\Phi), \quad \zeta \in \mathbb{T}, \quad 0 \le j \le n-1$$

(if $\min\{m,n\} \le j \le n-1$, we assume that $t_j(\Phi) = 0$). Suppose that $\mathfrak{V}$ is another superoptimal weight. Then

$$s_j\big(\mathfrak{V}(\zeta)\big) \le t_j^2(\Phi), \quad 0 \le j \le \ n-1. \tag{6.2}$$

By Theorem 6.1, there exists a matrix function G in $H^\infty(\mathbb{M}_{m,n})$ such that $(\Phi - G)^*(\Phi - G) \le \mathfrak{V}$. It follows now from (6.2) that

$$s_j\big((\Phi - G)(\zeta)\big) \le t_j(\Phi), \quad \zeta \in \mathbb{T}, \quad 0 \le j \le n-1,$$

and so G is a superoptimal approximation of Φ. Hence, $G = F$ and

$$s_j\big(\mathfrak{V}(\zeta)\big) = s_j\big(\mathfrak{W}(\zeta)\big), \quad \zeta \in \mathbb{T}, \quad 0 \le j \le n-1.$$

Since both $\mathfrak{W}(\zeta)$ and $\mathfrak{V}(\zeta)$ are positive semi-definite, it follows that $\mathfrak{V} = \mathfrak{W}$. ■

7. Thematic Indices

In §5 we have defined the notion of factorization indices (or thematic indices) of a thematic (or partial thematic) factorization. A natural question arises of whether the thematic indices depend on the choice of a factorization or whether they are uniquely determined by the matrix function Φ itself.

Example. Consider the matrix function $\Phi = \begin{pmatrix} \bar z^2 & \mathbb{O} \\ \mathbb{O} & \bar z^6 \end{pmatrix}$. It admits the following thematic factorizations:

$$\begin{aligned}
\Phi &= \begin{pmatrix} \mathbf{1} & \mathbb{O} \\ \mathbb{O} & \mathbf{1} \end{pmatrix}\begin{pmatrix} \bar z^2 & \mathbb{O} \\ \mathbb{O} & \bar z^6 \end{pmatrix}\begin{pmatrix} \mathbf{1} & \mathbb{O} \\ \mathbb{O} & \mathbf{1} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\mathbf{1}}{\sqrt2} & -\frac{z^5}{\sqrt2} \\ \frac{\bar z^5}{\sqrt2} & \frac{\mathbf{1}}{\sqrt2} \end{pmatrix}\begin{pmatrix} \bar z & \mathbb{O} \\ \mathbb{O} & \bar z^7 \end{pmatrix}\begin{pmatrix} \frac{\bar z}{\sqrt2} & \frac{\mathbf{1}}{\sqrt2} \\ -\frac{\mathbf{1}}{\sqrt2} & \frac{z}{\sqrt2} \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{O} & \mathbf{1} \\ \mathbf{1} & \mathbb{O} \end{pmatrix}\begin{pmatrix} \bar z^6 & \mathbb{O} \\ \mathbb{O} & \bar z^2 \end{pmatrix}\begin{pmatrix} \mathbb{O} & \mathbf{1} \\ \mathbf{1} & \mathbb{O} \end{pmatrix}.
\end{aligned}$$

The superoptimal singular values of Φ are $t_0 = t_1 = 1$. The indices of the first factorization are 2, 6; the indices of the second are 1, 7; and the indices of the third are 6, 2. Note that for all above factorizations the sum of the indices is 8.

We show in this section that the sum of the indices of a thematic factorization that correspond to the superoptimal singular values equal to a specific value does not depend on the choice of a thematic factorization. As we have observed in the above example such sums are equal to 8. We also obtain similar results for partial thematic factorizations.

In §10 we show that among thematic factorizations one can choose a so-called monotone thematic factorization and we prove in §9 that the indices of a monotone thematic factorization are uniquely determined by the function itself.

Let Φ be a matrix function in $L^\infty(\mathbb{M}_{m,n})$ and let

$$\sigma_0 > \sigma_1 > \cdots > \sigma_l$$

be all *distinct* superoptimal singular values of Φ.

Suppose that $1 \le r \le \min\{m,n\}$, $\|H_\Phi\|_{\mathrm{e}} < t_{r-1}$, and $t_r < t_{r-1}$. Let $F \in \boldsymbol{\Omega}_{r-1}$ (see the Introduction to this chapter for the definition of the sets $\boldsymbol{\Omega}_j$). Consider a partial thematic factorization (5.1) of $\Phi - F$. Recall that the thematic indices k_j, $0 \le j \le r-1$, are defined by (5.4). Consider the following numbers:

$$\nu_\varkappa = \sum_{\{j: t_j = \sigma_\varkappa\}} k_j, \quad \sigma_\varkappa \ge t_{r-1}. \tag{7.1}$$

The following theorem is the main result of the section.

Theorem 7.1. *Suppose that $1 \le r \le \min\{m,n\}$, $\|H_\Phi\|_{\mathrm{e}} < t_{r-1}$, and $t_r < t_{r-1}$. The numbers $\nu_\varkappa$ for $\sigma_\varkappa \ge t_{r-1}$ do not depend on the choice of a partial thematic factorization of $\Phi - F$.*

In particular, if $\|H_\Phi\|_{\mathrm{e}}$ is less than the smallest nonzero superoptimal singular value of Φ and Q is the superoptimal approximation of Φ, then $\Phi - Q$ admits a thematic factorization (5.5). In this case the numbers $\nu_\varkappa$ are defined in (7.1) for $0 \le \varkappa \le l$.

Theorem 7.2. *Suppose that $\|H_\Phi\|_{\mathrm{e}}$ is less than the smallest nonzero superoptimal singular value of Φ and let $Q \in H^\infty(\mathbb{M}_{m,n})$ be the superoptimal approximation of Φ. Then the numbers $\nu_\varkappa$, $0 \le \varkappa \le l$, do not depend on the choice of a thematic factorization of $\Phi - Q$.*

Clearly, Theorem 7.2 is a special case of Theorem 7.1. To prove Theorem 7.1, we need some preparations.

Suppose that $\mathfrak{V}$ is an admissible weight for H_Φ. We say that a nonzero vector function $\xi \in H^2(\mathbb{C}^n)$ is a *maximizing vector for* $\mathfrak{V}$ if

$$\|H_\Phi \xi\|_2^2 = (\mathfrak{V}\xi, \xi).$$

Lemma 7.3. *Let $\mathfrak{V}$ be an admissible weight for H_Φ and let $\xi \in H^2(\mathbb{C}^n)$ be a maximizing vector for $\mathfrak{V}$. Then $\Phi\xi \in H^2_-(\mathbb{C}^m)$, i.e., $H_\Phi \xi = \Phi\xi$.*

Proof. We have

$$(\mathfrak{V}\xi, \xi) = \|H_\Phi \xi\|_2^2 = \|\mathbb{P}_- \Phi\xi\|_2^2 \le \|\Phi\xi\|_2^2 \le (\mathfrak{V}\xi, \xi).$$

It follows that $\|\mathbb{P}_- \xi\|_2 = \|\Phi\xi\|_2$, which implies the result. ■

Note that for $\mathfrak{V} = c\boldsymbol{I}_n$, Lemma 7.3 has been proved in Theorem 2.2.3.

Given an admissible weight $\mathfrak{V}$ for H_Φ, we put

$$\mathcal{E}(\mathfrak{V}) = \{\xi \in H^2(\mathbb{C}^n) : \ \|H_\Phi \xi\|_2^2 = (\mathfrak{V}\xi, \xi)\}.$$

It is easy to see that $\xi \in \mathcal{E}(\mathfrak{V})$ if and only if

$$\xi \in \operatorname{Ker}(T_{\mathfrak{V}} - H_\Phi^* H_\Phi),$$

and so $\mathcal{E}(\mathfrak{V})$ is a closed subspace of $H^2(\mathbb{C}^n)$.

Given $\sigma \ge 0$, we define the function Λ_σ on $\mathbb{R}$ by

$$\Lambda_\sigma(s) = \begin{cases} s, & s \ge \sigma^2, \\ \sigma^2, & s < \sigma^2. \end{cases}$$

Theorem 7.4. *Suppose that Φ satisfies the hypotheses of Theorem 7.1. Then for $\sigma \ge t_{r-1}$*

$$\sum_{\{j: t_j \ge \sigma\}} k_j = \dim \mathcal{E}\big(\Lambda_\sigma((\Phi - F)^*(\Phi - F))\big). \tag{7.2}$$

Clearly, $(\Phi - F)^*(\Phi - F)$ is an admissible weight for H_Φ. Let us first deduce Theorem 7.1 from Theorem 7.4.

Proof of Theorem 7.1. Put $\mathfrak{W} \stackrel{\text{def}}{=} (\Phi - F)^*(\Phi - F)$. It follows immediately from (7.2) that

$$\nu_0 = \dim \mathcal{E}\big(\Lambda_{\sigma_0}(\mathfrak{W})\big), \quad \nu_1 = \dim \ E\big(\Lambda_{\sigma_1}(\mathfrak{W})\big) \ominus \mathcal{E}\big(\Lambda_{\sigma_0}(\mathfrak{W})\big), \quad \text{etc.} \quad ■$$

Proof of Theorem 7.4. It is easy to see that $\mathcal{E}(\Lambda_\sigma(\mathfrak{W}))$ is constant on $(\sigma_{\varkappa+1}, \sigma_\varkappa]$. Thus it is sufficient to prove that if $\sigma_\varkappa \ge t_{r-1}$, then

$$\sum_{\{j: t_j \ge \sigma_\varkappa\}} k_j = \dim \mathcal{E}(\Lambda_{\sigma_\varkappa}(\mathfrak{W})).$$

We argue by induction on r. Suppose that $r = 1$. It is easy to see that $\Lambda_{\sigma_0}(\mathfrak{W}) = t_0^2 \boldsymbol{I}_n$.

We have

$$\Phi - F = W_0^* \begin{pmatrix} t_0 u_0 & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V_0^*,$$

where V_0 and W_0^{t} are thematic matrix functions of the form

$$V_0 = \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix}, \quad W_0^{\mathrm{t}} = \begin{pmatrix} \boldsymbol{w} & \Xi \end{pmatrix}. \tag{7.3}$$

By Theorem 1.8 we can change F so that

$$\|\Psi\|_{L^\infty} = t_1 < t_0. \tag{7.4}$$

Thus we may assume that (7.4) holds.

Since $\operatorname{ind} T_{u_0} = k_0$, it follows that that $\dim \operatorname{Ker} T_{u_0} = k_0$. Let $\chi \in \operatorname{Ker} T_{u_0}$ and $\xi = \chi \boldsymbol{v}$. We have

$$\begin{aligned}(\Phi - F)\xi &= W_0^* \begin{pmatrix} t_0 u_0 & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} \begin{pmatrix} \chi \\ \mathbb{O} \end{pmatrix} \\ &= W_0^* \begin{pmatrix} t_0 u_0 \chi \\ \mathbb{O} \end{pmatrix} = t_0 u_0 \chi \bar{\boldsymbol{w}} \in H^2(\mathbb{C}^m),\end{aligned}$$

since $\chi \in \operatorname{Ker} T_{u_0}$. It is easy to see now that $\xi \in \mathcal{E}(\Lambda_{\sigma_0}(\mathfrak{W}))$, and so $\dim \mathcal{E}(\Lambda_{\sigma_0}(\mathfrak{W})) \ge k_0$.

Let us prove the opposite inequality. Suppose that $\xi \in \mathcal{E}(\Lambda_{\sigma_0}(\mathfrak{W}))$ and $\xi \ne \mathbb{O}$. Then ξ is a maximizing vector of H_Φ. It follows that only the upper entry of the column function $V^*\xi$ is nonzero. Denote this upper entry by χ. We have

$$\xi = VV^*\xi = \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix} \begin{pmatrix} \chi \\ \mathbb{O} \end{pmatrix} = \chi \boldsymbol{v} \in H^2(\mathbb{C}^n).$$

Since $\boldsymbol{v}$ is co-outer, it follows from Lemma 1.4 that $\chi \in H^2$. Again,

$$(\Phi - F)\xi = t_0 u_0 \chi \bar{\boldsymbol{w}}.$$

Hence, $\xi \in \mathcal{E}(\Lambda_{\sigma_0}(\mathfrak{W}))$ if and only if $\chi \in \operatorname{Ker} T_{u_0}$. It follows that $\dim \mathcal{E}(\Lambda_{\sigma_0}(\mathfrak{W})) \le k_0$.

Suppose now that the theorem is proved for $r - 1$. We have

$$\Phi - F = W_0^* \begin{pmatrix} t_0 u_0 & \mathbb{O} \\ \mathbb{O} & \Phi_\circ \end{pmatrix} V_0^*,$$

where $\Phi_\circ$ is an $(m-1) \times (n-1)$ matrix function that has the following partial thematic factorization:

$$\Phi_\circ = \breve{W}_1^* \cdots \begin{pmatrix} \boldsymbol{I}_{r-2} & \mathbb{O} \\ \mathbb{O} & \breve{W}_{r-1}^* \end{pmatrix} D \begin{pmatrix} \boldsymbol{I}_{r-2} & \mathbb{O} \\ \mathbb{O} & \breve{V}_{r-1}^* \end{pmatrix} \cdots \breve{V}_1^*,$$

where

$$D = \begin{pmatrix} t_1 u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \cdots & t_{r-1} u_{r-1} & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \Psi \end{pmatrix},$$

and the u_j, $\breve{V}_j$, and $\breve{W}_j$ are defined by (5.1) and (5.2). Put $\mathfrak{W}_\circ = \Phi_\circ^* \Phi_\circ$.

By the inductive hypothesis, the theorem holds for $\Phi_\circ$. Hence, if $\sigma_\varkappa \ge t_{r-1}$, then

$$N \stackrel{\text{def}}{=} \dim \mathcal{E}(\Lambda_{\sigma_\varkappa}(\mathfrak{W}_\circ)) = \sum_{\{j \ge 1 : t_j \ge \sigma_\varkappa\}} k_j.$$

By Lemma 7.3, $\xi \in \mathcal{E}(\Lambda_{\sigma_\varkappa}(\mathfrak{W}_\circ))$ if and only if

$$\Phi_\circ \xi \in H^2_-(\mathbb{C}^{m-1}) \quad \text{and} \quad \|\Phi_\circ \xi\|_2^2 = (\Lambda_\sigma(\mathfrak{W}_\circ)\xi, \xi).$$

Let $\xi_1, \xi_2, \cdots, \xi_N$ be a basis in $\mathcal{E}(\Lambda_{\sigma_\varkappa}(\mathfrak{W}_\circ))$ and let $\eta_\iota = H_{\Phi_\circ}\xi_\iota$. By Lemma 4.7, there exist scalar functions χ_ι, $1 \le \iota \le N$, in H^2 such that

$$W_0^* \begin{pmatrix} \chi_\iota \\ \eta_\iota \end{pmatrix} \in H^2_-(\mathbb{C}^m).$$

By Theorem 4.5, the matrix function Θ in (7.3) is left invertible. Let A be a matrix function in $H^\infty(\mathbb{M}_{n-1,n})$ such that $A\Theta = \boldsymbol{I}_{n-1}$. As in the proof of Theorem 4.1 we put

$$\xi_\iota^\# = A^{\mathrm{t}}\xi_\iota + q_\iota \boldsymbol{v},$$

where q_ι is a scalar function in H^2 satisfying

$$\mathbb{P}_+(t_0u_0q_\iota + t_0u_0\boldsymbol{v}^*A^{\mathrm{t}}\xi_j) = \chi_j.$$

Put $\eta_\iota^\# = H_\Phi\xi_\iota^\#$. We have

$$\eta_\iota^\# = \mathbb{P}_-W_0^* \begin{pmatrix} t_0u_0 & \mathbb{O} \\ \mathbb{O} & \Phi_\circ \end{pmatrix} V_0^*\xi_\iota^\# = \mathbb{P}_-W_0^* \begin{pmatrix} \chi_\iota + \omega_\iota \\ \eta_\iota \end{pmatrix},$$

where as in the proof of Theorem 4.1

$$\omega_\iota = \mathbb{P}_-(t_0u_0q_\iota + t_0u_0\boldsymbol{v}^*A^{\mathrm{t}}\xi_\iota).$$

As we have explained in the proof of Theorem 4.1

$$\mathbb{P}_-W_0^* \begin{pmatrix} \chi_\iota + \omega_\iota \\ \eta_\iota \end{pmatrix} = W_0^* \begin{pmatrix} \chi_\iota + \omega_\iota \\ \eta_\iota \end{pmatrix},$$

and so $\eta_\iota^\# = \Phi\xi_\iota^\#$.

Since the matrix function W is unitary-valued, we have

$$\begin{aligned} \|\Phi\xi_\iota^\#\|_2^2 &= \left\| W^* \begin{pmatrix} t_0u_0 & \mathbb{O} \\ \mathbb{O} & \Phi_\circ \end{pmatrix} \begin{pmatrix} \boldsymbol{v}^* \\ \Theta^{\mathrm{t}} \end{pmatrix} \xi_\iota^\# \right\|_2^2 \\ &= \left\| W^* \begin{pmatrix} t_0u_0 & \mathbb{O} \\ \mathbb{O} & \Phi_\circ \end{pmatrix} \begin{pmatrix} q_\iota + v^*A^{\mathrm{t}}\xi_\iota \\ \xi_\iota \end{pmatrix} \right\|_2^2 \\ &= t_0^2\|u_0(q_\iota + v^*A^t\xi_\iota)\|_2^2 + \|\Phi_\circ\xi_\iota\|_2^2 \\ &= t_0^2\|q_\iota + v^*A^t\xi_\iota\|_2^2 + (\Lambda_{\sigma_\varkappa}(\mathfrak{W}_\circ)\xi_\iota, \xi_\iota) \end{aligned}$$

(the last equality holds because $\xi_\iota \in \mathcal{E}(\Lambda_{\sigma_\varkappa}(\mathfrak{W}_\circ))$).

Consider the weight $\mathfrak{V}$ defined by

$$\mathfrak{V} = \begin{pmatrix} t_0^2 & \mathbb{O} \\ \mathbb{O} & \mathfrak{W}_\circ \end{pmatrix}.$$

Since

$$V^*\xi_\iota^\# = \begin{pmatrix} q_\iota + v^*A^{\mathrm{t}}\xi_\iota \\ \xi_\iota \end{pmatrix},$$

we have

$$\|\Phi\xi_\iota^\#\|_2^2 = (\Lambda_{\sigma_\varkappa}(\mathfrak{V})V^*\xi_\iota^\#, V^*\xi_\iota^\#) = (\Lambda_{\sigma_\varkappa}(\mathfrak{W})\xi_\iota^\#, \xi_\iota^\#) \tag{7.5}$$

(the last equality follows from the definitions of $\mathfrak{W}$ and $\mathfrak{V}$). Since $H_\Phi \xi_\iota^\# = \Phi \xi_\iota^\#$, it follows from (7.5) that $\xi_\iota^\# \in \mathcal{E}(\Lambda_{\sigma_\varkappa}(\mathfrak{W}))$.

We can add now another k_0 linear independent vectors of $\mathcal{E}(\Lambda_{\sigma_\varkappa}(\mathfrak{W}))$. Let $f_1, \cdots, f_{k_0}$ be a basis of $\operatorname{Ker} T_{u_0}$. Obviously, $f_j \boldsymbol{v} \in \mathcal{E}(\Lambda_{\sigma_\varkappa}(\mathfrak{W}))$. Let us show that the vectors $\xi_1^\#, \cdots, \xi_N^\#$, $f_1\boldsymbol{v}, \cdots, f_{k_0}\boldsymbol{v}$ are linearly independent. It is sufficient to prove that if $f \in \operatorname{Ker} T_{u_0}$ and $f\boldsymbol{v} + \sum_{\iota=1}^N c_\iota \xi_\iota^\# = \mathbb{O}$, then $f = \mathbb{O}$ and $c_\iota = 0$, $1 \le \iota \le N$. We have

$$V^*\left(f\boldsymbol{v} + \sum_{\iota=1}^N c_\iota \xi_\iota^\#\right) = \begin{pmatrix} f \\ \mathbb{O} \end{pmatrix} + \sum_{\iota=1}^N c_\iota \begin{pmatrix} \boldsymbol{v}^* A^t \xi_\iota + q_\iota \\ \xi_\iota \end{pmatrix} = \mathbb{O}. \tag{7.6}$$

Since the ξ_ι are linearly independent, it follows that $c_\iota = 0$, $1 \le \iota \le N$, which in turn implies that $f = \mathbb{O}$.

This proves that

$$\sum_{\{j: t_j \ge \sigma_\varkappa\}} k_j \le \dim \mathcal{E}(\Lambda_{\sigma_\varkappa}(\mathfrak{W})).$$

Let us prove the opposite inequality.

Denote by $\mathcal{E}_0$ the set of vectors in $\mathcal{E}(\Lambda_{\sigma_\varkappa}(\mathfrak{W}))$ of the form $f\boldsymbol{v}$, where f is a scalar function in H^2. It is easy to see that $f\boldsymbol{v} \in \mathcal{E}_0$ if and only if $f \in \operatorname{Ker} T_{u_0}$. It remains to show that there are at most

$$\sum_{\{j \ge 1: t_j \ge \sigma_\varkappa\}} k_j$$

vectors $\breve{\xi}_\iota$ in $\mathcal{E}(\Lambda_{\sigma_\varkappa}(\mathfrak{W}))$ that are linearly independent modulo $\mathcal{E}_0$. Let $\breve{\eta}_\iota \stackrel{\text{def}}{=} H_\Phi \breve{\xi}_\iota$. By Lemma 7.3, $\breve{\eta}_\iota = \Phi \xi_\iota$. Define the functions $\xi_\iota \in L^2(\mathbb{C}^{n-1})$ and $\eta_\iota \in L^2(\mathbb{C}^{m-1})$ by

$$V^* \breve{\xi}_\iota = \begin{pmatrix} \gamma_\iota \\ \xi_\iota \end{pmatrix}, \quad W \breve{\eta}_\iota = \begin{pmatrix} \delta_\iota \\ \eta_\iota \end{pmatrix}.$$

Since $V = \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix}$ and the $\breve{\xi}_\iota$ are linearly independent modulo $\mathcal{E}_0$, it follows that the ξ_ι are linearly independent. To complete the proof, it is sufficient to show that $\xi_\iota \in \mathcal{E}(\Lambda_{\sigma_\varkappa}(\mathfrak{W}_\circ))$.

Since $\breve{\eta}_\iota = \Phi \breve{\xi}_\iota$, we have $\eta_\iota = \Phi_\circ \xi_\iota$ and $\delta_\iota = t_0 u_0 \gamma_\iota$. It follows from the definitions of V and W that $\xi_\iota \in L^2(\mathbb{C}^{n-1})$ and $\eta_\iota \in H^2_-(\mathbb{C}^{m-1})$.

We have

$$\begin{aligned} \left(\Lambda_{\sigma_\varkappa}(\mathfrak{W})\breve{\xi}_\iota, \breve{\xi}_\iota\right) &= \left(\Lambda_{\sigma_\varkappa}(V^*\mathfrak{W}V)V^*\breve{\xi}_\iota, V^*\breve{\xi}_\iota\right) \\ &= \left(\begin{pmatrix} t_0^2 & \mathbb{O} \\ \mathbb{O} & \Lambda_{\sigma_\varkappa}(\mathfrak{W}_\circ) \end{pmatrix} \begin{pmatrix} \gamma_\iota \\ \xi_\iota \end{pmatrix}, \begin{pmatrix} \gamma_\iota \\ \xi_r \end{pmatrix}\right) \\ &= t_0^2 \|\gamma_\iota\|_2^2 + \left(\Lambda_{\sigma_\varkappa}(\mathfrak{W}_\circ)\xi_\iota, \xi_\iota\right). \end{aligned}$$

On the other hand,

$$\left(\Lambda_{\sigma_\varkappa}(\mathfrak{W})\breve{\xi}_\iota, \breve{\xi}_\iota\right) = \|\Phi\breve{\xi}_\iota\|_2^2 = \|\breve{\eta}_\iota\|_2^2 = \|\eta_\iota\|_2^2 + \|\delta_\iota\|_2^2 = \|\Phi_\circ\xi_\iota\|_2^2 + t_0^2\|\gamma_\iota\|_2^2.$$

Therefore $(\Lambda_{\sigma_\varkappa}(\mathfrak{W}_\circ)\xi_\iota, \xi_\iota) = \|\Phi_\circ\xi_\iota\|_2^2$, which implies $\xi_\iota \in \mathcal{E}(\Lambda_{\sigma_\varkappa}(\mathfrak{W}_\circ))$. ■

8. Inequalities Involving Hankel and Superoptimal Singular Values

In this section we prove that under the hypotheses of Theorem 5.1 the superoptimal singular values $t_j(\Phi)$ for $0 \le j \le r-1$ are majorized by the singular values $s_j(H_\Phi)$ of the Hankel operator H_Φ. In fact, we prove a considerably stronger result. Assuming that Φ satisfies the hypotheses of Theorem 5.1, we consider the *extended t-sequence* for Φ:

$$\underbrace{t_0, \cdots, t_0}_{k_0}, \underbrace{t_1, \cdots, t_1}_{k_1}, \cdots, \underbrace{t_{r-1}, \cdots, t_{r-1}}_{k_{r-1}} \tag{8.1}$$

in which t_j repeats k_j times. We denote the terms of the extended sequence by

$$\breve{t}_0, \breve{t}_1, \cdots, \breve{t}_{k_0+\cdots+k_{r-1}-1}.$$

Although the indices k_j may depend on the choice of a partial thematic factorization, it follows from Theorem 7.1 that the extended t-sequence is uniquely determined by the matrix function Φ. We show in this section that the terms of the extended t-sequence for Φ are majorized by the singular values of H_Φ.

In fact, we start this section with even a stronger inequality.

Theorem 8.1. *Let Φ be a badly approximable matrix function in $L^\infty(\mathbb{M}_{m,n})$ such that $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$ and suppose that*

$$\Phi = W^*\begin{pmatrix} t_0u_0 & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix}V^*, \tag{8.2}$$

where V and W^{t} are thematic matrix functions, u_0 is a unimodular function such that T_{u_0} is Fredholm, and $k_0 = \operatorname{ind} T_{u_0} > 0$. Then

$$s_j(H_\Psi) \le s_{j+k_0}(H_\Phi), \quad j \in \mathbb{Z}_+.$$

Proof. Clearly, it is sufficient to prove the following fact. Let $\mathcal{L}$ be a subspace of $H^2(\mathbb{C}^{n-1})$ such that $\|H_\Psi\xi\|_2 \ge s\|\xi\|_2$, for every $\xi \in \mathcal{L}$, where $0 < s \le t_0$. Then there exists a subspace $\mathcal{M}$ of $H^2(\mathbb{C}^n)$ such that $\dim\mathcal{M} \ge \dim\mathcal{L} + k_0$ and $\|H_\Phi g\|_2 \ge s\|g\|_2$ for every $f \in \mathcal{M}$.

Let ξ_ι, $1 \le \iota \le N$, be a basis in $\mathcal{L}$. Put $\eta_\iota = H_\Psi\xi_\iota$. By Lemma 4.7, there exist scalar functions χ_ι in H^2 such that $W^*\begin{pmatrix}\chi_\iota \\ \eta_\iota\end{pmatrix} \in H^2_-(\mathbb{C}^m)$. As in the proof of Theorem 7.4 we define the functions $\xi_\iota^\# \in H^2(\mathbb{C}^n)$ by

$$\xi_\iota^\# = A^t\xi_\iota + q_\iota\boldsymbol{v},$$

where q_ι is a scalar function in H^2 satisfying

$$\mathbb{P}_+(t_0 u_0 q_\iota + t_0 u_0 \boldsymbol{v}^* A^t \xi_\iota) = \chi_\iota.$$

We can now define $\mathcal{M}$ by

$$\mathcal{M} = \operatorname{span}\{\xi_\iota^\# + f\boldsymbol{v} : \ 1 \le \iota \le N, \ f \in \operatorname{Ker} T_{u_0}\}.$$

Let us show that $\dim \mathcal{M} = N + k_0$. Since $\dim \operatorname{Ker} T_{u_0} = k_0$, it is sufficient to prove that if $x\boldsymbol{v} + \sum_{\iota=1}^{N} c_\iota \xi_\iota^\# = \mathbb{O}$, then $x = \mathbb{O}$ and $c_\iota = 0$, $1 \le \iota \le N$. This follows immediately from (7.6).

To complete the proof it remains to show that $\|H_\Phi g\|_2 \ge s\|g\|_2$ for

$$g = f\boldsymbol{v} + \sum_{\iota=1}^{N} c_\iota \xi_\iota^\#.$$

Let $\xi = \sum_{\iota=1}^{N} c_\iota \xi_\iota$, $\eta = H_\Psi \xi$, $q = \sum_{\iota=1}^{N} c_\iota q_\iota$, and $\xi^\# = \sum_{\iota=1}^{N} c_\iota \xi_\iota^\#$.

We have

$$\begin{aligned}
W^* \begin{pmatrix} t_0 u_0 & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^* g &= W^* \begin{pmatrix} t_0 u_0 & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^* (f\boldsymbol{v} + q\boldsymbol{v} + A^t \xi) \\
&= W^* \begin{pmatrix} t_0 u_0 & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} \begin{pmatrix} f + q + \boldsymbol{v}^* A^t \xi \\ \xi \end{pmatrix} \\
&= W^* \begin{pmatrix} t_0 u_0 f + t_0 u_0 q + t_0 u_0 \boldsymbol{v}^* A^t \xi \\ \Psi \xi \end{pmatrix}.
\end{aligned}$$

Since $f \in \operatorname{Ker} T_{u_0}$, it follows that

$$W^* \begin{pmatrix} t_0 u_0 f \\ \mathbb{O} \end{pmatrix} \in H^2_-(\mathbb{C}^m).$$

On the other hand, it has been shown in the proof of Theorem 4.1 that

$$W^* \begin{pmatrix} t_0 u_0 q + t_0 u_0 \boldsymbol{v}^* A^t \xi \\ \mathbb{O} \end{pmatrix} \in H^2_-(\mathbb{C}^m).$$

It follows that

$$H_\Phi g = W^* \begin{pmatrix} t_0 u_0 f + t_0 u_0 q + t_0 u_0 \boldsymbol{v}^* A^t \xi \\ H_\Psi \xi \end{pmatrix}.$$

Therefore

$$\|H_\Phi g\|_2^2 = |t_0|^2 \|f + q + \boldsymbol{v}^* A^t \xi\|_2^2 + \|\eta\|_2^2.$$

We have

$$\|g\|_2^2 = \|V^* g\|_2^2 = \|f + q + \boldsymbol{v}^* A^t \xi\|_2^2 + \|\xi\|_2^2.$$

Since $s \le t_0$ and $\|\eta\|_2 \ge s\|\xi\|_2$, it follows that $\|H_\Phi g\|_2^2 \ge s^2 \|g\|_2^2$. ■

Now we are in a position to prove that the terms of the extended t-sequence defined by (8.1) are majorized by the singular values of the Hankel operator.

Theorem 8.2. *Suppose that* $\Phi \in L^\infty(\mathbb{M}_{m,n})$, $\|H_\Phi\|_e < t_{r-1}$, *and* $t_r < t_{r-1}$. *Then*

$$\breve{t}_j \le s_j(H_\Phi), \quad 0 \le j \le k_0 + k_1 + \cdots + k_{r-1} - 1. \tag{8.3}$$

In particular, if $\|H_\Phi\|_e$ is less than the smallest nonzero superoptimal singular value of Φ, then inequality (8.3) holds for all terms of the extended t-sequence that correspond to all nonzero superoptimal singular values of Φ. This is certainly the case if $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$.

Proof. Without loss of generality we may assume that $\Phi \in \boldsymbol{\Omega}_{r-1}$ (see the Introduction to this chapter for the definition of the sets $\boldsymbol{\Omega}_j$). We argue by induction on r. Consider the factorization (8.2). Let $f \in \operatorname{Ker} T_{u_0}$. It is easy to see that

$$H_\Phi f\boldsymbol{v} = t_0 u_0 f\overline{\boldsymbol{w}} \in H^2_-(\mathbb{C}^m).$$

It follows that $\|H_\Phi\|_2 = t_0\|f\boldsymbol{v}\|_2$, which proves that

$$s_j(H_\Phi) = t_0, \quad 0 \le j \le k_0 - 1. \tag{8.4}$$

Hence, the result holds for $r = 1$. By the inductive hypothesis, the result holds for Ψ, and so by Theorem 8.1,

$$\breve{t}_j \le s_j(H_\Phi), \quad k_0 \le j \le k_0 + k_1 + \cdots + k_{r-1} - 1.$$

Together with (8.4) this proves the theorem. ■

Corollary 8.3. *Suppose that* Φ *satisfies the hypotheses of Theorem 8.2. Then*

$$t_j(\Phi) \le s_j(H_\Phi), \quad 0 \le j \le r - 1.$$

9. Invariance of Residual Entries

We continue in this section studying invariance properties of (partial) thematic factorizations. We show that if a matrix function Φ admits a partial thematic factorization of the form (5.1), then the residual entry Ψ in (5.1) is uniquely determined by the function Φ itself modulo constant unitary factors. In the next section we introduce the notion of a monotone (partial) thematic factorization and we use the results of this section to prove that the indices of a monotone (partial) thematic factorization are uniquely determined by the function Φ itself.

Lemma 9.1. *Let* Φ *be an* $m \times n$ *matrix of the form*

$$\Phi = W^* \begin{pmatrix} u & 0 \\ 0 & \Psi \end{pmatrix} V^*,$$

where $m, n \ge 2$, $u \in \mathbb{C}$, $\Psi \in \mathbb{M}_{m-1,n-1}$, *and*

$$V = \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix} \in \mathbb{M}_{n,n}, \quad W = \begin{pmatrix} \boldsymbol{w} & \Xi \end{pmatrix}^t \in \mathbb{M}_{m,m}$$

are unitary matrices such that $\boldsymbol{v} \in \mathbb{M}_{n,1}$ *and* $\boldsymbol{w} \in \mathbb{M}_{m,1}$. *Then*

$$\Psi = \Xi^* \Phi \overline{\Theta}.$$

Proof. We have

$$\begin{aligned}\Xi^*\Phi\overline{\Theta} &= \Xi^* W^* \begin{pmatrix} u & 0 \\ 0 & \Psi \end{pmatrix} V^*\overline{\Theta} \\ &= \Xi^* \begin{pmatrix} \overline{\boldsymbol{w}} & \Xi \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & \Psi \end{pmatrix} \begin{pmatrix} \boldsymbol{v}^* \\ \Theta^{\mathrm{t}} \end{pmatrix} \overline{\Theta} \\ &= \begin{pmatrix} 0 & I_{m-1} \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & \Psi \end{pmatrix} \begin{pmatrix} 0 \\ I_{n-1} \end{pmatrix} = \Psi. \quad \blacksquare\end{aligned}$$

Corollary 9.2. *Let Φ be an $m \times n$ matrix of the form*

$$\Phi = W_0^* \cdots W_{r-1}^* \begin{pmatrix} \varphi_0 & 0 & \cdots & 0 & 0 \\ 0 & \varphi_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \varphi_{r-1} & 0 \\ 0 & 0 & \cdots & 0 & \Psi \end{pmatrix} V_{r-1}^* \cdots V_0^*,$$

where $r < \min\{m, n\}$, $\varphi_0, \varphi_1, \cdots, \varphi_{r-1} \in \mathbb{C}$,

$$V_j = \begin{pmatrix} I_j & 0 \\ 0 & \breve{V}_j \end{pmatrix}, \quad W_j = \begin{pmatrix} I_j & 0 \\ 0 & \breve{W}_j \end{pmatrix},$$

are unitary matrices such that

$$V_j = \begin{pmatrix} \boldsymbol{v}_j & \overline{\Theta_j} \end{pmatrix}, \quad W_j = \begin{pmatrix} \boldsymbol{w}_j & \overline{\Xi_j} \end{pmatrix}^t, \quad 0 \le j \le r-1,$$

$\boldsymbol{v}_j \in \mathbb{M}_{n-j,1}$, $\boldsymbol{w}_j \in \mathbb{M}_{m-j,1}$. Then

$$\Psi = \Xi_{r-1}^* \cdots \Xi_1^* \Xi_0^* \Phi \overline{\Theta_0 \Theta_1 \cdots \Theta_{r-1}}.$$

Proof. The result follows immediately from Lemma 9.1 by induction. $\blacksquare$

Theorem 9.3. *Suppose that a matrix function $\Phi \in L^\infty(\mathbb{M}_{m,n})$ admits partial thematic factorizations*

$$\Phi = W_0^* \cdots W_{r-1}^* \begin{pmatrix} t_0 u_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_1 u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & t_{r-1} u_{r-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \Psi \end{pmatrix} V_{r-1}^* \cdots V_0^*,$$

and

$$\Phi = \left(W_0^\heartsuit\right)^* \cdots \left(W_{r-1}^\heartsuit\right)^* \begin{pmatrix} t_0 u_0^\heartsuit & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \cdots & t_{r-1} u_{r-1}^\heartsuit & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \Psi^\heartsuit \end{pmatrix} \left(V_{r-1}^\heartsuit\right)^* \cdots \left(V_0^\heartsuit\right)^*.$$

Then there exist constant unitary matrices $\mathfrak{U}_1 \in \mathbb{M}_{n-r,n-r}$ and $\mathfrak{U}_2 \in \mathbb{M}_{m-r,m-r}$ such that

$$\Psi^\heartsuit = \mathfrak{U}_2 \Psi \mathfrak{U}_1.$$

Recall that by the definition of a partial thematic factorization, Ψ must satisfy (5.3), and this is very important.

Proof. Let

$$V_j = \begin{pmatrix} I_j & \mathbb{O} \\ \mathbb{O} & \breve{V}_j \end{pmatrix}, \quad W_j = \begin{pmatrix} I_j & \mathbb{O} \\ \mathbb{O} & \breve{W}_j \end{pmatrix},$$

and

$$V_j^\heartsuit = \begin{pmatrix} I_j & \mathbb{O} \\ \mathbb{O} & \breve{V}_j^\heartsuit \end{pmatrix}, \quad W_j^\heartsuit = \begin{pmatrix} I_j & \mathbb{O} \\ \mathbb{O} & \breve{W}_j^\heartsuit \end{pmatrix},$$

where

$$\breve{V}_j = \begin{pmatrix} \boldsymbol{v}_j & \overline{\Theta_j} \end{pmatrix}, \qquad \breve{W}_j = \begin{pmatrix} \boldsymbol{w}_j & \overline{\Xi_j} \end{pmatrix}^{\mathrm{t}}, \qquad 0 \le j \le r-1,$$

$$\breve{V}_j^\heartsuit = \begin{pmatrix} \boldsymbol{v}_j^\heartsuit & \overline{\Theta_j^\heartsuit} \end{pmatrix}, \qquad \breve{W}_j^\heartsuit = \begin{pmatrix} \boldsymbol{w}_j^\heartsuit & \overline{\Xi_j^\heartsuit} \end{pmatrix}^{\mathrm{t}}, \qquad 0 \le j \le r-1.$$

Here $\breve{V}_0 \stackrel{\mathrm{def}}{=} V_0$, $\breve{W}_0 \stackrel{\mathrm{def}}{=} V_0$, $\breve{V}_0^\heartsuit \stackrel{\mathrm{def}}{=} V_0^\heartsuit$, and $\breve{W}_0^\heartsuit \stackrel{\mathrm{def}}{=} W_0^\heartsuit$.

We need the following lemma.

Lemma 9.4.

$$\Theta_0\Theta_1 \cdots \Theta_{r-1} H^2(\mathbb{C}^{n-r}) = \Theta_0^\heartsuit\Theta_1^\heartsuit \cdots \Theta_{r-1}^\heartsuit H^2(\mathbb{C}^{n-r}) \tag{9.1}$$

and

$$\Xi_0\Xi_1 \cdots \Xi_{r-1} H^2(\mathbb{C}^{m-r}) = \Xi_0^\heartsuit\Xi_1^\heartsuit \cdots \Xi_{r-1}^\heartsuit H^2(\mathbb{C}^{m-r}). \tag{9.2}$$

Let us first complete the proof of Theorem 9.3. Consider the inner matrix functions

$$\Theta = \Theta_0\Theta_1 \cdots \Theta_{r-1}, \quad \Theta^\heartsuit = \Theta_0^\heartsuit\Theta_1^\heartsuit \cdots \Theta_{r-1}^\heartsuit$$

and

$$\Xi = \Xi_0\Xi_1 \cdots \Xi_{r-1}, \quad \Xi^\heartsuit = \Xi_0^\heartsuit\Xi_1^\heartsuit \cdots \Xi_{r-1}^\heartsuit.$$

By Lemma 9.4, $\Theta H^2(\mathbb{C}^{n-r}) = \Theta^\heartsuit H^2(\mathbb{C}^{n-r})$. In this case there exists a constant unitary matrix $Q_1 \in \mathbb{M}_{n-r,n-r}$ such that $\Theta^\heartsuit = \Theta Q_1$. Indeed, Θ and $\Theta^\heartsuit$ determine the same invariant subspace under multiplication by z; see Appendix 2.3. Similarly, there exists a constant unitary matrix $Q_2 \in \mathbb{M}_{m-r,m-r}$ such that $\Xi^\heartsuit = \Xi Q_2$.

By Corollary 9.2,

$$\Psi = \Xi^*\Phi\overline{\Theta}, \quad \Psi^\heartsuit = (\Xi^\heartsuit)^*\Phi\overline{\Theta^\heartsuit}.$$

Hence,

$$\Psi^\heartsuit = Q_2^*\Xi^*\Phi\overline{\Theta Q_1} = Q_2^*\Psi\overline{Q_1}. \quad \blacksquare$$

Proof of Lemma 9.4. It is sufficient to prove (9.1). Indeed, (9.2) follows from (9.1) applied to Φ^{t}.

It is easy to see that without loss of generality we may assume that $\|\Psi\|_{L^\infty} < t_{r-1}$. Indeed, we can subtract from Φ a matrix function in $\boldsymbol{\Omega}_{r-1}$ (see the Introduction to this chapter), and it follows from Theorem 1.8 that the resulting function admits a partial thematic factorization with the same unitary-valued function V_j and W_j, $0 \le j \le r-1$, and residual entry whose

L^∞ norm is less than t_{r-1}. It is also easy to see that if $\|\Psi\|_{L^\infty} < t_{r-1}$, then $\|\Psi^\heartsuit\|_{L^\infty}$ must also be less than t_{r-1}.

Consider the subspace $\mathcal{L}$ of $H^2(\mathbb{C}^n)$ defined by

$$\mathcal{L} = \left\{ f \in H^2(\mathbb{C}^n) : \; V_{r-1}^{\mathrm{t}} \cdots V_1^{\mathrm{t}} V_0^{\mathrm{t}} f = \begin{pmatrix} \mathbb{O} \\ \vdots \\ \mathbb{O} \\ * \\ \vdots \\ * \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}\mathbb{O}\\ \vdots\\ \mathbb{O}\end{matrix}}\right\} r \\ \\ \\ \\ \end{matrix} \right\},$$

i.e., $\mathcal{L}$ consists of vector functions $f \in H^2(\mathbb{C}^n)$ such that the first r components of the vector function $V_{r-1}^{\mathrm{t}} \cdots V_1^{\mathrm{t}} V_0^{\mathrm{t}} f$ are zero.

We define the real function ρ on $\mathbb{R}$ by

$$\rho(x) = \begin{cases} x, & x \ge t_{r-1}^2, \\ 0, & x < t_{r-1}^2, \end{cases}$$

and consider the operator $M : H^2(\mathbb{C}^n) \to L^2(\mathbb{C}^n)$ of multiplication by the matrix function $\rho\big(\Phi^{\mathrm{t}}\overline{\Phi}\big)$:

$$Mf = \rho\big(\Phi^{\mathrm{t}}\overline{\Phi}\big) f, \quad f \in H^2(\mathbb{C}^n).$$

Let us show that

$$\mathcal{L} = \operatorname{Ker} M. \tag{9.3}$$

We have

$$\Phi^{\mathrm{t}}\overline{\Phi} = \overline{V_0 V_1 \cdots V_{r-1}} \begin{pmatrix} t_0^2 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \cdots & t_{r-1}^2 & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \Psi^{\mathrm{t}}\overline{\Psi} \end{pmatrix} V_{r-1}^{\mathrm{t}} \cdots V_1^{\mathrm{t}} V_0^{\mathrm{t}},$$

and since $\|\Psi^{\mathrm{t}}\overline{\Psi}\|_{L^\infty} < t_{r-1}^2$, it follows that

$$\rho\big(\Phi^{\mathrm{t}}\overline{\Phi}\big) = \overline{V_0 V_1 \cdots V_{r-1}} \begin{pmatrix} t_0^2 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \cdots & t_{r-1}^2 & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \end{pmatrix} V_{r-1}^{\mathrm{t}} \cdots V_1^{\mathrm{t}} V_0^{\mathrm{t}}.$$

Since all matrix functions V_j are unitary-valued, this implies (9.3).

Thus the subspace $\mathcal{L}$ is uniquely determined by the function Φ and does not depend on the choice of a partial thematic factorization. It is easy to see that to complete the proof of Lemma 9.4, it is sufficient to prove the following lemma.

Lemma 9.5.

$$\mathcal{L} = \Theta_0 \Theta_1 \cdots \Theta_{r-1} H^2(\mathbb{C}^{n-r}). \tag{9.4}$$

Proof. We show by induction on r that (9.4) holds even without the assumption that $\|H_\Psi\| < t_{r-1}$ (note that this assumption is very important in the proof of (9.3)).

Suppose that $r = 1$. Then

$$\mathcal{L} = \left\{ f \in H^2(\mathbb{C}^n) :\ V_0^{\mathrm{t}} f = \begin{pmatrix} \mathbb{O} \\ * \\ \vdots \\ * \end{pmatrix} \right\}.$$

Obviously, if $f \in \Theta_0 H^2(\mathbb{C}^{n-1})$, then $f \in \mathcal{L}$. Suppose now that $f \in \mathcal{L}$. We have

$$V_0^{\mathrm{t}} f = \begin{pmatrix} \mathbb{O} \\ g \end{pmatrix}, \quad g \in L^2(\mathbb{C}^{n-1}).$$

Then

$$f = \overline{V_0} \begin{pmatrix} \mathbb{O} \\ g \end{pmatrix} = \begin{pmatrix} \overline{\boldsymbol{v}_0} & \Theta_0 \end{pmatrix} \begin{pmatrix} \mathbb{O} \\ g \end{pmatrix} = \Theta_0 g.$$

By Lemma 1.4, $g \in H^2(\mathbb{C}^{n-1})$, which proves the result for $r = 1$.

Suppose now that $r \ge 2$. By the inductive hypothesis,

$$\mathcal{L} = \left\{ \Theta_0 \cdots \Theta_{r-2} g :\ V_{r-1}^{\mathrm{t}} \cdots V_0^{\mathrm{t}} \Theta_0 \cdots \Theta_{r-2} g = \begin{pmatrix} \mathbb{O} \\ \vdots \\ \mathbb{O} \\ * \\ \vdots \\ * \end{pmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} \mathbb{O} \\ \vdots \\ \mathbb{O} \end{matrix}}\right\} r \\ \vphantom{\begin{matrix} * \\ \vdots \\ * \end{matrix}} \end{array} \right\}$$

(here $g \in H^2(\mathbb{C}^{n-r+1})$). It follows from the definition of thematic matrix functions that

$$V_{r-2}^{\mathrm{t}} \cdots V_0^{\mathrm{t}} \Theta_0 \cdots \Theta_{r-2} = \begin{pmatrix} \mathbb{O} & \cdots & \mathbb{O} \\ \vdots & \ddots & \vdots \\ \mathbb{O} & \cdots & \mathbb{O} \\ \mathbf{1} & \cdots & \mathbb{O} \\ \vdots & \ddots & \vdots \\ \mathbb{O} & \cdots & \mathbf{1} \end{pmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} \mathbb{O} \\ \vdots \\ \mathbb{O} \end{matrix}}\right\} r-1 \\ \left.\vphantom{\begin{matrix} \mathbf{1} \\ \vdots \\ \mathbb{O} \end{matrix}}\right\} n-r+1 \end{array} = \begin{pmatrix} \mathbb{O} \\ \boldsymbol{I}_{n-r+1} \end{pmatrix}.$$

Hence,

$$\mathcal{L} = \left\{ \Theta_0 \cdots \Theta_{r-2} g :\ g \in H^2(\mathbb{C}^{n-r+1}),\ \begin{pmatrix} \boldsymbol{v}_{r-1}^{\mathrm{t}} \\ \Theta_{r-1}^* \end{pmatrix} g = \begin{pmatrix} \mathbb{O} \\ * \\ \vdots \\ * \end{pmatrix} \right\}.$$

Since the result has already been proved for $r = 1$,

$$\mathcal{L} = \left\{ \Theta_0 \cdots \Theta_{r-2} g :\ g \in \Theta_{r-1} H^2(\mathbb{C}^{n-r}) \right\} = \Theta_0 \cdots \Theta_{r-1} H^2(\mathbb{C}^{n-r}). \quad \blacksquare$$

10. Monotone Thematic Factorizations and Invariance of Thematic Indices

We have seen in §7 that the matrix function $\Phi = \begin{pmatrix} \bar{z}^2 & \mathbb{O} \\ \mathbb{O} & \bar{z}^6 \end{pmatrix}$ has thematic factorizations with different thematic indices. We have shown in §7 that under the hypotheses of Theorem 5.1 the sum of thematic indices that correspond to all superoptimal singular values equal to a positive specific value $t \ge t_{r-1}$ does not depend on the choice of a thematic factorization. In other words, for each positive superoptimal singular value $t \ge t_{r-1}$ the number

$$\nu \stackrel{\text{def}}{=} \sum_{\{j:t_j=t\}} k_j$$

does not depend on the choice of a (partial) thematic factorization.

A natural question arises of whether we can arbitrarily distribute the number ν between the indices k_j with $t_j = t$ by choosing an appropriate thematic factorization (recall that the k_j must be positive integers).

In this section we show that the answer to this question is negative.

Definition. A (partial) thematic factorization

$$W_0^* \cdots W_{r-1}^* \begin{pmatrix} t_0 u_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_1 u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & t_{r-1} u_{r-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \Psi \end{pmatrix} V_{r-1}^* \cdots V_0^* \tag{10.1}$$

is called *monotone* if for any positive superoptimal singular value $t \ge t_{r-1}$ the thematic indices $k_l, k_{l+1}, \cdots, k_s$ that correspond to all superoptimal singular values equal to t satisfy

$$k_l \ge k_{l+1} \ge \cdots \ge k_s. \tag{10.2}$$

Here $t_l, t_{l+1}, \cdots, t_s$ are the superoptimal singular values equal to t.

We prove in this section that if a matrix function $\Phi \in L^\infty(\mathbb{M}_{m,n})$ satisfies the hypotheses of Theorem 5.1 and $F \in \boldsymbol{\Omega}_{r-1}$, then $\Phi - F$ possesses a monotone partial thematic factorization of the form (5.1). We also show that the indices of a monotone partial thematic factorization are uniquely determined by the function Φ itself and do not depend on the choice of a partial thematic factorization. The same results also hold for thematic factorizations when $\|H_\Phi\|_{\mathrm{e}}$ is less than the smallest nonzero superoptimal singular value of Φ. In particular, this is the case if $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$.

If we look at the thematic factorizations for the function $\Phi = \begin{pmatrix} \bar{z}^2 & \mathbb{O} \\ \mathbb{O} & \bar{z}^6 \end{pmatrix}$ considered at the beginning of §7, we see that only the third thematic factorization is monotone. It will follow from the results of this section that the thematic indices of any monotone thematic factorization must be equal to 6, 2. In particular, there are no thematic factorizations with indices 7, 1. Note that it is important that the indices in (10.2) are arranged in the *nonincreasing* order. The first two thematic factorizations have different thematic indices 2, 6 and 1, 7 that are arranged in the increasing order.

Theorem 10.1. *Suppose that* $\Phi \in L^\infty(\mathbb{M}_{m,n})$ *and* $r \le \min\{m,n\}$ *is a positive integer such that the superoptimal singular values of* Φ *satisfy*

$$t_{r-1} > t_r, \quad t_{r-1} > \|H_\Phi\|_{\mathrm{e}}.$$

Then Φ *admits a monotone partial thematic factorization of the form* (10.1).

Proof. Clearly, $\|H_{z^j\Phi}\| = \operatorname{dist}_{L^\infty}(\Phi, \bar{z}^j H^\infty(\mathbb{M}_{m,n}))$, and it is easy to see that

$$\lim_{j\to\infty} \|H_{z^j\Phi}\| = \operatorname{dist}_{L^\infty}\big(\Phi, (H^\infty + C)(\mathbb{M}_{m,n})\big) = \|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|.$$

Put

$$\boldsymbol{\iota}(H_\Phi) \stackrel{\text{def}}{=} \min\{j \ge 0 : \ \|H_{z^j\Phi}\| < \|H_\Phi\|\}.$$

Obviously, $\boldsymbol{\iota}(H_\Phi)$ depends only on the Hankel operator H_Φ and does not depend on the choice of its symbol.

We need three lemmas.

Lemma 10.2. *Let* Φ *be a matrix function in* $L^\infty(\mathbb{M}_{m,n})$ *such that* $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$. *Suppose that*

$$\Phi = W^* \begin{pmatrix} tu & \mathbb{O} \\ \mathbb{O} & \Delta \end{pmatrix} V^*, \tag{10.3}$$

where V *and* W^{t} *are thematic matrix functions of sizes* $n \times n$ *and* $m \times m$, $t > 0$, $\|\Delta\|_{L^\infty} \le t$, *and* u *is a unimodular function such that* T_u *is Fredholm. Then* $\operatorname{ind} T_u \le \iota(H_\Phi)$.

Lemma 10.3. *Let* Φ *be a badly approximable matrix function in* $L^\infty(\mathbb{M}_{m,n})$ *such that* $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$. *Then* Φ *admits a representation* (10.3) *with thematic matrix functions* V *and* W^{t}, $t = t_0 = \|H_\Phi\|$, *and a unimodular function* u *such that* T_u *is Fredholm and*

$$\operatorname{ind} T_u = \iota(H_\Phi).$$

Lemma 10.4. *Let* $\Phi \in L^\infty(\mathbb{M}_{m,n})$ *be a matrix function of the form*

$$\Phi = W^* \begin{pmatrix} u & \mathbb{O} \\ \mathbb{O} & \Delta \end{pmatrix} V^*,$$

where V *and* W^{t} *are thematic matrix functions of sizes* $n\times n$ *and* $m\times m$ *such that the Toeplitz operators* T_V *and* $T_{W^{\mathrm{t}}}$ *are invertible,* u *is a unimodular function such that* T_u *is Fredholm,* $\operatorname{ind} T_u = 0$, $\|H_\Delta\| \le 1$, *and* $\|H_\Delta\|_{\mathrm{e}} < 1$. *If* $\|H_\Phi\| < 1$, *then* $\|H_\Delta\| < 1$.

Let us first complete the proof of Theorem 10.1. We argue by induction on r. For $r = 1$ the result is trivial. Suppose now that $r > 1$. By Lemma

10.3, Φ admits a representation

$$\Phi = W^* \begin{pmatrix} t_0 u_0 & \mathbb{O} \\ \mathbb{O} & \Delta \end{pmatrix} V^*,$$

where V and W^{t} are thematic functions, $\|\Delta\|_{L^\infty} \le t_0$, and u_0 is a unimodular function such that T_{u_0} is Fredholm and $\operatorname{ind} T_{u_0} = \boldsymbol{\iota}(H_\Phi)$. By Theorem 4.1,

$$\|H_\Delta\|_{\mathrm{e}} \le \|H_\Phi\|_{\mathrm{e}}. \tag{10.4}$$

By the inductive hypothesis, Δ admits a monotone partial thematic factorization of the form

$$\Delta = \breve{W}_1^* \cdots \begin{pmatrix} \boldsymbol{I}_{r-2} & \mathbb{O} \\ \mathbb{O} & \breve{W}_{r-1}^* \end{pmatrix} D \begin{pmatrix} \boldsymbol{I}_{r-2} & \mathbb{O} \\ \mathbb{O} & \breve{V}_{r-1}^* \end{pmatrix} \cdots \breve{V}_1^*,$$

where

$$D = \begin{pmatrix} t_1 u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \cdots & t_{r-1} u_{r-1} & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \Psi \end{pmatrix},$$

and the $\breve{V}_j$ and $\breve{W}_j$ are as in (5.2). Clearly, $t_1 = \|\Delta\|_{L^\infty}$. If $t_1 < t_0$, then it is obvious that the above factorization of Δ leads to a monotone partial thematic factorization of Φ.

Suppose now that $t_1 = t_0$. To prove that the above factorization of Δ leads to a monotone partial thematic factorization of Φ, we have to establish the inequality $\operatorname{ind} T_{u_0} \ge \operatorname{ind} T_{u_1}$. By Lemma 10.2, $\boldsymbol{\iota}(H_\Delta) \ge \operatorname{ind} T_{u_1}$, and it suffices to prove the inequality

$$\boldsymbol{\iota}(H_\Phi) = \operatorname{ind} T_{u_0} \ge \boldsymbol{\iota}(H_\Delta).$$

Put $\boldsymbol{\iota} \stackrel{\text{def}}{=} \boldsymbol{\iota}(H_\Phi)$. We have

$$z^{\boldsymbol{\iota}} \Phi = W^* \begin{pmatrix} t_0 z^{\boldsymbol{\iota}} u_0 & \mathbb{O} \\ \mathbb{O} & z^{\boldsymbol{\iota}} \Delta \end{pmatrix} V^*.$$

Clearly, $\operatorname{ind} T_{z^{\boldsymbol{\iota}} u_0} = 0$. By the definition of $\boldsymbol{\iota}$, $\|H_{z^{\boldsymbol{\iota}}\Phi}\| < \|H_\Phi\| = t_0$. It is easy to see that

$$\|H_{z^{\boldsymbol{\iota}}\Delta}\|_{\mathrm{e}} = \|H_\Delta\|_{\mathrm{e}} < t_0$$

by (10.4). It follows from Lemma 10.4 that $\|H_{z^{\boldsymbol{\iota}}\Delta}\| < t_0$, which means that $\boldsymbol{\iota}(H_\Delta) \le \boldsymbol{\iota}$. ∎

Proof of Lemma 10.2. Let $k = \operatorname{ind} T_u$. Clearly, it is sufficient to consider the case $k > 0$. By Theorem 2.5, Φ is badly approximable and $\|H_\Phi\| = t$. We have

$$z^{k-1} \Phi = W^* \begin{pmatrix} t z^{k-1} u & \mathbb{O} \\ \mathbb{O} & z^{k-1} \Delta \end{pmatrix} V^*.$$

Then $\operatorname{wind}(z^{k-1}u) = -1$, and again by Theorem 2.5, $z^{k-1}\Phi$ is badly approximable and $\|\Phi\|_{L^\infty} = t$. Hence,

$$\|H_{z^{k-1}\Phi}\| = \|z^{k-1}\Phi\|_{L^\infty} = \|\Phi\|_{L^\infty} = t = \|H_\Phi\|,$$

and so $\boldsymbol{\iota}(H_\Phi) \ge k$. ■

Proof of Lemma 10.3. Put $\boldsymbol{\iota} \stackrel{\text{def}}{=} \boldsymbol{\iota}(H_\Phi)$. Then

$$\|H_{z^{\boldsymbol{\iota}-1}\Phi}\| = \|H_\Phi\| = \|\Phi\|_{L^\infty} = \|z^{\boldsymbol{\iota}-1}\Phi\|_{L^\infty},$$

and so $z^{\boldsymbol{\iota}-1}\Phi$ is badly approximable. Clearly,

$$\|H_{z^{\boldsymbol{\iota}-1}\Phi}\|_{\mathrm{e}} = \|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\| = \|H_{z^{\boldsymbol{\iota}-1}\Phi}\|.$$

By Theorem 2.2, $z^{\boldsymbol{\iota}-1}\Phi$ admits a representation

$$z^{\boldsymbol{\iota}-1}\Phi = W^* \begin{pmatrix} t\omega & \mathbb{O} \\ \mathbb{O} & \Delta \end{pmatrix} V^*,$$

where $t = \|H_\Phi\|$, ω is a unimodular function such that $\operatorname{ind} T_\omega > 0$, V and W^{t} are thematic functions, and $\|\Delta\|_{L^\infty} \le t$. Therefore

$$\Phi = W^* \begin{pmatrix} t\bar{z}^{\boldsymbol{\iota}-1}\omega & \mathbb{O} \\ \mathbb{O} & \bar{z}^{\boldsymbol{\iota}-1}\Delta \end{pmatrix} V^*.$$

Let $u = \bar{z}^{\boldsymbol{\iota}-1}\omega$. Clearly, $\operatorname{ind} T_u \ge \boldsymbol{\iota}$. Finally, by Lemma 10.2, $\operatorname{ind} T_u = \boldsymbol{\iota}$. ■

Proof of Lemma 10.4. Let

$$V = \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix}, \quad W^{\mathrm{t}} = \begin{pmatrix} \boldsymbol{w} & \Xi \end{pmatrix}.$$

By the remark after Theorem 4.5, there exist $A \in H^\infty(\mathbb{M}_{n-1,n})$ and $B \in H^\infty(\mathbb{M}_{m-1,m})$ such that $A\Theta = \boldsymbol{I}_{n-1}$ and $B\Xi = \boldsymbol{I}_{m-1}$. By Theorem 1.8, without loss of generality we may assume that $\|\Delta\|_{L^\infty} \le 1$.

Suppose that $\|H_\Delta\| = 1$. Since $\|H_\Delta\|_{\mathrm{e}} < 1$, there exists a nonzero function $g \in H^2(\mathbb{C}^{n-1})$ such that $\|H_\Delta g\|_2 = \|g\|_2$. Then $\Delta g \in H^2_-(\mathbb{C}^{m-1})$ and $\|\Delta(\zeta)g(\zeta)\|_{\mathbb{C}^{m-1}} = \|g(\zeta)\|_{\mathbb{C}^{n-1}}$ for almost all $\zeta \in \mathbb{T}$.

Let

$$f = A^{\mathrm{t}}g + q\boldsymbol{v},$$

where q is a scalar function in H^2. We want to find such a q that $\|H_\Phi f\|_2 = \|f\|_2$. Note that f is a nonzero function since

$$V^* f = \begin{pmatrix} \boldsymbol{v}^* \\ \Theta^{\mathrm{t}} \end{pmatrix} (A^{\mathrm{t}}g + q\boldsymbol{v}) = \begin{pmatrix} \boldsymbol{v}^* A^{\mathrm{t}} g + q \\ g \end{pmatrix}$$

and $g \ne 0$.

We have

$$\begin{aligned} \Phi f &= W^* \begin{pmatrix} u & \mathbb{O} \\ \mathbb{O} & \Delta \end{pmatrix} \begin{pmatrix} \boldsymbol{v}^* A^{\mathrm{t}} g + q \\ g \end{pmatrix} \\ &= \begin{pmatrix} \overline{\boldsymbol{w}} & \Xi \end{pmatrix} \begin{pmatrix} u\boldsymbol{v}^* A^{\mathrm{t}} g + uq \\ \Delta g \end{pmatrix} \\ &= \overline{\boldsymbol{w}}(u\boldsymbol{v}^* A^{\mathrm{t}} g + uq) + \Xi\Delta g. \end{aligned}$$

Since W^* and V^* are unitary-valued and $\|\Delta(\zeta)g(\zeta)\|_{\mathbb{C}^{m-1}} = \|g(\zeta)\|_{\mathbb{C}^{n-1}}$, it follows that $\|\Phi(\zeta)f(\zeta)\|_{\mathbb{C}^m} = \|f(\zeta)\|_{\mathbb{C}^n}$. It remains to choose q so that $\Phi f \in H^2_-(\mathbb{C}^m)$.

Since W^* is a unitary-valued matrix function, we have

$$\boldsymbol{I}_m = \begin{pmatrix} \overline{\boldsymbol{w}} & \Xi \end{pmatrix} \begin{pmatrix} \boldsymbol{w}^{\mathrm{t}} \\ \Xi^* \end{pmatrix} = \overline{\boldsymbol{w}}\boldsymbol{w}^{\mathrm{t}} + \Xi\Xi^*.$$

Hence,

$$\Xi = \Xi(B\Xi)^* = \Xi\Xi^* B^* = (\boldsymbol{I}_m - \overline{\boldsymbol{w}}\boldsymbol{w}^{\mathrm{t}})B^*.$$

It follows that

$$\begin{aligned} \Phi f &= \overline{\boldsymbol{w}}(u\boldsymbol{v}^* A^{\mathrm{t}} g + uq) + (\boldsymbol{I}_m - \overline{\boldsymbol{w}}\boldsymbol{w}^{\mathrm{t}})B^*\Delta g \\ &= \overline{\boldsymbol{w}}(u\boldsymbol{v}^* A^{\mathrm{t}} g + uq - \boldsymbol{w}^{\mathrm{t}} B^*\Delta g) + B^*\Delta g. \end{aligned}$$

Clearly, $B^*\Delta g \in H^2_-(\mathbb{C}^m)$, and so it suffices to find $q \in H^2$ such that

$$u\boldsymbol{v}^* A^{\mathrm{t}} g + uq - \boldsymbol{w}^{\mathrm{t}} B^*\Delta g \in H^2_-,$$

which is equivalent to the condition

$$T_u q = \mathbb{P}_+(\boldsymbol{w}^{\mathrm{t}} B^*\Delta g - u\boldsymbol{v}^* A^{\mathrm{t}} g).$$

The existence of such a q follows from the fact that T_u is Fredholm and $\operatorname{ind} T_u = 0$. ■

Corollary 10.5. *Let Φ be a very badly approximable matrix function in $L^\infty(\mathbb{M}_{m,n})$ such that $\|H_\Phi\|_{\mathrm{e}}$ is less than the smallest nonzero superoptimal singular value of Φ. Then Φ admits a monotone thematic factorization.*

Corollary 10.6. *Let Φ be a very badly approximable matrix function in $(H^\infty + C)(\mathbb{M}_{m,n})$. Then Φ admits a monotone thematic factorization.*

We are going to prove now that the indices of a monotone thematic factorization are uniquely determined by the function itself. We need the following lemma.

Lemma 10.7. *Suppose that a matrix function $\Phi \in L^\infty(\mathbb{M}_{m,n})$ admits a factorization of the form*

$$\Phi = W_0^* \cdots W_{r-1}^* \begin{pmatrix} tu_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & tu_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & tu_{r-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \Psi \end{pmatrix} V_{r-1}^* \cdots V_0^*,$$

where the V_j and W_j are of the form (5.2), *$\|H_\Psi\| < t$, and the u_j are unimodular functions such that T_{u_j} is Fredholm and $\operatorname{ind} T_{u_j} \le 0$. If $\|H_\Phi\|_{\mathrm{e}} < t$, then $\|H_\Phi\| < t$.*

Proof. We argue by induction on r. Let $r = 1$. We have

$$\Phi = W^* \begin{pmatrix} tu & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^*,$$

where V and W^{t} are thematic matrix functions, u is a unimodular function such that T_u is Fredholm, $\operatorname{ind} T_u \le 0$, and $\|H_\Psi\| < t$. It follows from Theorem 1.8 that we may subtract from Ψ a best analytic approximation without changing H_Φ, and so we may assume that $\|\Psi\|_{L^\infty} < t$. Without loss of generality we may also assume that $t = 1$.

Suppose that $\|H_\Phi\| = 1$. Since $\|H_\Phi\|_{\mathrm{e}} < 1$, there exists a nonzero function $f \in H^2(\mathbb{C}^n)$ such that $\|H_\Phi f\|_2 = \|f\|_2$. Then $\|\Phi f\|_2 = \|f\|_2$ and since $\|\Psi\|_{L^\infty} < 1$, it follows that $V^* f$ has the form

$$V^* f = \begin{pmatrix} * \\ \mathbb{O} \\ \vdots \\ \mathbb{O} \end{pmatrix}. \tag{10.5}$$

Let $\boldsymbol{v}$ be the first column of V. Equality (10.5) means that for almost all $\zeta \in \mathbb{T}$ the remaining columns of $V(\zeta)$ are orthogonal to $f(\zeta)$ in $\mathbb{C}^n$. Since V is unitary-valued, it follows that $f = \xi \boldsymbol{v}$ for a scalar function $\xi \in L^2$. By Lemma 1.4, $\xi \in H^2$. Note that $\|f\|_{H^2(\mathbb{C}^n)} = \|\xi\|_{H^2}$.

We have

$$\Phi f = W^* \begin{pmatrix} u\xi \\ \mathbb{O} \\ \vdots \\ \mathbb{O} \end{pmatrix} = u\xi \overline{\boldsymbol{w}},$$

where $\boldsymbol{w}$ is the first column of W^{t}. Since f is a maximizing vector of H_Φ, we have $u\xi\overline{\boldsymbol{w}} \in H^2_-(\mathbb{C}^n)$. Again, by Lemma 1.4, we find that $u\xi \in H^2_-$, i.e., $\xi \in \operatorname{Ker} T_u$. However, T_u has trivial kernel since $\operatorname{ind} T_u \le 0$. We have obtained a contradiction.

Suppose now that $r > 1$. Again, we may assume that $\|\Psi\|_{L^\infty} < t$. Let d be a negative integer such that $d < \operatorname{ind} T_{u_j}$, $0 \le j \le r-1$. Then

$$z^d \Phi = W_0^* \cdots W_{r-1}^* \begin{pmatrix} tz^d u_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & tz^d u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & tz^d u_{r-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & z^d \Psi \end{pmatrix} V_{r-1}^* \cdots V_0^*$$

is a partial thematic factorization of $z^d \Phi$. Put

$$\Delta = \breve{W}_1^* \cdots \begin{pmatrix} \boldsymbol{I}_{r-2} & \mathbb{O} \\ \mathbb{O} & \breve{W}_{r-1}^* \end{pmatrix} D \begin{pmatrix} \boldsymbol{I}_{r-2} & \mathbb{O} \\ \mathbb{O} & \breve{V}_{r-1}^* \end{pmatrix} \cdots \breve{V}_1^*,$$

where

$$D = \begin{pmatrix} t_1 u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \cdots & t_{r-1} u_{r-1} & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \Psi \end{pmatrix},$$

and the $\breve{V}_j$ and $\breve{W}_j$ are as in (5.2). Since obviously, $\|H_{z^d\Delta}\|_{\mathrm{e}} = \|H_\Delta\|_{\mathrm{e}}$ for any $d \in \mathbb{Z}$, it follows from Theorem 4.1 that $\|H_{z^d\Delta}\|_{\mathrm{e}} < t$, and so by the inductive hypotheses, $\|H_\Delta\| < t$. We have

$$\Phi = W_0^* \begin{pmatrix} tu & \mathbb{O} \\ \mathbb{O} & \Delta \end{pmatrix} V_0^*.$$

The result follows now from the case $r = 1$, which has already been established. ■

Theorem 10.8. *Let Φ be a badly approximable function in $L^\infty(\mathbb{M}_{m,n})$ such that $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$ and let r be the number of superoptimal singular values of Φ equal to $t_0 = \|H_\Phi\|$. Consider a monotone partial thematic factorization of Φ with indices*

$$k_0 \ge \cdots \ge k_{r-1} \tag{10.6}$$

corresponding to the superoptimal singular values equal to t_0. Let $\varkappa \ge 0$. Then

$$\dim\{f \in H^2(\mathbb{C}^n) : \|H_{z^\varkappa\Phi} f\|_2 = t_0\|f\|_2\} = \sum_{\{j\in[0,r-1]:k_j>\varkappa\}} k_j - \varkappa. \tag{10.7}$$

Proof. Let

$$\Phi = \begin{pmatrix} t_0u_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_0u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & t_0u_{r-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \Psi \end{pmatrix}$$

be a partial thematic factorization of Φ with indices satisfying (10.6). If $\varkappa \ge k_0$, then (10.7) holds by Lemma 10.7. Suppose now that $\varkappa < k_0$. Let

$$q = \max\{j \in [0, r-1] : \ k_j > \varkappa\}.$$

Clearly, the function $z^\varkappa\Phi$ admits the following representation:

$$z^\varkappa\Phi = W_0^* \cdots W_q^* \begin{pmatrix} t_0z^\varkappa u_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_0z^\varkappa u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & t_0z^\varkappa u_q & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \Delta \end{pmatrix} V_q^* \cdots V_0^*,$$

where Δ is a matrix function satisfying the hypotheses of Lemma 10.7. By Lemma 10.7, $\|H_\Delta\| < t_0$. Let $R \in H^\infty$ be a matrix function such that $\|\Delta - R\|_{L^\infty} < t_0$. It follows easily from Theorem 1.8 by induction on q that if we perturb Δ by a bounded analytic matrix function, $z^\varkappa\Phi$ also changes by an analytic matrix function. In particular, we can find a matrix function

$G \in H^\infty$ such that

$$
\begin{aligned}
&z^\varkappa \Phi - G \\
&= W_0^* \cdots W_q^* \begin{pmatrix} t_0 z^\varkappa u_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_0 z^\varkappa u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & t_0 z^\varkappa u_q & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \Delta - R \end{pmatrix} V_q^* \cdots V_0^*.
\end{aligned}
$$

By Theorem 7.4,

$$
\dim\{f \in H^2(\mathbb{C}^n) : \ \|H_{z^\varkappa \Phi - G} f\|_2 = t_0 \|f\|_2\} = \sum_{\{j \in [0, r-1] : k_j > \varkappa\}} k_j - \varkappa.
$$

Equality (10.7) follows now from the obvious fact that $H_{z^\varkappa \Phi - G} = H_{z^\varkappa \Phi}$. ■

We can now deduce from (10.7) the following result.

Theorem 10.9. *Suppose that $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and $q \le \min\{m, n\}$ is a positive integer such that the superoptimal singular values of Φ satisfy*

$$
t_{q-1} > t_q, \quad t_{q-1} > \|H_\Phi\|_{\mathrm{e}}
$$

and Φ admits a monotone partial thematic factorization

$$
\Phi = W_0^* \cdots W_{q-1}^* \begin{pmatrix} t_0 u_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_1 u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & t_{q-1} u_{q-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \Delta \end{pmatrix} V_{q-1}^* \cdots V_0^*.
$$

Then the indices of this factorization are uniquely determined by the function Φ itself.

Proof. Let r be the number of superoptimal singular values equal to $\|H_\Phi\|$. Then Φ admits the following partial thematic factorization

$$
\Phi = W_0^* \cdots W_{r-1}^* \begin{pmatrix} t_0 u_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_1 u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & t_{r-1} u_{r-1} & \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \Psi \end{pmatrix} V_{r-1}^* \cdots V_0^*,
$$

where

$$\Psi = \breve{W}_r^* \cdots \begin{pmatrix} \boldsymbol{I}_{q-r-1} & \mathbb{O} \\ \mathbb{O} & \breve{W}_{q-1}^* \end{pmatrix} D \begin{pmatrix} \boldsymbol{I}_{q-r-1} & \mathbb{O} \\ \mathbb{O} & \breve{V}_{q-1}^* \end{pmatrix} \cdots \breve{V}_r^*,$$

$$D = \begin{pmatrix} t_r u_r & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \cdots & t_{q-1}u_{q-1} & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \Delta \end{pmatrix},$$

and the $\breve{V}_j$ and $\breve{W}_j$ are as in (5.2).

By Theorem 9.3, Ψ is determined uniquely by Φ modulo constant unitary factors. Hence, it is sufficient to show that the indices $k_0, \cdots, k_{r-1}$ are uniquely determined by Φ.

It follows easily from (10.7) that

$$k_0 = \min\left\{\varkappa : \ \dim\{f \in H^2(\mathbb{C}^n) : \ \|H_{z^\varkappa \Phi} f\|_2 = t_0\|f\|_2\} = 0\right\}.$$

Now let d be the number of indices among $k_0, \cdots, k_{r-1}$ that are to equal to k_0. It follows easily from (10.7) that

$$d = \dim\{f \in H^2(\mathbb{C}^n) : \ \|H_{z^{k_0-1}\Phi} f\|_2 = t_0\|f\|_2\}.$$

Next, if $d < r$, then it follows from (10.7) that

$$k_d = \min\left\{\varkappa : \ \dim\{f \in H^2(\mathbb{C}^n) : \ \|H_{z^\varkappa \Phi} f\|_2 = t_0\|f\|_2\} = d(k_0 - \varkappa)\right\}.$$

Similarly, we can determine the multiplicity of the index k_d, then the next largest index, etc. ■

Corollary 10.10. *Suppose that Φ is very badly approximable function in $L^\infty(\mathbb{M}_{m,n})$ such that $\|H_\Phi\|_{\mathrm{e}}$ is less than the largest nonzero superoptimal singular value of Φ. Then the indices of a monotone thematic factorization of Φ are uniquely determined by Φ.*

Corollary 10.11. *Let Φ be a very badly approximable function in $(H^\infty + C)(\mathbb{M}_{m,n})$. Then the indices of a monotone thematic factorization of Φ are uniquely determined by Φ.*

11. Construction of Superoptimal Approximation and the Corona Problem

In §§2–4 we have given an algorithm to find a superoptimal approximation of a matrix function $\Phi \in L^\infty(\mathbb{M}_{m,n})$ that satisfies certain natural conditions. Recall that if $\min\{m,n\} > 1$, we start with a best approximation $F \in H^\infty(\mathbb{M}_{m,n})$ and find a factorization

$$\Phi - F = W^* \begin{pmatrix} t_0 u & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^*,$$

where V and W^{t} are thematic matrix functions of the form

$$V = \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix}, \quad W^{\mathrm{t}} = \begin{pmatrix} \boldsymbol{w} & \Xi \end{pmatrix}. \tag{11.1}$$

Then it has been shown in §2 that the problem of finding the superoptimal approximation of Φ reduces to the problem of finding the superoptimal approximation of Ψ that has a smaller size.

However, the problem is that first we have to find a best approximation F and the procedure discussed in §§2–4 does not give any hint how to find such a best approximation F. This seems even more disappointing since we want to prove in §12 that the nonlinear operator of superoptimal approximation has hereditary properties and to do it we have to find a best approximation F that inherits properties of Φ. Fortunately, to use the above scheme, F does not have to be a best approximation. It is sufficient to assume that $F \in H^\infty(\mathbb{M}_{m,n})$ and satisfies certain equations. We show in this section how to find such a function F by solving certain corona problems.

Throughout this section we assume that $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$. Recall that to find thematic matrix functions V and W^{t}, we start with a maximizing vector $\boldsymbol{f}$ of H_Φ and define $\boldsymbol{g} = t_0^{-1}\bar{z}\overline{H_\Phi \boldsymbol{f}}$. The vector functions $\boldsymbol{f}$ and $\boldsymbol{g}$ admit factorizations

$$\boldsymbol{f} = \vartheta_1 h \boldsymbol{v}, \quad \boldsymbol{g} = \vartheta_2 h \boldsymbol{w},$$

where ϑ_1 and ϑ_2 are scalar inner functions, h is a scalar outer, function in H^2, and $\boldsymbol{v}$ and $\boldsymbol{w}$ are inner and co-outer column functions. The entry u in (11.1) must satisfy $u = \bar{z}\bar{\vartheta}_1\bar{\vartheta}_2\bar{h}/h$.

If F is a best approximation of Φ, then by Theorem 2.2.3,

$$H_\Phi \boldsymbol{f} = (\Phi - F)\boldsymbol{f} = t_0 \overline{z\boldsymbol{g}}, \quad H_\Phi^* \overline{z\boldsymbol{g}} = (\Phi - F)^* \overline{z\boldsymbol{g}} = t_0 \boldsymbol{f},$$

and so

$$F\boldsymbol{f} = \Phi\boldsymbol{f} - t_0\overline{z\boldsymbol{g}} = \mathbb{P}_+\Phi\boldsymbol{f}, \quad F^*\overline{z\boldsymbol{g}} = \Phi^*\overline{z\boldsymbol{g}} - t_0\boldsymbol{f} = \mathbb{P}_-\Phi^*\overline{z\boldsymbol{g}}.$$

Thus F must satisfy the following equations

$$F\boldsymbol{f} = T_\Phi \boldsymbol{f}, \quad F^{\mathrm{t}}\boldsymbol{g} = T_{\Phi^{\mathrm{t}}}\boldsymbol{g}.$$

It follows that

$$F\boldsymbol{v} = \varphi_1, \quad F^{\mathrm{t}}\boldsymbol{w} = \varphi_2, \tag{11.2}$$

where

$$\varphi_1 \stackrel{\mathrm{def}}{=} \frac{\bar{\vartheta}_1 T_\Phi \boldsymbol{f}}{h}, \quad \varphi_2 \stackrel{\mathrm{def}}{=} \frac{\bar{\vartheta}_2 T_{\Phi^{\mathrm{t}}}\boldsymbol{g}}{h}. \tag{11.3}$$

Clearly, (11.2) implies that $\varphi_1 \in H^\infty(\mathbb{C}^m)$ and $\varphi_2 \in H^\infty(\mathbb{C}^n)$. It also follows from (11.2) that

$$\varphi_2^{\mathrm{t}}\boldsymbol{v} = \boldsymbol{w}^{\mathrm{t}}\varphi_1, \tag{11.4}$$

since both functions are equal to $\boldsymbol{w}^{\mathrm{t}}F\boldsymbol{v}$.

Theorem 11.1. *Suppose that $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and $\|H_\Phi\|_e < \|H_\Phi\|$. Let F be an arbitrary matrix function in $H^\infty(\mathbb{M}_{m,n})$ satisfying* (11.2). *Then $\Phi - F$ satisfies* (11.1) *for some matrix function $\Psi \in L^\infty(\mathbb{M}_{m-1,n-1})$. A matrix function $G \in H^\infty(\mathbb{M}_{m-1,n-1})$ is a superoptimal approximation of Ψ if and only if $F + \Xi G \Theta^{\mathrm{t}}$ is superoptimal approximation of Φ.*

Proof. We have

$$\begin{aligned} W(\Phi - F)V &= \begin{pmatrix} \boldsymbol{w}^{\mathrm{t}} \\ \Xi^* \end{pmatrix} (\Phi - F) \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{w}^{\mathrm{t}}(\Phi - F)\boldsymbol{v} & \boldsymbol{w}^{\mathrm{t}}(\Phi - F)\overline{\Theta} \\ \Xi^*(\Phi - F)\boldsymbol{v} & \Xi^*(\Phi - F)\overline{\Theta} \end{pmatrix}. \end{aligned}$$

It follows from (11.2) that

$$\begin{aligned} \Xi^*(\Phi - F)\boldsymbol{v} &= \Xi^*\Phi\boldsymbol{v} - \Xi^* F\boldsymbol{v} = \Xi^*\Phi\boldsymbol{v} - \Xi^*\frac{\bar{\vartheta}_1 T_\Phi \boldsymbol{f}}{h} \\ &= \frac{\bar{\vartheta}_1}{h}\Xi^*(\Phi\boldsymbol{f} - T_\Phi\boldsymbol{f}) = \frac{\bar{\vartheta}_1}{h}\Xi^* H_\Phi\boldsymbol{f} \\ &= t_0\frac{\bar{\vartheta}_1}{h}\Xi^*\overline{z\boldsymbol{g}} = t_0\frac{\bar{\vartheta}_1\bar{\vartheta}_2\bar{h}}{h}\Xi^*\overline{\boldsymbol{w}} = \mathbb{O}, \end{aligned}$$

since the matrix function W is unitary-valued.

Similarly,

$$\boldsymbol{w}^{\mathrm{t}}(\Phi - F)\overline{\Theta} = (\Theta^*(\Phi^{\mathrm{t}} - F^{\mathrm{t}})\boldsymbol{w})^{\mathrm{t}} = \mathbb{O}.$$

Next,

$$\boldsymbol{w}^{\mathrm{t}}(\Phi - F)\boldsymbol{v} = \frac{\bar{\vartheta}_1}{h}\boldsymbol{w}^{\mathrm{t}} H_\Phi\boldsymbol{f} = t_0\frac{\bar{\vartheta}_1}{h}\boldsymbol{w}^{\mathrm{t}}\bar{z}\boldsymbol{g} = t_0\frac{\bar{\vartheta}_1\bar{\vartheta}_2\bar{h}}{h} = t_0 u.$$

This proves (11.1) with $\Psi = \Xi^*(\Phi - F)\overline{\Theta}$.

The rest of the theorem follows from Theorem 1.8 in the same way as it was done in the proof of Theorem 2.4. ■

Now we are going to find a matrix function $F \in H^\infty(\mathbb{M}_{m,n})$ that satisfies (11.2). Consider the vector functions $\boldsymbol{v}$ and $\boldsymbol{w}$. By Theorem 4.3, the Toeplitz operators $T_{\overline{\boldsymbol{v}}}$ and $T_{\overline{\boldsymbol{w}}}$ are left invertible. By Theorem 3.6.1, the column functions $\boldsymbol{v}$ and $\boldsymbol{w}$ are left invertible in H^∞. In other words, we can solve the corona problems with data $\boldsymbol{v}$ and $\boldsymbol{w}$, and find vector functions $\psi_1 \in H^\infty(\mathbb{C}^n)$ and $\psi_2 \in H^\infty(\mathbb{C}^m)$ such that

$$\psi_1^{\mathrm{t}}\boldsymbol{v} = \psi_2^{\mathrm{t}}\boldsymbol{w} = \mathbf{1}, \tag{11.5}$$

where as usual $\mathbf{1}$ is the function identically equal to 1.

Theorem 11.2. *Let*

$$F = \varphi_1\psi_1^{\mathrm{t}} + \psi_2\varphi_2^{\mathrm{t}} - \psi_2\varphi_2^{\mathrm{t}}\boldsymbol{v}\psi_1^{\mathrm{t}}, \tag{11.6}$$

where φ_1 and φ_2 are given by (11.3), *and ψ_1 and ψ_2 satisfy* (11.5). *Then $F \in H^\infty(\mathbb{M}_{m,n})$ and F satisfies* (11.2).

Proof. Clearly, $F \in H^\infty(\mathbb{M}_{m,n})$. We have

$$F\boldsymbol{v} = \varphi_1\psi_1^{\mathrm{t}}\boldsymbol{v} + \psi_2\varphi_2^{\mathrm{t}}\boldsymbol{v} - \psi_2\varphi_2^{\mathrm{t}}\boldsymbol{v}\psi_1^{\mathrm{t}}\boldsymbol{v}.$$

By (11.5), $\varphi_1\psi_1^{\mathrm{t}}\boldsymbol{v} = \varphi_1$. By (11.4) and (11.5), $\psi_2\varphi_2^{\mathrm{t}}\boldsymbol{v} = \psi_2\boldsymbol{w}^{\mathrm{t}}\varphi_1 = \varphi_1$. Finally, $\psi_2\varphi_2^{\mathrm{t}}\boldsymbol{v}\psi_1^{\mathrm{t}}\boldsymbol{v} = \varphi_1\psi_1^{\mathrm{t}}\boldsymbol{v} = \varphi_1$ by (11.5). This proves that $F\boldsymbol{v} = \varphi_1$.

Let us prove the second equality in (11.2). We have

$$F^{\mathrm{t}}\boldsymbol{w} = \psi_1\varphi_1^{\mathrm{t}}\boldsymbol{w} + \varphi_2\psi_2^{\mathrm{t}}\boldsymbol{w} - \psi_1\boldsymbol{v}^{\mathrm{t}}\varphi_2\psi_2^{\mathrm{t}}\boldsymbol{w}.$$

By (11.4) and (11.5), $\psi_1\varphi_1^{\mathrm{t}}\boldsymbol{w} = \psi_1\boldsymbol{v}^{\mathrm{t}}\varphi_2 = \varphi_2$. By (11.5), $\varphi_2\psi_2^{\mathrm{t}}\boldsymbol{w} = \varphi_2$. Finally, by (11.5), $\psi_1\boldsymbol{v}^{\mathrm{t}}\varphi_2\psi_2^{\mathrm{t}}\boldsymbol{w} = \varphi_2$, which completes the proof of the theorem. ■

The procedure described above reduces the problem of finding a superoptimal approximation of Φ to the corresponding problem for Ψ. If $\|H_\Phi\|_{\mathrm{e}}$ is less than the smallest nonzero superoptimal singular value of Φ, we can iterate this procedure as we have done in §§3–4 and eventually find the superoptimal approximation of Φ. An advantage of the algorithm given in this section is that it does not require finding a best approximation. Instead, it depends on the solution of certain corona problems and we shall see in the next two sections that this will be beneficial for the study of hereditary properties and continuity properties of the nonlinear operator of superoptimal approximation. We refer the reader to Jones [1], where a constructive solution of the corona problem is given.

12. Hereditary Properties of Superoptimal Approximation

Theorem 3.3 enables us to define the nonlinear *operator* $\mathcal{A}^{(m,n)}$ *of superoptimal approximation* on the class $(H^\infty + C)(\mathbb{M}_{m,n})$. Namely, if $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$ and $Q \in H^\infty(\mathbb{M}_{m,n})$ is the superoptimal approximation of Φ by analytic matrix functions, we put

$$\mathcal{A}^{(m,n)}\Phi = Q.$$

In Chapter 7 we have introduced the notation $\mathcal{A}$ for the operator of best approximation by scalar analytic functions. Clearly, $\mathcal{A} = \mathcal{A}^{(1,1)}$. We are also going to use the notation $\mathcal{A} = \mathcal{A}^{(m,n)}$ if this does not lead to confusion.

Consider now a space X of functions on $\mathbb{T}$. We study in this section the question for which spaces X the condition $\Phi \in X$ implies that $\mathcal{A}\Phi \in X$. (By definition a matrix function belongs to X if all its entries belong to X.)

Recall that a function space is called hereditary for the operator $\mathcal{A}^{(1,1)}$ of best approximation by analytic functions if $\mathcal{A}^{(1,1)}X \subset X$. We say that X is called *completely hereditary* if $\mathcal{A}^{(m,n)}X(\mathbb{M}_{m,n}) \subset X(\mathbb{M}_{m,n})$ for all positive integers m and n.

It is convenient to enlarge the domain of the operator $\mathcal{A}^{(m,n)}$ to the space $VMO(\mathbb{M}_{m,n})$. Indeed, if $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$ and $F \in H^\infty(\mathbb{M}_{m,n})$, then $\mathcal{A}^{(m,n)}(\Phi + F) = \mathcal{A}^{(m,n)}\Phi + F$. Suppose now that $\Psi \in VMO(\mathbb{M}_{m,n})$. Then Ψ admits a representation $\Psi = \Phi + G$, where Φ is continuous and $G \in VMOA(\mathbb{M}_{m,n})$ and we put

$$\mathcal{A}\Psi = \mathcal{A}\Phi + G.$$

It is easy to see that the operator of superoptimal approximation is well defined on VMO. Thus we can consider the problem of whether a function space X is completely hereditary as far as $X \subset VMO$.

We show in this section that both linear $\mathcal{R}$-spaces (see §7.1) and decent function spaces (see §7.2) are completely hereditary. We also show that $\mathcal{A}^{(m,n)}$ is bounded on (quasi-)Banach $\mathcal{R}$-spaces, i.e., there exists $c_{m,n} > 0$ such that

$$\|\mathcal{A}^{(m,n)}\Phi\|_X \le c_{m,n}\|\Phi\|_X$$

(see §13.1 where the notation $\|\Phi\|_X$ is explained). We also prove in this section that if $\Phi \in X$, then $\Phi - \mathcal{A}\Phi$ has a thematic factorization with all factors in X. Note, however, that we use different methods to treat $\mathcal{R}$-spaces and decent function spaces.

Let us start with the following lemma.

Lemma 12.1. *Suppose that X is a linear $\mathcal{R}$-space or X is a decent function space. Let V be a balanced matrix function of the form*

$$V = \begin{pmatrix} \Upsilon & \overline{\Theta} \end{pmatrix},$$

where Υ and Θ are inner matrix functions. If $\Upsilon \in X$, then $\Theta \in X$.

If X is a (quasi-)Banach $\mathcal{R}$-space, then

$$\|\Theta\|_X \le \text{const}\,\|\Upsilon\|_X.$$

Note that in this section constants may depend on the size of matrix functions.

Proof. By Theorem 1.3, T_{V^*} has dense range. By the assumptions, $\mathbb{P}_- V^* \in X$. The result now follows from Theorem 13.1.1. ■

First we prove that linear $\mathcal{R}$-spaces are completely hereditary. Let Φ be an $m \times n$ matrix function in VMO. Then there exists an $m \times n$ matrix function $F \in BMOA$ such that $\Phi - F$ is bounded on $\mathbb{T}$ and $\|\Phi - F\|_{L^\infty} = \|H_\Phi\|$. In other words, $\Phi - F$ is a badly approximable matrix function. If $\mathbb{P}_-\Phi \ne \mathbb{O}$, then $\Phi - F$ admits a factorization

$$\Phi - F = W^* \begin{pmatrix} t_0 u & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^*, \tag{12.1}$$

where V and W^{t} are thematic matrix functions of the form

$$V = \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix}, \quad W^{\mathrm{t}} = \begin{pmatrix} \boldsymbol{w} & \overline{\Xi} \end{pmatrix}. \tag{12.2}$$

Lemma 12.2. *Let X be a linear $\mathcal{R}$-space and let Φ be a matrix function in X. Then $u \in X$, $\boldsymbol{v} \in X$, and $\boldsymbol{w} \in X$. If X is a (quasi-)Banach $\mathcal{R}$-space, then*

$$\|u\|_X \le \text{const}\,\|\Phi\|_X, \quad \|\boldsymbol{v}\|_X \le \text{const}\,\|\Phi\|_X, \quad \|\boldsymbol{w}\|_X \le \text{const}\,\|\Phi\|_X. \tag{12.3}$$

Proof. Clearly, we may assume that $\mathbb{P}_-\Phi \ne \mathbb{O}$. It follows from (12.1) that

$$\boldsymbol{w}^{\mathrm{t}}(\Phi - F)\boldsymbol{v} = t_0 u.$$

We have

$$t_0\mathbb{P}_+\bar{z}\bar{u} = \mathbb{P}_+\boldsymbol{w}^*(\bar{z}\overline{\Phi} - \bar{z}\overline{F})\overline{\boldsymbol{v}} = \mathbb{P}_+\boldsymbol{w}^*\bar{z}\overline{\Phi}\overline{\boldsymbol{v}} = \mathbb{P}_+\boldsymbol{w}^*(\mathbb{P}_+\bar{z}\overline{\Phi})\overline{\boldsymbol{v}}.$$

It follows from Lemma 7.1.1 that $\mathbb{P}_+\bar{z}\bar{u} \in X$. Hence,

$$\mathbb{P}_-u = \bar{z}\overline{\mathbb{P}_+\bar{z}\bar{u}} \in X.$$

By Theorem 2.2, u has the form $u = \bar{\vartheta}\bar{h}/h$, where ϑ is an inner function and h is an outer function in H^2. It follows from Theorem 4.4.10 that T_u has dense range in H^2. Thus by Theorem 7.1.4, $u \in X$.

It follows from (12.1) that

$$W(\Phi - F) = \begin{pmatrix} t_0u & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^*.$$

Take the first row to get

$$\boldsymbol{w}^{\mathrm{t}}(\Phi - F) = t_0u\boldsymbol{v}^*.$$

Consequently,

$$t_0\mathbb{P}_-u\boldsymbol{v}^* = t_0\overline{z\mathbb{P}_+\bar{z}\bar{u}\boldsymbol{v}^{\mathrm{t}}} = \overline{z\mathbb{P}_+\bar{z}\boldsymbol{w}^*\overline{(\Phi - F)}} = \overline{z\mathbb{P}_+\bar{z}\boldsymbol{w}^*\overline{\mathbb{P}_-\Phi}}.$$

Again, it follows from Lemma 7.1.1 that $\mathbb{P}_-u\boldsymbol{v}^* \in X$. Clearly,

$$\mathbb{P}_+u\boldsymbol{v}^* = T_{\boldsymbol{v}^*}u \in X$$

by Lemma 7.1.1. Hence $u\boldsymbol{v}^* \in X$, and since $\bar{u} \in X$, it follows from Lemma 7.1.1 that $\boldsymbol{v}^* = \bar{u}u\boldsymbol{v}^* \in X$. Therefore $\boldsymbol{v} \in X$. If we apply the same argument to Φ^{t}, we find that $\boldsymbol{w} \in X$.

It is easy to see that in case X is a (quasi-)Banach $\mathcal{R}$-space, the above reasonings show that (12.3) holds. ■

Now we can prove that heredity result for linear $\mathcal{R}$-space.

Theorem 12.3. *Let X be a linear $\mathcal{R}$-space. Then X is completely hereditary. If X is a (quasi-)Banach $\mathcal{R}$-space, then the operators $\mathcal{A}^{(m,n)}$ are bounded on $X(\mathbb{M}_{m,n})$.*

Proof. We may assume that $\Phi \in L^\infty$. Indeed, for any $\Phi \in X$ we can write

$$\Phi = \Phi_1 + \Phi_2 \tag{12.4}$$

for some $\Phi_1 \in X \cap L^\infty$ and $\Phi_2 \in VMO_A$. It is sufficient to find such a representation for each entry of Φ. For a scalar function φ in X we can write

$$\varphi = (\varphi - \mathcal{A}\varphi) + \mathcal{A}\varphi.$$

The fact that $\varphi - \mathcal{A}\varphi \in X$ follows from Theorem 7.1.6. If Φ is represented as in (12.4), then clearly, $\mathcal{A}\Phi = \mathcal{A}\Phi_1 + \Phi_2$.

Suppose first that $\min\{m, n\} = 1$. Without loss of generality we may assume that $m \le n$. Since the scalar case has already been treated in §7.1, consider the case $n > 1$. If $t_0 = 0$, the result is trivial. Otherwise, let F be the best approximation of Φ (it is unique by Theorem 2.2). Then

$$\Phi - F = W^* \begin{pmatrix} t_0 u & \mathbb{O} \end{pmatrix}$$

(see comments after the statement of Theorem 2.2). By Lemmas 12.1 and 12.2, u and W belong to X, which implies that $F \in X$.

Suppose now that $d = \min\{m, n\} > 1$. We argue by induction on d. Suppose that $F \in H^\infty(\mathbb{M}_{m,n})$ is the superoptimal approximation of Φ by bounded analytic matrix functions. Then $\Phi - F$ admits a factorization (12.1) with very badly approximable matrix function $\Psi \in L^\infty(\mathbb{M}_{m-1,n-1})$. It follows from (12.1) that

$$W(\Phi - F)V = \begin{pmatrix} t_0 u & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix}.$$

If we look at the lower right entries of this equality, we obtain

$$\Psi = \Xi^*(\Phi - F)\overline{\Theta},$$

and so

$$\mathbb{P}_-\Psi = \bar{z}(\mathbb{P}_+\bar{z}\Psi^*)^* = \bar{z}(\mathbb{P}_+\bar{z}\Theta^{\mathrm{t}}\Phi^*\Xi)^* - \bar{z}(\mathbb{P}_+\bar{z}\Theta^{\mathrm{t}}F^*\Xi)^*.$$

By Lemma 12.1, Θ and Ξ belong to X, and so by Lemma 7.1.1, $\bar{z}(\mathbb{P}_+\bar{z}\Theta^{\mathrm{t}}\Phi^*\Xi)^* \in X$. Again, since $F \in H^\infty$, it follows from Lemma 7.1.1 that $\bar{z}(\mathbb{P}_+\bar{z}\Theta^{\mathrm{t}}F^*\Xi)^*$, which proves that $\mathbb{P}_-\Psi \in X$.

Since Ψ is very badly approximable, $\mathcal{A}\mathbb{P}_-\Psi = -\mathbb{P}_+\Psi$, and by the inductive hypothesis, $\mathcal{A}\mathbb{P}_-\Psi \in X$, which proves that $\Psi \in X$. It follows now from (12.1) that $F \in X$.

It is easy to see that if X is a (quasi-)Banach $\mathcal{R}$-space, then the above reasonings show that $\|F\|_X \le \text{const}\,\|\Phi\|_X$, i.e., the operator of superoptimal approximation is bounded on $X(\mathbb{M}_{m,n})$. ■

Corollary 12.4. *Let Φ be a rational matrix function with poles outside $\mathbb{T}$. Then $\mathcal{A}\Phi$ is also rational.*

Corollary 12.5. *Let Φ be a matrix function in $H^\infty + C$. Then $\Phi - \mathcal{A}\Phi \in QC$.*

Proof. We have

$$\mathcal{A}\Phi = \mathcal{A}\mathbb{P}_-\Phi + \mathbb{P}_+\Phi,$$

and so

$$\Phi - \mathcal{A}\Phi = \mathbb{P}_-\Phi - \mathcal{A}\mathbb{P}_-\Phi.$$

Clearly, $\mathbb{P}_-\Phi \in VMO$ and since VMO is an $\mathcal{R}$-space, it follows from Theorem 12.3 that $\mathcal{A}\mathbb{P}_-\Phi \in VMO$. The result follows from the fact that $QC = L^\infty \cap VMO$ (see Appendix 2.5). ■

We proceed now to the case of decent function spaces. To prove that linear $\mathcal{R}$-spaces X are completely hereditary we have used essentially the following property:

$$\varphi \in X_+, \quad f \in H^\infty \quad \Longrightarrow \quad \mathbb{P}_+\bar{f}\varphi \in X. \tag{12.5}$$

Many classical decent spaces (e.g., Hölder spaces, Besov spaces) possess property (12.5). However, there are important decent spaces that do not satisfy (12.5). For example, the space $\mathcal{F}\ell^1$ of functions with absolutely converging Fourier series does not satisfy (12.5). Indeed, if (12.5) holds for $X = \mathcal{F}\ell^1$, then for any $f \in H^\infty$ there exists a constant c such that $\|\mathbb{P}_+\bar{f}\varphi\|_{\mathcal{F}\ell^1} \le c\|\varphi\|_{\mathcal{F}\ell^1}$. Now it remains to take $\varphi = z^n$ to see that f must belong to $\mathcal{F}\ell^1$.

To prove that decent spaces are completely hereditary we use properties of maximizing vectors of Hankel operators.

Theorem 12.6. *Let X be a decent function space. Then X is completely hereditary.*

Proof. Let Φ be an $m \times n$ matrix function in X. We want to prove that $\mathcal{A}\Phi \in X$. If $\mathbb{P}_-\Phi = \mathbb{O}$, then $\mathcal{A}\Phi = \Phi$ and the result is trivial. Suppose now that $\mathbb{P}_-\Phi \ne \mathbb{O}$.

By Theorem 13.2.2, there is a maximizing vector $\boldsymbol{f}$ of H_Φ such that $\boldsymbol{f}(\zeta) \ne \mathbb{O}$ for any $\zeta \in \mathbb{T}$. It admits a factorization of the form

$$\boldsymbol{f} = \vartheta_1 h \boldsymbol{v},$$

where ϑ_1 is a scalar inner function, h is a scalar outer function, and $\boldsymbol{v}$ is an inner and co-outer column function (see §2). Let $\boldsymbol{g} = t_0^{-1}\bar{z}\overline{H_\Phi \boldsymbol{f}}$. Then $\boldsymbol{g}$ is a maximizing vector of $H_{\Phi^{\rm t}}$ and $\boldsymbol{g}$ admits a factorization of the form

$$\boldsymbol{g} = \vartheta_2 h \boldsymbol{w},$$

where ϑ_2 is a scalar inner function and $\boldsymbol{w}$ is an inner and co-outer column function (see §2). By Theorem 4.2, ϑ_1 and ϑ_2 are finite Blaschke products. By the axiom (A4) (see §7.2), ϑ_1 and ϑ_2 belong to X.

Since $|h(\zeta)| = \|\boldsymbol{f}(\zeta)\|_{\mathbb{C}^n}$, it follows that h is invertible in X, and so $\boldsymbol{v} = h^{-1}\bar{\vartheta}_1\boldsymbol{f} \in X$. Clearly, $\boldsymbol{g} \in X$, and so $\boldsymbol{w} = h^{-1}\bar{\vartheta}_2\boldsymbol{g} \in X$. By Lemma 12.1, the thematic matrix functions V and $W^{\rm t}$ defined by (12.2) belong to X.

We argue by induction on $d = \min\{m, n\}$. If $d = 1$, then the unique best approximation F is determined by (12.1) and, obviously, $F \in X$.

Suppose that $d > 1$. Let F be the matrix function defined in (11.6), i.e.,

$$F = \varphi_1\psi_1^{\rm t} + \psi_2\varphi_2^{\rm t} - \psi_2\varphi_2^{\rm t}\boldsymbol{v}\psi_1^{\rm t},$$

where

$$\varphi_1 \stackrel{\rm def}{=} \frac{\bar{\vartheta}_1 T_\Phi \boldsymbol{f}}{h}, \quad \varphi_2 \stackrel{\rm def}{=} \frac{\bar{\vartheta}_2 T_{\Phi^{\rm t}} \boldsymbol{g}}{h},$$

and ψ_1 and ψ_2 are solutions of the corona problems

$$\psi_1^{\mathrm{t}} \boldsymbol{v} = \psi_2^{\mathrm{t}} \boldsymbol{w} = \mathbf{1}. \tag{12.6}$$

Clearly, φ_1 and φ_2 belong to X. It follows from Theorem 7.2.6 (see also its proof) that there exist solutions ψ_1 and ψ_2 of (12.6) that belong to X_+. It follows that $F \in X$. By Theorem 11.1, $\Phi - F$ satisfies (12.1) for some $\Psi \in L^\infty(\mathbb{M}_{m-1,n-1})$. Since Φ, V, W, $F \in X$, it follows that $\Psi \in X$. By Theorem 11.1, the problem of finding the superoptimal approximation of Φ reduces to the problem of finding the superoptimal approximation of Ψ. The result now follows from the inductive hypothesis. ■

It is easy to see that the proofs of Theorems 12.3 and 12.6 imply the following result.

Theorem 12.7. *Suppose that X is a linear $\mathcal{R}$-space or X is a decent function space. Let Φ be a matrix function in X. Then $\Phi - \mathcal{A}\Phi$ has a thematic factorization with all factors in X.*

In fact, it follows easily from the proof of Theorem 12.3 that in the case of a linear $\mathcal{R}$-space X all factors of *each* thematic factorization of $\Phi - \mathcal{A}\Phi$ are in X.

Corollary 12.8. *Suppose that Φ is a matrix function in $H^\infty + C$. Then all factors of each thematic factorization of $\Phi - \mathcal{A}\Phi$ belong to QC.*

13. Continuity Properties of Superoptimal Approximation

In this section we study the important problem of continuity of the operator of superoptimal approximation $\mathcal{A} = \mathcal{A}^{(m,n)}$ in the norm of decent function spaces X. As in the scalar case the continuity problem is important in applications since to compute a superoptimal approximation in practice, one has to be sure that a minor deviation of the initial data does not lead to a huge perturbatioon of the result. In other words, one needs continuous dependence of the solution on the initial matrix function.

Recall that in the scalar case a function $\varphi \in X$ with $\mathbb{P}_-\varphi \neq \mathbb{O}$ is a continuity point of $\mathcal{A}$ in the norm of X if and only if $\|H_\Phi\|$ is a singular value of H_Φ of multiplicity 1 (see Theorem 7.11.6). We obtain analogs of that result for the operator of superoptimal approximation.

We start with sufficient conditions. First we consider the case of an $m \times n$ matrix function Φ such that $t_{\min\{m,n\}-1}(\Phi) \neq 0$. We will see later that in the case $t_{\min\{m,n\}-1}(\Phi) = 0$ the result depends on the decent space X. We conclude the section with necessary conditions.

Theorem 13.1. *Let X be a decent function space and let $\Phi \in X(\mathbb{M}_{m,n})$. Suppose that $t_{\min\{m,n\}-1}(\Phi) \neq 0$. If $\Phi - \mathcal{A}\Phi$ has a thematic factorization* *with indices*

$$k_0 = k_1 = \cdots = k_{\min\{m,n\}-1} = 1, \tag{13.1}$$

then Φ *is a continuity point of the operator* $\mathcal{A}^{(m,n)}$ *of superoptimal approximation in* $X(\mathbb{M}_{m,n})$.

By Theorem 7.2, (13.1) implies that all thematic factorizations of $\Phi - \mathcal{A}\Phi$ have indices equal to 1. To prove Theorem 13.1, we use the construction of the superoptimal approximation based on the corona theorem (see §11). We need several lemmas.

Lemma 13.2. *Suppose that* $\Phi \in X(\mathbb{M}_{m,n})$ *and* $\Phi - \mathcal{A}\Phi$ *has a thematic factorization whose indices* k_j *are equal to* 1 *whenever* $t_j(\Phi) = \|H_\Phi\|$. *Then all maximizing vectors of* H_Φ *and of* $H_{\Phi^{\mathrm{t}}}$ *are co-outer and do not vanish on* $\mathbb{T}$.

Proof. Clearly, it follows from Theorem 7.2 that for each thematic factorization of $\Phi - \mathcal{A}\Phi$ the indices k_j are equal to 1 whenever $t_j(\Phi) = \|H_\Phi\|$.

Let $\boldsymbol{f} \in H^2(\mathbb{C}^n)$ be a maximizing vector of H_Φ. We have to prove that $\boldsymbol{f}$ is co-outer and $\boldsymbol{f}(\zeta) \ne \mathbb{O}$ for any $\zeta \in \mathbb{T}$. Put $\boldsymbol{g} = t_0^{-1}\bar{z}\overline{H_\Phi \boldsymbol{f}}$. Let us first show that $\boldsymbol{f}$ and $\boldsymbol{g}$ are co-outer. We have

$$\boldsymbol{f} = \vartheta_1 h \boldsymbol{v}, \quad \boldsymbol{g} = \vartheta_2 h \boldsymbol{w},$$

where ϑ_1 and ϑ_2 are scalar inner functions, and $\boldsymbol{v}$ and $\boldsymbol{w}$ are inner and co-outer vector functions. Consider thematic completions

$$V = \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix} \quad \text{and} \quad W^{\mathrm{t}} = \begin{pmatrix} \boldsymbol{w} & \overline{\Xi} \end{pmatrix}$$

of $\boldsymbol{v}$ and $\boldsymbol{w}$. Then by Theorem 2.2, for any best approximation $F \in H^\infty(\mathbb{M}_{m,n})$ the matrix function $\Phi - F$ admits a factorization

$$\Phi - F = W^* \begin{pmatrix} t_0 u_0 & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^*, \tag{13.2}$$

where $\|\Psi\|_{L^\infty} \le t_0$ and $u_0 = \bar{z}\overline{\vartheta}_1\overline{\vartheta}_2\bar{h}/h$. Then the index $k_0 = \operatorname{ind} T_{u_0}$ of this factorization is greater than 1 if ϑ_1 or ϑ_2 is not constant.

Suppose now that $\boldsymbol{f}$ vanishes on $\mathbb{T}$. Without loss of generality we may assume that $\boldsymbol{f}(1) = \mathbb{O}$. Then by Theorem 13.2.2, the function $\boldsymbol{f}_\# = (1-z)^{-1}\boldsymbol{f} \in X_+(\mathbb{C}^n)$ is also a maximizing vector of H_Φ. Then the outer function $h_\# = h/(z-1)$ belongs to X and

$$u_0 = \bar{z}\overline{\vartheta}_1\overline{\vartheta}_2\frac{\bar{h}}{h} = \bar{z}\overline{\vartheta}_1\overline{\vartheta}_2\frac{\bar{z}-1}{z-1}\frac{\bar{h}_\#}{h_\#} = -\bar{z}^2\overline{\vartheta}_1\overline{\vartheta}_2\frac{\bar{h}_\#}{h_\#},$$

and again the factorization index k_0 that corresponds to the factorization (13.2) is at least 2, which leads to a contradiction. ■

For a matrix function $G \in X(\mathbb{M}_{m,n})$ we consider the Toeplitz operator $T_G : H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^m)$. Obviously, T_G acts as a bounded linear operator from $X_+(\mathbb{C}^n)$ to $X_+(\mathbb{C}^m)$. To avoid confusion, we denote by T_G^X the Toeplitz operator T_G as an operator from $X_+(\mathbb{C}^n)$ to $X_+(\mathbb{C}^m)$. Similarly, by H_G^X we denote the Hankel operator H_G as an operator from $X_+(\mathbb{C}^n)$

to $X_-(\mathbb{C}^m)$ and by H_G^{*X} the operator H_G^* as an operator from $X_-(\mathbb{C}^m)$ to $X_+(\mathbb{C}^n)$.

Lemma 13.3. *Let $n > 1$ and let v be an inner and co-outer vector function in $X_+(\mathbb{C}^n)$. Then 0 is an isolated point of spectrum of the operators $T_{\bar v}^X T_{v^{\mathrm t}}^X$ on $X_+(\mathbb{C}^n)$ and $T_{\bar v}T_{v^{\mathrm t}}$ on $H^2(\mathbb{C}^n)$.*

Proof. Let us prove the lemma for the operator $T_{\bar v}^X T_{v^{\mathrm t}}^X$. The proof for the operator $T_{\bar v}T_{v^{\mathrm t}}$ is even simpler. Let us first show that $0 \in \sigma(T_{\bar v}^X T_{v^{\mathrm t}}^X)$. Indeed, let $\begin{pmatrix} v & \overline{\Upsilon} \end{pmatrix}$ be a thematic matrix function. By Theorems 13.5.1 and 1.3, $\Upsilon \in X$. Clearly, the columns of Υ belong to $\operatorname{Ker} T_{v^{\mathrm t}}^X$, and so $0 \in \sigma(T_{\bar v}^X T_{v^{\mathrm t}}^X)$.

Consider the operator $T_{v^{\mathrm t}}^X T_{\bar v}^X$. By Lemma 5.4.6, it is sufficient to show that it is invertible. We have

$$T_{v^{\mathrm t}}^X T_{\bar v}^X = I - H_{v^{\mathrm t}}^X H_{\bar v}^X.$$

The operator $H_{v^{\mathrm t}}^X H_{\bar v}^X$ is compact on $X_+(\mathbb{C}^n)$ (see the axiom (A3) in §7.2), and so it is sufficient to show that $\operatorname{Ker} T_{v^{\mathrm t}}^X T_{\bar v}^X = \{\mathbb{O}\}$. Let $\varphi \in \operatorname{Ker} T_{v^{\mathrm t}}^X T_{\bar v}^X$. Then $H_{v^{\mathrm t}} H_{\bar v}\varphi = \varphi$. Since $\|\bar v\|_{L^\infty(\mathbb{C}^n)} = 1$, it follows that $\bar v\varphi \in H^2_-(\mathbb{C}^n)$. Thus $\bar\varphi v^{\mathrm t} H^2(\mathbb{C}^n) \subset zH^1$. Since v is co-outer, $v^{\mathrm t} H^2(\mathbb{C}^n)$ is dense in H^2, and so $\bar\varphi H^2(\mathbb{C}^n) \subset zH^1$. Consequently, $\varphi = \mathbb{O}$. ■

For an inner and co-outer function $v \in H^\infty(\mathbb{C}^n)$ we denote by L_v the kernel of $T_{v^{\mathrm t}}$ and by P_v the orthogonal projection from $H^2(\mathbb{C}^n)$ onto L_v. Similarly, we denote by L_v^X the kernel of $T_{v^{\mathrm t}}^X$. Clearly, $L_v = \operatorname{Ker} T_{\bar v}T_{v^{\mathrm t}}$ and $L_v^X = \operatorname{Ker} T_{\bar v}^X T_{v^{\mathrm t}}^X$.

Consider a simple closed positively oriented Jordan curve γ that lies in the resolvent sets of $T_{\bar v}T_{v^{\mathrm t}}$ and $T_{\bar v}^X T_{v^{\mathrm t}}^X$, encircles zero, but does not wind round any other point of the spectra of $T_{\bar v}T_{v^{\mathrm t}}$ and $T_{\bar v}^X T_{v^{\mathrm t}}^X$. Clearly,

$$P_v = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (\zeta I - T_{\bar v}T_{v^{\mathrm t}})^{-1} d\zeta.$$

Consider the projection P_v^X from $X_+(\mathbb{C}^n)$ onto L_v^X defined by

$$P_v^X = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \left(\zeta I - T_{\bar v}^X T_{v^{\mathrm t}}^X\right)^{-1} d\zeta. \tag{13.3}$$

Obviously, $P_v^X\varphi = P_v\varphi$ for $\varphi \in X_+(\mathbb{C}^n)$.

Suppose now that $\{v_j\}_{j\ge1}$ is a sequence of inner and co-outer functions in $X_+(\mathbb{C}^n)$, which converges to v in the norm. Then $T_{v_j^{\mathrm t}}^X T_{\bar v_j}^X \to T_{v^{\mathrm t}}^X T_{\bar v}^X$ in the operator norm of $X_+(\mathbb{C}^n)$. As in the proof of Lemma 13.3, $T_{v^{\mathrm t}}^X T_{\bar v}^X$ is invertible, and hence there is a neighborhood $\mathcal{O}$ of zero that lies in the resolvent set of $T_{v^{\mathrm t}}^X T_{\bar v}^X$ and of $T_{v_j^{\mathrm t}}^X T_{\bar v_j}^X$ for all sufficiently large j. Without loss of generality we may assume that this holds for all values of j. Choose a simple closed contour γ lying in $\mathcal{O}$ and winding round 0. Then 0 is the only point inside or on γ of the spectra of $T_{\bar v_j}T_{v_j^{\mathrm t}}$ and $T_{\bar v_j}^X T_{v_j^{\mathrm t}}^X$. We can therefore define projections P_v, P_v^X, P_{v_j}, $P_{v_j}^X$ by integrals as above, all using the same contour γ. It is then easy to see from (13.3) that $P_{v_j}^X \to P_v^X$ in the operator norm.

Lemma 13.4. *Let $V = \begin{pmatrix} v & \overline{\Theta} \end{pmatrix}$ be a thematic matrix function, where $v \in H^\infty(\mathbb{C}^n)$ and $\Theta \in H^\infty(\mathbb{M}_{n,n-1})$. Suppose that $\{v_j\}_{j\ge 1}$ is a sequence of inner and co-outer functions that converges to v in $X_+(\mathbb{C}^n)$. There exist inner co-outer functions Θ_j such that $V_j \stackrel{\text{def}}{=} \begin{pmatrix} v_j & \overline{\Theta}_j \end{pmatrix}$ is thematic and $\|V - V_j\|_{X(\mathbb{M}_{n,n})} \to 0$.*

Proof. It has been shown in the proof of Theorem 1.1 that v has a thematic completion $\overline{\Theta}$ such that the columns of Θ have the form $P_v C_1, P_v C_2, \cdots, P_v C_{n-1}$, where $C_1, C_2, \cdots, C_{n-1}$ are constant column functions. By Theorem 1.2, the same is true for all thematic completions. Hence, the columns of Θ have the form $P_v C_k$ for some constants C_k. Consider the subspace of $H^2(\mathbb{C}^n)$

$$P_v\mathbb{C}^n \stackrel{\text{def}}{=} \{P_v C : \ C \in \mathbb{C}^n\},$$

where we identify $C \in \mathbb{C}^n$ with a constant function in $H^2(\mathbb{C}^n)$. This space has the remarkable property that the pointwise and H^2 inner products coincide on it. That is, if $\varphi_k = P_v C_k,\ k = 1, 2$, where $C_1,\ C_2 \in \mathbb{C}^n$, then

$$(\varphi_1, \varphi_2)_{H^2(\mathbb{C}^n)} = (\varphi_1(\zeta), \varphi_2(\zeta))_{\mathbb{C}^n}$$

for almost all $\zeta \in \mathbb{T}$. Indeed, $P_v\mathbb{C}^n = \Theta H^2(\mathbb{C}^{n-1})$ (see the proof of Theorem 1.1), and for any $C \in \mathbb{C}^n$

$$P_v C = \Theta \mathbb{P}_+ \Theta^* C = \Theta\Theta(0)^* C.$$

Thus

$$\begin{aligned}
(\varphi_1, \varphi_2)_{H^2(\mathbb{C}^n)} &= (P_v C_1, P_v C_2)_{H^2(\mathbb{C}^n)} = (\Theta\Theta(0)^* C_1, \Theta\Theta(0)^* C_2)_{H^2(\mathbb{C}^n)} \\
&= (\Theta(0)^* C_1, \Theta(0)^* C_2)_{H^2(\mathbb{C}^{n-1})} = (\Theta(0)^* C_1, \Theta(0)^* C_2)_{\mathbb{C}^{n-1}} \\
&= (\Theta(\zeta)\Theta(0)^* C_1, \Theta(\zeta)\Theta(0)^* C_2)_{\mathbb{C}^n} \\
&= (\varphi_1(\zeta), \varphi_2(\zeta))_{\mathbb{C}^n}
\end{aligned}$$

for almost all $\zeta \in \mathbb{T}$. It follows that any unit vector in $P_v\mathbb{C}^n$ is an inner column function, and any orthonormal sequence (with respect to the inner product of $H^2(\mathbb{C}^n)$) of vectors in $P_v\mathbb{C}^n$ constitutes the columns of an inner function.

Now let $P_v C_k,\ 1 \le k \le n-1$, be the columns of Θ, and consider the functions $P_{v_j} C_1, P_{v_j} C_2, \cdots, P_{v_j} C_{n-1}$. Clearly,

$$\|P_v C_k - P_{v_j} C_k\|_{X(\mathbb{C}^n)} = \left\|P_v^X C_k - P_{v_j}^X C_k\right\|_{X(\mathbb{C}^n)} \to 0 \quad \text{as} \quad j \to \infty.$$

It follows that for large values of j the inner products $(P_{v_j} C_{k_1}, P_{v_j} C_{k_2})_{H^2(\mathbb{C}^n)}$ are small for $k_1 \ne k_2$ and are close to 1 if $k_1 = k_2$. We shall show that the desired Θ_j can be obtained by orthonormalizing the $P_{v_j} C_k$.

Pick $M > 1$ such that $\|P_{v_j} C_k\|_{X(\mathbb{C}^n)} \le M$, $j \ge 1$, $1 \le k \le n-1$. By the equivalence of norms on finite-dimensional spaces there exists $K > 0$ such that for any $(n-1)\times(n-1)$ matrix $T=\{t_{kl}\}$,

$$\max|\,t_{kl}\,|\le\ \|T\|\ \le\ K\cdot\max|\,t_{kl}\,| \tag{13.4}$$

(here $\|T\|$ is the operator norm of T on $\mathbb{C}^{n-1}$).

Let $0<\varepsilon<1$. Choose j_0 such that $j\ge j_0$ implies

$$\|P_{v_j}C_k-P_vC_k\|_{X(\mathbb{C}^n)}<\frac{\varepsilon}{2},\quad k=1,\cdots,n-1, \tag{13.5}$$

and

$$|\,(P_{v_j}C_k,P_{v_j}C_l)-\delta_{kl}\,|<\frac{\varepsilon}{2KnM},\quad k,l=1,\cdots,n-1, \tag{13.6}$$

(here δ_{kl} is the Kronecker symbol).

Fix $j\ge j_0$ and let $T:\mathbb{C}^{n-1}\to P_{v_j}\mathbb{C}^n$ be the operator that maps the kth standard basis vector e_k of $\mathbb{C}^{n-1}$ to $P_{v_j}C_k$. The matrix of the operator T^*T on $\mathbb{C}^{n-1}$ is the Gram matrix $(P_{v_j}C_k,\ P_{v_j}C_l)$, and so by (13.4) and (13.6) we have

$$\|T^*T-I\|<\frac{\varepsilon}{2nM}<\frac{1}{2}.$$

By diagonalization,

$$\left\|(T^*T)^{-\frac12}-I\right\|<\frac{\varepsilon}{2nM}.$$

Let $\{\tau_{kl}\}$ be the matrix of $(T^*T)^{-\frac12}$, then

$$|\,\tau_{kl}-\delta_{kl}\,|<\frac{\varepsilon}{2nM}.$$

Let $T=U(T^*T)^{\frac12}$ be the polar decomposition of T, where U is a unitary operator from $\mathbb{C}^{n-1}$ onto $P_{v_j}\mathbb{C}^n$. Then the vectors $Ue_1,\cdots,Ue_{n-1}$ are orthonormal in $P_{v_j}\mathbb{C}^n$. Let Θ_j be the $n\times(n-1)$ matrix function with columns $Ue_1,\dots,Ue_{n-1}$. By the remark above, Θ_j is inner. By the fact that $P_{v_j}\mathbb{C}^n\subset L_{v_j}$, the columns of $\overline{\Theta}_j$ are pointwise orthogonal to v_j. Hence,

$$V_j\stackrel{\text{def}}{=}\begin{pmatrix} v_j & \overline{\Theta}_j\end{pmatrix}$$

is unitary-valued. Furthermore, the jth column Ue_j of Θ_j satisfies

$$\begin{aligned}
\|P_{v_j}C_k-Ue_k\|_{X(\mathbb{C}^n)} &= \|Te_k-T(T^*T)^{-\frac12}e_k\|_{X(\mathbb{C}^n)}\\
&=\left\|Te_k-\begin{pmatrix}Te_1 & \cdots & Te_{n-1}\end{pmatrix}\begin{pmatrix}\tau_{1k}\\ \vdots\\ \tau_{n-1,k}\end{pmatrix}\right\|_{X(\mathbb{C}^n)}\\
&\le|\,\tau_{1j}\,|\cdot\|Te_1\|_{X(\mathbb{C}^n)}+\cdots+|\,\tau_{n-1,j}\,|\cdot\|Te_{n-1}\|_{X(\mathbb{C}^n)}\\
&\le(n-1)\frac{\varepsilon}{2nM}M<\frac{\varepsilon}{2}.
\end{aligned}$$

On combining this inequality with (13.5), we obtain

$$\begin{aligned}\|P_v C_k - Ue_k\|_{X(\mathbb{C}^n)} &\le \|P_v C_k - P_{v_j} C_k\|_{X(\mathbb{C}^n)} + \|P_{v_j} C_k - Ue_k\|_{X(\mathbb{C}^n)}\\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

That is, the jth column of Θ_j tends to the jth column of Θ with respect to the norm of $X(\mathbb{C}^n)$. Hence $V_j \to V$ in $X(\mathbb{M}_{n,n})$. Finally, it follows from the proof of Theorem 1.1 that Θ is co-outer. ■

We need on more elementary fact.

Lemma 13.5. *Let E and F be Banach spaces, let $T : E \to F$ be a surjective continuous linear mapping, and let $x \in E$, $y \in F$ be such that $Tx = y$. For any $\varepsilon > 0$ there exists $\delta > 0$ such that, whenever $T_\# \in \mathcal{B}(E, F)$ and $\|T_\# - T\| < \delta$, the equation $T_\# x' = y$ has a solution x' satisfying $\|x' - x\| < \varepsilon$.*

Proof. We can suppose $\|x\| = 1$. By the Open Mapping Theorem there exists $c > 0$ such that the ball of radius c in F is contained in the image under T of the unit ball in E. Then

$$\|T^* l\| \ge c\|l\| \quad \text{for all} \quad l \in F^*.$$

Let $\delta = c\min\{1, \varepsilon\}/2$. Suppose $\|T_\# - T\| < \delta$. For any $l \in F^*$

$$\begin{aligned}\|T_\#^* l\| &= \|T^* l + (T_\# - T)^* l\| \ge \|T^* l\| - \|T_\# - T\| \cdot \|l\|\\ &\ge c\|l\| - \frac{c}{2}\|l\| = \frac{c}{2}\|l\|.\end{aligned}$$

Thus $T_\#$ maps the closed unit ball of E to a superset of the closed ball of radius $c/2$ in F. Since

$$\|(T - T_\#)x\| \le \|T - T_\#\| < \delta,$$

it follows that there exists $\xi \in E$ such that

$$\|\xi\| < \frac{2\delta}{c} \le \varepsilon$$

and $T_\#\xi = (T - T_\#)x$. Then $x' \stackrel{\text{def}}{=} x + \xi$ has the stated properties:

$$T_\# x' = T_\# x + (T - T_\#)x = Tx = y,$$

$$\|x - x'\| = \|\xi\| < \varepsilon. \quad ■$$

In the next lemma we show that if we have a solution $\xi \in X_+(\mathbb{C}^n)$ of the corona problem

$$\varphi^{\mathrm{t}}\xi = \mathbf{1}$$

for $\varphi \in X_+(\mathbb{C}^n)$ and ψ is a small perturbation of φ, then there is a solution $\eta \in X_+(\mathbb{C}^n)$ of the corona problem

$$\psi^{\mathrm{t}}\eta = \mathbf{1}, \tag{13.7}$$

which is a small peturbation of ξ.

Lemma 13.6. *Let ξ, $\varphi \in X_+(\mathbb{C}^n)$ be such that $\varphi^{\mathrm{t}}\xi = \mathbf{1}$ and let $\varepsilon > 0$. There exists $\delta > 0$ such that, for any $\psi \in X_+(\mathbb{C}^n)$ satisfying $\|\varphi - \psi\|_X < \delta$, there exists $\eta \in X_+(\mathbb{C}^n)$ such that $\|\xi - \eta\|_X < \varepsilon$ and $\psi^{\mathrm{t}}\eta = \mathbf{1}$.*

Proof. Consider the operator $T = T^X_{\varphi^{\mathrm{t}}} : X_+(\mathbb{C}^n) \to X_+$:

$$T\chi = \varphi^{\mathrm{t}}\chi.$$

Clearly, T is surjective and $T\xi = \mathbf{1}$. By Lemma 13.5, there exists $\delta > 0$ such that for an operator $T_\# : X_+(\mathbb{C}^n) \to X_+$ with $\|T - T_\#\| < \delta$ there exists a function $\eta \in X_+(\mathbb{C}^n)$ such that $\|\xi - \eta\|_X < \varepsilon$ and $T_\#\eta = \mathbf{1}$. Suppose now that $\psi \in X_+(\mathbb{C}^n)$ and $\|\varphi - \psi\|_X < \delta$. Then $\|T^X_{\varphi^{\mathrm{t}}} - T^X_{\psi^{\mathrm{t}}}\| < \delta$, and we can put $T_\# = T^X_{\psi^{\mathrm{t}}}$. Clearly, η is a desired of the corona problem (13.7). ■

Now we are ready to prove Theorem 13.1.

Proof of Theorem 13.1. Suppose that $n \le m$. We proceed by induction on n. Let $\{\Phi_j\}_{j\ge1}$ be a sequence of functions in X such that $\|\Phi - \Phi_j\|_{X(\mathbb{M}_{m,n})} \to 0$. We shall show that some subsequence of $\{\mathcal{A}\Phi_j\}_{j\ge1}$ converges to $\mathcal{A}\Phi$ in the norm of X: this will suffice to establish the continuity of $\mathcal{A}$ at Φ. Let $\boldsymbol{f}_j$ be a maximizing vector for the operator H_{Φ_j} on $H^2(\mathbb{C}^n)$. We can take it that the norm of $\boldsymbol{f}_j$ in $X_+(\mathbb{C}^n)$ is equal to 1:

$$\|\boldsymbol{f}_j\|_{X(\mathbb{C}^n)} = 1, \quad \|H_{\Phi_j}\boldsymbol{f}_j\|_{H^2_-(\mathbb{C}^m)} = \|H_{\Phi_j}\| \cdot \|\boldsymbol{f}_j\|_{H^2(\mathbb{C}^n)}.$$

Let $\mathcal{I}$ be a positively oriented Jordan contour that winds once round the largest eigenvalue t_0^2 of $H^*_\Phi H_\Phi$, contains no eigenvalues, and encircles no other eigenvalues.

It is easy to see from axioms (A1)–(A4) that the operators $H^{*X}_{\Phi_j} H^X_{\Phi_j}$ converge to $H^{*X}_\Phi H^X_\Phi$ in the operator norm of $X_+(\mathbb{C}^n)$. It follows that for large values of j there are no points of the spectrum of $H^{*X}_{\Phi_j} H^X_{\Phi_j}$ on $\mathcal{I}$. Let

$$\mathcal{P} = \frac{1}{2\pi\mathrm{i}} \oint_{\mathcal{I}} \left(\zeta I - H^{*X}_\Phi H^X_\Phi\right)^{-1} d\zeta$$

and

$$\mathcal{P}_j = \frac{1}{2\pi\mathrm{i}} \oint_{\mathcal{I}} \left(\zeta I - H^{*X}_{\Phi_j} H^X_{\Phi_j}\right)^{-1} d\zeta.$$

It is easy to see that $\mathcal{P}_j \to \mathcal{P}$ in the operator norm of $X_+(\mathbb{C}^n)$.

Clearly, if $\mathcal{P}\boldsymbol{f}_j$ is nonzero, it is a maximizing vector of $H^{*X}_\Phi H^X_\Phi$ and

$$\|\boldsymbol{f}_j - \mathcal{P}\boldsymbol{f}_j\|_{X(\mathbb{C}^n)} = \|\mathcal{P}_j\boldsymbol{f}_j - \mathcal{P}\boldsymbol{f}_j\|_{X(\mathbb{C}^n)} \to 0, \quad k \to \infty.$$

The vectors $\mathcal{P}\boldsymbol{f}_j$ belong to the finite-dimensional subspace of maximizing vectors of $H^*_\Phi H_\Phi$. Therefore there exists a convergent subsequence of the sequence $\{\mathcal{P}\boldsymbol{f}_j\}_{j\ge0}$. Without loss of generality we may assume that the sequence $\{\mathcal{P}\boldsymbol{f}_j\}_{j\ge0}$ converges in $X_+(\mathbb{C}^n)$ to a vector $\boldsymbol{f}$, which is a maximizing vector of $H^*_\Phi H_\Phi$. Obviously, $\|\boldsymbol{f}_j - \boldsymbol{f}\|_{X(\mathbb{C}^n)} \to 0$ as $j \to \infty$.

We also need the other Schmidt vectors corresponding to $\boldsymbol{f}$ and $\boldsymbol{f}_j$. We may assume that $\|H_{\Phi_j}\| \ne 0$ for all j. Let

$$\boldsymbol{g} = t_0^{-1}\bar{z}\overline{H_\Phi\boldsymbol{f}}, \quad \boldsymbol{g}_j = \|H_{\Phi_j}\|^{-1}\bar{z}\overline{H_{\Phi_j}\boldsymbol{f}_j}$$

in $X_+(\mathbb{C}^m)$. Since $H_{\Phi_j} \to H_\Phi$ in the norm of $\mathcal{B}(X_+(\mathbb{C}^n), X_-(\mathbb{C}^m))$ and $\boldsymbol{f}_j \to \boldsymbol{f}$ in $X_+(\mathbb{C}^n)$, it follows that $\boldsymbol{g}_j \to \boldsymbol{g}$ in $X_+(\mathbb{C}^m)$. By Lemma 13.2, the vector-functions $\boldsymbol{f}$ and $\boldsymbol{g}$ are co-outer and do not vanish on $\mathbb{T}$. Hence, they do not vanish on the closed unit disk $\operatorname{clos}\mathbb{D}$. Since convergence in X_+ implies uniform convergence on $\operatorname{clos}\mathbb{D}$, it follows that for sufficiently large values of j the vector functions $\boldsymbol{f}_j$ and $\boldsymbol{g}_j$ do not vanish on $\operatorname{clos}\mathbb{D}$, and so they are co-outer.

Now let us show that Theorem 13.1 holds when $n = 1$. In this case $\boldsymbol{f}$ and $\boldsymbol{f}_j$ are scalar functions in X. By Theorem 2.2.3, $|\boldsymbol{f}(\zeta)| = \|\boldsymbol{g}(z)\|_{\mathbb{C}^m}$ a.e. on $\mathbb{T}$. By continuity, equality holds at all points of $\mathbb{T}$. By Lemma 13.2, $\boldsymbol{f}$ is invertible in X. By virtue of the continuity of inversion in Banach algebras we deduce that $\{\boldsymbol{f}_j^{-1}\}$ converges to $\boldsymbol{f}^{-1}$ in X. Again by Theorem 2.2.3,

$$(\Phi - \mathcal{A}\Phi)\boldsymbol{f} = \|H_\Phi\|\overline{z\boldsymbol{g}}, \quad (\Phi_j - \mathcal{A}\Phi_j)\boldsymbol{f}_j = \|H_{\Phi_j}\|\overline{z\boldsymbol{g}_j},$$

and hence

$$\Phi - \mathcal{A}\Phi = \|H_\Phi\|\frac{\overline{z\boldsymbol{g}}}{\boldsymbol{f}}, \qquad \Phi^{(k)} - \mathcal{A}\Phi^{(k)} = \|H_{\Phi^{(k)}}\|\frac{\overline{z\boldsymbol{g}_j}}{\boldsymbol{f}_j},$$

for sufficiently large j. From these equalities it is clear that $\mathcal{A}\Phi_j \to \mathcal{A}\Phi$ in X. Thus the case $n = 1$ is established.

Now consider $n > 1$ and suppose the theorem is true for $n - 1$. We prove the induction step by block diagonalization of $\Phi - \mathcal{A}\Phi$. Let $\boldsymbol{f}$ and $\boldsymbol{g}$ be as above and let h be a scalar outer function such that $|h(\zeta)| = \|\boldsymbol{f}(\zeta)\|_{\mathbb{C}^n}$, $\zeta \in \mathbb{T}$. Then $|h(\zeta)| = \|\boldsymbol{g}(\zeta)\|_{\mathbb{C}^m}$, $\zeta \in \mathbb{T}$. We can choose h as follows:

$$h = e^{\rho + \mathrm{i}\tilde{\rho}},$$

where

$$\rho(\zeta) = \log\|\boldsymbol{f}(\zeta)\|_{\mathbb{C}^n}$$

and $\tilde{\rho}$ is the harmonic conjugate of ρ. It follows easily from (A1)–(A4) that $\|\boldsymbol{f}\|^2_{\mathbb{C}^n} \in X$. By Lemma 13.2, $\|\boldsymbol{f}\|^2_{\mathbb{C}^n}$ does not vanish on $\mathbb{T}$, and so its spectrum in the Banach algebra X is a compact interval of the positive real numbers. By the analytic functional calculus, $\rho = \frac{1}{2}\log\|\boldsymbol{f}\|^2_{\mathbb{C}^n} \in X$. By (A1) we also have $\tilde{\rho} \in X$. Thus $h = e^{\rho + \mathrm{i}\tilde{\rho}} \in X$. The above construction also makes it clear that if we define in a similar way the scalar outer functioons h_j so that $|h_j| = \|\boldsymbol{f}_j\|_{\mathbb{C}^n}$, then h_j converges to h in X.

Note also that since $|h| = \|\boldsymbol{f}\|_{\mathbb{C}^n}$ is bounded away from zero, h is invertible in X and $1/h_j \to 1/h$ in X. We have

$$\boldsymbol{f} = h\boldsymbol{v}, \quad \boldsymbol{g} = h\boldsymbol{w}, \quad \boldsymbol{f}_j = h_j\boldsymbol{v}_j, \quad \text{and} \quad \boldsymbol{g}_j = h_j\boldsymbol{w}_j,$$

where $\boldsymbol{v}$, $\boldsymbol{w}$, $\boldsymbol{v}_j$, and $\boldsymbol{w}_j$ are inner and co-outer column functions if j is sufficiently large. Then $\boldsymbol{v}_j \to \boldsymbol{v}$ and $\boldsymbol{w}_j \to \boldsymbol{w}$ in X_+ as $j \to \infty$.

By Theorem 13.4, we can find thematic matrix functions

$$\begin{aligned} V &= \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix}, & W^t &= \begin{pmatrix} \boldsymbol{w} & \overline{\Xi} \end{pmatrix}, \\ V_j &= \begin{pmatrix} \boldsymbol{v}_j & \overline{\Theta}_j \end{pmatrix}, & W_j^t &= \begin{pmatrix} \boldsymbol{w}_j & \overline{\Xi}_j \end{pmatrix} \end{aligned}$$

such that $V_j \to V$ and $W_j \to W$ in X. A fortiori,

$$\Theta_j \to \Theta, \quad \text{and} \quad \Xi_j \to \Xi$$

in $X(\mathbb{M}_{n,n-1})$ and $X(\mathbb{M}_{m,m-1})$, respectively.

We have explained in §11 that if a matrix function $F \in H^\infty(\mathbb{M}_{m,n})$ satisfies the equations

$$F\boldsymbol{f} = T_\Phi \boldsymbol{f} \quad \text{and} \quad F^{\mathrm{t}}\boldsymbol{g} = T_{\Phi^{\mathrm{t}}}\boldsymbol{g}, \tag{13.8}$$

then $\Phi - F$ admits a factorization

$$\Phi - F = W^* \begin{pmatrix} t_0 u_0 & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^*,$$

where $u_0 = \bar{z}\bar{h}/h$. Moreover, to find the superoptimal approximation of Φ, it is sufficient to find a superoptimal approximation of Ψ:

$$\Phi - \mathcal{A}\Phi = W^* \begin{pmatrix} t_0 u_0 & \mathbb{O} \\ \mathbb{O} & \Psi - \mathcal{A}\Psi \end{pmatrix} V^*. \tag{13.9}$$

Put

$$\varphi_1 = \frac{T_\Phi \boldsymbol{f}}{h}, \quad \varphi_2 = \frac{T_{\Phi^{\mathrm{t}}}\boldsymbol{g}}{h}.$$

Clearly, $\varphi_1 \in X_+(\mathbb{C}^m)$ and $\varphi_2 \in X_+(\mathbb{C}^n)$. As we have observed in §11, the corona problems

$$\psi_1^{\mathrm{t}}\boldsymbol{v} = \mathbf{1} \quad \text{and} \quad \psi_2^{\mathrm{t}}\boldsymbol{w} = \mathbf{1} \tag{13.10}$$

are solvable in H^∞. By Theorem 7.2.6, the corona problems (13.10) have solutions $\psi_1 \in X_+(\mathbb{C}^n)$ and $\psi_2 \in X_+(\mathbb{C}^m)$.

Finally, by Theorem 11.2, the matrix function $F \in X_+(\mathbb{M}_{m,n})$ defined by

$$F = \varphi_1\psi_1^{\mathrm{t}} + \psi_2\varphi_2^{\mathrm{t}} - \psi_2\varphi_2^{\mathrm{t}}\boldsymbol{v}\psi_1^{\mathrm{t}}$$

satisfies equations (13.8).

Similarly, for each sufficiently large j we can consider the functions

$$\varphi_{1,j} = \frac{T_\Phi \boldsymbol{f}_j}{h_j}, \quad \varphi_{2,j} = \frac{T_{\Phi^{\mathrm{t}}}\boldsymbol{g}_j}{h_j}.$$

By Lemma 13.6, there exist solutions $\psi_{1,j}$ and $\psi_{2,j}$ of the corona problems

$$\psi_{1,j}^{\mathrm{t}}\boldsymbol{v}_j = \mathbf{1} \quad \text{and} \quad \psi_{2,j}^{\mathrm{t}}\boldsymbol{w}_j = \mathbf{1}$$

such that

$$\lim_{j\to\infty} \|\psi_1 - \psi_{1,j}\|_{X(\mathbb{C}^n)} = 0 \quad \text{and} \quad \lim_{j\to\infty} \|\psi_2 - \psi_{2,j}\|_{X(\mathbb{C}^m)} = 0.$$

Put

$$F_j = \varphi_{1,j}\psi_{1,j}^{\mathrm{t}} + \psi_{2,j}\varphi_{2,j}^{\mathrm{t}} - \psi_{2,j}\varphi_{2,j}^{\mathrm{t}}\boldsymbol{v}\psi_{1,j}^{\mathrm{t}}.$$

Clearly, $F_j \to F$ in $X_+(\mathbb{M}_{m,n})$.

We have

$$\Phi_j - F_j = W_j^* \begin{pmatrix} t_0(\Phi_j)u_{0,j} & \mathbb{O} \\ \mathbb{O} & \Psi_j \end{pmatrix} V^*,$$

where for large values of j, $u_{0,j} = \bar{z}\bar{h}_j/h_j$. It follows that $\Psi_j \to \Psi$ in $X(\mathbb{M}_{m-1,n-1})$. Clearly, $u_{0,j} \to u_0$ in X. We have

$$\Phi_j - \mathcal{A}\Phi_j = W^* \begin{pmatrix} t_0(\Phi_j)u_{0,j} & \mathbb{O} \\ \mathbb{O} & \Psi_j - \mathcal{A}\Psi_j \end{pmatrix} V^*. \tag{13.11}$$

By the inductive hypothesis $\mathcal{A}$ is continuous at Ψ, and hence $\mathcal{A}\Psi_j \to \mathcal{A}\Psi$ in $X(\mathbb{M}_{m-1,n-1})$. It follows that $\mathcal{A}\Phi_j \to \mathcal{A}\Phi$ in $X(\mathbb{M}_{m,n})$. ■

Consider now the case when $t_{\min\{m,n\}-1}(\Phi) = 0$. One can see by considering diagonal matrix functions like

$$\begin{pmatrix} \bar{z} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}$$

that it is important whether the scalar operator $\mathcal{A}$ of best approximation is continuous at 0 in X, or equivalently whether $\mathcal{A}$ is bounded on X. This is not always so for decent spaces (see §7.7), and so the conclusion of Theorem 13.1 is not true if the condition $t_{\min\{m,n\}-1}(\Phi) \neq 0$ is relaxed. There is one case when this can be done.

Theorem 13.7. *Let X be the Besov space B_1^1 and let $\Phi \in X(\mathbb{M}_{m,n})$. If $\Phi - \mathcal{A}\Phi$ has a thematic factorization, in which the indices corresponding to nonzero superoptimal singular values are all equal to 1, then Φ is a continuity point of the operator $\mathcal{A}$ of superoptimal approximation in $X(\mathbb{M}_{m,n})$.*

Proof. Since B_1^1 is a Banach $\mathcal{R}$-space, the fact that this statement is true in the case $\Phi = \mathbb{O}$ follows from Theorem 12.3. Recall that X is also a decent function space. Let Φ have superoptimal singular values $t_0, \dots, t_{\min\{m,n\}-1}$. Let r be the number of nonzero superoptimal singular values of Φ: $r = \inf\{j : t_j = 0\}$. We prove the result by induction on r. As in the proof of Theorem 13.1, let $\{\Phi_j\}_{j\ge1}$ be a sequence of functions in X such that $\|\Phi - \Phi_j\|_{X(\mathbb{M}_{m,n})} \to 0$.

If $r = 0$, then $\Phi = \mathcal{A}\Phi \in H^\infty(\mathbb{M}_{m,n})$. Since $\Phi_j \to \Phi$ in X, by Theorem 12.3, $\mathcal{A}(\Phi_j - \Phi) \to \mathbb{O}$, and since $\Phi \in H^\infty(\mathbb{M}_{m,n})$, $\mathcal{A}(\Phi_j - \Phi) = \mathcal{A}\Phi_j - \Phi$. Thus $\mathcal{A}\Phi_j \to \Phi = \mathcal{A}\Phi$. Hence, $\mathcal{A}$ is continuous at Φ.

Now consider $r \ge 1$ and suppose the assertion holds for $r-1$. Since $t_0 \neq 0$, the compact operator H_Φ is not zero, and so $H_\Phi^* H_\Phi$ has finite-dimensional eigenspace corresponding to t_0^2. We now proceed as in the proof of Theorem 13.1: pick Schmidt vectors $\boldsymbol{f}$, $\boldsymbol{f}_j$, $\boldsymbol{g}$, $\boldsymbol{g}_j$, thematic functions V, V_j, $W^{\rm t}$, $W_j^{\rm t}$, and the matrix functions F, F_j, Ψ, Ψ_j exactly as described above. Once again (13.9) and (13.11) hold and the indices corresponding to any nonzero superoptimal singular value in any thematic factorization of $\Psi - \mathcal{A}\Psi$ are all 1. Moreover, the superoptimal singular values of Ψ are $t_1, \dots, t_{\min\{m,n\}-1}$, so that Ψ has $r-1$ nonzero superoptimal singular values. By the inductive hypothesis, $\mathcal{A}\Psi_j \to \mathcal{A}\Psi$ in $X(\mathbb{M}_{m-1,n-1})$. Equalities (13.9) and (13.11) now show that $\mathcal{A}\Phi_j \to \mathcal{A}\Phi$ in $X(\mathbb{M}_{m,n})$ as $j \to \infty$. Thus $\mathcal{A}$ is continuous at Φ. ■

We proceed now to necessary conditions. We can prove that the sufficient condition established in Theorem 13.1 is also necessary for $n \times n$ matrix functions Φ with $t_{n-1}(\Phi) \neq 0$.

Theorem 13.8. *Let X be a decent function space, let $\Phi \in X(\mathbb{M}_{n,n})$, and suppose that the $t_{n-1}(\Phi) \neq 0$. If $\mathcal{A}$ is continuous at Φ, then all indices in any thematic factorization of $\Phi - \mathcal{A}\Phi$ are equal to 1.*

The proof of Theorem 13.8 is based on the following lemma.

Lemma 13.9. *Let $\Phi \in X(\mathbb{M}_{n,n})$ and let $\varepsilon > 0$. Suppose that $t_{n-1}(\Phi) \neq 0$. Then there exists $\Phi_{\#} \in X(\mathbb{M}_{n,n})$ such that $\|\Phi - \Phi_{\#}\|_X < \varepsilon$, $t_{n-1}(\Phi_{\#}) \neq 0$, and all n thematic indices of $\Phi_{\#} - \mathcal{A}\Phi_{\#}$ are equal to 1.*

First we prove another lemma.

Lemma 13.10. *Let $\Phi \in X(\mathbb{M}_{n,n})$. Then for any $\varepsilon > 0$ there exists a matrix function Q in $X(\mathbb{M}_{n,n})$ such that $\|H_\Phi\| < \|H_Q\|$, $\|\Phi - Q\|_X < \varepsilon$, and* $\operatorname{rank}(H_\Phi - H_Q) = 1$.

Proof. Let $\boldsymbol{f}$ be a maximizing vector of H_Φ of norm 1 and let $\boldsymbol{g} = \|H_\Phi\|^{-1}\bar{z}\overline{H_\Phi \boldsymbol{g}}$. Choose a point $\zeta \in \mathbb{D}$ at which $\boldsymbol{f}(\zeta) \neq \mathbb{O}$ and $\boldsymbol{g}(\zeta) \neq \mathbb{O}$. We define the matrix function Q by

$$Q(z)\xi = \Phi(z)\xi + \delta(z-\zeta)^{-1}(\boldsymbol{f}(\zeta))^*\xi \cdot \overline{\boldsymbol{g}(\zeta)}, \quad \xi \in \mathbb{C}^n,$$

where $\delta > 0$.

Clearly, $\operatorname{rank}(H_\Phi - H_Q) = 1$ and $\|\Phi - Q\|_X < \varepsilon$ if δ is small enough. We have

$$\begin{aligned}
(H_Q \boldsymbol{f}, \bar{z}\overline{\boldsymbol{g}}) &= (H_\Phi \boldsymbol{f}, \bar{z}\overline{\boldsymbol{g}}) + \delta\big((z-\zeta)^{-1}((\boldsymbol{f}(\zeta))^*\boldsymbol{f}) \cdot \overline{\boldsymbol{g}(\zeta)}, \bar{z}\overline{\boldsymbol{g}}\big)_{H^2(\mathbb{C}^n)} \\
&= \|H_\Phi\| + \delta\big(((\boldsymbol{f}(\zeta))^*\boldsymbol{f}) \cdot (\boldsymbol{g}^{\mathrm{t}}\,\overline{\boldsymbol{g}(\zeta)}\ ), k_\zeta\big)_{H^2} \\
&= \|H_\Phi\| + \delta\|\boldsymbol{f}(\zeta)\|^2_{\mathbb{C}^n}\|\boldsymbol{g}(\zeta)\|^2_{\mathbb{C}^n},
\end{aligned}$$

where $k_\zeta(\zeta) = (1 - \bar{\zeta}z)^{-1}$ is the reproducing kernel of H^2. Since $\|\boldsymbol{f}\|_{H^2} = \|\boldsymbol{g}\|_{H^2} = 1$, it follows that $\|H_Q\| > \|H_\Phi\|$. ■

Proof of Lemma 13.9. Suppose that Q is a matrix function in $X(\mathbb{M}_{m,n})$ such that $\operatorname{rank}(H_\Phi - H_Q) = 1$ and $\|H_Q\| > \|H_\Phi\|$ (the existence of such a Q follows from Lemma 13.10). Then clearly,

$$s_1(H_Q) \le s_0(H_\Phi) < s_0(H_Q).$$

Hence, the singular value $s_0(H_Q)$ has multiplicity 1, and it follows from Theorem 7.4 (in the case $\sigma = t_0(Q)$) that the index k_0 of any thematic factorization of $Q - \mathcal{A}Q$ equals 1.

We argue by induction on n. If $n = 1$, we apply Lemma 13.10 and put $\Phi_{\#} = Q$. Suppose now that the result holds for $n-1$. Applying Lemma 13.10 to Φ, we can find a matrix function Q such that $\|\Phi - Q\|_X < \varepsilon/2$ and the index k_0 of any thematic factorization of $Q - \mathcal{A}Q$ is 1. Consider a partial thematic factorization of $Q - AQ$ of the form

$$Q - \mathcal{A}Q = W^* \begin{pmatrix} t_0(Q)u_0 & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^*,$$

where V and W^{t} are thematic matrix functions of the form

$$V = \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix}, \quad W^{\mathrm{t}} = \begin{pmatrix} \boldsymbol{w} & \overline{\Xi} \end{pmatrix}$$

and $\|\Psi\|_{L^\infty} < t_0(H_Q)$. Then u_0, V, W, $\Psi \in X$ (see §12).

By the inductive hypothesis, for any $\delta > 0$ there exists a matrix function $\Psi_\# \in X(\mathbb{M}_{n-1,n-1})$ such that $\|\Psi - \Psi_\#\|_X < \delta$ and all $n-1$ indices of any thematic factorization of $\Psi_\# - \mathcal{A}\Psi_\#$ are equal to 1. Clearly, if δ is sufficiently small, then $\|\Psi_\#\|_{L^\infty} < t_0(Q)$.

Put

$$G = Q - \mathcal{A}Q - W^* \begin{pmatrix} t_0(Q)u_0 & \mathbb{O} \\ \mathbb{O} & \Psi_\# \end{pmatrix} V^*.$$

If δ is sufficiently small, then obviously, $\|G\|_X < \varepsilon/2$. We have

$$\begin{aligned} Q - G - \mathcal{A}Q &= W^* \begin{pmatrix} t_0(Q)u_0 & \mathbb{O} \\ \mathbb{O} & \Psi_\# \end{pmatrix} V^* \\ &= W^* \begin{pmatrix} t_0(Q)u_0 & \mathbb{O} \\ \mathbb{O} & \Psi_\# - \mathcal{A}\Psi_\# \end{pmatrix} V^* + \Xi(\mathcal{A}\Psi_\#)\Theta^{\mathrm{t}}. \end{aligned}$$

Hence,

$$Q - G - \mathcal{A}Q - \Xi(\mathcal{A}\Psi_\#)\Theta^{\mathrm{t}} = W^* \begin{pmatrix} t_0(Q)u_0 & \mathbb{O} \\ \mathbb{O} & \Psi_\# - \mathcal{A}\Psi_\# \end{pmatrix} V^*.$$

The matrix function

$$W^* \begin{pmatrix} t_0(Q)u_0 & \mathbb{O} \\ \mathbb{O} & \Psi_\# - \mathcal{A}\Psi_\# \end{pmatrix} V^*$$

is very badly approximable (see §5), and so $\mathcal{A}Q + \Xi(\mathcal{A}\Psi_\#)\Theta^{\mathrm{t}}$ is the superoptimal approximation of $Q - G$. Now put $\Phi_\# = Q - G$. Then $\|\Phi - \Phi_\#\|_X < \varepsilon$ and $\Phi_\# - \mathcal{A}\Phi_\#$ has a thematic factorization with all n indices equal to 1. Thus all n indices of any thematic factorization of $\Phi_\# - \mathcal{A}\Phi_\#$ are equal to 1. ■

Proof of Theorem 13.8. Suppose that a thematic factorization of $\Phi - \mathcal{A}\Phi$ has an index greater than 1. By Corollary 1.7, the determinant of a thematic matrix function is constant and so the function $\det(\Phi - \mathcal{A}\Phi)$ has constant modulus and $\operatorname{wind}\det(\Phi - \mathcal{A}\Phi) < -n$. By Lemma 13.9, there exists a sequence $\{\Phi_j\}_{j\ge 0}$ of matrix functions in $X(\mathbb{M}_{n,n})$ converging to Φ in X and such that all indices of any thematic factorizations of $\Phi_j - \mathcal{A}\Phi_j$ are equal to 1. Again, the functions $\det(\Phi_j - \mathcal{A}\Phi_j)$ have constant modulus. Clearly, $\operatorname{wind}\det(\Phi_j - \mathcal{A}\Phi_j) = -n$. Suppose that $\lim_{j\to\infty} \|\mathcal{A}\Phi_j - \mathcal{A}\Phi\|_X = 0$. It follows that

$$\lim_{j\to\infty} \det(\Phi_j - \mathcal{A}\Phi_j) = \det(\Phi - \mathcal{A}\Phi) \quad \text{in} \quad X,$$

which is impossible by Lemma 7.11.5. ■

Remark. The proof shows a slightly stronger result. If $t_{n-1}(\Phi) \ne 0$ and the mapping $\mathcal{A}$ as a mapping from $X(\mathbb{M}_{n,n})$ to $BMO(\mathbb{M}_{n,n})$ is continuous, then all indices of any thematic factorization of $\Phi - \mathcal{A}\Phi$ are equal to 1.

We proceed now to the case when an $m \times n$ matrix function Φ has zero superoptimal singular value $t_{\min\{m,n\}-1}$. As we have already mentioned the result depends on the boundedness of the scalar best approximation operator $\mathcal{A}$ on X.

Theorem 13.11. *Suppose that X is one of the Besov spaces B_p^s, $s > 1/p$, or the Hölder–Zygmund spaces λ_α, Λ_α, $\alpha > 0$, or $X = \mathcal{F}\ell^1$ is the space of functions with absolutely converging Fourier series. If $\Phi \in X(\mathbb{M}_{m,n})$ and $t_{\min\{m,n\}-1} = 0$, then $\mathcal{A}$ is discontinuous at Φ in the norm of X.*

Proof. It is shown in §7.7 that $\mathcal{A}$ is unbounded on these spaces. We can suppose that $m \le n$. Suppose that r is the smallest integer for which $t_r = 0$. Consider first the case $r = 0$. Then $\Phi \in X_+$ and so $\mathcal{A}\Phi = \Phi$. Clearly, it is sufficient to show that Φ is discontinuous at $\mathbb{O}$. Since the scalar operator $\mathcal{A}$ of best approximation is unbounded on X, there exists a sequence $\{\varphi_j\}_{j\ge 0}$ of scalar functions in X such that $\|\varphi_j\|_X \to 0$ and $\|\mathcal{A}\varphi_j\|_X \ge \text{const}$. Consider the matrix functions Φ_j defined by

$$\Phi_j = \begin{pmatrix} \varphi_j & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \end{pmatrix}.$$

Clearly, $\|\Phi_j\|_X \to 0$,

$$\mathcal{A}\Phi_j = \begin{pmatrix} \mathcal{A}\varphi_j & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \end{pmatrix},$$

and so $\|\mathcal{A}\Phi_j\|_X \ge \text{const}$, which proves that $\mathcal{A}$ is discontinuous at $\mathbb{O}$.

Suppose now that $r \ge 1$. Consider a thematic factorisation

$$\Phi - \mathcal{A}\Phi = W_0^* \cdots W_{r-1}^* \begin{pmatrix} t_0 u_0 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \cdots & t_{r-1}u_{r-1} & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \end{pmatrix} V_{r-1}^* \cdots V_0^*.$$

Again, since $\mathcal{A}$ is unbounded on X, we may pick a sequence $\{\psi_j\}_{j\ge 0}$ of scalar functions X such that $\|\psi_j\|_X \to 0$ but $\|\mathcal{A}\psi_j\|_X \ge \text{const}$. Let

$$\Phi_j = \mathcal{A}\Phi + W_0^* \cdots W_{r-1}^* \begin{pmatrix} t_0 u_0 & \cdots & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbb{O} & \cdots & t_{r-1}u_{r-1} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \psi_j & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} & \mathbb{O} \end{pmatrix} V_{r-1}^* \cdots V_0^*.$$

Clearly, $\|\Phi - \Phi_j\|_X \to 0$ as $j \to \infty$. If we solve the superoptimal analytic approximation problem for Φ_j by successive diagonalization, then for the first r stages it proceeds exactly as for Φ. It follows that

$$\begin{aligned}&\Phi_j - \mathcal{A}\Phi_j\\ &= W_0^* \cdots W_{r-1}^* \begin{pmatrix} t_0 u_0 & \cdots & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbb{O} & \cdots & t_{r-1}u_{r-1} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \psi_j - \mathcal{A}\psi_j & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} & \mathbb{O} \end{pmatrix} V_{r-1}^* \cdots V_0^*.\end{aligned}$$

Thus

$$\mathcal{A}\Phi - \mathcal{A}\Phi_j = W_0^* \cdots W_{r-1}^* \begin{pmatrix} \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \mathcal{A}\psi_j & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} & \mathbb{O} \end{pmatrix} V_{r-1}^* \cdots V_0^*.$$

Since $\|\mathcal{A}\psi_j\|_X \geq 1$, it cannot be true that $\mathcal{A}\Phi_j \to \mathcal{A}\Phi$ in X. Thus $\mathcal{A}$ is discontinuous on X at Φ. ■

14. Unitary Interpolants of Matrix Functions

Let Φ be a matrix function in $L^\infty(\mathbb{M}_{n,n})$. We study in this section the problem of finding unitary-valued symbols of the Hankel operator H_Φ. In other words, we want to find $n \times n$ unitary-valued functions U such that

$$\hat{U}(j) = \hat{\Phi}(j), \quad j < 0. \tag{14.1}$$

A unitary-valued function U satisfying (14.1) is called a *unitary interpolant of* Φ. (It would be more appropriate to call it a unitary extrapolant.) Clearly, (14.1) means that $H_\Phi = H_U$ and so for Φ to have a unitary interpolant it is necessary that

$$\|H_\Phi\| = \operatorname{dist}_{L^\infty}\big(\Phi, H^\infty(\mathbb{M}_{n,n})\big) \leq 1. \tag{14.2}$$

We are interested in this section in unitary interpolants U of Φ such that the Toeplitz operator T_U on $H^2(\mathbb{C}^n)$ is Fredholm. By Theorem 3.4.6, T_U is Fredholm if and only if $\|H_U\|_{\mathrm{e}} < 1$ and $\|H_{U^*}\|_{\mathrm{e}} < 1$. Hence, for Φ to have a unitary interpolant U with Fredholm T_U it is necessary that

$$\|H_\Phi\|_{\mathrm{e}} = \operatorname{dist}_{L^\infty}\big(\Phi, (H^\infty + C)(\mathbb{M}_{n,n})\big) < 1. \tag{14.3}$$

Clearly, if U is a unitary interpolant of matrix function Φ satisfying (14.3), then T_U is Fredholm if and only if

$$\|H_{U^*}\|_{\mathrm{e}} < 1. \tag{14.4}$$

We show in this section that if X is an $\mathcal{R}$-space or X is a decent functiion space and $\Phi \in X$, then all unitary interpolants U of Φ that satisfy (14.4) also belong to X.

If T_U is Fredholm, the unitary-valued function U admits, by Theorem 3.5.2, a Wiener–Hopf factorization of the form

$$\Psi = Q_2^* \begin{pmatrix} z^{d_0} & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & z^{d_1} & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & z^{d_{n-1}} \end{pmatrix} Q_1,$$

where $d_0, \cdots, d_{n-1} \in \mathbb{Z}$ are the *Wiener–Hopf indices*, and Q_1 and Q_2 are functions invertible in $H^2(\mathbb{M}_{n,n})$. Here we start enumeration with 0 for technical reasons. Moreover, we can always assume that the indices are arranged in the nondecreasing order:

$$d_0 \le d_1 \le \cdots \le d_{n-1}$$

in which case the indices d_j are uniquely determined by the function Ψ (see Theorem 3.5.10).

Let us now state the main results of the section. By Theorem 10.1, if Φ satisfies conditions (14.2) and (14.3) and $Q \in H^\infty(\mathbb{M}_{n,n})$ is a best approximation of Φ by bounded analytic matrix functions, then $\Phi - Q$ admits a monotone partial thematic factorization of the form

$$\Phi - Q = W_0^* \cdots W_{r-1}^* \begin{pmatrix} u_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & u_{r-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \Psi \end{pmatrix} V_{r-1}^* \cdots V_0^*,$$

where $\|\Psi\|_{L^\infty} \le 1$ and $\|H_\Psi\| < 1$ (here r is the number of superoptimal singular values of Φ equal to 1; it may certainly happen that $r = 0$, in which case $\Psi = \Phi - Q$). We denote by k_j, $0 \le j \le r-1$, the thematic indices of the above factorization. Recall that by Theorem 10.9, the indices k_j are uniquely determined by Φ.

Theorem 14.1. *Let Φ be a matrix function in $L^\infty(\mathbb{M}_{n,n})$ such that $\|H_\Phi\|_{\mathrm{e}} < 1$. Then Φ has a unitary interpolant U satisfying $\|H_{U^*}\|_{\mathrm{e}} < 1$ if and only if $\|H_\Phi\| \le 1$.*

If U is a unitary interpolant of a matrix function Φ and conditions (14.2), (14.3), and (14.4) hold, we denote by d_j, $0 \le j \le n-1$, the Wiener–Hopf factorization indices of U arranged in the nondecreasing order:

$$d_0 \le d_1 \le \cdots \le d_{n-1}.$$

Recall that the indices d_j are uniquely determined by the function U.

Theorem 14.2. *Let Φ be a matrix function in $L^\infty(\mathbb{M}_{n,n})$ such that $\|H_\Phi\| \le 1$ and $\|H_\Phi\|_{\mathrm{e}} < 1$. Let r be the number of superoptimal singular values of Φ equal to 1. Then each unitary interpolant U of Φ satisfying $\|H_{U^*}\|_{\mathrm{e}} < 1$ has precisely r negative Wiener–Hopf indices. Moreover, $d_j = -k_j$, $0 \le j \le r-1$.*

In particular, Theorem 14.2 says that the negative Wiener–Hopf indices of a unitary interpolant U of Φ that satisfies (14.4) are uniquely determined by Φ.

Theorem 14.3. *Let Φ and r satisfy the hypotheses of Theorem 14.2. Then for any sequence of integers $\{d_j\}_{r\le j\le n-1}$ satisfying*

$$0 \le d_r \le d_{r+1} \le \cdots \le d_{n-1}$$

there exists a unitary interpolant U of Φ such that $\|H_{U^*}\|_{\mathrm{e}} < 1$ and the nonnegative Wiener–Hopf factorization indices of U are $d_r, d_{r+1}, \cdots, d_{n-1}$.

Note that Theorem 14.1 follows immediately from Theorem 14.3.

Theorem 14.4. *Let Φ satisfy the hypotheses of Theorem 14.2. Then Φ has a unique unitary interpolant if and only if all superoptimal singular values of Φ are equal to 1.*

Proof of Theorem 14.2. Suppose that U is a unitary interpolant of Φ that satisfies (14.4). Put $G = U - \Phi \in H^\infty(\mathbb{M}_{n,n})$.

Let $\varkappa \ge 0$. Let us show that if $f \in H^2(\mathbb{C}^n)$, then $\|H_{z^\varkappa\Phi} f\|_2 = \|f\|_2$ if and only if $f \in \operatorname{Ker} T_{z^\varkappa U}$.

Indeed, if $f \in \operatorname{Ker} T_{z^\varkappa U}$, then $z^\varkappa U f \in H^2_-(\mathbb{C}^n)$. It follows that

$$\|H_{z^\varkappa\Phi} f\|_2 = \|H_{z^\varkappa(\Phi+G)} f\|_2 = \|\mathbb{P}_-(z^\varkappa U f)\|_2 = \|z^\varkappa U f\|_2 = \|f\|_2.$$

Conversely. suppose that $\|H_{z^\varkappa\Phi} f\|_2 = \|f\|_2$. Then $\|H_{z^\varkappa U} f\|_2 = \|f\|_2$. Hence, $z^\varkappa U f \in H^2_-(\mathbb{C}^n)$, and so $f \in \operatorname{Ker} T_{z^\varkappa U}$.

It follows from Corollary 3.5.8 applied to the function $z^\varkappa U$ that

$$\dim \operatorname{Ker} T_{z^\varkappa U} = \sum_{\{j\in[0,n-1]:-d_j>\varkappa\}} -d_j - \varkappa.$$

By Theorem 10.8,

$$\dim\{f \in H^2(\mathbb{C}^n) : \|H_{z^\varkappa\Phi} f\|_2 = t_0\|f\|_2\} = \sum_{\{j\in[0,r-1]:k_j>\varkappa\}} k_j - \varkappa.$$

Hence,

$$\sum_{\{j\in[0,n-1]:-d_j>\varkappa\}} -d_j - \varkappa = \sum_{\{j\in[0,r-1]:k_j>\varkappa\}} k_j - \varkappa, \quad \varkappa \ge 0. \tag{14.5}$$

It is easy to see from (14.5) that U has r negative Wiener–Hopf factorization indices and $d_j = -k_j$ for $0 \le j \le r-1$. ■

To prove Theorem 14.3, we need two lemmas.

Lemma 14.5. *Let $\Psi \in L^\infty(\mathbb{M}_{m,m})$ and $\|H_\Psi\| < 1$. Then for any nonnegative integers d_j, $0 \le j \le m-1$, there exists a unitary interpolant U of Ψ that admits a representation*

$$U = W_0^* \cdots W_{m-1}^* \begin{pmatrix} u_0 & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & u_1 & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & u_{m-1} \end{pmatrix} V_{m-1}^* \cdots V_0^*, \tag{14.6}$$

where

$$W_j = \begin{pmatrix} \boldsymbol{I}_j & \mathbb{O} \\ \mathbb{O} & \breve{W}_j \end{pmatrix}, \quad V_j = \begin{pmatrix} \boldsymbol{I}_j & \mathbb{O} \\ \mathbb{O} & \breve{V}_j \end{pmatrix}, \quad 1 \le j \le m-1,$$

V_0, W_0^t, $\breve{V}_j$, $\breve{W}_j^t$ are thematic matrix functions, and $u_0, \cdots, u_{m-1}$ are unimodular functions such that the Toeplitz operators T_{u_j} are Fredholm and

$$\operatorname{ind} T_{u_j} = -d_j, \quad 0 \le j \le m-1.$$

Proof of Lemma 14.5. We argue by induction on m. Assume first that $m = 1$. Let ψ be a scalar function in L^∞ such that $\|H_\psi\| < 1$. Without loss of generality we may assume that $\|\psi\|_\infty < 1$. Consider the function $\bar{z}^{d_0+1}\psi$. Clearly, $\|H_{\bar{z}^{d_0+1}\psi}\| < 1$. It is easy to see that there exists $c \in \mathbb{R}$ such that

$$\|H_{\bar{z}^{d_0+1}\psi + c\bar{z}}\| = 1.$$

Since $H_{c\bar{z}}$ has finite rank, it is easy to see that

$$\|H_{\bar{z}^{d_0+1}\psi + c\bar{z}}\|_{\mathrm{e}} = \|H_{\bar{z}^{d_0+1}\psi}\|_{\mathrm{e}} < 1 = \|H_{\bar{z}^{d_0+1}\psi + c\bar{z}}\|.$$

It follows from Theorems 1.1.4 and 7.5.5 that $\bar{z}^{d_0+1}\psi + c\bar{z}$ has a unique best approximation g by H^∞ functions, the error function $u = \bar{z}^{d_0+1}\psi + c\bar{z} - g$ is unimodular, T_u is Fredholm, and $\operatorname{ind} T_u > 0$. On the other hand,

$$\operatorname{dist}_{L^\infty}(zu, H^\infty) = \|H_{\bar{z}^{d_0}\psi + c - zg}\| = \|H_{\bar{z}^{d_0}\psi}\| < 1,$$

and so zu is not badly approximable. Hence, $\operatorname{ind} T_{zu} \le 0$. It follows that $\operatorname{ind} T_u = 1$. We have

$$z^{d_0+1}u = \psi + cz^{d_0} - z^{d_0+1}g.$$

Put $u_0 = z^{d_0+1}u$. Then $\psi - u_0 = z^{d_0+1}g - cz^{d_0} \in H^\infty$. Hence, u_0 is a unitary interpolant of ψ and $\operatorname{ind} T_{u_0} = -d_0 - 1 + \operatorname{ind} T_u = -d_0$.

Suppose now that the lemma is proved for $(m-1) \times (m-1)$ matrix functions. By Theorem 1.8, without loss of generality we may assume that $\|\Psi\|_{L^\infty} < 1$. Then $\|H_{\bar{z}^{d_0+1}\Psi}\| < 1$. As in the scalar case, there exists $c \in \mathbb{R}$ such that

$$\|H_{\bar{z}^{d_0+1}\Psi + c\bar{z}I_m}\| = 1.$$

Clearly,

$$\|H_{\bar{z}^{d_0+1}\Psi + c\bar{z}I_m}\|_{\mathrm{e}} < 1.$$

Let G be a best approximation of $\bar{z}^{d_0+1}\Psi + c\bar{z}I_m$ by H^∞ matrix functions. By Theorems 2.2 and 4.1, $\bar{z}^{d_0+1}\Psi + c\bar{z}I_m - G$ admits a representation

$$\bar{z}^{d_0+1}\Psi + c\bar{z}I_m - G = W^* \begin{pmatrix} u & \mathbb{O} \\ \mathbb{O} & \Upsilon \end{pmatrix} V^*, \tag{14.7}$$

where V and W^{t} are thematic matrices, $\|\Upsilon\|_{L^\infty} \le 1$, u is a unimodular function such that T_u is Fredholm, and $\operatorname{ind} T_u > 0$. It is easy to see that $\|H_{\bar{z}^{d_0+1}\Psi + c\bar{z}I_m - G}\|_{\mathrm{e}} < 1$. By Theorem 4.1, $\|H_\Upsilon\|_{\mathrm{e}} < 1$.

Let us show that $\operatorname{ind} T_u = 1$. Suppose that $\operatorname{ind} T_u > 1$. Then

$$z(\bar{z}^{d_0+1}\Psi + c\bar{z}I_m - G) = W^* \begin{pmatrix} zu & \mathbb{O} \\ \mathbb{O} & z\Upsilon \end{pmatrix} V^*$$

is still badly approximable (see Theorem 2.5). Hence,

$$\begin{aligned} 1 &= \|z(\bar{z}^{d_0+1}\Psi + c\bar{z}I_m - G)\|_{L^\infty} \\ &= \|H_{\bar{z}^{d_0}\Psi + cI_m - zG}\| = \|H_{\bar{z}^{d_0}\Psi}\| \le \|\bar{z}^{d_0}\Psi\|_{L^\infty} < 1. \end{aligned}$$

We have got a contradiction. Multiplying both sides of (14.7) by z^{d_0+1}, we obtain

$$\Psi + cz^{d_0}I_m - z^{d_0+1}G = W^* \begin{pmatrix} z^{d_0+1}u & \mathbb{O} \\ \mathbb{O} & z^{d_0+1}\Upsilon \end{pmatrix} V^*.$$

Put $u_0 = z^{d_0+1}u$. Clearly, $\operatorname{ind} T_{u_0} = -d_0 - 1 + \operatorname{ind} T_u = -d_0$.

Let us show that $\|H_{z^{d_0+1}\Upsilon}\| < 1$. Consider the following factorization:

$$\bar{z}^{d_0}\Psi + cI_m - zG = W^* \begin{pmatrix} zu & \mathbb{O} \\ \mathbb{O} & z\Upsilon \end{pmatrix} V^*.$$

Clearly, $\|H_{\bar{z}^{d_0}\Psi + cI_m - zG}\| = \|H_{\bar{z}^{d_0}\Psi}\| < 1$ and $\|H_{z\Upsilon}\|_{\mathrm{e}} = \|H_\Upsilon\|_{\mathrm{e}} < 1$. By Lemma 10.4, $\|H_{z\Upsilon}\| < 1$. Hence,

$$\|H_{z^{d_0+1}\Upsilon}\| \le \|H_{z\Upsilon}\| < 1.$$

We can now apply the inductive hypothesis to $z^{d_0+1}\Upsilon$. Finally, by Lemma 1.8, there exists a function $F \in H^\infty(\mathbb{M}_{m,m})$ such $\Psi - F$ admits a desired representation. ■

Lemma 14.6. *Let U be an $n \times n$ matrix function of the form* (14.6), *where the V_j and W_j are as in Lemma 14.5, the u_j are unimodular functions such that the operators T_{u_j} are Fredholm whose indices are arbitrary integers. If $\|H_U\|_{\mathrm{e}} < 1$, then $\|H_{U^*}\|_{\mathrm{e}} < 1$.*

Proof of Lemma 14.6. Since

$$\|H_U\|_{\mathrm{e}} = \|H_{z^l U}\|_{\mathrm{e}}, \quad \|H_{U^*}\|_{\mathrm{e}} = \|H_{z^l U^*}\|_{\mathrm{e}}$$

for any $l \in \mathbb{Z}$, we may assume without loss of generality that $\operatorname{ind} T_{u_j} > 0$, $0 \le j \le n-1$. In this case (14.6) is a thematic factorization of U. By Theorem 5.4, the Toeplitz operator T_U has dense range in $H^2(\mathbb{C}^n)$. The result now follows from Theorem 4.4.11. ■

Proof of Theorem 14.3. If $r = n$ and Q is a best approximation of Φ by bounded analytic matrix functions, then Q is a superoptimal approximation of Φ. It follows from (14.3) that $\Phi - Q$ admits a thematic factorization (see Theorem 5.2), and so $U = \Phi - Q$ is a unitary interpolant of Φ. By Lemma 14.6, U satisfies (14.4).

Suppose now that $r < n$ and Q is a best approximation of Φ by bounded analytic matrix functions. Then $\Phi - Q$ admits a thematic factorization of the form

$$\Phi - Q = W_0^* \cdots W_{r-1}^* \begin{pmatrix} u_0 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \cdots & u_{r-1} & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \Psi \end{pmatrix} V_{r-1}^* \cdots V_0^*,$$

where

$$W_j = \begin{pmatrix} \boldsymbol{I}_j & \mathbb{O} \\ \mathbb{O} & \breve{W}_j \end{pmatrix}, \quad V_j = \begin{pmatrix} \boldsymbol{I}_j & \mathbb{O} \\ \mathbb{O} & \breve{V}_j \end{pmatrix}, \quad 1 \le j \le r-1,$$

V_0, W_0^{t}, $\breve{V}_j$, $\breve{W}_j^{\mathrm{t}}$ are thematic matrix functions, $\|\Psi\|_{L^\infty} \le 1$, $\|H_\Psi\| < 1$, $u_0, \cdots, u_{r-1}$ are unimodular functions such that the Toeplitz operators T_{u_j} are Fredholm, and

$$\operatorname{ind} T_{u_0} \ge \operatorname{ind} T_{u_1} \ge \cdots \ge \operatorname{ind} T_{u_{r-1}} > 0$$

(see Theorem 10.1).

We can now apply Lemma 14.5 to Ψ and find a matrix function $G \in H^\infty(\mathbb{M}_{n-r,n-r})$ such that

$$\Psi - G = \breve{W}_r^* \cdots \begin{pmatrix} \boldsymbol{I}_{n-1-r} & \mathbb{O} \\ \mathbb{O} & \breve{W}_{n-1} \end{pmatrix} D \begin{pmatrix} \boldsymbol{I}_{n-1-r} & \mathbb{O} \\ \mathbb{O} & \breve{V}_{n-1} \end{pmatrix} \cdots \breve{V}_r^*,$$

where

$$D = \begin{pmatrix} u_r & \cdots & \mathbb{O} \\ \vdots & \ddots & \vdots \\ \mathbb{O} & \cdots & u_{n-1} \end{pmatrix},$$

and the $\breve{V}_j$, $\breve{W}_j^{\mathrm{t}}$ are thematic matrix functions, $u_r, \cdots, u_{n-1}$ are unimodular functions such that the operators T_{u_j} are Fredholm and

$$\operatorname{ind} T_{u_j} = -d_j, \quad r \le j \le n-1.$$

Using Theorem 1.8, we can inductively find a matrix function $F \in H^\infty(\mathbb{M}_{n,n})$ such that $\Phi - F$ admits a factorization

$$\Phi - F = W_0^* \cdots W_{n-1}^* \begin{pmatrix} u_0 & \cdots & \mathbb{O} \\ \vdots & \ddots & \vdots \\ \mathbb{O} & \cdots & u_{n-1} \end{pmatrix} V_{n-1}^* \cdots V_0^*,$$

where

$$W_j = \begin{pmatrix} \boldsymbol{I}_j & \mathbb{O} \\ \mathbb{O} & \breve{W}_j \end{pmatrix}, \quad V_j = \begin{pmatrix} \boldsymbol{I}_j & \mathbb{O} \\ \mathbb{O} & \breve{V}_j \end{pmatrix}, \quad r \le j \le n-1.$$

Clearly, $\Phi - F$ is a unitary-valued function. By Lemma 14.6, it satisfies (14.4). To prove that the Wiener–Hopf factorization indices of $\Phi - F$ are equal to

$$-\operatorname{ind} T_{u_0}, -\operatorname{ind} T_{u_1}, \cdots, -\operatorname{ind} T_{u_{n-1}},$$

it is sufficient to apply Theorem 14.2 to the matrix function $\bar{z}^{d_{n-1}+1}(\Phi - F)$. ■

Proof of Theorem 14.4. Suppose that all superoptimal singular values of Φ are equal to 1. Let U be a unitary interpolant of Φ. Clearly, $\Phi - U$ is the superoptimal approximation of Φ, which is unique because of conditions (14.2) and (14.3).

If Φ has superoptimal singular values less than 1, then by Theorem 14.3, Φ has infinitely many unitary interpolants satisfying (14.4). ■

We proceed now to the heredity problem for unitary interpolants. Suppose that the initial matrix function Φ belongs to a certain function space X. We study the question of whether it is possible to obtain results similar to Theorems 14.1–14.4 for unitary interpolants that belong to the same class X. We obtain such results for the linear $\mathcal{R}$-spaces and the decent function spaces. Recall that if U is a unitary-valued function in VMO, then H_U and H_{U^*} are compact, and so T_U is Fredholm. Clearly, it is sufficient to prove an analog of Theorem 14.3.

Theorem 14.7. *Suppose that X is either a linear $\mathcal{R}$-space or a decent function space Let Φ be a matrix function in $X(\mathbb{M}_{n,n})$ such that $\|H_\Phi\| \le 1$. Let r be the number of superoptimal singular values of Φ equal to 1. Then for any integers $\{d_j\}_{r \le j \le n-1}$ such that*

$$0 \le d_r \le d_{r+1} \le \cdots \le d_{n-1}$$

there exists a unitary interpolant $U \in X(\mathbb{M}_{n,n})$ whose nonnegative Wiener–Hopf factorization indices are $d_r, d_{r+1}, \cdots, d_{n-1}$.

Note that any unitary interpolant $U \in X(\mathbb{M}_{n,n})$ must satisfy (14.4), and so by Theorem 14.2, U must have precisely r negative Wiener–Hopf factorization indices that are uniquely determined by Φ.

Proof of Theorem 14.7. By Theorems 14.2 and 14.3, it is sufficient to prove that under the hypotheses of the theorem any unitary interpolant U of Φ that satisfies (14.4) must belong to X.

If U is a unitary interpolant of Φ that satisfies (14.4), then $\|H_U\|_{\mathrm{e}} = 0$, and so T_U is Fredholm. Hence, the desired result follows immediately from Theorems 13.1.2 and 13.5.2. ■

The following special case of Theorem 14.7 is most important.

Theorem 14.8. *Let Φ be a matrix function in $(H^\infty + C)(\mathbb{M}_{n,n})$ such that $\|H_\Phi\| \le 1$. Let r be the number of superoptimal singular values of Φ equal to 1. Then for any integers $\{d_j\}_{r \le j \le n-1}$ such that*

$$0 \le d_r \le d_{r+1} \le \cdots \le d_{n-1}$$

there exists a unitary interpolant $U \in QC(\mathbb{M}_{n,n})$ whose nonnegative Wiener–Hopf factorization indices are $d_r, d_{r+1}, \cdots, d_{n-1}$.

Proof. Let $X = VMO$. Then X is an $\mathcal{R}$-space. The condition $\Phi \in H^\infty + C$ implies $\mathbb{P}_-\Phi \in X$. Let U be a unitary interpolant of Φ that satisfies the conclusion of Theorem 14.3. By Theorem 14.8, $U \in VMO$. The result follows now from the identity

$$QC = VMO \cap L^\infty$$

(see Appendix 2.5). ■

15. Canonical Factorizations

In this section we modify the notion of a thematic factorization and introduce so-called canonical factorizations. This allows us to obtain another method of finding the superoptimal approximation for matrix functions satisfying the same sufficient conditions. Unlike thematic factorizations, the factors in canonical factorizations are uniquely determined modulo constant unitary factors. We also consider partial canonical factorizations of badly approximable functions and establish for them the same invariance properties. We characterize the badly approximable matrix functions in terms of partial canonical factorizations and the very badly approximable matrix functions in terms of canonical factorizations. Finally, we obtain in this section heredity results for canonical and partial canonical factorizations.

To obtain a thematic factorization, we have started with a maximizing vector of the corresponding Hankel operators. To obtain a canonical factorization, we consider all maximizing vectors of the same Hankel operator.

Theorem 15.1. *Suppose that $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$. Let $\mathcal{M}$ be the minimal invariant subspace of multiplication by z on $H^2(\mathbb{C}^n)$ that contains all maximizing vectors of H_Φ. Then*

$$\mathcal{M} = \Upsilon H^2(\mathbb{C}^r), \tag{15.1}$$

where r is the number of superoptimal singular values of Φ equal to $\|H_\Phi\|$, and Υ is an inner and co-outer $n \times r$ matrix function.

Proof. Consider first the case $m = n$. Without loss of generality we may assume that $\|H_\Phi\| = 1$. It follows from Theorems 14.1 and 14.2 that there exists a unitary interpolant $\mathcal{U}$ of Φ such that the Toeplitz operator $T_{\mathcal{U}}$ is Fredholm and each such unitary interpolant has precisely r negative Wiener–Hopf indices. Consider a Wiener–Hopf factorization of $\mathcal{U}$

$$\mathcal{U} = Q_2^* \begin{pmatrix} z^{d_1} & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & z^{d_2} & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & z^{d_n} \end{pmatrix} Q_1^{-1}, \tag{15.2}$$

where Q_1 and Q_2 are matrix functions invertible in $H^2(\mathbb{M}_{n,n})$, and $d_1 \le d_2 \le \cdots \le d_n$. Since $\mathcal{U}$ has r negative Wiener–Hopf indices, we have

$$d_1 \le \cdots \le d_r < 0 \le d_{r+1} \le \cdots \le d_n.$$

Clearly, $H_\Phi = H_{\mathcal{U}}$. It is also easy to see that a nonzero function $f \in H^2(\mathbb{C}^n)$ is a maximizing vector of H_Φ if an only if $f \in \operatorname{Ker} T_{\mathcal{U}}$. It is easy to see from (15.2) that

$$\operatorname{Ker} T_{\mathcal{U}} = \left\{ Q_1 \begin{pmatrix} q_1 \\ \vdots \\ q_r \\ \mathbb{O} \\ \vdots \\ \mathbb{O} \end{pmatrix} : \ q_j \in \boldsymbol{\mathcal{P}}_+, \ \deg q_j < -d_j, \ 1 \le j \le r \right\}. \tag{15.3}$$

(Recall that $\boldsymbol{\mathcal{P}}_+$ is the set of analytic polynomials.) Since $\mathcal{M}$ is the minimal invariant subspace of multiplication by z that contains $\operatorname{Ker} T_{\mathcal{U}}$, it follows from (15.3) that

$$\mathcal{M} = \operatorname{clos}_{H^2(\mathbb{C}^n)} \left\{ Q_1 \begin{pmatrix} q_1 \\ \vdots \\ q_r \\ \mathbb{O} \\ \vdots \\ \mathbb{O} \end{pmatrix} : \ q_j \in \boldsymbol{\mathcal{P}}_+, \ 1 \le j \le r \right\}. \tag{15.4}$$

Since $Q_1(\zeta)$ is an invertible matrix for all $\zeta \in \mathbb{D}$, it follows easily from (15.4) that $\dim\{f(\zeta) : \ f \in \mathcal{M}\} = r$ for all $\zeta \in \mathbb{D}$. Therefore the z-invariant subspace $\mathcal{M}$ has the form $\mathcal{M} = \Upsilon H^2(\mathbb{C}^r)$, where Υ is an $n \times r$ inner matrix function (see Appendix 2.3). It remains to prove that Υ is co-outer.

Denote by $Q_\heartsuit$ the matrix function obtained from Q_1 by deleting the last $n-r$ columns. It is easy to see that Υ is an inner part of $Q_\heartsuit$. Let $Q_\heartsuit = \Upsilon F$, where F is an $r \times r$ outer matrix function.

Denote by $Q_\spadesuit$ the matrix function obtained from Q_1^{-1} by deleting the last $n - r$ rows. Clearly, $Q_\spadesuit(\zeta) Q_\heartsuit(\zeta) = I_r$ for almost all $\zeta \in \mathbb{T}$.

We have

$$\boldsymbol{I}_r = Q_\spadesuit Q_\heartsuit = Q_\spadesuit \Upsilon F,$$

and so

$$\boldsymbol{I}_r = F^{\mathrm{t}} \Upsilon^{\mathrm{t}} Q_\spadesuit^{\mathrm{t}}.$$

Both F^{t} and $\Upsilon^{\mathrm{t}} Q_\spadesuit^{\mathrm{t}}$ are $r \times r$ matrix functions. Hence,

$$\boldsymbol{I}_r = \Upsilon^{\mathrm{t}} Q_\spadesuit^{\mathrm{t}} F^{\mathrm{t}}.$$

It follows that Υ^{t} is outer, and so Υ is co-outer.

Consider now the case $m < n$. Let $\Phi_\#$ be the matrix function obtained from Φ by adding $n-m$ zero rows. It is easy to see that the Hankel operators

H_Φ and $H_{\Phi_\#}$ have the same maximizing vectors. This reduces the problem to the case $m = n$.

Finally, assume that $m > n$. Let $\Phi_\flat$ be the matrix function obtained from Φ by adding $m - n$ zero columns. It is easy to see that f is a maximizing vector of $H_{\Phi_\flat}$ if and only if it can be obtained from a maximizing vector of H_Φ by adding $m-n$ zero coordinates. Let $\mathcal{M}_\flat$ be the minimal invariant subspace of multiplication by z on $H^2(\mathbb{C}^m)$ that contains all maximizing vectors of $H_{\Phi_\flat}$. Clearly, the number of superoptimal singular values of $\Phi_\flat$ equal to 1 is still r. Therefore there exists an $m \times r$ inner and co-outer matrix function $\Upsilon_\flat$ such that

$$\mathcal{M}_\flat = \Upsilon_\flat H^2(\mathbb{C}^r).$$

It is easy to see that the last $m - n$ rows of $\Upsilon_\flat$ are zero. Denote by Υ the matrix function obtained from $\Upsilon_\flat$ by deleting the last $m - n$ zero rows. Obviously, Υ is an inner and co-outer $n \times r$ matrix function and $\mathcal{M} = \Upsilon H^2(\mathbb{C}^r)$. ■

We need the following result.

Lemma 15.2. *Suppose that Φ satisfies the hypotheses of Theorem 15.1 and $\mathcal{M}$ is given by* (15.1). *If $\|\Phi\|_{L^\infty} = \|H_\Phi\|$ and f is a nonzero vector function in $\mathcal{M}$, then $f(\zeta)$ is a maximizing vector of $\Phi(\zeta)$ for almost all $\zeta \in \mathbb{T}$.*

Proof. By Theorem 2.2.3, if f is a maximizing vector of H_Φ, then $f(\zeta)$ is a maximizing vector of Φ for almost all $\zeta \in \mathbb{T}$ and $\|\Phi(\zeta)\|_{\mathbb{M}_{m,n}} = \|H_\Phi\|$ almost everywhere. Without loss of generality we may assume that $\|H_\Phi\| = 1$.

Let L be the set of vector functions of the form

$$q_1 g_1 + \cdots + q_M g_M,$$

where $q_j \in \boldsymbol{\mathcal{P}}_+$ and the g_j are maximizing vectors of H_Φ. By definition, $\mathcal{M}$ is the norm closure of L. Since the $g_j(\zeta)$ are maximizing vectors of $\Phi(\zeta)$ for almost all $\zeta \in \mathbb{T}$, it follows that for $g \in L$, $g(\zeta)$ is a maximizing vector of $\Phi(\zeta)$ almost everywhere on $\mathbb{T}$. Let $\{f_j\}$ be a sequence of vector functions in L that converges to $f \in \mathcal{M}$ in $H^2(\mathbb{C}^n)$. Clearly,

$$\begin{aligned}
\int_{\mathbb{T}} \|\Phi(\zeta)f(\zeta)\|^2_{\mathbb{C}^m} d\boldsymbol{m}(\zeta) &= \lim_{j\to\infty} \int_{\mathbb{T}} \|\Phi(\zeta)f_j(\zeta)\|^2_{\mathbb{C}^m} d\boldsymbol{m}(\zeta) \\
&= \lim_{j\to\infty} \int_{\mathbb{T}} \|f_j(\zeta)\|^2_{\mathbb{C}^m} d\boldsymbol{m}(\zeta) \\
&= \int_{\mathbb{T}} \|f(\zeta)\|^2_{\mathbb{C}^m} d\boldsymbol{m}(\zeta),
\end{aligned}$$

and since obviously, $\|\Phi(\zeta)f(\zeta)\|_{\mathbb{C}^m} \le \|f(\zeta)\|_{\mathbb{C}^m}$ almost everywhere on $\mathbb{T}$, it follows that $f(\zeta)$ is a maximizing vector of $\Phi(\zeta)$ for almost all $\zeta \in \mathbb{T}$. ■

As we have observed in §2, a function $f \in H^2(\mathbb{C}^n)$ is a maximizing vector of H_Φ if and only if $g = \bar{z}\overline{H_\Phi f}$ is a maximizing vector of $H_{\Phi^{\mathrm{t}}}$. It is easy to see that the matrix functions Φ and Φ^{t} have the same superoptimal singular values. Let $\mathcal{N}$ be the minimal invariant subspace of multiplication by z on $H^2(\mathbb{C}^m)$ that contains all maximizing vectors of $H_{\Phi^{\mathrm{t}}}$. By Theorem 15.1, there exists an inner and co-outer matrix function $\Omega \in H^\infty(\mathbb{M}_{m,r})$ such that

$$\mathcal{N} = \Omega H^2(\mathbb{C}^r).$$

By Theorem 1.1, Υ and Ω have balanced completions, i.e., there exist inner and co-outer matrix functions $\Theta \in H^\infty(\mathbb{M}_{n,n-r})$ and $\Xi \in H^\infty(\mathbb{M}_{m,m-r})$ such that

$$\mathcal{V} \stackrel{\mathrm{def}}{=} \begin{pmatrix} \Upsilon & \overline{\Theta} \end{pmatrix} \quad \text{and} \quad \mathcal{W}^{\mathrm{t}} \stackrel{\mathrm{def}}{=} \begin{pmatrix} \Omega & \overline{\Xi} \end{pmatrix} \tag{15.5}$$

are unitary-valued matrix functions.

Theorem 15.3. *Let $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and $\|H_\Phi\|_{\mathrm{e}} < t_0 = \|H_\Phi\|$. Let r be the number of superoptimal singular values of Φ equal to t_0. Suppose that F is a best approximation of Φ by analytic matrix functions. Then $\Phi - F$ admits a factorization of the form*

$$\Phi - F = \mathcal{W}^* \begin{pmatrix} t_0 U & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} \mathcal{V}^*, \tag{15.6}$$

where $\mathcal{V}$ and $\mathcal{W}$ are given by (15.5), *U is an $r \times r$ unitary-valued very badly approximable matrix function such that $\|H_U\|_{\mathrm{e}} < 1$, and Ψ is a matrix-function in $L^\infty(\mathbb{M}_{m-r,n-r})$ such that $\|\Psi\|_{L^\infty} \le t_0$ and $\|H_\Psi\| = t_r(\Phi) < \|H_\Phi\|$. Moreover, U is uniquely determined by the choice of Υ and Ω and does not depend on the choice of F.*

Proof. Without loss of generality we may assume that $\|H_\Phi\| = 1$. It follows from Lemma 15.2 that the columns of $\Upsilon(\zeta)$ are maximizing vectors of $\Phi(\zeta) - F(\zeta)$ for almost all $\zeta \in \mathbb{T}$. Similarly, the columns of $\Omega(\zeta)$ are maximizing vectors of $\Phi^{\mathrm{t}}(\zeta) - F^{\mathrm{t}}(\zeta)$ almost everywhere on $\mathbb{T}$.

We need two elementary lemmas.

Lemma 15.4. *Let $A \in \mathbb{M}_{m,n}$ and $\|A\| = 1$. Suppose that $v_1, \cdots, v_r$ is an orthonormal family of maximizing vectors of A and $w_1, \cdots, w_r$ is an orthonormal family of maximizing vectors of A^t. Then*

$$\begin{pmatrix} w_1 & \cdots & w_r \end{pmatrix}^{\mathrm{t}} A \begin{pmatrix} v_1 & \cdots & v_r \end{pmatrix}$$

is a unitary matrix.

Lemma 15.5. *Let A be a matrix in $\mathbb{M}_{m,n}$ such that $\|A\| = 1$ and A has the form*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is a unitary matrix. Then A_{12} and A_{21} are the zero matrices.

Both lemmas are obvious. Let us complete the proof of Theorem 15.3.

Consider the matrix function

$$\begin{pmatrix} U & X \\ Y & \Psi \end{pmatrix} \stackrel{\mathrm{def}}{=} \mathcal{W}(\Phi - F)\mathcal{V}.$$

Here $U \in L^\infty(\mathbb{M}_{r,r})$, $X \in L^\infty(\mathbb{M}_{r,n-r})$, $Y \in L^\infty(\mathbb{M}_{m-r,r})$, and $\Psi \in L^\infty(\mathbb{M}_{m-r,n-r})$. If we apply Lemma 15.4 to the matrices $(\Phi - F)(\zeta)$, $\zeta \in \mathbb{T}$, and the columns of $\Upsilon(\zeta)$ and $\Omega(\zeta)$, we see $U = \Omega^{\mathrm{t}}(\Phi - F)\Upsilon$ is unitary-valued. By Lemma 15.5, X and Y are the zero matrix functions, which proves (15.6).

Let us prove that $\|H_U\|_{\mathrm{e}} < 1$. Since

$$U = \Omega^{\mathrm{t}}(\Phi - F)\Upsilon, \tag{15.7}$$

we have

$$\begin{aligned}
\|H_U\|_{\mathrm{e}} &= \operatorname{dist}_{L^\infty}\big(U, (H^\infty + C)(\mathbb{M}_{r,r})\big) \\
&= \operatorname{dist}_{L^\infty}\big(\Omega^{\mathrm{t}}\Phi\Upsilon, (H^\infty + C)(\mathbb{M}_{r,r})\big) \\
&\le \operatorname{dist}_{L^\infty}\big(\Phi, (H^\infty + C)(\mathbb{M}_{m,n})\big) = \|H_\Phi\|_{\mathrm{e}} < 1.
\end{aligned}$$

Now it is time to show that U is very badly approximable. Denote by $\mathcal{L}$ the minimal invariant subspace of multiplication by z that contains all maximizing vectors of H_U. Suppose that f is a maximizing vector of H_Φ. Then $f = \Upsilon\varphi$ for some $\varphi \in H^2(\mathbb{C}^r)$. Clearly, $H_\Phi f$ is a maximizing vector of H_Φ^*. By Theorem 2.2.3, $H_\Phi f = (\Phi - F)f$. Hence, $\bar{z}\overline{\Phi - F}\bar{f}$ is a maximizing vector of $H_{\Phi^{\mathrm{t}}}$. Therefore $\bar{z}\overline{\Phi - F}\bar{f} \in \Omega H^2(\mathbb{C}^r)$, and so

$$\Omega^{\mathrm{t}}(\Phi - F)f = \Omega^{\mathrm{t}}(\Phi - F)\Upsilon\varphi = U\varphi \in H^2_-(\mathbb{C}^r).$$

It follows that φ is a maximizing vector of H_U and $\|H_U\| = 1$. Therefore

$$\Upsilon H^2(\mathbb{C}^r) \subset \Upsilon\mathcal{L}.$$

Hence, $\mathcal{L} = H^2(\mathbb{C}^r)$, and by Theorem 15.1, $t_0(U) = \cdots = t_{r-1}(U) = 1$. It follows that U is very badly approximable. Hence, the zero matrix function is the only best approximation of U by analytic matrix functions.

This uniqueness property together with (15.7) implies that U does not depend on the choice of the best approximation F.

It is evident from (15.6) that $\|\Psi\|_{L^\infty} \le 1$. It remains to prove that $\|H_\Psi\| = t_r(\Phi)$.

Suppose that $F_\$$ is another best approximation of Φ by bounded analytic matrix functions. Then as we have already proved, $\Phi - F_\$$ can be represented as

$$\Phi - F_\$ = \mathcal{W}^* \begin{pmatrix} U & \mathbb{O} \\ \mathbb{O} & \Psi_\$ \end{pmatrix} \mathcal{V}^*,$$

where $\Psi_\$$ is a matrix function in $L^\infty(\mathbb{M}_{m-r,n-r})$ such that $\|\Psi_\$\|_{L^\infty} \le 1$. Clearly,

$$s_j((\Phi - F_\$)(\zeta)) = 1, \quad 0 \le j \le r - 1,$$

and

$$s_r((\Phi - F_\$)(\zeta)) = \|\Psi_\$(\zeta)\|_{\mathbb{M}_{m-r,n-r}}$$

for almost all $\zeta \in \mathbb{T}$.

By Theorem 1.8, a matrix function $G \in H^\infty(\mathbb{M}_{m,n})$ is a best approximation of Φ if and only if there exists $Q \in H^\infty(\mathbb{M}_{m-r,n-r})$ such that $\|\Psi - Q\|_{L^\infty} \le 1$ and

$$\Phi - G = \mathcal{W}^* \begin{pmatrix} U & \mathbb{O} \\ \mathbb{O} & \Psi - Q \end{pmatrix} \mathcal{V}^*.$$

This proves that $\|H_\Psi\| = t_r(\Phi)$. ■

Corollary 15.6. *Under the hypotheses of Theorem 15.3 the Toeplitz operator T_U on $H^2(\mathbb{C}^r)$ is Fredholm.*

Proof. By Theorem 5.4, T_{zU} has dense range in $H^2(\mathbb{C}^r)$. By Theorem 4.4.11, $H^*_{\bar{z}U^*}H_{\bar{z}U^*}$ is unitarily equivalent to the restriction of $H^*_{zU}H_{zU}$ to the subspace

$$\{f \in H^2(\mathbb{C}^r) : \ \|H_{zU}f\|_2 = \|f\|_2\}.$$

Since $\|H_U\|_{\mathrm{e}} < 1$, this subspace is finite-dimensional, and so

$$\|H_{U^*}\|_{\mathrm{e}} = \|H_{\bar{z}U^*}\|_{\mathrm{e}} = \lim_{j\to\infty} s_j(H_{\bar{z}U^*}) = \lim_{j\to\infty} s_j(H_{zU}) = \|H_{zU}\|_{\mathrm{e}} = \|H_U\|_{\mathrm{e}}.$$

The result now follows from Theorem 3.4.6. ■

Factorizations of the form (15.6) with Ψ satisfying

$$\|\Psi\|_{L^\infty} \le t_0 \quad \text{and} \quad \|H_\Psi\| < t_0$$

form a special class of *partial canonical factorizations*. The matrix function Ψ is called the *residual entry* of the partial canonical factorization. The notion of a partial canonical factorization in the general case will be defined later in this section.

The following theorem together with Theorem 15.3 gives a characterization of the badly approximable matrix functions Φ satisfying the condition $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$.

Theorem 15.7. *Let $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$. Suppose that Φ admits a representation of the form*

$$\Phi = \mathcal{W}^* \begin{pmatrix} \sigma U & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} \mathcal{V}^*,$$

where $\sigma > 0$, $\mathcal{V}$ and $\mathcal{W}^{\mathrm{t}}$ are r-balanced matrix functions, U is a very badly approximable unitary-valued $r\times r$ matrix function such that $\|H_U\|_{\mathrm{e}} < 1$, and $\|\Psi\|_{L^\infty} \le \sigma$. Then Φ is badly approximable and $t_0(\Phi) = \cdots = t_{r-1}(\Phi) = \sigma$.

Proof. Suppose that $\mathcal{V}$ and $\mathcal{W}$ are given by (15.5). Let $\varphi \in H^2(\mathbb{C}^r)$ be a maximizing vector of H_U. Then it is easy to see that

$$\|H_\Phi \Upsilon\varphi\|_2 = \sigma\|\Upsilon\varphi\|,$$

while

$$\|H_\Phi\| \le \|\Phi\|_{L^\infty} = \sigma.$$

Hence, $\|H_\Phi\| = \sigma$, $\Upsilon\varphi$ is a maximizing vector of H_Φ, and Φ is badly approximable.

Since U is very badly approximable, it follows from Theorem 15.1 that the minimal invariant subspace of multiplication by z on $H^2(\mathbb{C}^r)$ that contains all maximizing vectors of H_U is the space $H^2(\mathbb{C}^r)$ itself. Let $\mathcal{M}$ be the minimal invariant subspace of multiplication by z on $H^2(\mathbb{C}^n)$ that contains all maximizing vectors of H_Φ. Since Υ is co-outer, it follows that the matrix $\Upsilon(\zeta)$ has rank r for all $\zeta \in \mathbb{D}$. Hence,

$$\dim\{f(\zeta) : \; f \in \mathcal{M}\} \ge r \quad \text{for all} \quad \zeta \in \mathbb{D}.$$

It follows now from Theorem 15.1 that $t_0(\Phi) = \cdots = t_{r-1}(\Phi)$. ■

We also need the following version of the converse to Theorem 15.3.

Theorem 15.8. *Let $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$. Suppose that Φ admits a representation of the form*

$$\Phi = \mathcal{W}^* \begin{pmatrix} \sigma U & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} \mathcal{V}^*,$$

where $\sigma > 0$, $\mathcal{V}$ and $\mathcal{W}^{\mathrm{t}}$ are r-balanced matrix functions of the form (15.5), *U is a very badly approximable unitary-valued $r \times r$ matrix function such that $\|H_U\|_{\mathrm{e}} < 1$, and $\|H_\Psi\| < \sigma$. Then $\Upsilon H^2(\mathbb{C}^r)$ is the minimal invariant subspace of multiplication by z on $H^2(\mathbb{C}^n)$ that contains all maximizing vectors of H_Φ and $\Omega H^2(\mathbb{C}^r)$ is the minimal invariant subspace of multiplication by z on $H^2(\mathbb{C}^m)$ that contains all maximizing vectors of $H_{\Phi^{\mathrm{t}}}$.*

Proof. By Theorem 1.8, we may assume without loss of generality that $\|\Psi\|_{L^\infty} < s$. We need the following lemma.

Lemma 15.9. *Suppose that Φ satisfies the hypotheses of Theorem 15.8. A function f in $H^2(\mathbb{C}^n)$ is a maximizing vector of H_Φ if and only if $f = \Upsilon g$, where $g \in H^2(\mathbb{C}^r)$ and g is a maximizing vector of H_U.*

Let us first complete the proof of Theorem 15.8. Since U is very badly approximable, we have $t_0(U) = \cdots = t_{r-1}(U) = 1$. By Theorem 15.1, the minimal invariant subspace of multiplication by z on $H^2(\mathbb{C}^r)$ that contains all maximizing vectors of H_U is $H^2(\mathbb{C}^r)$. It now follows from Lemma 15.9 that the minimal invariant subspace of multiplication by z on $H^2(\mathbb{C}^n)$ that contains all maximizing vectors of H_Φ is $\Upsilon H^2(\mathbb{C}^r)$. To complete the proof, we can apply this result to the matrix function Φ^{t}. ■

Proof of Lemma 15.9. First of all, by Theorem 15.7, $\|H_\Phi\| = \sigma$. Without loss of generality we may assume that $\sigma = 1$. It has been proved in the proof of Theorem 15.7 that if g is a maximizing vector of H_U, then Υg is a maximizing vector of H_Φ.

Suppose now that f is a maximizing vector of H_Φ. We have

$$\begin{aligned} \Phi f &= \mathcal{W}^* \begin{pmatrix} U & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} \mathcal{V}^* f \\ &= \mathcal{W}^* \begin{pmatrix} U & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} \begin{pmatrix} \Upsilon^* f \\ \Theta^{\mathrm{t}} f \end{pmatrix} \\ &= \mathcal{W}^* \begin{pmatrix} U\Upsilon^* f \\ \Psi\Theta^{\mathrm{t}} f \end{pmatrix}. \end{aligned}$$

Since $\mathcal{W}^*$ is unitary-valued and $\|\Psi\|_{L^\infty} < 1$, it follows that $\Theta^{\mathrm{t}} f = 0$. Put $g = \Upsilon^* f \in L^2(\mathbb{C}^r)$. We have

$$f = \mathcal{V}\mathcal{V}^* f = \mathcal{V}\begin{pmatrix} \Upsilon^* \\ \Theta^{\mathrm{t}} \end{pmatrix} f = \begin{pmatrix} \Upsilon & \overline{\Theta} \end{pmatrix}\begin{pmatrix} \Upsilon^* f \\ 0 \end{pmatrix} = \Upsilon\Upsilon^* f = \Upsilon g.$$

Since Υ is co-outer, it follows from Lemma 1.4 that $g \in H^2(\mathbb{C}^r)$.

We have

$$\Phi f = \mathcal{W}^*\begin{pmatrix} U\Upsilon^*\Upsilon g \\ \mathbb{O} \end{pmatrix} = \begin{pmatrix} \overline{\Omega} & \Xi \end{pmatrix}\begin{pmatrix} Ug \\ \mathbb{O} \end{pmatrix} = \begin{pmatrix} \overline{\Omega} Ug \\ \mathbb{O} \end{pmatrix}.$$

Clearly, f is a maximizing vector of H_Φ if and only if $\overline{\Omega} Ug \in H^2_-(\mathbb{C}^r)$, which is equivalent to the condition $\bar{z}\Omega\overline{Ug} \in H^2(\mathbb{C}^r)$. Since Ω^{t} is outer, it follows from Lemma 1.4 that $\bar{z}\overline{Ug} \in H^2(\mathbb{C}^r)$, which is equivalent to the fact that $Ug \in H^2_-(\mathbb{C}^r)$. But the latter just means that $g \in \operatorname{Ker} T_U$, and so g is a maximizing vector of H_U. ■

Suppose now that the matrix function Ψ in the factorization (15.6) also satisfies the condition $\|H_\Psi\|_{\mathrm{e}} < \|H_\Psi\|$. Then we can continue this process, find a best analytic approximation G of Ψ and factorize $\Psi - G$ as in (15.6). If we are able to continue this diagonalization process until the very end, we construct the unique superoptimal approximation Q of Φ and obtain a canonical factorization of $\Phi - Q$.

Therefore we need an estimate of $\|H_\Psi\|_{\mathrm{e}}$. If $r = 1$, then by Theorem 4.1, $\|H_\Psi\|_{\mathrm{e}} \le \|H_\Phi\|_{\mathrm{e}}$. We want to obtain the same inequality for an arbitrary r. We could try to generalize the proof of Theorem 4.1 to the case of an arbitrary r. However, we are going to choose another way. We would like to deduce this result for an arbitrary r from the corresponding result in the case $r = 1$.

Suppose that Φ and F satisfy the hypotheses of Theorem 15.3. Then $\Phi - F$ admits a factorization (15.6). On the other hand,q it follows from Theorem 5.1 that $\Phi - F$ admits a partial thematic factorization of the form

$$\Phi - F = W_0^* \cdots W_{r-1}^* \begin{pmatrix} t_0 u_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_0 u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & t_0 u_{r-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \Delta \end{pmatrix} V_{r-1}^* \cdots V_0^*, \tag{15.8}$$

where $\|\Delta\|_{L^\infty} \le t_0$ and $\|H_\Delta\| < t_0$,

$$V_j = \begin{pmatrix} \boldsymbol{I}_j & \mathbb{O} \\ \mathbb{O} & \breve{V}_j \end{pmatrix}, \quad W_j = \begin{pmatrix} \boldsymbol{I}_j & \mathbb{O} \\ \mathbb{O} & \breve{W}_j \end{pmatrix}, \quad 0 \le j \le r-1,$$

with thematic matrix functions $\breve{V}_j, \breve{W}_j^{\mathrm{t}}$, and the u_j are unimodular functions such that the Toeplitz operators T_{u_j} are Fredholm with $\operatorname{ind} T_{u_j} > 0$. For $j = 0$ we assume that $\breve{V}_0 = V_0$ and $\breve{W}_0 = W_0$.

Suppose that

$$\breve{V}_j = \begin{pmatrix} \boldsymbol{v}_j & \overline{\Theta}_j \end{pmatrix}, \quad \breve{W}_j^{\mathrm{t}} = \begin{pmatrix} \boldsymbol{w}_j & \Xi_j \end{pmatrix}, \quad 0 \le j \le r-1, \tag{15.9}$$

where $\boldsymbol{v}_j, \Theta_j, \boldsymbol{w}_j, \Xi_j$ are inner and co-outer. Recall that $\mathcal{V}$ and $\mathcal{W}$ have the form (15.5).

Theorem 15.10. *Under the above hypotheses there exist constant unitary matrices $\mathfrak{U}_1 \in \mathbb{M}_{n-r,n-r}$ and $\mathfrak{U}_2 \in \mathbb{M}_{m-r,m-r}$ such that*

$$\Theta = \Theta_0 \Theta_1 \cdots \Theta_{r-1} \mathfrak{U}_1 \tag{15.10}$$

and

$$\Xi = \Xi_0 \Xi_1 \cdots \Xi_{r-1} \mathfrak{U}_2. \tag{15.11}$$

Proof. It follows from Theorem 1.8 that if we replace F with another best approximation G, the matrix function $\Phi - G$ will still admit factorizations of the forms (15.6) and (15.8) with the same matrix functions $\mathcal{V}, \mathcal{W}, V_j, W_j$. Hence, we may assume that $F \in \boldsymbol{\Omega}_r$ (see the Introduction to this chapter). Then the matrix functions Ψ in (15.6) and Δ in (15.8) satisfy the inequalities

$$\|\Psi\|_{L^\infty} < t_0 \quad \text{and} \quad \|\Delta\|_{L^\infty} < t_0.$$

Define the function $\rho : \mathbb{R} \to \mathbb{R}$ by

$$\rho(t) = \begin{cases} t, & t \ge t_0^2, \\ 0, & t < t_0^2. \end{cases}$$

Consider the operator $M : H^2(\mathbb{C}^n) \to L^2(\mathbb{C}^n)$ of multiplication by the matrix function $\rho\big((\Phi - F)^{\mathrm{t}}(\overline{\Phi - F})\big)$. It was proved in Lemmas 9.4 and 9.5 that

$$\operatorname{Ker} M = \Theta_0 \Theta_1 \cdots \Theta_{r-1} H^2(\mathbb{C}^{n-r}). \tag{15.12}$$

On the other hand, it follows from (15.6) that

$$(\Phi - F)^{\mathrm{t}}(\overline{\Phi - F}) = \overline{\mathcal{V}} \begin{pmatrix} t_0^2 \boldsymbol{I}_r & \mathbb{O} \\ \mathbb{O} & \Psi^{\mathrm{t}} \overline{\Psi} \end{pmatrix} \mathcal{V}^{\mathrm{t}}$$

and since $\|\Psi^{\mathrm{t}} \overline{\Psi}\|_{L^\infty} < t_0^2$, we have

$$\rho\big((\Phi - F)^{\mathrm{t}}(\overline{\Phi - F})\big) = \overline{\mathcal{V}} \begin{pmatrix} t_0^2 \boldsymbol{I}_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} \mathcal{V}^{\mathrm{t}} = \overline{\mathcal{V}} \begin{pmatrix} t_0^2 \boldsymbol{I}_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} \begin{pmatrix} \Upsilon^{\mathrm{t}} \\ \Theta^* \end{pmatrix}.$$

By Theorem 1.1, $\operatorname{Ker} T_{\Upsilon^{\mathrm{t}}} = \Theta H^2(\mathbb{C}^{n-r})$. It is easy to see now that $\operatorname{Ker} M = \Theta H^2(\mathbb{C}^{n-r})$. Together with (15.12) this yields

$$\Theta_0 \Theta_1 \cdots \Theta_{r-1} H^2(\mathbb{C}^{n-r}) = \Theta H^2(\mathbb{C}^{n-r}),$$

which means that both inner functions Θ and $\Theta_0 \Theta_1 \cdots \Theta_{r-1}$ determine the same invariant subspace of multiplication by z on $H^2(\mathbb{C}^n)$. Therefore there exists a constant unitary function $\mathfrak{U}_1$ such that (15.10) holds (see Appendix 2.3). To prove (15.11), we can apply (15.10) to $(\Phi - F)^{\mathrm{t}}$. ■

Corollary 15.11. *Let Ψ and Δ be the matrix functions in the factorizations* (15.6) *and* (15.8). *Then*

$$\Delta = \mathfrak{U}_2 \Psi \mathfrak{U}_1^t, \tag{15.13}$$

where $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are unitary matrices from (15.10) *and* (15.11).

Proof. By Corollary 9.2,

$$\Delta = \Xi_{r-1}^* \cdots \Xi_1^* \Xi_0^* (\Phi - F) \overline{\Theta_0 \Theta_1 \cdots \Theta_{r-1}}.$$

By Theorem 15.10,

$$\Delta = \mathfrak{U}_2 \Xi^* (\Phi - F) \overline{\Theta} \mathfrak{U}_1^{\mathrm{t}}.$$

On the other hand, it is easy to see from (15.6) that

$$\Psi = \Xi^* (\Phi - F) \overline{\Theta}, \tag{15.14}$$

which implies (15.13). ■

Now we are in a position to estimate $\|H_\Psi\|_{\mathrm{e}}$ for the residual entry Ψ in the factorization (15.6).

Theorem 15.12. *Let Φ be a function in $L^\infty(\mathbb{M}_{m,n})$ such that $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$. Then for the residual entry Ψ in the partial canonical factorization* (15.6) *the following inequality holds:*

$$\|H_\Psi\|_{\mathrm{e}} \le \|H_\Phi\|_{\mathrm{e}}.$$

Proof. Iterating Theorem 4.1, we find that $\|H_\Delta\|_{\mathrm{e}} \le \|H_\Phi\|_{\mathrm{e}}$. The result now follows from (15.13). ■

Consider now the unitary-valued matrix function U in the partial canonical factorization (15.6). By Corollary 15.6, the Toeplitz operator T_U is Fredholm. We are now going to evaluate the index of T_U in terms of the indices

$$k_j = \operatorname{ind} T_{u_j}, \quad 0 \le j \le r - 1.$$

of the partial thematic factorization (15.8).

Theorem 15.13. *Let $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$. Then the entry U of of the diagonal block matrix function in the partial canonical factorization* (15.6) *satisfies*

$$\operatorname{ind} T_U = \dim \operatorname{Ker} T_U = k_0 + k_1 + \cdots + k_{r-1}.$$

Proof. Without loss of generality we may assume that $\|H_\Phi\| = 1$. By Theorem 15.3, U is very badly approximable, $\|H_U\|_{\mathrm{e}} < \|H_U\| = 1$. Therefore by Theorem 5.4, the Toeplitz operator T_{zU} has dense range in $H^2(\mathbb{C}^r)$. Hence, $\operatorname{Ker} T_U^* = \{\mathbb{O}\}$, and so $\operatorname{ind} T_U = \dim \operatorname{Ker} T_U$.

By Theorem 7.4,

$$\dim\{f \in H^2(\mathbb{C}^n) : \ \|H_\Phi f\|_2 = \|f\|_2\} = k_0 + k_1 + \cdots + k_{r-1} \tag{15.15}$$

Let us show that the left-hand side of (15.15) is equal to $\dim \operatorname{Ker} T_U$. Indeed, if $g \in H^2(\mathbb{C}^r)$, then $g \in \operatorname{Ker} T_U$ if and only if g is a maximizing vector of H_U. By Lemma 15.9,

$$\dim\{g \in H^2(\mathbb{C}^r) : \|H_U g\|_2 = \|g\|_2\} = \dim\{f \in H^2(\mathbb{C}^n) : \|H_\Phi f\|_2 = \|f\|_2\},$$

which proves the result. ■

We obtain now an inequality between the singular values of H_Φ and the singular values of H_Ψ that generalizes Theorem 8.1.

Theorem 15.14. *Let $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$. Suppose that $F \in H^\infty(\mathbb{M}_{m,n})$ is a best approximation of Φ by bounded analytic functions and $\Phi - F$ is represented by the partial canonical factorization* (15.6). *Then*

$$s_j(H_\Psi) \le s_{k+j}(H_\Phi), \quad j \ge 0, \tag{15.16}$$

where $k \stackrel{\text{def}}{=} \operatorname{ind} T_U$.

Proof. Consider the partial thematic factorization (15.8). Let $k_j \stackrel{\text{def}}{=} \operatorname{ind} T_{u_j}$, $0 \le j \le r-1$, be the indices of this factorization. We have

$$\Phi - F = W_0^* \begin{pmatrix} t_0 u_0 & \mathbb{O} \\ \mathbb{O} & \Phi^{[1]} \end{pmatrix} V_0^*, \tag{15.17}$$

where $\Phi^{[1]}$ is given by the partial thematic factorization

$$\Phi^{[1]} = \breve{W}_1^* \cdots \begin{pmatrix} \boldsymbol{I}_{r-2} & \mathbb{O} \\ \mathbb{O} & \breve{W}_{r-1}^* \end{pmatrix} D \begin{pmatrix} \boldsymbol{I}_{r-2} & \mathbb{O} \\ \mathbb{O} & \breve{V}_{r-1}^* \end{pmatrix} \cdots \breve{V}_1^*$$

and

$$D = \begin{pmatrix} t_0 u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \cdots & t_0 u_{r-1} & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \Delta \end{pmatrix}.$$

Now we can apply Theorem 8.1 to the factorization (15.17) and find that

$$s_j(H_{\Phi^{[1]}}) \le s_{k_0+j}(H_\Phi), \quad j \ge 0.$$

Then we can apply Theorem 8.1 to the above partial thematic factorization of $\Phi^{[1]}$, etc. After applying Theorem 8.1 r times we obtain the inequality

$$s_j(H_\Delta) \le s_{k_0+\cdots+k_{r-1}+j}(H_\Phi), \quad j \ge 0.$$

The result now follows from Corollary 15.11 and Theorem 15.13. ■

Now we are going to prove that under the hypotheses of Theorem 15.1 the Toeplitz operators $T_{\mathcal{V}}$, $T_{\mathcal{V}^{\mathrm{t}}}$, $T_{\mathcal{W}}$, and $T_{\mathcal{W}^{\mathrm{t}}}$ are invertible. Then we deduce that the matrix functions Υ, Θ, Ω, and Ξ are left invertible in H^∞. Recall that in the case $r = 1$ those facts have been proved in §4.

Theorem 15.15. *Let* $\Phi \in L^\infty(\mathbb{M}_{m,n})$ *and suppose that* $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$. *Let* $\mathcal{V}$ *and* $\mathcal{W}^t$ *be the* r*-balanced matrix functions in the partial canonical factorization* (15.6). *Then the Toeplitz operators* $T_{\mathcal{V}}$, $T_{\mathcal{V}^{\mathrm{t}}}$, $T_{\mathcal{W}}$, *and* $T_{\mathcal{W}^{\mathrm{t}}}$ *are invertible.*

Proof. As in the case of $r = 1$ it follows from Theorem 1.3 that

$$\|H_{\overline{\Upsilon}}\| = \|H_{\overline{\Theta}}\|$$

(see Lemma 4.6) and as in the case $r = 1$ to prove that $T_{\mathcal{V}}$ and $T_{\mathcal{V}^{\mathrm{t}}}$ are invertible, it is sufficient to show that

$$\|H_{\overline{\Theta}}\| < 1$$

(see Lemma 4.6).

Consider the partial thematic factorization (15.8). By Lemma 4.6, the Toeplitz operators $T_{\breve{V}_j}$ are invertible for $0 \le j \le r-1$. Since the $\breve{V}_j$ are 1-balanced, this is equivalent to the fact that $\|H_{\overline{\Theta}_j}\| < 1$, $0 \le j \le r-1$ (see the proof of Lemma 4.6).

By Theorem 15.10, $\Theta = \Theta_0 \cdots \Theta_{r-1}\mathfrak{U}$ for some constant unitary matrix $\mathfrak{U}$. Clearly, to show that $\|H_{\overline{\Theta}}\| < 1$, it is sufficient to prove the following lemma.

Lemma 15.16. *Let* Θ_1 *be an* $n \times k$ *inner matrix function and let* Θ_2 *be a* $k \times l$ *inner matrix function such that* $\|H_{\overline{\Theta}_1}\| < 1$ *and* $\|H_{\overline{\Theta}_2}\| < 1$. *Let* $\Theta = \Theta_1\Theta_2$. *Then* $\|H_{\overline{\Theta}}\| < 1$.

Proof of Lemma 15.16. Let $\sigma < 1$ be a positive number such that $\|H_{\overline{\Theta}_1}\| < \sigma$ and $\|H_{\overline{\Theta}_2}\| < \sigma$. Let $f \in H^2(\mathbb{C}^l)$. We have

$$\begin{aligned}
\|H_{\overline{\Theta}} f\|_2^2 &= \|\mathbb{P}_-\overline{\Theta_1\Theta_2} f\|_2^2 \\
&= \|\mathbb{P}_-\overline{\Theta_1}\mathbb{P}_-\overline{\Theta_2} f + \mathbb{P}_-\overline{\Theta_1}\mathbb{P}_+\overline{\Theta_2} f\|_2^2 \\
&= \|\overline{\Theta_1}\mathbb{P}_-\overline{\Theta_2} f + \mathbb{P}_-\overline{\Theta_1}\mathbb{P}_+\overline{\Theta_2} f\|_2^2 \\
&= \|\mathbb{P}_-\overline{\Theta_2} f + \Theta_1^{\mathrm{t}}\mathbb{P}_-\overline{\Theta_1}\mathbb{P}_+\overline{\Theta_2} f\|_2^2.
\end{aligned}$$

We claim that the functions $\mathbb{P}_-\overline{\Theta_2} f$ and $\Theta_1^{\mathrm{t}}\mathbb{P}_-\overline{\Theta_1}\mathbb{P}_+\overline{\Theta_2} f$ are orthogonal. Indeed, let $\varphi \in H^2_-(\mathbb{C}^k)$ and $\psi \in H^2(\mathbb{C}^k)$. We have

$$(\varphi, \Theta_1^{\mathrm{t}}\mathbb{P}_-\overline{\Theta_1}\psi) = (\overline{\Theta_1}\varphi, \mathbb{P}_-\overline{\Theta_1}\psi) = (\overline{\Theta_1}\varphi, \overline{\Theta_1}\psi) = (\varphi, \psi) = 0,$$

since $\overline{\Theta_1}$ takes isometric values almost everywhere on $\mathbb{T}$. It follows that

$$\begin{aligned}
\|H_{\overline{\Theta}} f\|_2^2 &= \|\mathbb{P}_-\overline{\Theta_2} f\|_2^2 + \|\Theta_1^{\mathrm{t}}\mathbb{P}_-\overline{\Theta_1}\mathbb{P}_+\overline{\Theta_2} f\|_2^2 \\
&\le \|\mathbb{P}_-\overline{\Theta_2} f\|_2^2 + \|\mathbb{P}_-\overline{\Theta_1}\mathbb{P}_+\overline{\Theta_2} f\|_2^2 \\
&= \|\mathbb{P}_-\overline{\Theta}_2 f\|_2^2 + \|H_{\overline{\Theta_1}}\mathbb{P}_+\overline{\Theta_2} f\|_2^2 \\
&\le \|\mathbb{P}_-\overline{\Theta}_2 f\|_2^2 + \sigma^2\|\mathbb{P}_+\overline{\Theta}_2 f\|_2^2 \\
&= \sigma^2(\|\mathbb{P}_-\overline{\Theta}_2 f\|_2^2 + \|\mathbb{P}_+\overline{\Theta}_2 f\|_2^2) + (1-\sigma^2)\|\mathbb{P}_-\overline{\Theta}_2 f\|_2^2 \\
&= \sigma^2\|\overline{\Theta_2} f\|_2^2 + (1-\sigma^2)\|H_{\overline{\Theta}_2} f\|_2^2 \\
&\le \sigma^2\|f\|_2^2 + \sigma^2(1-\sigma^2)\|f\|_2^2 = (2\sigma^2 - \sigma^4)\|f\|_2^2.
\end{aligned}$$

The result now follows from the trivial inequality $2\sigma^2 - \sigma^4 < 1$. ■

Corollary 15.17. *Under the hypotheses of Theorem 15.15 the matrix functions* Υ, Θ, Ω, *and* Ξ *are left invertible in* H^∞.

Proof. We have shown in the proof of Theorem 15.15 that $\|H_{\overline{\Upsilon}}\| < 1$ and $\|H_{\overline{\Theta}}\| < 1$. By Theorem 3.4.4, the Toeplitz operators $T_{\overline{\Upsilon}}$ and $T_{\overline{\Theta}}$ are left invertible. It now follows from Theorem 3.6.1 that Υ and Θ are left invertible in H^∞. To prove the left invertibility of Ω and Ξ, we can replace Φ with Φ^{t}. ■

Let us proceed now to the construction of a canonical factorization for very badly approximable matrix functions.

Given $\Phi \in L^\infty(\mathbb{M}_{m,n})$, consider the sequence $\{t_j\}$ of its superoptimal singular values. Suppose that

$$t_0 = \cdots = t_{r_1-1} > t_{r_1} = \cdots = t_{r_2-1} > \cdots > t_{r_{\iota-1}} = \cdots = t_{r_\iota-1} \tag{15.18}$$

are all nonzero superoptimal singular values of Φ, i.e., t_0 has multiplicity r_1 and t_{r_j} has multiplicity $r_{j+1} - r_j$, $1 \le j \le \iota - 1$.

By Theorem 15.3, if $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$ and F_0 is a best approximation of Φ by a bounded analytic matrix function, then $\Phi - F_0$ admits a factorization

$$\Phi - F_0 = \mathcal{W}_0^* \begin{pmatrix} t_0 U_0 & \mathbb{O} \\ \mathbb{O} & \Phi^{[1]} \end{pmatrix} \mathcal{V}_0^*,$$

where $\mathcal{V}_0$ and $\mathcal{W}_0^{\mathrm{t}}$ are r_1-balanced matrix functions, U_0 is a very badly approximable unitary-valued matrix function such that $\|H_{U_0}\|_{\mathrm{e}} < 1$, and $\Phi^{[1]}$ is a matrix function in $L^\infty(\mathbb{M}_{m-r_1,n-r_1})$ such that $\|H_{\Phi^{[1]}}\| = t_{r_1}$.

By Theorem 15.12, $\|H_{\Phi^{[1]}}\|_{\mathrm{e}} \le \|H_\Phi\|_{\mathrm{e}}$. If t_{r_1} is still greater than $\|H_\Phi\|_{\mathrm{e}}$, we can apply Theorem 15.3 to $\Phi^{[1]}$ and find that for a best approximation G_1 of $\Phi^{[1]}$ the matrix function $\Phi^{[1]} - G_1$ admits a factorization

$$\Phi^{[1]} - G_1 = \breve{\mathcal{W}}_1^* \begin{pmatrix} t_{r_1} U_1 & \mathbb{O} \\ \mathbb{O} & \Phi^{[2]} \end{pmatrix} \breve{\mathcal{V}}_1^*,$$

where $\breve{\mathcal{V}}_1$ and $\breve{\mathcal{W}}_1^{\mathrm{t}}$ are $(r_2 - r_1)$-balanced matrix functions, U_1 is a very badly approximable unitary-valued matrix function of size $(r_2 - r_1) \times (r_2 - r_1)$ such that $\|H_{U_1}\|_{\mathrm{e}} < 1$, and $\Phi^{[2]}$ is a matrix function in $L^\infty(\mathbb{M}_{m-r_2,n-r_2})$ such that $\|H_{\Phi^{[2]}}\| = t_{r_2}$.

We can now apply Theorem 1.8 and find a matrix function $F_1 \in H^\infty(\mathbb{M}_{m,n})$ such that

$$\Phi - F_1 = \mathcal{W}_0^* \mathcal{W}_1^* \begin{pmatrix} t_0 U_0 & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_{r_1} U_1 & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \Phi^{[2]} \end{pmatrix} \mathcal{V}_1^* \mathcal{V}_0^*,$$

where

$$\mathcal{V}_1 = \begin{pmatrix} \boldsymbol{I}_{r_1} & \mathbb{O} \\ \mathbb{O} & \breve{\mathcal{V}}_1 \end{pmatrix} \quad \text{and} \quad \mathcal{W}_1 = \begin{pmatrix} \boldsymbol{I}_{r_1} & \mathbb{O} \\ \mathbb{O} & \breve{\mathcal{W}}_1 \end{pmatrix}.$$

If t_{r_2} is still greater than $\|H_\Phi\|_{\mathrm{e}}$, we can continue this process and apply Theorem 15.3 to $\Phi^{[2]}$. Suppose now that $t_{r_{d-1}} > \|H_\Phi\|_{\mathrm{e}}$, $2 \le d \le \iota$. Then continuing the above process and applying Theorem 1.8, we can find a function $F \in H^\infty(\mathbb{M}_{m,n})$ such that $\Phi - F$ admits a factorization

$$\Phi - F$$

$$= \mathcal{W}_0^* \cdots \mathcal{W}_{d-1}^* \begin{pmatrix} t_0 U_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_{r_1} U_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & t_{r_{d-1}} U_{d-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \Phi^{[d]} \end{pmatrix} \mathcal{V}_{d-1}^* \cdots \mathcal{V}_0^*, \tag{15.19}$$

where the U_j are $(r_{j+1} - r_j) \times (r_{j+1} - r_j)$ very badly approximable unitary-valued functions such that $\|H_{U_j}\|_{\mathrm{e}} < 1$,

$$\mathcal{V}_j = \begin{pmatrix} \boldsymbol{I}_{r_j} & \mathbb{O} \\ \mathbb{O} & \breve{\mathcal{V}}_j \end{pmatrix} \quad \text{and} \quad \mathcal{W}_j = \begin{pmatrix} \boldsymbol{I}_{r_j} & \mathbb{O} \\ \mathbb{O} & \breve{\mathcal{W}}_j \end{pmatrix}, \quad 1 \le j \le d-1, \tag{15.20}$$

$\breve{\mathcal{V}}_j$ and $\breve{\mathcal{W}}_j^{\mathrm{t}}$ are $(r_{j+1} - r_j)$-balanced matrix functions, and Φ is a matrix function satisfying

$$\left\|\Phi^{[d]}\right\|_{L^\infty} \le t_{r_{d-1}}, \quad \text{and} \quad \left\|H_{\Phi^{[d]}}\right\| < t_{r_{d-1}}. \tag{15.21}$$

Factorizations of the form (15.19) with $\mathcal{V}_j$ and $\mathcal{W}_j$ of the form (15.20) and $\Phi^{[d]}$ satisfying (15.21) are called *partial canonical factorizations*. The matrix function $\Phi^{[d]}$ is called the *residual entry* of the partial canonical factorization (15.19).

Finally, if $\|H_\Phi\|_{\mathrm{e}}$ is less than the smallest nonzero superoptimal singular value $t_{r_\iota - 1}$, then we can complete this process and construct the unique superoptimal approximation of Φ by bounded analytic matrix functions. This proves the following theorem.

Theorem 15.18. *Let $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and suppose that the nonzero superoptimal singular values of Φ satisfy* (15.18). *If $\|H_\Phi\|_{\mathrm{e}} < t_{r_\iota - 1}$ and F is the unique superoptimal approximation of Φ by bounded analytic functions, then $\Phi - F$ admits a factorization*

$$\Phi - F = \mathcal{W}_0^* \cdots \mathcal{W}_{\iota-1}^* \begin{pmatrix} t_0 U_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_{r_1} U_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & t_{r_{\iota-1}} U_{\iota-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \end{pmatrix} \mathcal{V}_{\iota-1}^* \cdots \mathcal{V}_0^*, \tag{15.22}$$

where the $\mathcal{V}_j$, $\mathcal{W}_j$, and U_j are as above.

Note that the lower right entry of the diagonal matrix function on the right-hand side of (15.22) is the zero matrix function of size $(m - r_\iota) \times (n - r_\iota)$. Here it may happen that $m - r_\iota$ or $n - r_\iota$ can be zero, which would mean that the corresponding rows or columns do not exist.

Clearly, the left-hand side of (15.22) is a very badly approximable matrix function. Factorizations of the form (15.22) are called *canonical factorizations* of very badly approximable matrix functions.

The following result shows that the right-hand side of (15.22) is always a very badly approximable matrix function.

Theorem 15.19. *Let Φ be a matrix function in $L^\infty(\mathbb{M}_{m,n})$ such that $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$, ι is a positive integer, $r_1, \cdots, r_\iota$ are positive integers satisfying*

$$r_1 < r_2 < \cdots < r_\iota$$

and

$$\sigma_0 > \sigma_1 > \cdots > \sigma_{\iota-1} > 0.$$

Suppose that Φ admits a factorization

$$\Phi = \mathcal{W}_0^* \cdots \mathcal{W}_{\iota-1}^* \begin{pmatrix} \sigma_0 U_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \sigma_1 U_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & \sigma_{\iota-1} U_{\iota-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \end{pmatrix} \mathcal{V}_{\iota-1}^* \cdots \mathcal{V}_0^*,$$

in which the U_j, $\mathcal{V}_j$, and $\mathcal{W}_j$ are as above. Then Φ is very badly approximable and the superoptimal singular values of Φ are given by

$$t_\varkappa(\Phi) = \begin{cases} \sigma_0, & \varkappa < r_1, \\ \sigma_j, & r_j \le \varkappa < r_{j+1}, \\ 0, & \varkappa \ge r_\iota. \end{cases}$$

Proof. Let

$$\mathcal{V}_0 = \begin{pmatrix} \Upsilon & \overline{\Theta} \end{pmatrix} \quad \text{and} \quad \mathcal{W}_0 = \begin{pmatrix} \Omega & \overline{\Xi} \end{pmatrix}^{\mathrm{t}},$$

where Υ, Θ, Ω, and Ξ are inner and co-outer matrix functions.

By Theorem 15.8, the minimal invariant subspace of multiplication by z on $H^2(\mathbb{C}^n)$ that contains all maximizing vectors of H_Φ is $\Upsilon H^2(\mathbb{C}^{r_1})$ and the minimal invariant subspace of multiplication by z on $H^2(\mathbb{C}^m)$ that contains all maximizing vectors of $H_{\Phi^{\mathrm{t}}}$ is $\Omega H^2(\mathbb{C}^{r_1})$. By Theorem 15.7, Φ is badly approximable and

$$t_0(\Phi) = \cdots = t_{r_1-1} = \sigma_0.$$

By Theorem 1.8, we can reduce our problem to the function

$$\breve{\mathcal{W}}_1^* \cdots \begin{pmatrix} \boldsymbol{I}_{r_{\iota-1}-r_1} & \mathbb{O} \\ \mathbb{O} & \breve{\mathcal{W}}_{\iota-1}^* \end{pmatrix} D \begin{pmatrix} \boldsymbol{I}_{r_{\iota-1}-r_1} & \mathbb{O} \\ \mathbb{O} & \breve{\mathcal{V}}_{\iota-1}^* \end{pmatrix} \cdots \breve{\mathcal{V}}_1^*,$$

where

$$D = \begin{pmatrix} \sigma_1 U_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \cdots & \sigma_{\iota-1}U_{\iota-1} & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \end{pmatrix}.$$

This function is also represented by a canonical factorization that makes it possible to continue this process and prove that Φ is very badly approximable and the superoptimal singular values of Φ satisfy the desired equality. ■

We can now demonstrate an advantage of canonical factorizations over thematic factorizations. Namely, we show that canonical factorizations possess certain invariance properties, i.e., the matrix functions U_j in the canonical factorization (15.22) are uniquely determined modulo unitary constant factors. Moreover, if

$$\mathcal{V}_0 = \begin{pmatrix} \Upsilon_0 & \overline{\Theta}_0 \end{pmatrix}, \quad \mathcal{W}_0^{\mathrm{t}} = \begin{pmatrix} \Omega_0 & \Xi_0 \end{pmatrix},$$

$$\breve{\mathcal{V}}_j = \begin{pmatrix} \Upsilon_j & \overline{\Theta}_j \end{pmatrix}, \quad \breve{\mathcal{W}}_j^{\mathrm{t}} = \begin{pmatrix} \Omega_j & \Xi_j \end{pmatrix}, \quad 1 \le j \le \iota - 1,$$

and the $\breve{\mathcal{V}}_j$ and $\breve{\mathcal{W}}_j$ are given by (15.20), then the matrix functions $\Upsilon_j, \Theta_j, \Omega_j$, and Ξ_j are also uniquely determined modulo constant unitary factors.

We start with partial canonical factorizations of the form (15.6). Suppose that a matrix function Φ in $L^\infty(\mathbb{M}_{m,n})$ satisfies $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$ and admits partial canonical factorizations

$$\Phi = \begin{pmatrix} \overline{\Omega} & \Xi \end{pmatrix} \begin{pmatrix} \sigma U & 0 \\ 0 & \Psi \end{pmatrix} \begin{pmatrix} \Upsilon & \overline{\Theta} \end{pmatrix}^* \tag{15.23}$$

and

$$\Phi = \begin{pmatrix} \overline{\Omega^\circ} & \Xi^\circ \end{pmatrix} \begin{pmatrix} \sigma^\circ U^\circ & 0 \\ 0 & \Psi^\circ \end{pmatrix} \begin{pmatrix} \Upsilon^\circ & \overline{\Theta^\circ} \end{pmatrix}^*, \tag{15.24}$$

where $\|\Psi\|_{L^\infty} \le \sigma$, $\|H_\Psi\| < \sigma$, $\|\Psi^\circ\|_{L^\infty} \le \sigma^\circ$, $\|H_{\Psi^\circ}\| < \sigma^\circ$, U is an $r \times r$ very badly approximable unitary-valued function such that $\|H_U\|_{\mathrm{e}} < 1$, U° is an $r^\circ \times r^\circ$ very badly approximable unitary-valued function such that $\|H_{U^\circ}\|_{\mathrm{e}} < 1$, $\begin{pmatrix} \Upsilon & \overline{\Theta} \end{pmatrix}$ and $\begin{pmatrix} \Omega & \Xi \end{pmatrix}^{\mathrm{t}}$ are r-balanced matrix functions, and $\begin{pmatrix} \Upsilon^\circ & \overline{\Theta^\circ} \end{pmatrix}$ and $\begin{pmatrix} \Omega^\circ & \overline{\Xi^\circ} \end{pmatrix}^{\mathrm{t}}$ are r°-balanced matrix functions.

Theorem 15.20. *Let Φ be a badly approximable function in $L^\infty(\mathbb{M}_{m,n})$ such that $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$. Suppose that Φ admits factorizations* (15.23) *and* (15.24). *Then $r = r^\circ$, $\sigma = \sigma^\circ$, and there exist unitary matrices $\mathfrak{V}^\#, \mathfrak{V}^\flat \in \mathbb{M}_{r,r}$, $\mathfrak{U}^\# \in \mathbb{M}_{n-r,n-r}$, $\mathfrak{U}^\flat \in \mathbb{M}_{m-r,m-r}$ such that*

$$\Upsilon^\circ = \Upsilon \mathfrak{V}^\#, \quad \Omega^\circ = \Omega \mathfrak{V}^\flat, \tag{15.25}$$

$$\Theta^\circ = \Theta \mathfrak{U}^\#, \quad \Xi^\circ = \Xi \mathfrak{U}^\flat, \tag{15.26}$$

and

$$U^\circ = (\mathfrak{V}^\flat)^{\rm t} U \mathfrak{V}^\#, \quad \Psi^\circ = (\mathfrak{U}^\flat)^* \Psi \overline{\mathfrak{U}^\#}. \tag{15.27}$$

Proof. By Theorem 15.7, $\sigma = \sigma^\circ = \|H_\Phi\|$. Next, by Theorem 15.8, the minimal invariant subspace of multiplication by z on $H^2(\mathbb{C}^n)$ that contains all maximizing vectors of H_Φ is equal to $\Upsilon H^2(\mathbb{C}^r)$ and at the same time it is equal to $\Upsilon^\circ H^2(\mathbb{C}^{r^\circ})$. It follows that $r = r^\circ$ and there exists a unitary matrix $\mathfrak{V}^\# \in \mathbb{M}_{r,r}$ such that $\Upsilon^\circ = \Upsilon \mathfrak{V}^\#$. Applying the same reasoning to $\Phi^{\rm t}$, we find a unitary matrix function $\mathfrak{V}^\flat \in \mathbb{M}_{r,r}$ such that $\Omega^\circ = \Omega \mathfrak{V}^\flat$, which proves (15.25).

By Theorem 1.1,

$$\Theta H^2(\mathbb{C}^{n-r}) = \operatorname{Ker} T_{\Upsilon^{\rm t}} \quad \text{and} \quad \Theta^\circ H^2(\mathbb{C}^{n-r}) = \operatorname{Ker} T_{(\Upsilon^\circ)^{\rm t}}.$$

By (15.25), $\operatorname{Ker} T_{\Upsilon^{\rm t}} = \operatorname{Ker} T_{(\Upsilon^\circ)^{\rm t}}$, which implies that there exists a unitary matrix $\mathfrak{U}^\# \in \mathbb{M}_{n-r,n-r}$ such that $\Theta^\circ = \Theta \mathfrak{U}^\#$. Applying the same reasoning to $\Phi^{\rm t}$, we find a unitary matrix $\mathfrak{U}^\flat \in \mathbb{M}_{m-r,m-r}$ such that $\Xi^\circ = \Xi \mathfrak{U}^\flat$, which proves (15.26).

By (15.7),

$$\sigma U = \Omega^{\rm t} \Phi \Upsilon \quad \text{and} \quad \sigma U^\circ = (\Omega^\circ)^{\rm t} \Phi \Upsilon^\circ.$$

This implies the first equality in (15.27).

Finally, by (15.14),

$$\Psi = \Xi^* \Phi \overline{\Theta} \quad \text{and} \quad \Psi^\circ = \Xi^{\circ *} \Phi \overline{\Theta^\circ},$$

which completes the proof of (15.27). ■

We can obtain similar results for arbitrary partial canonical factorizations and canonical factorizations. Let us consider in detail the following special case. Suppose that the matrix functions Ψ and Ψ° in (15.23) and (15.24) admit the following partial canonical factorizations:

$$\Psi = \begin{pmatrix} \overline{\Omega}_1 & \Xi_1 \end{pmatrix} \begin{pmatrix} \sigma_1 U_1 & 0 \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} \Upsilon_1 & \overline{\Theta}_1 \end{pmatrix}^* \tag{15.28}$$

and

$$\Psi^\circ = \begin{pmatrix} \overline{\Omega_1^\circ} & \Xi_1^\circ \end{pmatrix} \begin{pmatrix} \sigma_1^\circ U_1^\circ & 0 \\ 0 & \Delta^\circ \end{pmatrix} \begin{pmatrix} \Upsilon_1^\circ & \overline{\Theta_1^\circ} \end{pmatrix}^*, \tag{15.29}$$

where $\|\Delta\|_{L^\infty} \le \sigma_1$, $\|H_\Delta\| < \sigma_1$ and $\|\Delta^\circ\|_{L^\infty} < \sigma_1^\circ$, $\|H_{\Delta^\circ}\| < \sigma_1^\circ$.

By Theorem 15.20, $\Psi^\circ = (\mathfrak{U}^\flat)^* \Psi \overline{\mathfrak{U}^\#}$. It now follows from (15.29) that

$$\Psi = \begin{pmatrix} \overline{\overline{\mathfrak{U}^\flat \Omega_1^\circ}} & \mathfrak{U}^\flat \Xi_1^\circ \end{pmatrix} \begin{pmatrix} \sigma_1^\circ U_1^\circ & 0 \\ 0 & \Delta^\circ \end{pmatrix} \begin{pmatrix} \overline{\mathfrak{U}^\#} \Upsilon_1^\circ & \overline{\mathfrak{U}^\# \Theta_1^\circ} \end{pmatrix}^*, \tag{15.30}$$

which is another partial canonical factorization of Ψ.

We can now compare the factorizations (15.28) and (15.30). By Theorem 15.20, $\sigma_1^\circ = \sigma_1$ and there exist unitary matrices $\mathfrak{V}_1^\#, \mathfrak{V}_1^\flat, \mathfrak{U}_1^\#, \mathfrak{U}_1^\flat$ such that

$$\Upsilon_1^\circ = (\mathfrak{U}^\#)^{\rm t} \Upsilon_1 \mathfrak{V}_1^\#, \quad \Omega_1^\circ = (\mathfrak{U}^\flat)^{\rm t} \Omega_1 \mathfrak{V}_1^\flat,$$

$$\Theta_1^\circ = (\mathfrak{U}^\#)^*\Theta_1\mathfrak{U}_1^\#, \quad \Xi_1^\circ = (\mathfrak{U}^\flat)^*\Xi_1\mathfrak{U}_1^\flat,$$

and

$$U_1^\circ = (\mathfrak{V}_1^\flat)^{\mathrm{t}}U_1\mathfrak{V}_1^\#, \quad \Delta^\circ = (\mathfrak{U}_1^\flat)^*\Delta\overline{\mathfrak{U}_1{}^\#}.$$

It is easy to see that the same results hold in the case of arbitrary partial canonical factorizations as well as arbitrary canonical factorizations.

To conclude this section, we proceed to the heredity problem for (partial) canonical factorizations. We prove that if X is a linear $\mathcal{R}$-space or X is a decent function space and the initial matrix function Φ belongs to X, then all factors in its (partial) canonical factorizations also belong to X.

Theorem 15.21. *Suppose that X is a linear $\mathcal{R}$-space or X is a decent function space. Let Φ be a bounded $m \times n$ matrix function such that $\mathbb{P}_-\Phi$ is a nonzero matrix function in $X(\mathbb{M}_{m,n})$. If $F \in H^\infty(\mathbb{M}_{m,n})$ is a best approximation of Φ and $\Phi - F$ admits a partial canonical factorization*

$$\Phi - F = \begin{pmatrix} \overline{\Omega} & \Xi \end{pmatrix} \begin{pmatrix} t_0 U & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} \begin{pmatrix} \Upsilon & \overline{\Theta} \end{pmatrix}^*,$$

then $\Upsilon, \Theta, \Omega, \Xi, U, \mathbb{P}_-\Psi \in X$.

Proof. Assume without loss of generality that $t_0 = 1$. By Theorem 1.8, if we replace a best approximating function F with any other best approximation, we do not change $\mathbb{P}_-\Psi$. Thus we may assume that F is the unique superoptimal approximation of Φ by bounded analytic matrix functions. By Theorem 12.7, $\Phi - F$ belongs to X.

Let us first prove that $\Theta \in X$. Consider a partial thematic factorization of $\Phi - F$ of the form (15.8). Then the matrix functions V_j given by (15.9) belong to X (see Lemma 12.1 and the proofs of Theorems 12.3 and 12.6). In particular, the inner matrix functions Θ_j in (15.9) belong to X. Since $X \cap L^\infty$ is an algebra, it now follows from (15.10) that $\Theta \in X$.

Consider now the unitary-valued matrix function $\mathcal{V} = \begin{pmatrix} \Upsilon & \overline{\Theta} \end{pmatrix}$. By Theorem 1.3, the Toeplitz operator $T_{\mathcal{V}}$ has dense range in $H^2(\mathbb{C}^n)$. Therefore by Theorems 13.1.1 and 13.5.1, $\mathcal{V} \in X$, and so $\Upsilon \in X$.

If we apply the above reasoning to Φ^{t}, we prove that $\Xi \in X$ and $\Omega \in X$. It follows now from (15.7) that $U \in X$. Finally, it follows from (15.14) that $\mathbb{P}_-\Psi \in X$. ■

Remark. It can be shown that if X is a linear $\mathcal{R}$-space, then the X-norms of Υ, Θ, Ω, Ξ, U, $\mathbb{P}_-\Psi$ can be estimated in terms of the X-norm of $\mathbb{P}_-\Phi$.

Clearly, it follows from Theorem 15.21 that the same result holds for arbitrary partial canonical factorizations. In particular, the following theorem holds.

Theorem 15.22. *Let $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and F is the superoptimal approximation of Φ by bounded analytic matrix functions. If* (15.22) *is a canonical factorization of $\Phi - F$, then all factors on the right-hand side of* (15.22) *belong to X.*

Now consider separately the important case $X = VMO$, which is an $\mathcal{R}$-space.

Theorem 15.23. *Let $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$ and $\mathbb{P}_-\Phi \neq 0$. If F is a best approximation of Φ by bounded analytic functions and $\Phi - F$ admits a partial canonical factorization* (15.6)*, then $\mathcal{V}, \mathcal{W}, U \in QC$ and $\Psi \in H^\infty + C$.*

Theorem 15.23 follows immediately from Theorem 15.21 if we put $X = VMO$.

16. Very Badly Approximable Unitary-Valued Functions

We have seen in the previous section that an important role is played by the very badly approximable unitary-valued matrix functions U satisfying the condition

$$\|H_U\|_{\mathrm{e}} < 1.$$

In this section we obtain a characterization of this class and study some properties of such functions. Recall that by Corollary 15.6, for such a matrix function U the Toeplitz operator T_U is Fredholm.

Theorem 16.1. *Let U be an $n \times n$ unitary-valued matrix function such that $\|H_U\|_{\mathrm{e}} < 1$. The following are equivalent:*
(i) *U is very badly approximable;*
(ii) *the Toeplitz operator $T_{zU} : H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^n)$ has dense range in $H^2(\mathbb{C}^n)$;*
(iii) *the Toeplitz operator $T_{\bar{z}U^*} : H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^n)$ has trivial kernel;*
(iv) *the Toeplitz operator T_U is Fredholm and the indices of a Wiener–Hopf factorization of U are negative.*

Proof. Clearly (ii) and (iii) are equivalent. Next, by Theorem 5.4, (i) implies (ii).

Let us show that (ii) implies (iv). Clearly, it follows from (ii) that T_U has dense range. It follows easily Theorem 4.4.11 that $\|H_U\|_{\mathrm{e}} = \|H_{U^*}\|_{\mathrm{e}}$ and by Theorem 3.4.6, the operator T_U is Fredholm. Consider a Wiener–Hopf factorization of U:

$$U = \Psi_2^* \begin{pmatrix} z^{d_1} & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & z^{d_2} & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & z^{d_n} \end{pmatrix} \Psi_1, \tag{16.1}$$

where $\Psi_1^{\pm 1}, \Psi_2^{\pm 1} \in H^2(\mathbb{M}_{n,n})$. By Corollary 3.5.8, (iii) is equivalent to the fact that

$$d_j < 0, \quad 1 \le j \le n. \tag{16.2}$$

It remains to prove that (iv) implies (i). Clearly, U is a unitary interpolant of itself and since all the indices d_j are negative, it follows from Theorem 14.2 that $t_0(\Phi) = \cdots = t_{n-1}(\Phi) = 1$, and so U is very badly approximable. ■

Corollary 16.2. *Let X be a decent function space. Suppose that U is an $n \times n$ unitary-valued function in X. Then U is very badly approximable if and only if U admits a factorization* (16.1) *such that $\Psi_1^{\pm 1} \in X$, $\Psi_2^{\pm 1} \in X$, and* (16.2) *holds.*

Proof. The result follows immediately from Theorems 16.1 and 13.6.1. ■

Corollary 16.3. *Let U be a unitary-valued very badly approximable function satisfying $\|H_U\|_{\mathrm{e}} < 1$ and let $k = -(d_1 + \cdots + d_n)$, where the d_j are the indices of a Wiener–Hopf factorization* 16.1 *of U. Then*

$$s_j(H_{U^*}) = s_{j+k}(H_U), \quad j \in \mathbb{Z}_+.$$

Proof. It follows from Theorem 16.1 that T_U has dense range. Then by Theorem 4.4.11,

$$s_j(H_{U^*}) = s_{j+d}(H_U), \quad j \in \mathbb{Z}_+,$$

where $d = \dim\{f \in H^2(\mathbb{C}^n) : \ \|H_U f\|_2 = \|f\|_2\}$. Finally, it follows from Theorem 7.4 that $d = k$. ■

17. Superoptimal Meromorphic Approximation

In this section we study the problem of superoptimal approximation of a matrix function on $\mathbb{T}$ by meromorphic matrix functions of degree at most k, $k \in \mathbb{Z}_+$. Given $k \in \mathbb{Z}_+$, we denote by $H^\infty_{(k)}(m,n)$ the class of bounded $\mathbb{M}_{m,n}$-valued functions Q on $\mathbb{T}$ such that the McMillan degree of $\mathbb{P}_- Q$ is at most k. Recall that $Q \in H^\infty_{(k)}(m,n)$ if and only if $\operatorname{rank} H_Q \le k$ (see §2.5). We also write $H^\infty_{(k)}$ instead of $H^\infty_{(k)}(m,n)$ if this does not lead to confusion. As we have seen in §11.4, the problem of approximation by $H^\infty_{(k)}(m,n)$-functions is important in applications to the problem of model reduction. As in the scalar case the problem of approximation by $H^\infty_{(k)}(m,n)$ functions is called the *Nehari–Takagi* problem.

For $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and $k \in \mathbb{Z}_+$ we define the sets $\boldsymbol{\Omega}_j^{(k)}$, $0 \le j \le \min\{m,n\} - 1$, by

$$\boldsymbol{\Omega}_0^{(k)} = \left\{ Q \in H^\infty_{(k)}(m,n) : \ Q \text{ minimizes } \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} \|\Phi(\zeta) - Q(\zeta)\|_{\mathbb{M}_{m,n}} \right\},$$

$$\boldsymbol{\Omega}_j^{(k)} = \left\{ Q \in \boldsymbol{\Omega}_{j-1}^{(k)} : \ Q \text{ minimizes } \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} s_j\big(\Phi(\zeta) - Q(\zeta)\big) \right\}.$$

We put

$$t_j^{(k)} = t_j^{(k)}(\Phi) \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} s_j\big(\Phi(\zeta) - Q(\zeta)\big)$$

for $Q \in \mathbf{\Omega}_j^{(k)}$ and $0 \le j \le \min\{m,n\} - 1$. Note that the sets $\mathbf{\Omega}_j^{(0)}$ coincide with the sets $\mathbf{\Omega}_j$ defined in the introduction to this chapter.

A matrix function $F \in H^\infty_{(k)}(m,n)$ is called a *superoptimal approximation* of Φ by meromorphic matrix functions of degree at most k (or a superoptimal solution of the Nehari–Takagi problem) if $F \in \mathbf{\Omega}^{(k)}_{\min\{m,n\}-1}$.

Note that the matrix functions in $\mathbf{\Omega}_0^{(k)}$ are precisely the best approximations of Φ by meromorphic matrix functions of degree at most k. By Theorem 4.3.1 and the remark after it, $\mathbf{\Omega}_0^{(k)} \ne \varnothing$ and $\|H_\Phi - H_Q\| = s_k(H_\Phi)$ for any $Q \in \mathbf{\Omega}_0^{(k)}$.

It turns out, however, that unlike the case of analytic approximation the condition $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$ (and even any smoothness condition on Φ) does not guarantee the uniqueness of a superoptimal approximation by functions in $H^\infty_{(k)}(m,n)$. Indeed, suppose that $k = 1$ and consider the matrix function

$$\Phi = \begin{pmatrix} \bar z & \mathbb{O} \\ \mathbb{O} & \bar z \end{pmatrix}.$$

It is easy to see that $t_0^{(1)} = 1$ and $t_1^{(1)} = 0$. It is also clear that the functions

$$F_1 = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \bar z \end{pmatrix} \quad \text{and} \quad F_2 = \begin{pmatrix} \bar z & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}$$

are superoptimal approximations by functions in $H^\infty_{(1)}(2,2)$. In fact, it is easy to see that any function of the form $\bar z P$ with P a rank one projection on $\mathbb{C}^2$ is a superoptimal approximation of Φ.

We show in this section that we still have uniqueness under the additional assumption $s_k(H_\Phi) < s_{k-1}(H_\Phi)$. We obtain this result not only for $H^\infty + C$ matrix functions but also for matrix functions Φ with a sufficiently small norm $\|H_\Phi\|_{\mathrm{e}}$. We also obtain in this section a parametrization formula for all best approximations by matrix functions in $H^\infty_{(k)}$.

Theorem 17.1. *Let k be a positive integer and let Φ be a matrix function in $L^\infty(\mathbb{M}_{m,n})$ such that $\|H_\Phi\|_{\mathrm{e}}$ is less than the smallest nonzero number among the $t_j^{(k)}$, $0 \le j \le \min\{m,n\} - 1$. If $s_k(H_\Phi) < s_{k-1}(H_\Phi)$, then there exists a unique superoptimal approximation F of Φ by functions in $H^\infty_{(k)}(m,n)$. Moreover, for this F*

$$s_j\big((\Phi - F)(\zeta)\big) = t_j^{(k)}, \quad 0 \le j \le \min\{m,n\} - 1, \quad \zeta \in \mathbb{T}. \tag{17.1}$$

We need the following elementary fact.

Lemma 17.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let T be a bounded linear operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that $s_{k-1}(T) > s_k(T) > \|T\|_{\mathrm{e}}$. Let R be an operator of rank k such that $\|T - R\| = s_k(T)$. Then any Schmidt* *vector of T corresponding to the singular value $s_k(T)$ belongs to* $\operatorname{Ker} R$ *and is a maximizing vector of $T - R$.*

Proof. Let ξ be a unit Schmidt vector corresponding to $s_k(T)$ and let $\eta = (s_k(T))^{-1}T\xi$. Let $\xi_0, \cdots, \xi_{k-1}$ be orthonormal Schmidt vectors of T corresponding to the singular values $s_0(T), \cdots, s_{k-1}(T)$. Put $\eta_j \stackrel{\text{def}}{=} (s_j(T))^{-1}T\xi_j$, $0 \le j \le k-1$. Let $\xi_k \stackrel{\text{def}}{=} \xi$ and $\eta_k \stackrel{\text{def}}{=} \eta$. Clearly, $\xi_0, \cdots, \xi_k$ and $\eta_0, \cdots, \eta_k$ are orthonormal families. Denote by P the orthogonal projection from $\mathcal{H}_2$ onto $\operatorname{span}\{\eta_0, \cdots, \eta_k\}$.

Consider the operator $PR\big|\operatorname{span}\{\xi_0, \cdots, \xi_k\}$ whose rank is at most k. Then there exists a unit vector

$$x = \sum_{j=0}^{k} a_j\xi_j$$

that belongs to $\operatorname{Ker} PR$. It follows that $PTx = P(T-R)x$, and so $\|PTx\| \le s_k(T)$. We have

$$PTx = \sum_{j=0}^{k} a_j s_j(T)\eta_j \quad \text{and} \quad \|PTx\|^2 = \sum_{j=0}^{k} s_j(T)^2|a_j|^2,$$

and so

$$\|PTx\|^2 > s_k(T)^2 \sum_{j=0}^{k} |a_j|^2 > 1$$

unless $a_0 = \cdots = a_{k-1} = 0$. Thus $x = a_k\xi_k$ with $|a_k| = 1$. Since $PRx = \mathbb{O}$, Rx is orthogonal to $\operatorname{span}\{\eta_0, \cdots, \eta_k\}$. If $Rx \ne \mathbb{O}$, then

$$s_k(T)^2 \ge \|T - Rx\|^2 = \|Tx\|^2 + \|Rx\|^2 > \|Tx\|^2 = s_k^2(T).$$

Thus $Rx = R\xi = \mathbb{O}$, which proves the result. ■

Proof of Theorem 17.1. Let Q be any best approximation of Φ by matrix functions in $H_{(k)}$. As we have observed above,

$$\|\Phi - Q\|_{L^\infty} = s_k \stackrel{\text{def}}{=} s_k(H_\Phi) = t_0^{(k)}.$$

If $s_k = 0$, then $\operatorname{rank} H_\Phi \le k$, and so $\Phi \in H^\infty_{(k)}$. Clearly, the only Q that minimizes $\|\Phi - Q\|_{L^\infty}$ is Φ and the theorem holds in this case. Suppose that $s_k > 0$. By the hypotheses, $\|H_\Phi\|_{\mathrm{e}} < s_k$, and so H_Φ has a unit Schmidt vector $\boldsymbol{f}$ that corresponds to s_k. Let us prove that

$$H_\Phi \boldsymbol{f} = (\Phi - Q)\boldsymbol{f}. \tag{17.2}$$

By Lemma 17.2, $\boldsymbol{f}$ is a maximizing vector of $H_{\Phi-Q} = H_\Phi - H_Q$, and so

$$\begin{aligned} s_k &= \|H_{\Phi-Q}\| = \|H_{\Phi-Q}\boldsymbol{f}\| = \|\mathbb{P}_-(\Phi - Q)\boldsymbol{f}\| \\ &\le \|(\Phi - Q)\boldsymbol{f}\| \le \|\Phi - Q\|_{L^\infty}\|\boldsymbol{f}\| = s_k. \end{aligned}$$

It follows that

$$\|\mathbb{P}_-(\Phi - Q)\boldsymbol{f}\| = \|(\Phi - Q)\boldsymbol{f}\|,$$

and since by Lemma 17.2, $\boldsymbol{f} \in \operatorname{Ker} H_Q$, it follows that

$$(\Phi - Q)\boldsymbol{f} = \mathbb{P}_-(\Phi - Q)\boldsymbol{f} = H_{\Phi-Q}\boldsymbol{f} = H_\Phi \boldsymbol{f} - H_Q \boldsymbol{f} = H_\Phi \boldsymbol{f},$$

which proves (17.2).

Put

$$\boldsymbol{g} = s_k^{-1}\bar{z}\overline{H_\Phi \boldsymbol{f}} \in H^2(\mathbb{C}^m).$$

Since $H_\Phi \boldsymbol{f} = H_{\Phi-Q}\boldsymbol{f}$, it follows from Theorem 2.2.3 that $\|\boldsymbol{f}(\zeta)\|_{\mathbb{C}^n} = \|\boldsymbol{g}(\zeta)\|_{\mathbb{C}^m}$ almost everywhere on $\mathbb{T}$. Hence, $\boldsymbol{f}$ and $\boldsymbol{g}$ admit factorizations

$$\boldsymbol{f} = \vartheta_1 h \boldsymbol{v}, \quad \boldsymbol{g} = \vartheta_2 h \boldsymbol{w}, \tag{17.3}$$

where ϑ_1 and ϑ_2 are scalar inner functions, h is a scalar outer function in H^2, and $\boldsymbol{v}$ and $\boldsymbol{w}$ are inner and co-outer column functions.

Denote by $\mathcal{E}$ the set of matrix functions of the form $\Phi - Q$ where $Q \in \boldsymbol{\Omega}_0^{(k)}$. By (17.1), $E\boldsymbol{f} = H_\Phi \boldsymbol{f} = s_k \bar{z}\bar{\boldsymbol{g}} = t_0^{(k)}\bar{z}\bar{\boldsymbol{g}}$ for any $E \in \mathcal{E}$. It follows from (17.3) that

$$E\boldsymbol{v} = t_0^{(k)} u_0 \overline{\boldsymbol{w}}, \quad E \in \mathcal{E}, \tag{17.4}$$

where $u_0 = \bar{z}\bar{\vartheta}_1\bar{\vartheta}_2\bar{h}/h$.

If $n = 1$, then $\boldsymbol{v}$ is a nonzero scalar function in H^∞ and we have

$$E = t_0^{(k)}\boldsymbol{v}^{-1}u_0\overline{\boldsymbol{w}},$$

which uniquely determines E and proves the constancy of the singular values. By considering Φ^{t}, we obtain the result for $m = 1$. This proves the theorem in the case $\min\{m,n\} = 1$. Now consider the case $\min\{m,n\} > 1$ and suppose that the result holds for any lesser value of $\min\{m,n\}$.

By Theorem 1.1, there exist thematic completions

$$V = \begin{pmatrix} \boldsymbol{v} & \Theta \end{pmatrix} \quad \text{and} \quad W^{\mathrm{t}} = \begin{pmatrix} \boldsymbol{w} & \Xi \end{pmatrix}. \tag{17.5}$$

It is easily seen from (17.4) that the matrix function WEV has the form

$$WEV = \begin{pmatrix} t_0^{(k)}u_0 & * \\ * & * \end{pmatrix},$$

where V and W are defined by (17.5). Clearly, $\|WEV\|_{L^\infty} \le t_0^{(k)}$, u_0 is a scalar unimodular function, and so by Theorem 15.5, WEV has the form

$$WEV = \begin{pmatrix} t_0^{(k)}u_0 & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix}, \tag{17.6}$$

where $\Psi \in L^\infty(\mathbb{M}_{m-1,n-1})$ and $\|\Psi\|_{L^\infty} \le t_0^{(k)}$.

Denote now by $\breve{\mathcal{E}}$ the set of all matrix functions $\Phi - Q$ with $Q \in H^\infty_{(k)}(m,n)$ such that $W(\Phi - Q)V$ is of the form (17.6). Clearly, $\mathcal{E} \subset \breve{\mathcal{E}}$. It is easy to see that for $Q_\# \in H^\infty_{(k)}(m,n)$, the error function $E_\# = \Phi - Q_\#$ belongs to $\breve{\mathcal{E}}$ if and only if

$$E_\# \boldsymbol{f} = t_0^{(k)}\bar{z}\bar{\boldsymbol{g}}, \quad \boldsymbol{g}^{\mathrm{t}} E_\# = t_0^{(k)}\bar{z}\boldsymbol{f}^*.$$

Fix some $E_\# = \Phi - Q_\# \in \breve{\mathcal{E}}$. Then

$$Q_\# \boldsymbol{f} = \Phi \boldsymbol{f} - t_0^{(k)} \overline{z\boldsymbol{g}}, \quad \boldsymbol{g}^{\mathrm{t}} Q_\# = \boldsymbol{g}^{\mathrm{t}} \Phi - t_0^{(k)} \bar{z} \boldsymbol{f}^*.$$

For any $E = \Phi - Q \in \mathcal{E}$ we have

$$WEV = WE_\# V + W(Q_\# V - QV). \tag{17.7}$$

Suppose that

$$WE_\# V = \begin{pmatrix} t_0^{(k)} u_0 & \mathbb{O} \\ \mathbb{O} & \Psi_\# \end{pmatrix}.$$

Since WEV and $WE_\# V$ have the same first column and W is unitary-valued, it follows that QV and $Q_\# V$ have the same first column, say X. Let

$$QV = \begin{pmatrix} X & N \end{pmatrix} \quad \text{and} \quad Q_\# V = \begin{pmatrix} X & N_\# \end{pmatrix}. \tag{17.8}$$

It follows from (17.7) that

$$\begin{pmatrix} t_0^{(k)} u_0 & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} = \begin{pmatrix} t_0^{(k)} u_0 & \mathbb{O} \\ \mathbb{O} & \Psi_\# \end{pmatrix} + \begin{pmatrix} \mathbb{O} & W(N_\# - N) \end{pmatrix}.$$

The matrix functions N that can appear in (17.8) are precisely those $N \in L^\infty(\mathbb{M}_{m-1,n-1})$ satisfying

(i) $\begin{pmatrix} X & N \end{pmatrix} \in H^\infty_{(k)}(m,n)V$;

(ii) $W(N_\# - N) \in \begin{pmatrix} \mathbb{O} \\ L^\infty(\mathbb{M}_{m-1,n-1}) \end{pmatrix}$ and $\begin{pmatrix} \mathbb{O} \\ \Psi_\# \end{pmatrix} + W(N_\# - N)$

has L^∞ norm at most $t_0^{(k)}$.

We wish to find N that minimizes lexicographically the sequence

$$\operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} s_j\big(WEV(\zeta)\big), \quad 0 \le j \le \min\{m,n\} - 1. \tag{17.9}$$

We have

$$s_0\big(WEV(\zeta)\big) = t_0^{(k)}, \quad \zeta \in \mathbb{T}, \tag{17.10}$$

and

$$\operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} s_j\big(WEV(\zeta)\big) = \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} s_{j-1}\left(\begin{pmatrix} \mathbb{O} \\ \Psi_\#(\zeta) \end{pmatrix} + (W(N_\# - N))(\zeta)\right)$$

for $j \ge 1$. It follows from (ii) that

$$W(N_\# - N) = \begin{pmatrix} \mathbb{O} \\ \Xi^*(N_\# - N) \end{pmatrix}$$

(see (17.5)), and so to minimize (17.9) lexicographically, we have to minimize lexicographically the sequence

$$\operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} s_j\big((\Psi_\# + \Xi^*(N_\# - N))(\zeta)\big), \qquad 0 \le j \le \min\{m,n\} - 2.$$

Now we are going to parametrize the matrix functions N satisfying (i).

Lemma 17.3. *Let $X \in H^\infty_{(k)}(m,n)\boldsymbol{v}$. Then there exist $K \in H^\infty(\mathbb{M}_{m,n})$ and an $m \times m$ Blaschke–Potapov product B of degree $l \le k$ such that $\begin{pmatrix} X & N \end{pmatrix} \in H^\infty_{(k)}(m,n)V$ if and only if*

$$N \in B^*(K\overline{\Theta} + H^\infty_{(k-l)}(m,n-1)). \tag{17.11}$$

Proof. Pick an $m \times m$ Blaschke–Potapov product B of minimal degree such that $BX \in H^\infty(\mathbb{M}_{m,n})\boldsymbol{v}$. Let l be the degree of B. Then $l \le k$. Pick any $K \in H^\infty(\mathbb{M}_{m,n})$ such that $BX = K\boldsymbol{v}$.

Suppose that N is of the form (17.11), i.e., $N = B^*(K\overline{\Theta} + \Upsilon^* G)$, where Υ is an $m \times m$ Blaschke–Potapov product of degree at most $k-l$ and $G \in H^\infty(\mathbb{M}_{m,n-1})$. Then

$$\begin{aligned} \Upsilon B \begin{pmatrix} X & N \end{pmatrix} &= \begin{pmatrix} \Upsilon K\boldsymbol{v} & \Upsilon K\overline{\Theta} + G \end{pmatrix} \\ &= (\Upsilon K + G\Theta^{\mathrm{t}}) \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix} \in H^\infty(\mathbb{M}_{m,n})V. \end{aligned} \tag{17.12}$$

Hence, $\begin{pmatrix} X & N \end{pmatrix} \in H^\infty_{(k)}(m,n)V$.

Conversely, suppose that $\begin{pmatrix} X & N \end{pmatrix} \in H^\infty_{(k)}(m,n)V$. Then there exist an $m \times m$ Blaschke–Potapov product $\mathcal{O}$ of degree at most k and a matrix function $Z \in H^\infty(\mathbb{M}_{m,n})$ such that

$$\begin{pmatrix} X & N \end{pmatrix} = \mathcal{O}^* Z V. \tag{17.13}$$

Consideration of the first column of this equality yields $\mathcal{O}X = Z\boldsymbol{v}$. By the choice of B, $\mathcal{O}$ admits a factorization $\mathcal{O} = \Upsilon B$ for some Blaschke–Potapov product Υ of degree at most $k-l$. We have

$$(Z - \Upsilon K)\boldsymbol{v} = \mathcal{O}X - \Upsilon K\boldsymbol{v} = \Upsilon BX - \Upsilon K\boldsymbol{v} = \mathbb{O}$$

by the definition of K. It follows from Lemma 1.4 that $Z - \Upsilon K = G\Theta^{\mathrm{t}}$ for some $G \in H^\infty(\mathbb{M}_{m,n-1})$. Consideration of the second block column in (17.13) yields

$$Z\overline{\Theta} = \mathcal{O}N = \Upsilon BN,$$

whence

$$\Upsilon BN = (\Upsilon K + G\Theta^{\mathrm{t}})\overline{\Theta} = \Upsilon K\overline{\Theta} + G.$$

Thus $N = B^*(K\overline{\Theta} + \Upsilon^* G)$, and so N is of the form (17.11). ■

Note that if N is parametrized as in the lemma by

$$N = B^*(K\overline{\Theta} + \Upsilon^* G),$$

then it follows from (17.12) that

$$B \begin{pmatrix} X & N \end{pmatrix} = (K + \Upsilon^* G\Theta^{\mathrm{t}})V,$$

and by (17.8),

$$Q = B^*(K + \Upsilon^* G\Theta^{\mathrm{t}}) = B^*(K + D\Theta^{\mathrm{t}}), \tag{17.14}$$

where $D \stackrel{\mathrm{def}}{=} \Upsilon^* G \in H^\infty_{(k-l)}$.

Lemma 17.3 enables us to write N and $N_\#$ in the form

$$N = B^*(K\overline{\Theta} + D), \quad N_\# = B^*(K\overline{\Theta} + D_\#)$$

with D, $D_\# \in H^\infty_{(k-l)}(m,n-1)$. We can thus express conditions (i) and (ii) as follows:

(i′) $N = B^*(K\overline{\Theta} + D)$, $D \in H^\infty_{(k-l)}(m,n-1)$,

(ii′) $\boldsymbol{w}^{\mathrm{t}}B^*(D_\# - D) = \mathbb{O}$ and $\Psi_\# + \Xi^* B^*(D_\# - D)$ has L^∞ norm at most $t_0^{(k)}$.

Here $D_\#$ is a fixed element of $H^\infty_{(k-l)}(m,n-1)$.

We have to minimize lexicographically the sequence

$$\operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} s_j\big((\Psi_\# + \Xi^* B^*(D_\# - D))(\zeta)\big), \quad 0 \le j \le \min\{m,n\} - 2.$$

Note that if this sequence is minimized lexicographically, the norm condition in (ii′) automatically holds.

Now we are going to parametrize those $D \in H^\infty_{(k-l)}(m,n-1)$ that satisfy (ii′). Let us first reformulate the first condition in (ii′).

Define the $m \times 1$ column function $\boldsymbol{y}$ by $\boldsymbol{y} = (\operatorname{adj} B)^{\mathrm{t}}\boldsymbol{w}$. Since B is an inner matrix function, it follows that the matrix function $(\operatorname{adj} B)^{\mathrm{t}}$ is also inner, and so the column function $\boldsymbol{y}$ is inner as well. Multiplying the first condition in (ii′) by the scalar Blaschke product $\det B$, we find that it is equivalent to the condition $\boldsymbol{y}^{\mathrm{t}}(D_\# - D) = \mathbb{O}$. Let τ be a greatest common inner divisor of the entries of $\boldsymbol{y}$. Then the column function $\boldsymbol{x} \stackrel{\text{def}}{=} \bar{\tau}\boldsymbol{y}$ is an inner and co-outer column function. Clearly, the first condition in (ii′) is equivalent to

$$\boldsymbol{x}^{\mathrm{t}}(D_\# - D) = \mathbb{O}.$$

By Theorem 1.1, $\boldsymbol{x}$ admits a thematic completion $\begin{pmatrix} \boldsymbol{x} & \overline{\Sigma} \end{pmatrix}$.

Lemma 17.4. *Let $D_\# \in H^\infty_{(r)}(m,n-1)$ and let $\begin{pmatrix} \boldsymbol{x} & \overline{\Sigma} \end{pmatrix}$ be an $m \times m$ thematic matrix function. There exist $Y \in H^\infty(\mathbb{M}_{m,n-1})$ and an $(n-1)\times(n-1)$ Blaschke–Potapov product Λ of degree $q \le r$ such that for $D \in H^\infty_{(r)}(m,n-1)$, D satisfies*

$$\boldsymbol{x}^{\mathrm{t}}(D_\# - D) = \mathbb{O} \tag{17.15}$$

if and only if

$$D \in \big(Y + \Sigma H^\infty_{(r-q)}(m-1,n-1)\big)\Lambda^*. \tag{17.16}$$

Proof. Let Λ be a Blaschke–Potapov product of minimal degree such that

$$\boldsymbol{x}^{\mathrm{t}} D_\# \Lambda \in \boldsymbol{x}^{\mathrm{t}} H^\infty(\mathbb{M}_{m,n-1}).$$

Let q be the degree of Λ. Clearly, $q \le r$. We have

$$\boldsymbol{x}^{\mathrm{t}} D_\# = \boldsymbol{x}^{\mathrm{t}} Y \Lambda^* \quad \text{for some} \quad Y \in H^\infty(\mathbb{M}_{m,n-1}).$$

Suppose that (17.16) holds, i.e.,

$$D = (Y + \Sigma L)\Lambda^* \quad \text{for some} \quad L \in H^\infty_{(r-q)}(m-1,n-1).$$

Then

$$\boldsymbol{x}^{\mathrm{t}}(D_\# - D) = -\boldsymbol{x}^{\mathrm{t}}\Sigma L \Lambda^* = \mathbb{O}$$

since $\begin{pmatrix} \boldsymbol{x} & \overline{\Sigma} \end{pmatrix}$ is unitary-valued.

Conversely, suppose that (17.15) holds. Pick an $(n-1)\times(n-1)$ Blaschke–Potapov product Ω of degree at most r such that $D\Omega \in H^\infty(\mathbb{M}_{m,n-1})$. Then $\boldsymbol{x}^{\mathrm{t}} D_{\#}\Omega \in H^\infty(\mathbb{M}_{1,n-1})$, and so $\Omega = \Lambda\Delta$ for some Blaschke–Potapov product Δ of degree at most $r-q$. Since

$$\boldsymbol{x}^{\mathrm{t}}(D - Y\Lambda^*) = \mathbb{O},$$

it follows that $D\Lambda - Y = \Sigma L$ for some $L \in L^\infty(\mathbb{M}_{m-1.n-1})$. Then

$$D\Omega = D\Lambda\Delta = Y\Delta + \Sigma L\Delta \in H^\infty(\mathbb{M}_{m,n-1}), \tag{17.17}$$

and so

$$D = (Y + \Sigma L)\Lambda^*. \tag{17.18}$$

It follows from (17.17) that $\Sigma L\Delta \in H^\infty(\mathbb{M}_{m,n-1})$. Since Σ is co-outer, it follows from Lemma 1.4 that $L\Delta \in H^\infty(\mathbb{M}_{m-1,n-1})$, and so $L \in H^\infty_{(r-q)}(m-1, n-1)$. The result now follows from (17.18). ■

Consider now the $m \times (m-1)$ inner and co-outer matrix function Σ. It is uniquely determined by $\boldsymbol{x}$ modulo a left constant unitary factor. Let us construct such a matrix function Σ. Clearly, the matrix function $\begin{pmatrix} \boldsymbol{y} & \overline{\Sigma} \end{pmatrix}$ is unitary-valued if and only if $\begin{pmatrix} \boldsymbol{x} & \overline{\Sigma} \end{pmatrix}$ is. The matrix function

$$(\operatorname{adj} B)^{\mathrm{t}} W = (\operatorname{adj} B)^{\mathrm{t}} \begin{pmatrix} \boldsymbol{w} & \overline{\Xi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{y} & \overline{(\operatorname{adj} B^*)\Xi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{y} & (\det B)\overline{B\Xi} \end{pmatrix}$$

is clearly unitary-valued, and so therefore is $\begin{pmatrix} \boldsymbol{y} & \overline{B\Xi} \end{pmatrix}$. Consider the inner-outer factorization of $(B\Xi)^{\mathrm{t}}$:

$$(B\Xi)^{\mathrm{t}} = \Pi^{\mathrm{t}}\Sigma^{\mathrm{t}},$$

where Π is an $(m-1)\times(m-1)$ inner matrix function and Σ is an $m\times(m-1)$ inner and co-outer matrix function. Clearly, $\begin{pmatrix} \boldsymbol{y} & \overline{\Sigma} \end{pmatrix}$ is unitary-valued, and so $\begin{pmatrix} \boldsymbol{x} & \overline{\Sigma} \end{pmatrix}$ is thematic.

We have $B\Xi = \Sigma\Pi$. Thus

$$\Pi = \Sigma^* B\Xi. \tag{17.19}$$

By Lemma 17.4, there exist $Y \in H^\infty(\mathbb{M}_{m,n-1})$ and a Blaschke–Potapov product Λ of degree $q \le k - l$ such that any matrix function D satisfying (ii′) can be written as

$$D = (Y + \Sigma L)\Lambda^*$$

for $L \in H^\infty_{(k-l-q)}(m-1, n-1)$. In particular,

$$D_{\#} = (Y + \Sigma L_{\#})\Lambda^*$$

for some $L_{\#} \in H^\infty_{(k-l-q)}(m-1, n-1)$. We have to minimize lexicographically the sequence

$$\operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} s_j\big((\Psi_{\#} + \Xi^* B^*(D_{\#} - D))(\zeta)\big), \quad 0 \le j \le \min\{m, n\} - 2.$$

Clearly, by (17.19),

$$\Psi_{\#} + \Xi^* B^*(D_{\#} - D) = \Psi_{\#} + \Xi^* B^*\Sigma(L_{\#} - L)\Lambda^* = \Psi_{\#} + \Pi^*(L_{\#} - L)\Lambda^*,$$

and so the problem reduces to the problem to minimize lexicographically the sequence

$$\operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} s_j\big((\Pi\Psi_{\#}\Lambda + L_{\#} - L)(\zeta)\big), \quad 0 \le j \le \min\{m,n\} - 2,$$

as L varies over $H^\infty_{(k-l-q)}(m-1,n-1)$. It is easy to see that Q is a superoptimal approximation of Φ by meromorphic matrix functions in $H^\infty_{(k)}$ if and only if L is a superoptimal approximation of

$$\Phi_1 \stackrel{\text{def}}{=} \Pi\Psi_{\#}\Lambda + L_{\#}$$

by meromorphic matrix functions in $H^\infty_{(k-l-q)}$.

Let us show that this new approximation problem satisfies the hypotheses of Theorem 17.1 so that we can invoke the inductive hypotheses. Clearly, $t_j^{(k)}(\Phi) = t_{j-1}^{(k-l-q)}(\Phi_1)$, $1 \le j \le \min\{m,n\} - 1$. To verify the assumption on the essential norm of the Hankel operator, it is sufficient to show that

$$\|H_{\Phi_1}\|_{\mathrm{e}} \le \|H_\Phi\|_{\mathrm{e}}. \tag{17.20}$$

Since $\operatorname{rank} H_{L_{\#}} < \infty$, it follows that

$$\|H_{\Phi_1}\|_{\mathrm{e}} = \|H_{\Pi\Psi_{\#}\Lambda}\|_{\mathrm{e}}.$$

It is easy to see that

$$\big\|H_{\Pi\Psi_{\#}\Lambda}\big\|_{\mathrm{e}} = \big\|H_{\Pi\Psi_{\#}}\big|\Lambda H^2(\mathbb{M}_{n-1,n-1})\big\|_{\mathrm{e}}$$

and since the subspace $\Lambda H^2(\mathbb{M}_{n-1,n-1})$ has finite codimension in $H^2(\mathbb{M}_{n-1,n-1})$, it follows that

$$\big\|H_{\Pi\Psi_{\#}\Lambda}\big\|_{\mathrm{e}} = \big\|H_{\Pi\Psi_{\#}}\big\|_{\mathrm{e}}.$$

It is also clear that

$$\big\|H_{\Pi\Psi_{\#}}\big\|_{\mathrm{e}} = \big\|H_{(\Pi\Psi_{\#})^{\mathrm{t}}}\big\|_{\mathrm{e}} = \big\|H_{\Psi_{\#}^{\mathrm{t}}\Pi^{\mathrm{t}}}\big\|_{\mathrm{e}},$$

and we can apply the above reasoning again to prove that

$$\big\|H_{\Psi_{\#}^{\mathrm{t}}\Pi^{\mathrm{t}}}\big\|_{\mathrm{e}} = \|H_{\Psi_{\#}^{\mathrm{t}}}\|_{\mathrm{e}} = \|H_{\Psi_{\#}}\|_{\mathrm{e}}.$$

Next, $\operatorname{rank} H_{Q_{\#}} < \infty$, and so

$$\|H_\Phi\|_{\mathrm{e}} = \big\|H_{\Phi - Q_{\#}}\big\|_{\mathrm{e}}.$$

Finally, by Theorem 4.1,

$$\big\|H_{\Psi_{\#}}\big\|_{\mathrm{e}} \le \big\|H_{\Phi - Q_{\#}}\big\|_{\mathrm{e}},$$

which completes the proof of (17.20).

To apply the inductive hypotheses, it is sufficient to show that either $k - l - q = 0$ (in which case uniqueness is a consequence of Theorem 4.8) or

$$s_{k-l-q-1}(H_{\Phi_1}) > s_{k-l-q}(H_{\Phi_1}). \tag{17.21}$$

Suppose that both are false, i.e., $l + q < k$ and

$$s_{k-l-q-1}(H_{\Phi_1}) = s_{k-l-q}(H_{\Phi_1}).$$

Then there exists $L \in H^\infty_{(k-l-q-1)}$, which minimizes $\|\Phi_1 - L\|_{L^\infty}$ over $Q_1 \in H^\infty_{(k-l-q)}$. Thus L determines via (17.14) and (17.19) the matrix function

$$Q = B^*(K + (Y + \Sigma L)\Lambda^*\Theta^{\mathrm{t}}),$$

which is a best approximation of Φ by matrix functions in $H^\infty_{(k)}$. Since $L \in H^\infty_{(k-l-q-1)}$, and B and Λ are of degree l and q, respectively, it follows that $Q \in H^\infty_{(k-1)}$. Thus

$$\operatorname{dist}_{L^\infty}\big(\Phi, H^\infty_{(k-1)}(m,n)\big) = \operatorname{dist}_{L^\infty}\big(\Phi, H^\infty_{(k)}(m,n)\big),$$

and so

$$s_{k-1}(H_\Phi) = s_k(H_\Phi),$$

which contradicts the hypotheses of the theorem.

We have proved that either $k - l - q = 0$ or (17.21) holds. This allows us to apply the inductive hypothesis to Φ_1. Formula (17.1) also follows easily by induction (see (17.10)). This completes the proof. ■

The following result is an important consequence of Theorem 17.1.

Theorem 17.5. *Let k be a positive integer and let Φ be a matrix function in $(H^\infty + C)(\mathbb{M}_{m,n})$. If $s_k(H_\Phi) < s_{k-1}(H_\Phi)$, then there exists a unique superoptimal approximation F of Φ by functions in $H^\infty_{(k)}(m,n)$. Moreover, for this F*

$$s_j\big((\Phi - F)(\zeta)\big) = t_j^{(k)}, \quad 0 \le j \le \min\{m,n\} - 1, \quad \textit{for almost all} \quad \zeta \in \mathbb{T}.$$

Proof. Clearly, under the hypotheses of the theorem $\|H_\Phi\|_{\mathrm{e}} = 0$ and the result immediately follows from Theorem 17.1. ■

We conclude this section with a parametrization formula for the set

$$\boldsymbol{\Omega}_0^{(k)} = \{Q \in H^\infty_{(k)}(m,n) : \ \|\Phi - Q\|_{L^\infty} = s_k(H_\Phi)\}$$

of optimal solutions of the Nehari–Takagi problem.

Theorem 17.6. *Let k and Φ satisfy the hypotheses of Theorem 17.1 and suppose that m, $n \ge 2$. There exist matrix functions $K \in H^\infty(\mathbb{M}_{m,n})$, $Y \in H^\infty(\mathbb{M}_{m,n-1})$, an $m \times m$ Blaschke–Potapov product B of degree l, an $(n-1)\times(n-1)$ Blaschke–Potapov product Λ of degree q such that $l+q \le k$, inner and co-outer functions $\Theta \in H^\infty(\mathbb{M}_{n,n-1})$, $\Sigma \in H^\infty(\mathbb{M}_{m,m-1})$, and a matrix function $\Phi_1 \in L^\infty(\mathbb{M}_{m-1,n-1})$ such that the set of best approximations of Φ by matrix functions in $H^\infty_{(k)}(m,n)$ is equal to*

$$\left\{ B^*(K + (Y + \Sigma L)\Lambda^*\Theta^t) : \begin{array}{l} L \in H^\infty_{(k-l-q)}(m-1, n-1), \\ \|\Phi_1 - L\|_{L^\infty} \le s_k(H_\Phi) \end{array} \right\}.$$

The proof is contained in the proof of Theorem 17.1. The matrix functions K, Y, B, L, Θ, Σ, Φ_1 are defined in the proof of Theorem 17.1.

18. Analytic Approximation of Infinite Matrix Functions

In this section we study the problem of analytic approximation of infinite matrix functions. We assume that the matrices have size $\infty \times \infty$. Obviously, the cases of $m \times \infty$ or $\infty \times m$ matrix functions reduce to the case of $\infty \times \infty$ matrix functions. As usual we identify infinite matrices with operators on the sequence space ℓ^2 and we consider matrix functions that take values in the space $\mathcal{B}$ of bounded linear operators on ℓ^2. We denote by $\boldsymbol{\mathcal{C}}$ the space of compact operators on ℓ^2.

We consider here the class $L^\infty(\mathcal{B})$ of weakly measurable matrix functions Φ that take values in $\mathcal{B}$ and we equip $L^\infty(\mathcal{B})$ with the norm

$$\|\Phi\|_{L^\infty(\mathcal{B})} \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} \|\Phi(\zeta)\|_{\mathcal{B}}.$$

As in the case of finite matrix functions we consider the problem of approximation of Φ by functions in the space $H^\infty(\mathcal{B})$ of operator-valued analytic functions.

Note that instead of ℓ^2 we can consider an arbitrary separable infinite-dimensional Hilbert space $\mathcal{H}$ and study the problem of approximation of functions in $L^\infty(\mathcal{B}(\mathcal{H}))$ by functions in $H^\infty(\mathcal{B}(\mathcal{H}))$. However, we want to fix an orthonormal basis in $\mathcal{H}$, which makes it convenient to consider the space ℓ^2 with the standard orthonormal basis.

To define the notion of a superoptimal approximation for infinite matrix functions we have to consider the infinite sequence of sets $\boldsymbol{\Omega}_j$, $j \in \mathbb{Z}_+$, defined by

$$\boldsymbol{\Omega}_0 = \{Q \in H^\infty(\mathcal{B}) : \ F \text{ minimizes } \ \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} \|\Phi(\zeta) - F(\zeta)\|_{\mathcal{B}}\},$$

$$\boldsymbol{\Omega}_j = \left\{ Q \in \boldsymbol{\Omega}_{j-1} : \ F \text{ minimizes } \ \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} s_j\big(\Phi(\zeta) - F(\zeta)\big) \right\}.$$

The sequence $\{t_j\}_{j\ge 0} = \{t_j(\Phi)\}_{j \ge 0}$ of *superoptimal singular values* is defined by

$$t_j \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} s_j\big(\Phi(\zeta) - Q(\zeta)\big) \quad \text{for} \quad Q \in \boldsymbol{\Omega}_j, \quad j \in \mathbb{Z}_+.$$

We say that a matrix function $F \in H^\infty(\mathcal{B})$ is a *superoptimal approximation of* Φ by bounded analytic operator functions if

$$F \in \bigcap_{j \ge 0} \boldsymbol{\Omega}_j.$$

We prove in this section the uniqueness of a superoptimal approximation under the condition that the Hankel operator $H_\Phi : H^2(\ell^2) \to H^2_-(\ell^2)$ is compact, which is equivalent to the condition that $\Phi \in H^\infty(\mathcal{B}) + C(\boldsymbol{\mathcal{C}})$; see Theorem 2.4.1. We apply a method that can also be used to obtain an alternative proof of Theorems 3.3 and 3.4. We also obtain results on thematic indices similar to the results of §7 for finite matrix functions and inequalities involving superoptimal singular values similar to those obtained in §8. The following theorem is the main result of the section.

Theorem 18.1. *Let Φ be a function in $H^\infty(\mathcal{B}) + C(\boldsymbol{\mathcal{C}})$. Then there exists a unique superoptimal approximation $F \in H^\infty(\mathcal{B})$. Moreover, with such F*

$$s_j\big((\Phi - F)(\zeta)\big) = t_j(\Phi), \quad \textit{for almost all} \quad \zeta \in \mathbb{T}, \quad j \in \mathbb{Z}_+.$$

Proof. Suppose that $H_\Phi \neq 0$. As in the case of finite matrix functions we start with a maximizing vector $\boldsymbol{f}$ of H_Φ and factorize it:

$$\boldsymbol{f} = \vartheta_1 h \boldsymbol{v}, \tag{18.1}$$

where ϑ_1 is a scalar inner function, h is a scalar outer function in H^2, and $\boldsymbol{v}$ is an inner and co-outer column function. Consider the maximizing vector $\boldsymbol{g}$ of $H_{\Phi^{\mathrm{t}}}$ defined by

$$\boldsymbol{g} = \|H_\Phi\|^{-1} \bar{z}\overline{H_\Phi \boldsymbol{f}}. \tag{18.2}$$

As in the case of finite matrix functions (see §2) it can be shown that $\boldsymbol{g}$ admits a factorization

$$\boldsymbol{g} = \vartheta_2 h \boldsymbol{w},$$

where ϑ_2 is a scalar inner function, h is the same outer function, and $\boldsymbol{w}$ is an inner and co-outer column function.

We say that a matrix function $V \in L^\infty(\mathcal{B})$ is a *thematic* matrix function if $V(\zeta)$ is a unitary operator for almost all $\zeta \in \mathbb{T}$ and it has the form $\begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix}$, where $\boldsymbol{v}$ is an inner and co-outer column function and Θ is an inner and co-outer function in $H^\infty(\mathcal{B})$. First, we prove that as in the case of finite matrices (see §1) an inner and co-outer matrix function has a thematic completion.

Consider the subspace $\mathcal{L} = \operatorname{Ker} T_{\boldsymbol{v}^{\mathrm{t}}}$ of $H^2(\ell^2)$. It is invariant under multiplication by, z and so by the Beurling–Lax–Halmos theorem (see Appendix 2.3), it has the form $\mathcal{L} = \Theta H^2(\mathcal{K})$, where $\mathcal{K} = \ell^2$ or $\mathcal{K} = \mathbb{C}^m$ for some $m \in \mathbb{Z}_+$.

Lemma 18.2. *Let $\boldsymbol{v} = \{\boldsymbol{v}_j\}_{j \geq 0}$ be an inner column and let $\mathcal{L} = \operatorname{Ker} T_{\boldsymbol{v}^{\mathrm{t}}} = \Theta H^2(\mathcal{K})$. Then $\dim \operatorname{Ker} \Theta^*(\zeta) = 1$ for almost all $\zeta \in \mathbb{T}$, and so the matrix function $\begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix}$ is unitary-valued.*

Proof. Let us first show that $\dim \operatorname{Ker} \Theta^*(\zeta) \geq 1$. Consider the matrix function

$$U = \begin{pmatrix} \bar{\boldsymbol{v}} & \Theta \end{pmatrix}.$$

It is easy to see that $U(\zeta)$ is isometric a.e. on $\mathbb{T}$. It follows that the columns of $\Theta(\zeta)$ are orthogonal to $\bar{\boldsymbol{v}}(\zeta)$ a.e. on $\mathbb{T}$. Therefore $\bar{\varphi}(\zeta) \in \operatorname{Ker} \Theta^*(\zeta)$ a.e., which proves that $\dim \operatorname{Ker} \Theta^*(\zeta) \geq 1$.

To show that $\dim \operatorname{Ker} \Theta^*(\zeta) \le 1$ we assume without loss of generality that $\boldsymbol{v}_0 \ne \mathbb{O}$. Consider the matrix function

$$G = \begin{pmatrix} -\boldsymbol{v}_1 & -\boldsymbol{v}_2 & -\boldsymbol{v}_3 & \cdots \\ \boldsymbol{v}_0 & \mathbb{O} & \mathbb{O} & \cdots \\ \mathbb{O} & \boldsymbol{v}_0 & \mathbb{O} & \cdots \\ \mathbb{O} & \mathbb{O} & \boldsymbol{v}_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to see that $GH^2(\ell^2) \subset \mathcal{L} = \Theta H^2(\mathcal{K})$. The proof will be completed if we show that $\dim \operatorname{Ker} G^{\mathrm{t}}(\zeta) \le 1$ a.e. on $\mathbb{T}$. Assume that $c = \{c_j\}_{j\ge 0} \in \operatorname{Ker} G^{\mathrm{t}}(\zeta)$. We have

$$\begin{pmatrix} -\boldsymbol{v}_1(\zeta) & \boldsymbol{v}_0(\zeta) & \mathbb{O} & \mathbb{O} & \cdots \\ -\boldsymbol{v}_2(\zeta) & \mathbb{O} & \boldsymbol{v}_0(\zeta) & \mathbb{O} & \cdots \\ -\boldsymbol{v}_3(\zeta) & \mathbb{O} & \mathbb{O} & \boldsymbol{v}_0(\zeta) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \mathbb{O} \\ \mathbb{O} \\ \mathbb{O} \\ \vdots \end{pmatrix}. \tag{18.3}$$

If $\boldsymbol{v}_0(\zeta) \ne 0$, we have from (18.3)

$$c_j = \frac{\boldsymbol{v}_j(\zeta)}{\boldsymbol{v}_0(\zeta)} c_0, \quad j \ge 1,$$

which proves that the c_j with $j \ge 1$ are uniquely determined by c_0, and so $\dim \operatorname{Ker} G^{\mathrm{t}}(\zeta) \le 1$. ■

Lemma 18.3. *Under the hypothesis of Lemma 18.2 the function Θ is co-outer.*

Proof. Assume the contrary. Then $\Theta^{\mathrm{t}} = \mathcal{O}\mathcal{G}$, where $\mathcal{G}$ is an outer function and $\mathcal{O}$ is an inner function . It follows that $\Theta = \mathcal{G}^{\mathrm{t}}\mathcal{O}^{\mathrm{t}}$. Multiplying this equality by $(\mathcal{O}^{\mathrm{t}})^*$ on the right, we find that $\mathcal{G}^t = \Theta(\mathcal{O}^{\mathrm{t}})^*$. Since $(\mathcal{O}^{\mathrm{t}})^*(\zeta)$ is isometric a.e. on $\mathbb{T}$, it follows that $\mathcal{G}^{\mathrm{t}}$ is inner.

Consider the space $\mathcal{G}^{\mathrm{t}}H^2(\ell^2)$. Let us show that it is contained in $\operatorname{Ker} T_{\boldsymbol{v}^{\mathrm{t}}}$. Note first that

$$\Theta(\zeta)c \perp \bar{\boldsymbol{v}}(\zeta) \quad \text{a.e. on} \quad \mathbb{T} \quad \text{for any} \quad c \in \ell^2. \tag{18.4}$$

Indeed this follows from the fact that $\Theta g \subset \operatorname{Ker} T_{\varphi^t}$, where $g(\zeta) \equiv c$. Now let $f \in H^2(\ell^2)$. We have

$$(\mathcal{G}^{\mathrm{t}}(\zeta)f(\zeta), \bar{\boldsymbol{v}}(\zeta))_{\ell^2} = \big(\Theta(\zeta)(\mathcal{O}^{\mathrm{t}})^*(\zeta)f(\zeta), \bar{\boldsymbol{v}}(\zeta)\big)_{\ell^2} = 0$$

by (18.4).

It follows that

$$\mathcal{G}^{\mathrm{t}}H^2(\ell^2) = \Theta(\mathcal{O}^{\mathrm{t}})^*H^2(\ell^2) \subset \Theta H^2(\ell^2).$$

Multiplying the last inclusion on the left by Θ^*, we obtain

$$(\mathcal{O}^{\mathrm{t}})^*H^2(\ell^2) \subset H^2(\ell^2).$$

Clearly, this implies that $\mathcal{O}^{\mathrm{t}}$ is a constant unitary matrix. ■

Lemmas 18.2 and 18.3 immediately imply the following fact.

Corollary 18.4. *Let $\boldsymbol{v}$ be an inner and co-outer function in $H^\infty(\ell^2)$. Then there exists an inner and co-outer matrix function $\Theta \in H^\infty(\mathcal{B})$ such that the matrix function*

$$\begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix}$$

is thematic.

Theorem 18.5. *Let V be a thematic matrix function in $L^\infty(\mathcal{B})$. Then the Toeplitz operator T_V on $H^2(\ell^2)$ has dense range and trivial kernel.*

The proof of this theorem is exactly the same as the proof of Theorem 1.3.

Theorem 18.6. *Let V be a thematic matrix function in $L^\infty(\mathcal{B})$. Then the operators $H_V^* H_V$ and $H_{V^*}^* H_{V^*}$ on $H^2(\ell^2)$ are unitarily equivalent.*

Proof. This is an immediate consequence of Theorems 18.5 and 4.4.11. ■

Corollary 18.7. *Let $V = \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix}$ be a thematic matrix function. Suppose that the Hankel operator $H_{\bar{\boldsymbol{v}}}$ is compact. Then the Toeplitz operator T_V is invertible.*

Proof. Clearly, the condition that $H_{\bar{\boldsymbol{v}}}$ is compact is equivalent to the condition that H_{V^*} is compact. It follows from Theorem 18.6 that H_V is compact. We have

$$T_V^* T_V = I - H_V^* H_V \quad \text{and} \quad T_{V^*}^* T_{V^*} = I - H_{V^*}^* H_{V^*},$$

and so T_V is Fredholm. The result now follows now Theorem 18.5. ■

Theorem 18.8. *Let V and W^{t} be thematic matrix functions in $L^\infty(\mathcal{B})$. Then*

$$WH^\infty(\mathcal{B})V \bigcap \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & L^\infty(\mathcal{B}) \end{pmatrix} = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & H^\infty(\mathcal{B}) \end{pmatrix}.$$

The proof of Theorem 18.8 is exactly the same as the proof of Theorem 1.8 for finite matrix functions. ■

To prove Theorem 18.1, we take inner and co-outer column functions $\boldsymbol{v}$ and $\boldsymbol{w}$ defined in (18.1) and (18.2) and construct thematic completions

$$V = \begin{pmatrix} \boldsymbol{v} & \Theta \end{pmatrix} \quad \text{and} \quad W^{\mathrm{t}} = \begin{pmatrix} \boldsymbol{w} & \Xi \end{pmatrix} \quad , \tag{18.5}$$

which exist by Corollary 18.4. Let $Q \in H^\infty(\mathcal{B})$ be an arbitrary best approximation of Φ by bounded analytic functions. Then $\Phi - Q$ admits a factorization

$$\Phi - Q = W^* \begin{pmatrix} t_0 u_0 & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix} V^*, \tag{18.6}$$

where $u_0 \stackrel{\text{def}}{=} \bar{z}\bar{\vartheta}_1\bar{\vartheta}_2\bar{h}/h$ and Ψ is a function in $L^\infty(\mathcal{B})$ such that $\|\Psi\|_{L^\infty} \le t_0$. This can be proved in exactly the same way as in §2 for finite matrix functions. Moreover, as in the case of finite matrix functions the problem of finding a superoptimal approximation of Φ reduces via Theorem 18.8 to the problem of finding a superoptimal approximation of Ψ (see §2).

To be able to continue this process, we have to be sure that H_Ψ has a maximizing vector. This would be true if we knew that H_Ψ is compact. Fortunately, we can prove this.

Theorem 18.9. *Let $\Phi \in L^\infty(\mathcal{B})$ satisfy the hypotheses of Theorem 18.1. Let Ψ be as in* (18.6). *Then the Hankel operator H_Ψ is compact.*

We need the following analog of scalar Theorem 1.5.1.

Lemma 18.10. *The set $H^\infty(\mathcal{B}) + C(\boldsymbol{\mathcal{C}})$ is a closed subalgebra of $L^\infty(\mathcal{B})$.*

Proof. The fact that $H^\infty(\mathcal{B})+C(\boldsymbol{\mathcal{C}})$ is closed in $L^\infty(\mathcal{B})$ is an immediate consequence of Theorem 2.4.1. To prove that $H^\infty(\mathcal{B})+C(\boldsymbol{\mathcal{C}})$ is an algebra, it is sufficient to show that for $\boldsymbol{\mathcal{C}}$-valued trigonometric polynomials G_1 and G_2 and for D_1, $D_2 \in H^\infty(\mathcal{B})$ the function $(G_1 + D_1)(G_2 + D_2)$ belongs to $H^\infty(\mathcal{B}) + C(\boldsymbol{\mathcal{C}})$. We have

$$(G_1 + D_1)(G_2 + D_2) = G_1G_2 + G_1D_2 + D_1G_2 + D_1D_2.$$

Clearly, $G_1G_2 \in C(\boldsymbol{\mathcal{C}})$, $D_1D_2 \in H^\infty(\mathcal{B})$. It is also easy to see that $\mathbb{P}_-G_1D_2$ is a $\boldsymbol{\mathcal{C}}$-valued trigonometric polynomial and $\mathbb{P}_+G_1D_2 \in H^\infty(\mathcal{B})$. The same can be said about G_2D_1. ■

Proof of Theorem 18.9. The fact that $u_0 \in QC$ can be proved in exactly the same way as in the case of finite matrices (see Theorem 3.1). Let us show that the Hankel operator $H_{\boldsymbol{v}^*}$ is compact. It follows easily from (18.6) that

$$\bar{u}_0\boldsymbol{w}^{\mathrm{t}}(\Phi - Q) = t_0\boldsymbol{v}^*.$$

The compactness of $H_{\boldsymbol{v}^*}$ now follows from Lemma 18.10. This is equivalent to the fact that H_{V^*} is compact. By Theorem 18.6, the Hankel H_V is compact. If we apply the above reasoning to Φ^{t}, we find that H_W is compact. Thus both V and W belong to $H^\infty(\mathcal{B}) + C(\boldsymbol{\mathcal{C}})$.

By (18.6), we have

$$W(\Phi - Q)V = \begin{pmatrix} t_0u_0 & \mathbb{O} \\ \mathbb{O} & \Psi \end{pmatrix}.$$

By Lemma 18.10, the left-hand side of this equality belongs to $H^\infty(\mathcal{B}) + C(\boldsymbol{\mathcal{C}})$. Hence, $\Psi \in H^\infty(\mathcal{B}) + C(\boldsymbol{\mathcal{C}})$. ■

Theorem 18.9 allows us to apply the above procedure to Ψ and iterate this process. Let r be a positive integer. If $t_{r-1}(\Phi) = 0$, the process stops and we get a unique superoptimal approximation. Otherwise, we can take an arbitrary $Q \in \boldsymbol{\Omega}_{r-1}$. Then the function $\Phi - Q$ admits a factorization

$$\Phi - Q = W_0^* \cdots \begin{pmatrix} \boldsymbol{I}_{r-1} & \mathbb{O} \\ \mathbb{O} & W_{r-1}^* \end{pmatrix} D \begin{pmatrix} \boldsymbol{I}_{r-1} & \mathbb{O} \\ \mathbb{O} & V_{r-1}^* \end{pmatrix} \cdots V_0^*, \tag{18.7}$$

where

$$D = \begin{pmatrix} t_0 u_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_1 u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & t_{r-1} u_{r-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \Psi^{(r)} \end{pmatrix}. \tag{18.8}$$

Here $V_0, \cdots, V_{r-1}, W_0^{\mathrm{t}}, \cdots, W_{r-1}^{\mathrm{t}}$ are thematic matrix functions, $u_0, \cdots, u_{r-1}$ are unimodular functions in QC with negative winding numbers, $\Psi_r \in L^\infty(\mathcal{B})$ and $\|\Psi_r\|_{L^\infty} \le t_{r-1}$. Moreover, the functions $u_0, \cdots, u_{r-1}$, $V_0, \cdots, V_{r-1}, W_0, \cdots, W_{r-1}$ can be chosen the same for any $Q \in \boldsymbol{\Omega}_{r-1}$.

Let us first observe that a superoptimal approximation of Φ exists. Indeed, let $Q_r \in \boldsymbol{\Omega}_r$. Then $\{Q_r\}_{r\ge 0}$ is a bounded sequence in $H^\infty(\mathcal{B})$ and we can find a subsequence $\{Q_{r_k}\}_{k\ge 0}$ such that $\{Q_{r_k}(\zeta)\}_{k\ge 0}$ converges for any $\zeta \in \mathbb{D}$ in the weak operator topology to $F(\zeta)$ for some $F \in H^\infty(\mathcal{B})$. It is easy to see that F is a superoptimal approximation of Φ.

To prove that the superoptimal approximation is unique, it is sufficient to show that

$$\lim_{j\to\infty} t_j(\Phi) = 0. \tag{18.9}$$

Indeed, suppose that F_1 and F_2 are superoptimal approximations of Φ. Then for each positive integer r the matrix functions $\Phi - F_1$ and $\Phi - F_2$ admit factorizations of the form (18.7), i.e., for $i = 1, 2$ we have

$$\Phi - F_i = \Phi - Q = W_0^* \cdots \begin{pmatrix} \boldsymbol{I}_{r-1} & \mathbb{O} \\ \mathbb{O} & W_{m-1}^* \end{pmatrix} D_i \begin{pmatrix} \boldsymbol{I}_{r-1} & \mathbb{O} \\ \mathbb{O} & V_{m-1}^* \end{pmatrix} \cdots V_0^*,$$

where

$$D_i = \begin{pmatrix} t_0 u_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_1 u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & t_{r-1} u_{r-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \Psi_i^{(r)} \end{pmatrix}.$$

It follows that

$$F_1 - F_2 = W_0^* \cdots \begin{pmatrix} \boldsymbol{I}_{r-1} & \mathbb{O} \\ \mathbb{O} & W_{m-1}^* \end{pmatrix} (D_2 - D_1) \begin{pmatrix} \boldsymbol{I}_{r-1} & \mathbb{O} \\ \mathbb{O} & V_{m-1}^* \end{pmatrix} \cdots V_0^*,$$

and so

$$\begin{aligned} \|F_1 - F_2\|_{L^\infty} &= \|D_1 - D_2\|_{L^\infty} \\ &= \left\| \begin{pmatrix} \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \cdots & \mathbb{O} & \Psi_1^{(r)} - \Psi_2^{(r)} \end{pmatrix} \right\|_{L^\infty} \le 2t_{r-1}. \end{aligned}$$

Since r is arbitrary, it follows from (18.9) that $F_1 = F_2$.

Thus the uniqueness of a superoptimal approximation will be established as soon as we prove (18.9). To prove (18.9), we obtain an inequality between the superoptimal singular values $t_j(\Phi)$ and the singular values $s_j(H_\Phi)$ as has been done in §8. First, we need the following fact.

Theorem 18.11. *The matrix function* Θ *and* Ξ *in* (18.5) *are left invertible in* H^∞*, i.e., there exist matrix functions* A, $B \in H^\infty(\mathcal{B})$ *such that*

$$A(\zeta)\Theta(\zeta) = I \quad \textit{and} \quad B(\zeta)\Xi(\zeta) = I, \quad \zeta \in \mathbb{D}.$$

The result can be derived from Corollary 18.7 in exactly the same way as it has been done in the proof of Theorem 4.5 in the case of finite matrix functions.

We can introduce the *thematic indices* associated with the factorization (18.7):

$$k_j \stackrel{\text{def}}{=} -\operatorname{wind} u_j = \operatorname{ind} T_{u_j}, \quad 0 \le j \le r-1,$$

where the u_j are diagonal entries in (18.8). As in the case of finite matrix functions, $k_j > 0$.

Now suppose that $t_r < t_{r-1}$ in the factorization (18.7). Such factorizations are called *partial thematic factorizations*. It is easy to see that as in the case of finite matrix functions thematic indices of a partial thematic factorization may depend on the choice of a factorization (see §7). However, the following result holds.

Theorem 18.12. *Let* Φ *satisfy the hypotheses of Theorem 18.1, let* $t \ge t_{r-1} > t_r$*, and let the* k_j, $0 \le j \le r-1$*, be the thematic indices of a partial thematic factorization* (18.7)*. Then the numbers*

$$\sum_{j: t_j = t} k_j$$

do not depend on the choice of a partial thematic factorization.

The following theorem is an analog of Theorem 8.1.

Theorem 18.13. *Let* Φ *satisfy the hypotheses of Theorem 18.1 and let* Ψ *be the matrix function in the factorization* (18.6)*. Then*

$$s_j(H_\Psi) \le s_{j+k_0}(H_\Phi), \quad j \in \mathbb{Z}_+.$$

Theorems 18.12 and 18.13 can be deduced from Theorem 18.11 in the same way as it has been done in the proof of Theorems 7.1 and 8.1 in the case of finite matrix functions.

We can consider now the extended t-sequence associated with a partial thematic factorization (18.7):

$$\underbrace{t_0, \cdots, t_0}_{k_0}, \underbrace{t_1, \cdots, t_1}_{k_1}, \cdots, \underbrace{t_{r-1}, \cdots, t_{r-1}}_{k_{r-1}}$$

in which t_j repeats k_j times. As in §8 we denote the terms of the extended sequence by

$$\breve{t}_0, \breve{t}_1, \cdots, \breve{t}_{k_0+\cdots+k_{r-1}-1}.$$

It follows easily from Theorem 18.12 that the terms of the extended t-sequence are uniquely determined by Φ and do not depend on the choice of a partial thematic factorization.

Theorem 18.14. *Let Φ satisfy the hypotheses of Theorem 18.1. Then the terms of the extended t-sequence associated with a partial thematic factorization* (18.6) *satisfy*

$$\breve{t}_j \le s_j(H_\Phi), \quad 0 \le j \le k_0 + k_1 + \cdots + k_{r-1} - 1.$$

Theorem 18.14 can be deduced easily from Theorem 18.13 in exactly the same way as Theorem 8.2 has been deduced from Theorem 8.1.

Corollary 18.15. *Under the hypotheses of Theorem 18.1*

$$t_j(\Phi) \le s_j(H_\Phi), \quad j \in \mathbb{Z}_+.$$

Corollary 18.15 follows immediately from Theorem 18.14. ■

Now we are able to complete the proof of Theorem 18.1. Since H_Φ is compact, $s_j(H_\Phi) \to 0$ as $j \to \infty$, and so by Corollary 18.15, (18.9) holds, which completes the proof of the uniqueness of the superoptimal approximation F.

The fact that $s_j\big((\Phi - F)(\zeta)\big) = t_j$ almost everywhere on $\mathbb{T}$ follows immediately from factorization formula (18.7). ■

Remark. As in §10 we can introduce the notion of a monotone partial thematic factorization and prove that it is always possible to find a monotone partial thematic factorization of the form (18.6) and the thematic indices of a monotone partial thematic factorization are uniquely determined by the matrix function Φ.

We proceed now to the construction of a thematic factorization. Let Φ satisfy the hypotheses of Theorem 18.1. We can consider the infinite process of construction for each positive integer r a factorization (18.6). Doing so, we obtain a sequence of badly approximable unimodular functions $\{u_j\}_{j\ge 0}$, and sequences of thematic matrix functions $\{V_j\}_{j\ge 0}$ and $\{W_j^{\mathrm{t}}\}_{j\ge 0}$ such that for each positive integer r a factorization of the form (18.6) holds. Consider the infinite products

$$\boldsymbol{W}^* = W_0^* \begin{pmatrix} \mathbf{1} & \mathbb{O} \\ \mathbb{O} & W_1^* \end{pmatrix} \cdots \begin{pmatrix} \boldsymbol{I}_{r-1} & \mathbb{O} \\ \mathbb{O} & W_{r-1}^* \end{pmatrix} \cdots \tag{18.10}$$

and

$$\boldsymbol{V} = V_0 \begin{pmatrix} \mathbf{1} & \mathbb{O} \\ \mathbb{O} & V_1 \end{pmatrix} \cdots \begin{pmatrix} \boldsymbol{I}_{r-1} & \mathbb{O} \\ \mathbb{O} & V_{r-1} \end{pmatrix} \cdots. \tag{18.11}$$

If we identify functions in $L^\infty(\mathcal{B})$ with multiplication operators on $L^2(\ell^2)$, it is easy to see that both infinite products converge in the strong operator topology and define unitary-valued functions $\boldsymbol{V}$ and $\boldsymbol{W}$. Indeed, it is sufficient to verify the convergence on the vector functions $\{f_j\}_{j\ge 0}$ with only finitely many nonzero entries, and for such vector functions the convergence is obvious.

Theorem 18.16. *Let Φ satisfy the hypotheses of Theorem 18.1 and let F be the unique superoptimal approximation of Φ by bounded analytic matrix functions. Then $\Phi - F$ admits a factorization*

$$\Phi - F = \boldsymbol{W}^* \boldsymbol{D} \boldsymbol{V}^*,$$

where $\boldsymbol{V}$ and $\boldsymbol{W}$ are defined in (18.10) *and* (18.11), *and $\boldsymbol{D}$ is the diagonal matrix function defined by*

$$\boldsymbol{D} = \begin{pmatrix} t_0u_0 & \mathbb{O} & \mathbb{O} & \cdots \\ \mathbb{O} & t_1u_1 & \mathbb{O} & \cdots \\ \mathbb{O} & \mathbb{O} & t_2u_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{18.12}$$

Proof. For each positive integer r the function $\Phi - F$ admits a factorization of the form (18.6). We can now pass to the limit. The result follows from the strong convergence of the infinite products (18.10) and (18.11) and the obvious fact that the sequence of functions

$$\begin{pmatrix} t_0u_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_1u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & t_{r-1}u_{r-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \Psi^{(r)} \end{pmatrix}$$

converges to $\boldsymbol{D}$ in $L^\infty(\mathcal{B})$. ■

We conclude this chapter with a heredity result for the nonlinear operator of superoptimal approximation by infinite bounded analytic matrix functions. Recall that in §12 among other results it has been shown that if a finite matrix function Φ belongs to VMO and F is its superoptimal approximation, then F also belongs to VMO. This is equivalent to the fact that if H_Φ is compact, then $H_{(\Phi-F)^*}$ is also compact, or in other words, $(\Phi - F)^* \in H^\infty + C$. We prove that the same result holds for infinite matrix functions.

Theorem 18.17. *Let Φ satisfy the hypotheses of Theorem 18.1 and let F be the unique superoptimal approximation of Φ by bounded analytic matrix functions. Then $(\Phi - F)^* \in H^\infty(\mathcal{B}) + C(\boldsymbol{\mathcal{C}})$.*

Proof. Consider the sequence of functions

$$R_j = W_0^* \cdots \begin{pmatrix} \boldsymbol{I}_{r-1} & \mathbb{O} \\ \mathbb{O} & W_{j-1}^* \end{pmatrix} \mathcal{D}_j \begin{pmatrix} \boldsymbol{I}_{r-1} & \mathbb{O} \\ \mathbb{O} & V_{j-1}^* \end{pmatrix} \cdots V_0^*, \quad j \ge 1,$$

where

$$\mathcal{D}_j = \begin{pmatrix} t_0 u_0 & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_1 u_1 & \cdots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & t_{j-1} u_{j-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} \end{pmatrix}.$$

Obviously,

$$R_j = \boldsymbol{W}^* \mathcal{D}_j \boldsymbol{V}^*$$

and so

$$\|\Phi - F - R_j\|_{L^\infty} \le t_j \to 0 \quad \text{as} \quad j \to \infty. \tag{18.13}$$

Since the functions V_j and W_j belong to $H^\infty(\mathcal{B}) + C(\boldsymbol{\mathcal{C}})$ (see the proof of Theorem 18.9), it follows that $H_{R_j^*}$ is compact for each j. By (18.13),

$$\lim_{j\to\infty} \|H_{(\Phi - F)^*} - H_{R_j^*}\| = 0,$$

which implies the compactness of $H_{(\Phi-F)^*}$. ■

19. Back to the Adamyan–Arov–Krein Parametrization

In this section we use the results of §15 to obtain a version of the Adamyan–Arov–Krein parametrization of solutions of the Nehari problem for a Hankel operator $\Gamma : H^2(\mathbb{C}^n) \to H^2_-(\mathbb{C}^m)$ in the case $\|\Gamma\| = \rho$ and $\|\Gamma\|_{\mathrm{e}} < \|\Gamma\|$. We reduce this case to the case $\|\Gamma\| < \rho$ treated in §5.4 (see Theorem 5.4.16). Though we have parametrized in §5.5 the solutions of the Nehari problem in the most general case, we present here an alternative approach that works in this case and that is a straightforward application of partial canonical factorizations of badly approximable matrix functions.

Let $\Psi \in L^\infty(\mathbb{M}_{m,n})$ such that $\Gamma = H_\Psi$. We are looking for solutions of the Nehari problem with $\rho = \|H_\Psi\|$. Clearly, this is equivalent to the problem of parametrization of all best approximations of Ψ by analytic matrix functions in the L^∞ norm. We assume that $\|H_\Psi\|_{\mathrm{e}} < \|H_\Psi\|$. Let t_j, $0 \le j \le \min\{m,n\} - 1$, be the superoptimal singular values of Ψ. If $t_{\min\{m,n\}-1} = t_0$, it follows from the results of §15 that Ψ has a unique best approximation by analytic matrix functions.

Suppose now that $t_{\min\{m,n\}-1} < t_0 = \rho$. Then by Theorem 15.3, for any best approximation $F \in H^\infty(\mathbb{M}_{m,n})$ the matrix function $\Psi - F$ admits a partial canonical factorization

$$\Psi - F = \mathcal{W}^* \begin{pmatrix} \rho U & \mathbb{O} \\ \mathbb{O} & \Psi^{(1)} \end{pmatrix} \mathcal{V}^*,$$

where $\Psi^{(1)} \in L^\infty(\mathbb{M}_{m-r,n-r})$ and $\|H_{\Psi^{(1)}}\| < \rho$. Here r is the number of superoptimal singular values of Ψ equal to t_0 and

$$\mathcal{V} = \begin{pmatrix} \Upsilon & \overline{\Theta} \end{pmatrix} \quad \text{and} \quad \mathcal{W}^{\mathrm{t}} = \begin{pmatrix} \Omega & \overline{\Xi} \end{pmatrix}$$

are r-balanced unitary-valued functions.

By Theorem 1.8, Φ is a symbol of Γ with $\|\Phi\| \le \rho$ if and only if there exists a symbol $\Phi^{(1)}$ of $H_{\Psi^{(1)}}$ such that $\|\Phi^{(1)}\| \le \rho$ and

$$\Phi = \mathcal{W}^* \begin{pmatrix} \rho U & \mathbb{O} \\ \mathbb{O} & \Phi^{(1)} \end{pmatrix} \mathcal{V}^*. \tag{19.1}$$

By Theorem 5.4.16, the set of such matrix functions $\Phi^{(1)}$ admits the following parametrization:

$$\Phi^{(1)} = \rho\big(\boldsymbol{Q}_\rho(\bar{z}) + \boldsymbol{P}_{*\rho}(\bar{z})\mathcal{E}(z)\big)\big(\boldsymbol{P}_\rho(z) + \boldsymbol{Q}_{*\rho}(z)\mathcal{E}(z)\big)^{-1}, \tag{19.2}$$

where $\mathcal{E}$ is in the unit ball of $H^\infty(\mathbb{M}_{m-r,n-r})$, and the functions $\boldsymbol{P}_\rho$, $\boldsymbol{Q}_\rho$, $\boldsymbol{P}_{*\rho}$, and $\boldsymbol{Q}_{*\rho}$ have been defined in §5.4.

Thus we obtain the following version of the Adamyan–Arov–Krein parametrization.

Theorem 19.1. *Suppose that $\Psi \in L^\infty(\mathbb{M}_{m,n})$, $\|H_\Psi\|_{\mathrm{e}} < \|H_\Psi\| = \rho$, and r is the number of the superoptimal singular values of Ψ equal to ρ. Then under the above notation, Φ is a symbol of H_Ψ with $\|\Phi\| = \rho$ if and only if Φ admits a representation*

$$\Phi = \rho\big(\overline{\Omega} U \Upsilon^* + \Xi\big(\boldsymbol{Q}_\rho(\bar{z}) + \boldsymbol{P}_{*\rho}(\bar{z})\mathcal{E}(z)\big)\big(\boldsymbol{P}_\rho(z) + \boldsymbol{Q}_{*\rho}(z)\mathcal{E}(z)\big)^{-1}\Theta^{\mathrm{t}}\big)$$

with $\mathcal{E}$ is in the unit ball of $H^\infty(\mathbb{M}_{m-r,n-r})$.

Proof. The result follows straightforwardly from (19.1) and (19.2). ■

Concluding Remarks

The notion of a superoptimal approximation by analytic matrix functions was introduced in Young [2], where a uniqueness result for a class of $H^\infty + C$ functions satisfying a certain additional assumption was stated. However, the proof given in Young [2] contains an error. Note also that earlier in Davis [2] a similar notion had been used for the problem of completing matrix contractions (see §2.1).

The uniqueness of a superoptimal analytic approximation for $H^\infty + C$ functions (Theorem 3.3) was established in Peller and Young [1]. Later Treil [8] obtained another proof of the same result.

The results of §1 are taken from Alexeev and Peller [3]. The existence and uniqueness (modulo a constant unitary factor) of a balanced completion was established in Vasyunin [1]. Note that in the case of thematic matrix functions (i.e., in the case $r = 1$) the results of §1 had been obtained earlier in Peller and Young [1].

The results of §2 and §3 can be found in Peller and Young [1].

The results of §4 were obtained in Peller and Treil [2]; they deal with noncompact Hankel operators, which requires a considerably more sophisticated technique developed in Peller and Treil [2]. Note that the method of constructing the vectors $\xi_j^{\#}$ in the proof of Theorem 4.1 is based on the technique developed in Peller and Young [1] and [2]. In fact, in Peller and Treil [2] a more general problem was solved. Namely, similar results were obtained in Peller and Treil [2] for superoptimal solutions of the four block problem (see Concluding Remarks to Chapter 2).

The results of §5 were found in Peller and Young [1] in the case of functions in $H^\infty + C$ and in Peller and Treil [2] in the case when $\|H_\Phi\|_e$ is not too large. The notions of thematic factorizations and thematic indices were introduced in Peller and Young [1].

The notions of admissible and superoptimal weights were given in Treil [8]. The results of §6 were also established in Treil [8].

The fact that thematic indices may depend on the choice of a thematic factorization was observed in Peller and Young [1]. Theorem 7.1 was obtained in Peller and Young [2] in the case of $H^\infty + C$ functions. The notion of an extended t-sequence was introduced in Peller and Young [2]. Theorem 8.2 was proved in Peller and Young [2] in the case of $H^\infty + C$ functions. The stronger result, Theorem 8.1, was found in Peller and Treil [2].

The results of §9 and §10 were obtained in Alexeev and Peller [1].

The results of §11 were found in Peller and Young [3] (see also Peller and Young [5]). Note that in Peller and Young [3] constructive algorithms are discussed to find the superoptimal approximants.

The results of §12 were established in Peller and Young [1] for $\mathcal{R}$-spaces and for decent spaces under the assumption (12.5). Later in Peller [24] the same results were proved for arbitrary decent spaces. We also mention here the paper Peller [25] in which hereditary properties are discussed for the four block problem.

The continuity results were obtained in Peller and Young [5] and Peller [24].

Section 14 follows Alexeev and Peller [2]. In Dym and Gohberg [1] the authors considered the problem of finding unitary interpolants and describing their Wiener–Hopf indices for functions that belong to a function space that satisfies axioms similar to our axioms (A1)–(A4) of decent spaces. They stated the results that the negative Wiener–Hopf indices are uniquely determined by the function while the nonnegative indices can be arbitrary. Note, however, that the reasoning in Dym and Gohberg [1] contains a gap. Later in Dym and Gohberg [2] the problem of unitary interpolation was studied in a more general situation. Another approach to this problem was given by Ball [1]. In Alexeev and Peller [2] a new method was found that allows one to write explicit formulas for the negative Wiener–Hopf indices of unitary interpolants in terms of the thematic indices of monotone thematic factorizations and express the number of negative Wiener–Hopf indices as the number of superoptimal singular values equal to 1.

The results of §15 and §16 are taken from Alexeev and Peller [3].

Theorem 17.5 was proved first in Treil [8] by a different method. The proof given in §17 is more constructive; it was found in Peller and Young [4]. Theorem 17.1 is published here for the first time.

Theorem 18.1 was proved in Treil [8] in a different way. The more constructive proof given in §18 was found in Peller [22]. The proof of Lemma 18.2 was suggested by Vasyunin. Theorems 18.9, 18.16, and 18.17 were obtained in Peller [22]. Theorems 18.12, 18.13, and 18.14 were found in Peller and Treil [1] (to be more precise, the statement of Theorem 18.13 is added in proof in Peller and Treil [1]; its proof is the same as the proof given in Peller and Treil [2] for finite matrix functions).

The material of §19 is published here for the first time.

We also mention here the paper Woerdeman [1] in which the author studied superoptimal approximation of finite block matrices by triangular block matrices and found an analog of thematic factorizations. The results of Woerdeman generalize the results of Davis [2] for 2×2 block matrices.

15

Hankel Operators and Similarity to a Contraction

In this chapter Hankel operators are used to solve the problem of whether each polynomially bounded operator on Hilbert space is similar to a contraction. This problem has a long history.

Sz.-Nagy [1] showed that if T is an invertible operator on Hilbert space such that $\sup_{n\in\mathbb{Z}} \|T^n\| < \infty$, then T is *similar* to a unitary operator, i.e., there exists an invertible operator V such that VTV^{-1} is unitary. Sz.-Nagy posed a similar problem for operators T (not necessarily invertible) satisfying the condition

$$\sup_{n\in\mathbb{Z}_+} \|T^n\| < \infty.$$

Such operators are called *power bounded operators*. The problem was whether each power bounded operator is similar to a contraction. Recall that an operator R on Hilbert space is called a *contraction* if $\|R\| \le 1$.

It turned out, however, that there are power bounded operators that are not similar to contractions. The first example of such an operator was constructed by Foguel [1]. Other examples of power bounded operators not similar to contractions were found in Davie [1], Peller [5], and Bożeiko [1].

After the appearance of Foguel's counter-example it has become natural to try to impose a stronger condition on an operator under which the operator would have to be similar to a contraction. Von Neumann [1] proved that if T is a contraction on Hilbert space, then for any analytic polynomial φ

$$\|\varphi(T)\| \le \|\varphi\|_\infty = \max_{|\zeta|\le 1} |\varphi(\zeta)|,$$

where as usual $\varphi(T) \stackrel{\text{def}}{=} \sum_{j\ge 0} \hat{\varphi}(j)T^j$. There are many different proofs of von Neumann's inequality, see e.g., Sz.-Nagy and Foias [1], where the proof based on Sz.-Nagy's theorem on unitary dilations is given. It follows from von Neumann's inequality that if T is similar to a contraction, then T is *polynomially bounded*, i.e.,

$$\|\varphi(T)\| \le \text{const}\,\|\varphi\|_\infty$$

for any analytic polynomial φ. Lebow [1] showed that the operator constructed by Foguel is not polynomially bounded. Halmos [3] posed the question of whether each polynomially bounded operator is similar to a contraction.

Later Arveson [1–2] observed that the Sz.-Nagy dilation theorem implies that an operator similar to a contraction is not only polynomially bounded but also *completely polynomially bounded*, i.e.,

$$\left\|\begin{pmatrix} \varphi_{11}(T) & \varphi_{12}(T) & \cdots & \varphi_{1n}(T) \\ \varphi_{21}(T) & \varphi_{22}(T) & \cdots & \varphi_{2n}(T) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{n1}(T) & \varphi_{n2}(T) & \cdots & \varphi_{nn}(T) \end{pmatrix}\right\|$$

$$\le c\cdot\max_{|\zeta|\le 1}\left\|\begin{pmatrix} \varphi_{11}(\zeta) & \varphi_{12}(\zeta) & \cdots & \varphi_{1n}(\zeta) \\ \varphi_{21}(\zeta) & \varphi_{22}(\zeta) & \cdots & \varphi_{2n}(\zeta) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{n1}(\zeta) & \varphi_{n2}(\zeta) & \cdots & \varphi_{nn}(\zeta) \end{pmatrix}\right\|$$

for any positive integer n and any polynomial matrix $\{\varphi_{jk}\}_{1\le j,k\le n}$ with a constant c not depending on n. Paulsen [1] showed that the converse is also true, i.e., complete polynomial boundedness implies similarity to a contraction. However, the problem of whether polynomial boundedness implies similarity to a contraction remained open.

In Peller [5] the following operator was introduced. Let ψ be a function analytic in the unit disk $\mathbb{D}$. Consider the operator R_ψ on $\ell^2 \oplus \ell^2$ defined by

$$R_\psi = \begin{pmatrix} S^* & \Gamma_\psi \\ \mathbb{O} & S \end{pmatrix}, \tag{0.1}$$

where S is the shift operator on ℓ^2 and Γ_ψ is the Hankel operator on ℓ^2 with matrix $\{\hat{\psi}(j+k)\}_{j,k\ge 0}$ in the standard basis of ℓ^2. By the Nehari theorem, R_ψ is bounded if and only if $\psi \in BMOA$. Such operators were used in Peller [5] to construct power bounded operators that are not polynomially bounded (see §2). The operators R_ψ were considered independently by Foias and Williams (see Carlson, Clark, Foias, and Williams [1]). The reason the operators R_ψ do a nice job here is that one can very easily compute functions of such operators (see §1).

The hope was to find a function ψ such that R_ψ is polynomially bounded but not similar to a contraction. It was shown in Peller [11] that R_ψ is polynomially bounded if $\psi' \in BMOA$. Then Bourgain [1] showed that under the condition $\psi' \in BMOA$ the operator R_ψ is similar to a contraction. Finally, it was proved by Aleksandrov and Peller [1] that R_ψ is polynomially bounded if and only if $\psi' \in BMOA$, and so R_ψ is polynomially bounded if and only if it is similar to a contraction.

However, it was not the end of the story. Pisier [2] considered the operators R_ψ on the space $\ell^2(\mathcal{H}) \oplus \ell^2(\mathcal{H})$ of functions taking values in a Hilbert space $\mathcal{H}$ in which case S is the shift operator on $\ell^2(\mathcal{H})$, ψ is a $\mathcal{B}(\mathcal{H})$-valued function, and Γ_ψ is the operator on $\ell^2(\mathcal{H})$ with block Hankel matrix $\{\hat{\psi}(j+k)\}_{j,k\ge 0}$. He showed that there exists such a function ψ for which R_ψ is polynomially bounded but not similar to a contraction.

In §1 we characterize the polynomially bounded operators R_ψ in the scalar case and prove that R_ψ is polynomially bounded if and only if it is similar to a contraction. In §2 we describe the class of power bounded operators R_ψ and we show that among them there are operators that are not polynomially bounded. Finally, in §3 we construct polynomially bounded operators R_ψ with operator-valued ψ that are polynomially bounded but not similar to a contraction.

1. Operators R_ψ in the Scalar Case

The main result of this section describes the polynomially bounded operators R_ψ and says that R_ψ is polynomially bounded if and only if it is similar to a contraction. To prove this result we need a so-called weak factorization theorem for the class of functions that consists of the derivatives of H^1 functions.

First we prove an elementary lemma. It says how to compute functions of operators R_ψ. In fact, one of the most important features of the operators R_ψ is that the polynomials of R_ψ can be evaluated explicitly. Recall that the operators R_ψ are defined by (0.1).

As usual we identify the spaces ℓ^2 and H^2 in the natural way:

$$\{c_n\}_{n\ge 0} \quad \longleftrightarrow \quad \sum_{n\ge 0} c_n z^n.$$

The operators S and S^* are defined on the space of functions analytic in $\mathbb{D}$ by

$$Sf(z) = zf(z), \quad (S^*f)(z) = \frac{f(z) - f(0)}{z}.$$

Lemma 1.1. *Let $\psi \in BMOA$ and let φ be an analytic polynomial. Then*

$$\varphi(R_\psi) = \begin{pmatrix} S^* & \Gamma_{\varphi'(S^*)\psi} \\ \mathbb{O} & S \end{pmatrix}. \tag{1.1}$$

Proof. It is sufficient to prove the result in the case when $\varphi = z^n$. Using the formula

$$S^*\Gamma_\psi = \Gamma_\psi S = \Gamma_{S^*\psi},$$

one can easily verify (1.1) by induction. ■

The following theorem is the main result of this section.

Theorem 1.2. *Let ψ be a function analytic in $\mathbb{D}$. The following are equivalent:*

(i) *R_ψ is polynomially bounded;*
(ii) *R_ψ is similar to a contraction;*
(iii) *$\psi' \in BMOA$.*

To prove Theorem 1.2, we obtain a so-called weak factorization theorem. For $1 \le p \le \infty$ consider the class

$$\mathfrak{X}^p \stackrel{\text{def}}{=} \{f' : f \in H^p\}$$

and we endow $\mathfrak{X}^p$ with the norm

$$\|g\|_{\mathfrak{X}^p} \stackrel{\text{def}}{=} \|f\|_{H^p}, \quad f' = g, \quad f(0) = 0.$$

We also consider the subspace

$$\mathfrak{x}^\infty \stackrel{\text{def}}{=} \{f' : f \in C_A\}$$

of $\mathfrak{X}^\infty$.

It is well known (see Appendix 2.6) that the class $\mathfrak{X}^p$ admits the following description:

$$g \in \mathfrak{X}^p \quad \Longleftrightarrow \quad \int_{\mathbb{T}} \left(\int_0^1 |g(r\zeta)|^2 (1-r)\,dr \right)^{p/2} d\boldsymbol{m}(\zeta) < \infty. \tag{1.2}$$

The following weak factorization theorem will be used only for $p = 1$, but it is more natural to prove it for $1 \le p < \infty$.

Theorem 1.3. *Let $1 \le p < \infty$ and let g be a function analytic in $\mathbb{D}$. Then $g \in \mathfrak{X}^p$ if and only if there are functions $\xi_m \in \mathfrak{X}^\infty$ and $\eta_m \in H^p$, $1 \le m \le 4$, such that*

$$g = \sum_{m=1}^{4} \xi_m \eta_m. \tag{1.3}$$

Proof. Let us first prove that for $\xi \in \mathfrak{X}^\infty$ and $\eta \in H^p$ we have $\xi\eta \in \mathfrak{X}^p$. Let $\xi = \psi'$, where $\psi \in H^\infty$. We have

$$\xi\eta = \psi'\eta = (\psi\eta)' - \psi\eta'.$$

Clearly, $\psi\eta \in H^p$, and so $(\psi\eta)' \in \mathfrak{X}^p$. It is obvious from (1.2) that $\psi\eta' \in \mathfrak{X}^p$.

Now let g be an arbitrary function in $\mathfrak{X}^p$. Consider first the case $p > 1$. Then we are able to represent g in the form

$$g = \xi_1\eta_1 + \xi_2\eta_2,$$

with $\xi_m \in \mathfrak{X}^\infty$ and $\eta_m \in H^p$, $m = 1, 2$.

Let f be a function in H^p such that $f' = g$ and $f(0) = 0$. Then $\operatorname{Im} f$ is the harmonic conjugate of $\operatorname{Re} f$. Clearly, we can represent f as $f = f_1 - f_2$, where $f_j \in H^p$ and $\operatorname{Re} f_j \ge 0$ on $\mathbb{T}$, $j = 1, 2$. Indeed, if

$$(\operatorname{Re} f_1)(\zeta) = (\operatorname{Re} f)_+(\zeta) \stackrel{\text{def}}{=} \max\{0, \operatorname{Re} f(\zeta)\}$$

and $\operatorname{Im} f_1$ is the harmonic conjugate of $\operatorname{Re} f_1$, then $f_1 \in H^p$, since $1 < p < \infty$.

Hence it is sufficient to show that if $f \in H^p$ and $\operatorname{Re} f \ge 0$, then there exist $\xi \in \mathfrak{X}^\infty$ and $\eta \in H^p$ such that $f' = \xi\eta$. Put $\xi \stackrel{\text{def}}{=} (f^{\mathrm{i}})'$, $\eta \stackrel{\text{def}}{=} -\mathrm{i} f \cdot f^{-\mathrm{i}}$. Since $\operatorname{Re} f \ge 0$, it follows that f^{i} is invertible in H^∞, and so $\xi \in \mathfrak{X}^\infty$, $\eta \in H^p$. The equality $f' = \xi\eta$ is obvious.

Consider now the case $p = 1$. Let f be a function in H^1 such that $f' = g$. Then $f = uv$, where $u, v \in H^2$. We have already shown that there exist functions $\xi_m \in \mathfrak{X}^\infty$ and $\tau_m \in H^2$, $1 \le m \le 4$, such that

$$u' = \xi_1\tau_1 + \xi_2\tau_2, \quad v' = \xi_3\tau_3 + \xi_4\tau_4.$$

Clearly, (1.3) holds with

$$\eta_1 = \tau_1 v, \quad \eta_2 = \tau_2 v, \quad \eta_3 = \tau_3 u \quad \eta_4 = \tau_4 u. \quad \blacksquare$$

Remark. It is easy to see that $\|g\|_{\mathfrak{X}^p}$ is equivalent to

$$\inf \sum_{m=1}^{4} \|\xi_m\|_{\mathfrak{X}^\infty} \|\eta_m\|_{H^p},$$

where the infimum is taken over all function ξ_m and η_m satisfying (1.3).

Consider now the space X^p of functions f analytic in $\mathbb{D}$, which admit a representation

$$f = \sum_{m \ge 0} \xi_m \eta_m, \tag{1.4}$$

where $\xi_m \in \mathfrak{x}^\infty$, $\eta_m \in H^p$, and

$$\sum_{m \ge 0} \|\xi_m\|_{\mathfrak{x}^\infty} \|\eta_m\|_{H^p} < \infty. \tag{1.5}$$

For f in X^p we define $\|f\|_{X^p}$ to be the infimum of the sum in (1.5) over all representations of the form (1.4).

Theorem 1.4. *Let $1 \le p < \infty$. Then $\mathfrak{X}^p = X^p$ and the norms in $\mathfrak{X}^p$ and X^p are equivalent.*

Proof. It follows from Theorem 1.3 that $X^p \subset \mathfrak{X}^p$. Let us show that $\mathfrak{X}^p \subset X^p$.

For a function f analytic in $\mathbb{D}$ and for $0 < r < 1$ we denote by f_r the function defined by $f_r(\zeta) = f(r\zeta)$. Let $g \in \mathfrak{X}^p$. Then by the remark following Theorem 1.3 there exist functions $\xi_m \in \mathfrak{X}^\infty$ and $\eta_m \in H^p$,

$1 \le m \le 4$, such that (1.3) holds and

$$\sum_{m=1}^{4} \|\xi_m\|_{\mathfrak{X}^\infty} \|\eta_m\|_{H^p} \le \text{const} \|g\|_{\mathfrak{X}^p}.$$

We have

$$g_r = \sum_{m=1}^{4} (\xi_m)_r (\eta_m)_r.$$

Clearly, $(\xi_m)_r \in \mathfrak{x}^\infty$, $\|(\xi_m)_r\|_{\mathfrak{x}^\infty} \le \|\xi_m\|_{\mathfrak{x}^\infty}$, and $\|(\eta_m)_r\|_{H^p} \le \|\eta_m\|_{H^p}$. Therefore $\|g_r\|_{X^p} \le \text{const} \|g\|_{\mathfrak{X}^p}$. The result follows from the fact that $\|g - g_r\|_{\mathfrak{X}^p} \to 0$ as $r \to 1$. ■

Proof of Theorem 1.2. (i)⇔(iii). It follows from Lemma 1.1 that R_ψ is polynomially bounded if and only if

$$\|\Gamma_{\varphi'(S^*)\psi}\| \le \text{const} \|\varphi\|_\infty$$

for any analytic polynomial φ. By the Nehari theorem this is equivalent to the fact that

$$\|\varphi'(S^*)\psi\|_{BMOA} \le \text{const} \|\varphi\|_\infty,$$

which in turn is equivalent to the inequality

$$|\langle g, \varphi'(S^*)\psi \rangle| \le \text{const} \|\varphi\|_\infty \|g\|_{H^1}$$

for any analytic polynomials φ and g, where the pairing

$$\langle u, v \rangle \stackrel{\text{def}}{=} \sum_{n \ge 0} \hat{u}(n)\hat{v}(n), \quad u \in H^1, \quad v \in BMOA, \tag{1.6}$$

is defined at least in the case when u is a polynomial and extends by continuity.

We have

$$\langle g, \varphi'(S^*)\psi \rangle = \langle \varphi'(S)g, \psi \rangle = \langle \varphi' g, \psi \rangle. \tag{1.7}$$

Since $(H^1)^* = BMOA$ with respect to the pairing (1.6), it follows that

$$(\mathfrak{X}^1)^* = \{h : \ h' \in BMOA\}$$

with respect to the same pairing.

Suppose that $\psi' \in BMOA$. We have

$$|\langle \varphi' g, \psi \rangle| \le \text{const} \cdot (\|\psi'\|_{BMOA} + |\psi(0)|) \cdot \|\varphi' g\|_{\mathfrak{X}^1}. \tag{1.8}$$

By Theorem 1.3,

$$\|\varphi' g\|_{\mathfrak{X}^1} \le \text{const} \|\varphi\|_{H^\infty} \|g\|_{H^1}. \tag{1.9}$$

It follows now from (1.7), (1.8), and (1.9) that

$$\|\varphi(R_\psi)\| \le \text{const} \|\varphi\|_{H^\infty}.$$

Suppose now that R_ψ is polynomially bounded. Then it follows from (1.7) that

$$|\langle \varphi' g, \psi \rangle| \le \text{const} \|\varphi\|_\infty \|g\|_{H^1}$$

for any polynomials φ and g. Therefore the same inequality holds for arbitrary $\varphi \in \mathfrak{x}^\infty$ and $g \in H^1$. By Theorem 1.4, it follows that

$$|\langle h, \psi\rangle| \le \text{const}\, \|g\|_{\mathfrak{X}^1}$$

for an arbitrary function h in $\mathfrak{X}^1$. Hence, $\psi \in (\mathfrak{X}^1)^*$, and so $\psi' \in BMOA$.

Clearly, the implication (ii)$\Rightarrow$(i) in Theorem 1.2 is obvious. The remaining implication (iii)$\Rightarrow$(ii) is a consequence of the following theorem.

Theorem 1.5. *Let ψ be a function analytic in $\mathbb{D}$ such that $\psi' \in BMOA$. Then R_ψ is similar to the operator $S \oplus S^*$ on $H^2 \oplus H^2$.*

We need some preparation to prove Theorem 1.5.

Lemma 1.6. *Let f and g be functions in H^2. Then $f'g \in \mathfrak{X}_1$.*

Proof. By (1.2), we have to show that

$$\int_{\mathbb{T}} \left(\int_0^1 |(f'g)(r\zeta)|^2 (1-r)dr \right)^{1/2} d\boldsymbol{m}(\zeta) < \infty.$$

Consider the radial maximal function $g^{(*)}$ on $\mathbb{T}$ defined by

$$g^{(*)}(\zeta) \stackrel{\text{def}}{=} \sup_{0<r<1} |g(r\zeta)|, \quad \zeta \in \mathbb{T}.$$

Then

$$\|g^{(*)}\|_{L^2} \le \text{const}\, \|g\|_{H^2} \tag{1.10}$$

(see Appendix 2.1). Therefore

$$\begin{aligned}
&\int_{\mathbb{T}} \left(\int_0^1 |(f'g)(r\zeta)|^2 (1-r)dr \right)^{1/2} d\boldsymbol{m}(\zeta) \\
&\le \int_{\mathbb{T}} \left(\int_0^1 |f'(r\zeta)|^2 (1-r)dr \right)^{1/2} g^{(*)}(\zeta) d\boldsymbol{m}(\zeta) \\
&\le \left(\int_{\mathbb{T}} \int_0^1 |f'(r\zeta)|^2 (1-r)\, dr\, d\boldsymbol{m}(\zeta) \right)^{1/2} \|g^{(*)}\|_2 \\
&\le \text{const}\, \|f\|_2 \|g\|_2
\end{aligned}$$

by (1.2) and (1.10). ∎

Consider now the following subspaces of $H^2 \oplus H^2$:

$$\mathcal{K} = H^2 \oplus \{\mathbb{O}\} \quad \text{and} \quad \mathcal{L} = \operatorname{span}\left\{ R_\psi^n \begin{pmatrix} \mathbb{O} \\ \mathbf{1} \end{pmatrix} : \ n \in \mathbb{Z}_+ \right\}.$$

It is easy to see that

$$
\begin{aligned}
R_\psi^n \begin{pmatrix} \mathbb{O} \\ \mathbf{1} \end{pmatrix} &= \begin{pmatrix} (S^*)^n & n\Gamma_{(S^*)^{n-1}\psi} \\ \mathbb{O} & S^n \end{pmatrix} \begin{pmatrix} \mathbb{O} \\ \mathbf{1} \end{pmatrix} \\
&= \begin{pmatrix} n(S^*)^{n-1}\psi \\ z^n \end{pmatrix}. \qquad (1.11)
\end{aligned}
$$

Clearly, the subspaces $\mathcal{K}$ and $\mathcal{L}$ are invariant subspaces of R_ψ and $R_\psi \big| \mathcal{K} = S^* \oplus \mathbb{O}$. Thus to prove Theorem 1.5, it suffices to prove that $R_\psi \big| \mathcal{L}$ is similar to the shift operator S on H^2 and $H^2 \oplus H^2$ is a direct sum of $\mathcal{K}$ and Λ. The latter means that

$$\mathcal{K} + \mathcal{L} = H^2 \oplus H^2 \quad \text{and} \quad \mathcal{K} \cap \mathcal{L} = \{\mathbb{O}\}.$$

Theorem 1.7. *Let ψ be a function analytic in $\mathbb{D}$ such that $\psi' \in BMOA$. Then $H^2 \oplus H^2$ is a direct sum of $\mathcal{K}$ and $\mathcal{L}$ and $R_\psi \big| \mathcal{L}$ is similar to the shift operator S on H^2.*

We define the operator Q on the set of analytic polynomials by

$$Q \sum_{k \ge 0} c_k z^k = \sum_{k \ge 0} k c_k (S^*)^{k-1} \psi.$$

Lemma 1.8. *Let ψ be a function analytic in $\mathbb{D}$ such that $\psi' \in BMOA$. Then Q extends to a bounded operator on H^2.*

Proof. Let f and g be analytic polynomials. We have

$$\langle Qf, g \rangle = \sum_{k \ge 0} k \hat{f}(k) \langle (S^*)^{k-1}\psi, g \rangle = \sum_{k \ge 0} k \hat{f}(k) \langle \psi, z^{k-1} g \rangle = \langle \psi, f'g \rangle.$$

The result now follows from Lemma 1.6 and from the fact that $\mathfrak{X}_1^*$ can be identified with the space $\{\psi : \ \psi' \in BMOA\}$ with respect to the pairing (1.6). ■

Proof of Theorem 1.7. Let us first show that

$$\begin{pmatrix} Qf \\ f \end{pmatrix} = \sum_{k \ge 0} \hat{f}(k) R_\psi^k \begin{pmatrix} \mathbb{O} \\ \mathbf{1} \end{pmatrix} \qquad (1.12)$$

for any analytic polynomial f. Indeed, it follows from (1.11) that

$$\sum_{k \ge 0} \hat{f}(k) R_\psi^k \begin{pmatrix} \mathbb{O} \\ \mathbf{1} \end{pmatrix} = \sum_{k \ge 0} \hat{f}(k) \begin{pmatrix} k(S^*)^{k-1}\psi \\ z^k \end{pmatrix} = \begin{pmatrix} Qf \\ f \end{pmatrix}$$

by the definition of Q. Since Q is bounded and obviously,

$$\left\| \begin{pmatrix} Qf \\ f \end{pmatrix} \right\|_2 \ge \|f\|_2,$$

it follows that

$$\mathcal{L} = \left\{ \begin{pmatrix} Qf \\ f \end{pmatrix} : \ f \in H^2 \right\}.$$

Let us show that $H^2 \oplus H^2$ is a direct sum of $\mathcal{K}$ and $\mathcal{L}$. Let $f,\, g \in H^2$. We have

$$\begin{pmatrix} g \\ f \end{pmatrix} = \begin{pmatrix} g - Qf \\ \mathbb{O} \end{pmatrix} + \begin{pmatrix} Qf \\ f \end{pmatrix}.$$

On the other hand, if

$$\begin{pmatrix} g \\ f \end{pmatrix} \in \mathcal{K} \cap \mathcal{L},$$

then $g = Qf$ and $f = \mathbb{O}$, whence $g = \mathbb{O}$.

It remains to prove that $R_\psi \big| \mathcal{L}$ is similar to the shift operator S on H^2. Consider the operator $V : H^2 \to \mathcal{L}$ defined by

$$Vf = \begin{pmatrix} Qf \\ f \end{pmatrix} \in \mathcal{L}.$$

Clearly, V is a bounded linear operator, $\operatorname{Ker} V = \{\mathbb{O}\}$, and $\operatorname{Range} V = \mathcal{L}$. Now let f be an analytic polynomial. We have

$$VSf = \begin{pmatrix} Qzf \\ zf \end{pmatrix} = \begin{pmatrix} \sum\limits_{k \ge 0} (k+1)\hat{f}(k)(S^*)^k \psi \\ Sf \end{pmatrix}.$$

On the other hand,

$$\begin{aligned} R_\psi V f &= \begin{pmatrix} S^*Qf + \Gamma_\psi f \\ Sf \end{pmatrix} = \begin{pmatrix} S^*Qf \\ Sf \end{pmatrix} + \begin{pmatrix} \Gamma_\psi f \\ Sf \end{pmatrix} \\ &= \begin{pmatrix} \sum\limits_{k \ge 0} k\hat{f}(k)(S^*)^k \psi \\ Sf \end{pmatrix} + \begin{pmatrix} \sum\limits_{k \ge 0} \hat{f}(k)(S^*)^k \psi \\ Sf \end{pmatrix} = VSf. \end{aligned}$$

Thus $R_\psi \big| \mathcal{L} = VSV^{-1}$, which completes the proof. ■

2. Power Bounded Operators R_ψ

In this section we characterize the power bounded operators R_ψ. Comparing this characterization with the results of §1, we find among operators R_ψ many power bounded operators that are not polynomially bounded.

Theorem 2.1. *Let $\psi \in BMOA$. Then the operator R_ψ is power bounded if and only is ψ belongs to the Zygmund class Λ_1.*

We need the following result. Let $f \in BMOA$ and $\alpha > 0$. Then $f \in \Lambda_\alpha$ if and only if

$$\|(S^*)^n f\|_{BMOA} \le \operatorname{const} \cdot (1+n)^{-a}, \quad n \in \mathbb{Z}_+. \tag{2.1}$$

We refer the reader to Appendix 2.6, formula (A2.11), for the proof.

Proof of Theorem 2.1. By (1.1),

$$R_\psi^n = \begin{pmatrix} S^* & n\Gamma_{(S^*)^{n-1}\psi} \\ \mathbb{O} & S \end{pmatrix}.$$

Hence, R_ψ is power bounded if and only if

$$\left\|\Gamma_{(S^*)^n\psi}\right\| \le \text{const}\cdot(1+n)^{-1}, \quad n \in \mathbb{Z}_+.$$

By the Nehari theorem this is equivalent to the fact that

$$\|(S^*)^n\psi\|_{BMOA} \le \text{const}\cdot(1+n)^{-1}, \quad n \in \mathbb{Z}_+.$$

Now it remains to apply (2.1) to conclude that R_ψ is power bounded if and only if $\psi \in \Lambda_1$. ■

Now we are in a position to construct power bounded operators R_ψ that are not polynomially bounded.

Corollary 2.2. *Let ψ be a function analytic in $\mathbb{D}$ such that $\psi \in \Lambda_1$ but $\psi' \notin BMOA$. Then the operator R_ψ is power bounded but not polynomially bounded.*

Proof. The result follows immediately from Theorems 1.2 and 2.1. ■

Let us now explain how to find functions $\psi \in (\Lambda_1)_+$ such that $\psi' \notin BMOA$. Consider the function

$$f = \sum_{j=0}^{\infty} c_j z^{2^j}, \tag{2.2}$$

where $c_j \in \mathbb{C}$. It follows immediately from the description of Λ_1 in terms of convolutions with the polynomials W_j (see Appendix 2.6) that a function ψ of the form (2.2) belongs to Λ_1 if and only if $|c_j| \le \text{const}\cdot 2^{-j}$. On the other hand,

$$\psi' = \sum_{j=0}^{\infty} c_j 2^j z^{2^j-1},$$

and it follows from Paley's theorem (see Zygmund [1], Ch.XII, §7) that $\psi' \in BMOA$ if and only if $\{c_j 2^{-j}\}_{j\ge 0} \in \ell^2$. Thus if $\{c_j\}_{j\ge0}$ is a sequence of complex numbers such that

$$|c_j| \le \text{const}\cdot 2^{-j} \quad \text{and} \quad \sum_{j\ge0} |c_j 2^{-j}|^2 = \infty,$$

then R_ψ is power bounded but not polynomially bounded. For example, we can take the function $\psi = \sum\limits_{j\ge0} 2^{-j} z^{2^j}$.

3. Counterexamples

In this section we consider operators of the form R_ψ on spaces of operator-valued functions and we show that among such operators there are polynomially bounded but not similar to a contraction. To prove that an operator is not similar to a contraction, we use the Arveson–Paulsen criterion mentioned in the introduction to this chapter. Namely, we show that the operator is not completely polynomially bounded. We need in this section only the easy part of this criterion, and we give a proof of this easy part here.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let Ψ be an analytic function in $\mathbb{D}$ with values in the space $\mathcal{B}(\mathcal{H},\mathcal{K})$ of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. We assume that the block Hankel matrix $\Gamma_\Psi = \{\hat{\Psi}(j+k)\}_{j,k\ge 0}$ determines a bounded linear operator from $\ell^2(\mathcal{H})$ to $\ell^2(\mathcal{K})$. We denote here by $S_\mathcal{H}$ and $S_\mathcal{K}$ the shift operators on $\ell^2(\mathcal{H})$ and $\ell^2(\mathcal{K})$. Consider now the operator R_Ψ on $\ell^2(\mathcal{K}) \oplus \ell^2(\mathcal{H})$ defined by

$$R_\Psi = \begin{pmatrix} S_\mathcal{K}^* & \Gamma_\Psi \\ \mathbb{O} & S_\mathcal{H} \end{pmatrix}.$$

As in the case of scalar functions we have the following formula:

$$\varphi(R_\Psi) = \begin{pmatrix} \varphi(S_\mathcal{K}^*) & \Gamma_{\varphi'(S^*_{\mathcal{H},\mathcal{K}})\Psi} \\ \mathbb{O} & \varphi(S_\mathcal{H}) \end{pmatrix}, \tag{3.1}$$

where $S^*_{\mathcal{H},\mathcal{K}}$ is backward shift on the space of analytic $\mathcal{B}(\mathcal{H},\mathcal{K})$-valued functions defined by

$$(S^*_{\mathcal{H},\mathcal{K}}F)(z) = \frac{1}{z}(F(z) - F(0)).$$

In what follows we sometimes will omit subscripts and write simply S, S^* instead of $S_\mathcal{H}$ or $S^*_{\mathcal{H},\mathcal{K}}$. We identify in a natural way the spaces $\ell^2(\mathcal{H})$ and $\ell^2(\mathcal{K})$ with the Hardy classes $H^2(\mathcal{H})$ and $H^2(\mathcal{K})$ of vector-valued functions.

The main result of this section is the following.

Theorem 3.1. *Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. Then there exists a function $\Psi \in L^\infty(\mathcal{B}(\mathcal{H}))$ such that the operator R_Ψ on $H^2(\mathcal{H}) \oplus H^2(\mathcal{H})$ is polynomially bounded but not similar to a contraction.*

However, we begin with the case when $\mathcal{H} = \mathbb{C}$, $\mathcal{K}$ is an infinite-dimensional Hilbert space, and the operator-valued function has the following form:

$$\Xi = \sum_{j\ge 0} \alpha_j e_j z^j, \tag{3.2}$$

where $\alpha_j \in \mathbb{C}$ and $\{e_j\}_{j\ge 0}$ is an orthonormal basis of $\mathcal{K}$. Here we identify $\mathcal{B}(\mathbb{C},\mathcal{K})$ with $\mathcal{K}$ in a natural way. It turns out that for such functions Ξ the norm of the Hankel operator Γ_Ξ can be evaluated explicitly.

Theorem 3.2. *Let Ξ be the $\mathcal{K}$-valued function given by* (3.2). *Then Γ_Ξ is a bounded operator from H^2 to $H^2(\mathcal{K})$ if and only if $\{\alpha_j\} \in \ell^2$ and*

$$\|\Gamma_\Xi\| = \|\{\alpha_j\}_{j\ge 0}\|_{\ell^2}. \tag{3.3}$$

Proof. Suppose that Γ_Ξ is bounded. Then $\sum_{j\ge 0} \alpha_j e_j \in \mathcal{K}$, and so $\{\alpha_j\}_{j\ge 0} \in \ell^2$. It suffices to prove (3.3) for finitely supported sequences $\{\alpha_j\}_{j\ge 0}$. Let $\{\omega_{jk}\}_{j,k\ge 0}$ be the matrix of $\Gamma_\Xi^*\Gamma_\Xi$. It is easy to see that

$$\omega_{jk} = \begin{cases} \sum_{m\ge j} |\alpha_m|^2, & j = k, \\ 0, & j \ne k, \end{cases} \tag{3.4}$$

which implies (3.3). ■

Theorem 3.3. *Let* $\{\alpha_j\} \in \ell^2$ *and let* Ξ *be defined by* (3.2). *The following are equivalent:*

(i) R_Ξ *is power bounded;*
(ii) R_Ξ *is polynomially bounded;*
(iii) R_Ξ *is similar to a contraction;*
(iv) *the sequence* $\{\alpha_j\}_{j\ge 0}$ *satisfies*

$$\sup_{m\in\mathbb{Z}_+} m\left(\sum_{j\ge m}|\alpha_j|^2\right)^{1/2} < \infty. \tag{3.5}$$

Proof. Obviously, (iii)⇒(ii)⇒(i). Let us show that (i)⇒(iv). By (3.1) and Theorem 3.2,

$$\|R_\Xi^n\| \ge n\|\Gamma_{(S^*)^n\Xi}\| = n\left(\sum_{j\ge n}|\alpha_j|^2\right)^{1/2}, \quad n>0,$$

which proves (iv).

It remains to prove that (iv)⇒(iii). Consider the operator D of differentiation, which is defined on the dense subset of polynomials in H^2 by $Df = f'$. Let us show that $\Gamma_\Xi D$ extends to a bounded linear operator from H^2 to $H^2(\mathcal{K})$. Indeed,

$$(\Gamma_\Xi D)^*\Gamma_\Xi D = D^*\Gamma_\Xi^*\Gamma_\Xi D$$

and it follows from (3.4) that $(\Gamma_\Xi D)^*\Gamma_\Xi D$ has diagonal matrix with diagonal entries $0, \sum\limits_{j\ge 0}|\alpha_j|^2, \cdots, m^2\sum\limits_{j\ge m-1}|\alpha_j|^2, \cdots$, and so it is bounded.

Let V be the operator on $H^2(\mathcal{K})\oplus H^2$ defined by

$$V = \begin{pmatrix} I & -\Gamma_\Xi D \\ \mathbb{O} & I \end{pmatrix}.$$

It is straightforward to see that V is invertible and

$$V^{-1} = \begin{pmatrix} I & \Gamma_\Xi D \\ \mathbb{O} & I \end{pmatrix}.$$

We have

$$\begin{aligned} VR_\Xi V^{-1} &= \begin{pmatrix} I & -\Gamma_\Xi D \\ \mathbb{O} & I \end{pmatrix} \begin{pmatrix} S_{\mathcal{K}}^* & \Gamma_\Xi \\ \mathbb{O} & S \end{pmatrix} \begin{pmatrix} I & \Gamma_\Xi D \\ \mathbb{O} & I \end{pmatrix} \\ &= \begin{pmatrix} S_{\mathcal{K}}^* & \Gamma_\Xi + S_{\mathcal{K}}^*\Gamma_\Xi D - \Gamma_\Xi DS \\ \mathbb{O} & S \end{pmatrix} \\ &= \begin{pmatrix} S_{\mathcal{K}}^* & \Gamma_\Xi + \Gamma_\Xi SD - \Gamma_\Xi DS \\ \mathbb{O} & S \end{pmatrix} \\ &= \begin{pmatrix} S_{\mathcal{K}}^* & \Gamma_\Xi(I + SD - DS) \\ \mathbb{O} & S \end{pmatrix} = \begin{pmatrix} S_{\mathcal{K}}^* & \mathbb{O} \\ \mathbb{O} & S \end{pmatrix}, \end{aligned}$$

since obviously, $I + SD - DS = \mathbb{O}$. ■

Now we consider the operators R_Ψ with Ψ taking values in $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is an infinite-dimensional Hilbert space. Let us first find a condition sufficient for polynomial boundedness. Suppose that $\{Y_j\}_{j\ge 0}$ is a sequence of bounded linear operators on $\mathcal{H}$ such that

$$\left\| \sum_{j\ge 0} \lambda_j Y_j \right\| \le \left(\sum_{j\ge 0} |\lambda_j|^2 \right)^{1/2} \tag{3.6}$$

for any finitely supported sequence $\{\lambda_j\}_{j\ge 0}$.

Theorem 3.4. *Let $\{Y_j\}_{j\ge 0}$ be a sequence of operators on $\mathcal{H}$ satisfying* (3.6) *and let $\{\alpha_j\}_{j\ge 0}$ be a sequence of complex numbers satisfying* (3.5). *If Ψ is the operator-valued function defined by*

$$\Psi(z) = \sum_{j\ge 0} \alpha_j z^j Y_j,$$

then the operator R_Ψ is polynomially bounded.

We need the following lemma.

Lemma 3.5. *Let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{K}_1$, and $\mathcal{K}_2$ are Hilbert spaces and let Q be a bounded linear operator from $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ to $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$. If $\{\Omega_j\}_{j\ge 0}$ is a sequence in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $\{\Omega_{j+k}\}_{j,k\ge 0}$ determines a bounded linear operator from $\ell^2(\mathcal{H}_1)$ to $\ell^2(\mathcal{H}_2)$, then $\{Q\Omega_{j+k}\}_{j,k\ge 0}$ determines a bounded linear operator from $\ell^2(\mathcal{K}_1)$ to $\ell^2(\mathcal{K}_2)$ and*

$$\|\{Q\Omega_{j+k}\}_{j,k\ge 0}\| \le \|Q\| \cdot \|\{\Omega_{j+k}\}_{j,k\ge 0}\|.$$

Proof. By Theorem 2.2.2, there exists a function $\Phi \in L^\infty(\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2))$ such that $\hat{\Phi}(j) = \Omega_j$ and $\|\Phi\|_{L^\infty(\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2))} = \|\{\Omega_{j+k}\}_{j,k\ge 0}\|$. Consider the function $\Phi_\heartsuit$ defined by

$$\Phi_\heartsuit(\zeta) = Q\Phi(\zeta), \quad \zeta \in \mathbb{T}.$$

Clearly, $\hat{\Phi}_\heartsuit(j) = Q\hat{\Phi}(j)$, $j \in \mathbb{Z}_+$, and

$$\|\Phi_\heartsuit\|_{L^\infty(\mathcal{B}(\mathcal{K}_1,\mathcal{K}_2))} \le \|Q\| \cdot \|\Phi\|_{L^\infty(\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2))}. \quad ■$$

Proof of Theorem 3.4. Let $\mathcal{K}$ be the infinite-dimensional Hilbert space from Theorem 3.2. We apply Lemma 3.5 with $\mathcal{H}_1 = \mathbb{C}$, $\mathcal{H}_2 = \mathcal{K}$, $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{H}$. The operator Q is defined by $Q \sum_{j\ge 0} \alpha_j e_j = \sum_{j\ge 0} \alpha_j Y_j$. By (3.6), $\|Q\| \le 1$. To prove that R_Ψ is polynomially bounded, we have to show that $\left\|\Gamma_{\varphi'(S^*)\Psi}\right\| \le \text{const}\,\|\varphi\|_\infty$ for any polynomial φ. This is an immediate consequence of Lemma 3.5 and the inequality $\left\|\Gamma_{\varphi'(S^*)\Xi}\right\| \le \text{const}\,\|\varphi\|_\infty$ guaranteed by Theorem 3.3. ■

Among operator functions Ψ satisfying the hypotheses of Theorem 3.4 we are going to find those for which R_Ψ is not similar to a contraction. We are going to use *CAR sequences* (i.e., sequences satisfying the canonical anticommutation relations) of bounded linear operators $\{X_j\}_{j\ge 0}$ on Hilbert space satisfying

$$X_j X_k + X_k X_j = \mathbb{O}, \quad X_j X_k^* + X_k^* X_j = \begin{cases} I, & j = k, \\ \mathbb{O}, & j \ne k, \end{cases} \qquad j, k \in \mathbb{Z}_+. \tag{3.7}$$

We show later that such sequences do exist. First, we prove the following important property of CAR sequences.

Theorem 3.6. *Let $\{X_j\}_{j\ge 0}$ be a sequence of Hilbert space operators satisfying* (3.7). *Then*

$$\left\|\sum_{j\ge 0} \alpha_j X_j\right\| = \left(\sum_{j\ge 0} |\alpha_j|^2\right)^{1/2}.$$

Proof. Clearly, we may assume that $\{\alpha_j\}_{j\ge 0}$ is a finitely supported sequence. Let $X = \sum_{j\ge 0} \alpha_j X_j$. It follows from (3.7) that

$$X^2 = \mathbb{O} \quad \text{and} \quad X^*X + XX^* = \left(\sum_{j\ge 0} |\alpha_j|^2\right) I.$$

We have $\|X^*XX^*X\| = \|X\|^4$, and since $X^*X^*XX = \mathbb{O}$, we obtain

$$\begin{aligned} \|X\|^4 &= \|X^*(XX^*)X + X^*(X^*X)X\| \\ &= \left\|X^*\left(\left(\sum_{j\ge 0} |\alpha_j|^2\right) I\right) X\right\| = \left(\sum_{j\ge 0} |\alpha_j|^2\right) \|X\|^2. \quad ■ \end{aligned}$$

We are going to prove the following result.

Theorem 3.7. *Let $\{X_j\}_{j\ge 0}$ be a CAR system and let $\{\alpha_j\}_{j\ge 0}$ be a sequence of complex numbers such that*

$$\sum_{j\ge 0} j^2 |\alpha_j|^2 = \infty. \tag{3.8}$$

Let

$$\Psi = \sum_{j\ge 0} \alpha_j z^j X_j,$$

where $\{X_j\}_{j\ge 0}$ *is a CAR system. Suppose that* Γ_Ψ *is a bounded operator. Then* R_Ψ *is not similar to a contraction.*

Let us first deduce Theorem 3.1 from Theorem 3.7.

Proof of Theorem 3.1. Let $\{\alpha_j\}_{j\ge 0}$ be a sequence of complex numbers satisfying (3.5) and (3.8). Note that such sequences do exist. For example, we can take $\alpha_j = (j+1)^{-3/2}$. Then by Theorems 3.6 and 3.4, R_Ψ is polynomially bounded while according to Theorem 3.7, it is not similar to a contraction. ■

Now we are going to construct a CAR sequence. We are going to use Hilbert tensor products (see §8.2) and tensor products of operators on Hilbert spaces.

Lemma 3.8. *There exists a sequence* $\{X_j\}_{0\le j\le n-1}$ *of operators on* $\mathbb{C}^{2^n}$ *satisfying* (3.7).

Proof. As usual, we identify operators on $\mathbb{C}^{2^n}$ with matrices of size $2^n \times 2^n$. Consider the following 2×2 matrices:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now we can define the X_j by

$$X_j = \underbrace{A\otimes\cdots\otimes A}_{j} \otimes B \otimes \underbrace{I_2\otimes\cdots\otimes I_2}_{n-j-1}, \quad 0\le j\le n-1. \tag{3.9}$$

Clearly, the X_j can be considered as matrices of size $2^n\times 2^n$. Let us verify (3.7) for j, $k\le n-1$. The fact that $X_j^2 = \mathbb{O}$ follows immediately from the equality $B^2 = \mathbb{O}$. We have

$$X_j^* = \underbrace{A\otimes\cdots\otimes A}_{j} \otimes B^* \otimes \underbrace{I_2\otimes\cdots\otimes I_2}_{n-j-1}, \quad 0\le j\le n-1.$$

Then

$$X_j X_j^* = \underbrace{I_2\otimes\cdots\otimes I_2}_{j} \otimes BB^* \otimes \underbrace{I_2\otimes\cdots\otimes I_2}_{n-j-1}, \quad 0\le j\le n-1,$$

and

$$X_j^* X_j = \underbrace{I_2\otimes\cdots\otimes I_2}_{j} \otimes BB^* \otimes \underbrace{I_2\otimes\cdots\otimes I_2}_{n-j-1}, \quad 0\le j\le n-1.$$

The equality $X_jX_j^* + X_j^*X_j = I$ is a consequence of the obvious equality $BB^* + BB^* = I_2$.

Now let $j\ne k$. The equality $X_jX_k + X_kX_j = \mathbb{O}$ follows immediately from the equality $AB + BA = \mathbb{O}$. Finally, the fact that $X_jX_k^* + X_k^*X_j = \mathbb{O}$ is an immediate consequence of of the equality $AB^* + B^*A = \mathbb{O}$. ■

The following result shows that any CAR sequences of length n are in a sense isomorphic.

Lemma 3.9. *Let $X_0, \cdots, X_{n-1}$ and $X_0^\spadesuit, \cdots, X_{n-1}^\spadesuit$ be CAR sequences of of operators on Hilbert space. Let $\mathfrak{A}$ and $\mathfrak{A}^\spadesuit$ be the C^*-algebras generated by the X_j and by the $X_j^\spadesuit$, respectively. Then there exists a $*$-isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}^\spadesuit$ that takes X_j to $X_j^\spadesuit$, $0 \le j \le n-1$.*

We need another lemma.

Lemma 3.10. *Let $\{X_j\}$ be a CAR system and let $j_1, \cdots, j_m$ be a sequence of disjoint nonnegative integers. Then $X_{j_1} \cdots X_{j_m} \ne \mathbb{O}$.*

Proof. Let $Z = X_{j_2} \cdots X_{j_m} \ne \mathbb{O}$. Suppose that $X_{j_1} Z = \mathbb{O}$. Then

$$X_{j_1}^* X_{j_1} Z = \mathbb{O} \quad \text{and} \quad X_{j_1} X_{j_1}^* Z = (-1)^{m-1} X_{j_1}^* Z X_{j_1} = \mathbb{O}.$$

Hence,

$$Z = (X_{j_1} X_{j_1}^* + X_{j_1}^* X_{j_1}) Z = \mathbb{O},$$

and the result follows by induction. ■

Proof of Lemma 3.9. We define by induction finite subsets Δ_j, $j = 0, 1 \cdots, n$, of $\mathfrak{A}$. Put $\Delta_0 = I$,

$$\Delta_j = \Delta_{j-1} \cup \Delta_{j-1} X_j \cup X_j^* \Delta_{j-1} \cup X_j^* \Delta_{j-1} X_j, \quad j \ge 1.$$

It is easy to see that the sets Δ_j are finite and $\mathfrak{A}$ is the linear span of Δ_n. If we prove that Δ_n is linearly independent, we can construct in the same way the subset $\Delta_n^\spadesuit$ of $\mathfrak{A}^\spadesuit$, establish the natural one-to-one correspondence between the elements of Δ_n and $\Delta_n^\spadesuit$, and see that this one-to-one correspondence extends to a $*$-isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}^\spadesuit$.

We prove by induction the following fact that implies the linear independence of Δ_n: if $0 \le m < j_1 < \cdots < j_k$, then the set

$$X_{j_k}^* \cdots X_{j_1}^* \Delta_m X_{j_1} \cdots X_{j_k}$$

is linearly independent. The case $m = 0$ follows from Lemma 3.10. Suppose that we have already proved this for all numbers less than or equal to m and let $m + 1 < j_1 < \cdots < j_k$. Put $Z = X_{j_1} \cdots X_{j_k}$. Let D_0, D_1, D_2, and D_3 be elements of Δ_m such that

$$Z^* D_0 Z + Z^* D_1 X_{m+1} Z + Z^* X_{m+1}^* D_2 Z + Z^* X_{m+1}^* D_3 X_{m+1} Z = \mathbb{O}. \quad (3.10)$$

We have to prove that $D_0 = D_1 = D_2 = D_3 = \mathbb{O}$.

Since $Z X_{m+1} = (-1)^k X_{m+1} Z$ and $X_{m+1}^* Z^* = (-1)^k Z^* X_{m+1}^*$, we can multiply (3.10) by X_{m+1} on the right and by X_{m+1}^* on the left and find that

$$Z^* X_{m+1}^* D_0 X_{m+1} Z = \mathbb{O}.$$

By the inductive hypotheses, $D_0 = \mathbb{O}$. Now multiplying (3.10) by X_{m+1}^* on the left, we obtain

$$Z^* X_{m+1}^* D_1 X_{m+1} Z = \mathbb{O},$$

which again implies that $D_1 = \mathbb{O}$. Next, multiplying (3.10) by X_{m+1} on the right, we find that $D_3 = \mathbb{O}$, which in turn implies that $D_4 = \mathbb{O}$. ■

Now we are in a position to prove that there exists an infinite CAR sequence.

Theorem 3.11. *There exists a sequence $\{X_j\}_{j\ge0}$ of bounded linear operators on Hilbert space that satisfy* (3.7).

Proof. It follows from Lemma 3.9 that there exists a sequence of finite-dimensional C^*-algebras $*$-isometrically imbedded in each other: $\mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \mathfrak{A}_3 \cdots$ and such that $\mathfrak{A}_n$ is generated by a CAR sequence $X_0, X_1, \cdots, X_{n-1}$. This gives us an infinite CAR sequence $\{X_j\}_{j\ge0}$. ■

We need the following important property of CAR sequences.

Theorem 3.12. *Let $\{X_j\}_{j\ge0}$ be a CAR sequence. Then for every finitely supported sequence $\{\alpha_j\}_{j\ge0}$ the following inequality holds:*

$$\frac{1}{2}\sum_{j\ge0}|\alpha_j| \le \left\|\sum_{j\ge0}\alpha_j X_j \otimes X_j\right\| \le \sum_{j\ge0}|\alpha_j|. \tag{3.11}$$

Proof. Clearly, the right inequality is obvious. To establish the left one, we may assume that we deal with a finite CAR sequence $X_0, \cdots, X_{n-1}$. By Lemma 3.9, it is sufficient to prove this inequality for the sequence defined by (3.9). Then we can identify $X_j \otimes X_j$ with

$$\underbrace{(A\otimes A)\otimes\cdots\otimes(A\otimes A)}_{j}\otimes(B\otimes B)\otimes\underbrace{I_4\otimes\cdots\otimes I_4}_{n-j-1}.$$

Let e_1, e_2 be the standard orthonormal basis of $\mathbb{C}^2$. Consider the unit vectors

$$x_j = \frac{1}{\sqrt{2}}(e_1\otimes e_1 + \omega_j e_2\otimes e_2)$$

in $\mathbb{C}^2\otimes\mathbb{C}^2$, where $\omega_j\in\mathbb{C}$ and $\alpha_j\bar{\omega}_j = |\alpha_j|$. Clearly, $\|x_j\| = 1$. Let

$$x \stackrel{\text{def}}{=} x_0\otimes\cdots\otimes x_{n-1}.$$

It is easy to see that $A\otimes A(e_k\otimes e_k) = e_k\otimes e_k$, $k = 1, 2$, and $B\otimes B$ is a rank one partial isometry that takes $e_1\otimes e_2$ to $e_2\otimes e_2$. Now it is easy to verify that

$$\left(\sum_{j=0}^{n-1}\alpha_j(X_j\otimes X_j)x, x\right) = \sum_{j=0}^{n-1}\frac{\alpha_j\bar{\omega}_j}{2} = \frac{1}{2}\sum_{j=0}^{n-1}|\alpha_j|. \quad ■$$

To prove Theorem 3.7, we show that under its hypotheses the operator R_Ψ is not completely polynomially bounded (see the introduction to this chapter). Thus we need the easy part of the Arveson–Paulsen theorem (which is due to Arveson) stated in the introduction. We prove it here.

Theorem 3.13. *Let T be an operator on Hilbert space similar to a contraction. Then T is completely polynomially bounded.*

Recall that we identify here the space $\mathbb{M}_{n,n}$ of $n \times n$ matrices with the space of linear operators on $\mathbb{C}^n$. For a Hilbert space $\mathcal{H}$ we identify the tensor product $\mathcal{B}(\mathcal{H}) \otimes \mathbb{M}_{n,n}$ with the space of $n \times n$ operator matrices $\{T_{jk}\}_{j,k\ge0}$, $T_{jk} \in \mathcal{B}(\mathcal{H})$. Given a matrix $A = \{a_{jk}\}_{j,k\ge0} \in \mathbb{M}_{n,n}$, and a bounded linear operator T on $\mathcal{H}$, we identify $A \otimes T$ with the operator matrix $\{a_{jk}T\}_{j,k\ge0}$.

Proof of Theorem 3.13. It is easy to see from the definition of complete polynomial boundedness that we have to prove the following inequality:

$$\left\| \sum_{j=0}^{m} B_j \otimes T^j \right\|_{\mathbb{M}_{n,n}\otimes\mathcal{B}(\mathcal{H})} \le \operatorname{const} \cdot \max_{|\zeta|\le1} \left\| \sum_{j=0}^{m} \zeta^j B_j \right\|_{\mathbb{M}_{n,n}} \tag{3.12}$$

for any matrices $B_0, \cdots, B_m \in \mathbb{M}_{n,n}$.

It is evident that it suffices to prove the result for a contraction T on a Hilbert space $\mathcal{H}$. We use the Sz.-Nagy theorem on unitary dilations (see Appendix 1.5). Thus there exist a Hilbert space $\mathcal{K}$ such that $\mathcal{H} \subset \mathcal{K}$ and a unitary operator U on $\mathcal{K}$ such that $T^j = P_{\mathcal{H}} U^j \big| \mathcal{H}$ for any $j \in \mathbb{Z}_+$, where $P_{\mathcal{H}}$ is the orthogonal projection onto $\mathcal{H}$. Then $B_j \otimes T^j = P_{\mathbb{C}^n\otimes\mathcal{H}} B_j \otimes U^j$, where $P_{\mathbb{C}^n\otimes\mathcal{H}}$ is the orthogonal projection from $\mathbb{C}^n \otimes \mathcal{K}$ onto $\mathbb{C}^n \otimes \mathcal{H}$. Thus it suffices to prove (3.12) for unitary operators. The spectral theorem for unitary operators allows us to realize a unitary operator as multiplication by z on an L^2 space of vector functions (see Appendix 1.4). Thus the operator $\sum\limits_{j=0}^{m} B_j \otimes U^j$ becomes multiplication by the matrix function $\sum\limits_{j=0}^{m} z^j B_j$, which proves (3.12) for T replaced with U. ■

Now we are able to prove Theorem 3.7.

Proof of Theorem 3.7. Suppose that R_Ψ is similar to a contraction. Then R_Ψ must satisfy (3.12). Since

$$R_\Psi^j = \begin{pmatrix} (S^*)^j & j\Gamma_{(S^*)^{j-1}\Psi} \\ \mathbb{O} & S^j \end{pmatrix},$$

it follows that

$$\left\| \sum_{j=0}^{m} jB_j \otimes \Gamma_{(S^*)^{j-1}\Psi} \right\|_{\mathbb{M}_{n,n}\otimes\mathcal{B}(\mathcal{H})} \le \operatorname{const} \cdot \max_{|\zeta|\le1} \left\| \sum_{j=0}^{m} \zeta^j B_j \right\|_{\mathbb{M}_{n,n}} \tag{3.13}$$

for any matrices $B_j \in \mathbb{M}_{n,n}$. We consider now only the $(0,0)$ entry of $\Gamma_{(S^*)^{j-1}\Psi}$ which is $\alpha_j X_j$. Thus it follows from (3.13) that

$$\left\| \sum_{j=0}^{m} j\alpha_j B_j \otimes X_j \right\|_{\mathbb{M}_{n,n}\otimes\mathcal{B}(\mathcal{H})} \le \operatorname{const} \cdot \max_{|\zeta|\le1} \left\| \sum_{j=0}^{m} \zeta^j B_j \right\|_{\mathbb{M}_{n,n}} \tag{3.14}$$

for any matrices $B_j \in \mathbb{M}_{n,n}$. By Lemmas 3.8 and 3.9, we may assume that the operators X_j in (3.14) are realized as matrices in $\mathbb{M}_{2^{m+1},2^{m+1}}$. Now we put $n = 2^{m+1}$ and $B_j = j\bar{\alpha}_j X_j$ in (3.14) and we obtain

$$\left\| \sum_{j=0}^{m} j^2 |\alpha_j|^2 X_j \otimes X_j \right\|_{\mathbb{M}_{n,n} \otimes \mathcal{B}(\mathcal{H})} \le \text{const} \cdot \max_{|\zeta| \le 1} \left\| \sum_{j=0}^{m} j \bar{\alpha}_j \zeta^j X_j \right\|_{\mathbb{M}_{n,n}}.$$

It follows now from Theorems 3.6 and 3.12 that

$$\sum_{j=0}^{m} j^2 |\alpha_j|^2 \le \text{const} \left(\sum_{j=0}^{m} j^2 |\alpha_j|^2 \right)^{1/2},$$

and so

$$\sum_{j=0}^{m} j^2 |\alpha_j|^2 \le \text{const},$$

which contradicts (3.8). ■

Concluding Remarks

The operators R_Ψ were introduced in Peller [5]; see also Carlson, Clark, Foias, and Williams [1]. The implication (iii)⇒(i) of Theorem 1.2 was proved in Peller [11]. The implication (iii)⇒(ii) was obtained in Bourgain [1]. Bourgain's proof is based on Paulsen's theorem on similarity to a contraction. The proof of the implication (iii)⇒(ii) given in §1 is due to Stafney [1]. The implication (i)⇒(iii) was obtained in Aleksandrov and Peller [1]. The weak factorization Theorems 1.3 and 1.4 were found in Aleksandrov and Peller [1]. Note that the paper Aleksandrov and Peller [1] also contains weak factorization theorems for other function spaces. Theorems 1.5 and 1.7 are due to Stafney [1].

Note that before the results of Aleksandrov and Peller [2] Paulsen observed that R_ψ is similar to a contraction if and only if the matrix $\{(j-k)\hat{\psi}(j+k)\}_{j,k\ge 0}$ determines a bounded linear operator on ℓ^2 (unpublished). It can be obtained from the results of Janson and Peetre [2] that this matrix determines a bounded operator if and only if $\psi' \in BMOA$ (Janson and Peetre considered integral operators, but their technique also works for matrices). Thus these two results give another proof of the fact that R_ψ is similar to a contraction if and only if $\psi' \in BMOA$. Note that Petrovic observed (unpublished) that a result of Stafney [1] together with the observation by Paulsen mentioned above also implies that R_ψ is similar to a contraction if and only if $\psi' \in BMOA$. We also mention here the paper Ferguson [1], in which she gave another proof of Bourgain's result that the condition $\psi' \in BMOA$ implies similarity to a contraction.

The results of §2 were obtained in Peller [5].

The main result of §3, Theorem 3.1, was obtained in Pisier [2]. Namely, Pisier proved that if $\{X_j\}_{j\ge 0}$ is a CAR system and $\{\alpha_j\}_{j\ge 0}$ is a sequence of complex numbers satisfying (3.5) and 3.8 and $\Psi = \sum_{j\geq 0} \alpha_j z^j X_j$, then R_Ψ is polynomially bounded but not similar to a contraction. The proof given in Pisier [2] is rather complicated and involves martingales. In §3 we give the proof of Pisier's result that was found by Davidson and Paulsen [1]. We refer the reader to Pisier [1] for information about CAR sequences. We also mention the paper Kislyakov [2] in which a proof of Pisier's result was given that uses a technique of singular integrals. Another proof of Pisier's result was given in J.E. McCarthy [1].

Appendix 1
Operators on Hilbert Space

We collect in this appendix necessary information on linear operators on Hilbert space. We give here almost no proofs and we give references for more detailed information.

1. Singular Values and Operator Ideals

Let T be a bounded linear operator from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$. The *singular values* $s_n(T)$, $n \in \mathbb{Z}_+$, of T are defined by

$$s_n(T) \stackrel{\text{def}}{=} \inf\{\|T - K\| : \ K : \mathcal{H} \to \mathcal{K}, \ \operatorname{rank} K \le n\},$$

where $\operatorname{rank} K$ stands for the rank of K. Clearly, the sequence $\{s_n(T)\}_{n \ge 0}$ is nonincreasing and its limit

$$s_\infty(T) \stackrel{\text{def}}{=} \lim_{n \to \infty} s_n(T)$$

is equal to the *essential norm* of T, which is by definition

$$\|T\|_{\mathrm{e}} = \inf\{\|T - K\| : \ K \in \boldsymbol{\mathcal{C}}(\mathcal{H}, \mathcal{K})\},$$

where $\boldsymbol{\mathcal{C}}(\mathcal{H}, \mathcal{K})$ is the space of compact operators from $\mathcal{H}$ to $\mathcal{K}$.

It is easy to see that if T_1 and T_2 are operators from $\mathcal{H}$ to $\mathcal{K}$, then

$$s_{m+n}(T_1 + T_2) \le s_n(T_1) + s_m(T_2), \quad m, n \in \mathbb{Z}_+.$$

If A, T, and B are bounded linear Hilbert space operators such that the product ATB makes sense, then it can easily be seen that

$$s_n(ATB) \le \|A\| s_n(T) \|B\|, \quad n \in \mathbb{Z}_+.$$

If T is a self-adjoint operator and D is a nonnegative rank one operator, and $T_\# = T + D$, then

$$s_n(T_\#) \ge s_n(T) \ge s_{n+1}(T_\#) \ge s_{n+1}(T) \quad \text{for any} \quad n \in \mathbb{Z}_+.$$

The operator T is compact if and only if $\lim_{n\to\infty} s_n(T) = 0$. If T is a compact operator from $\mathcal{H}$ to $\mathcal{K}$, it admits a *Schmidt expansion*

$$Tx = \sum_{n\ge 0} s_n(T)(x, f_n)g_n, \quad x \in \mathcal{H},$$

where $\{f_n\}_{n\ge 0}$ is an orthonormal sequence in $\mathcal{H}$ and $\{g_n\}_{n\ge 0}$ is an orthonormal sequence in $\mathcal{K}$. Note that the sum is finite if and only if T has finite rank.

If $0 < p < \infty$, we denote by $\boldsymbol{S}_p(\mathcal{H},\mathcal{K})$ the *Schatten–von Neumann class* of operators T from $\mathcal{H}$ to $\mathcal{K}$ such that

$$\|T\|_{\boldsymbol{S}_p} \stackrel{\text{def}}{=} \left(\sum_{n\ge 0} s_n(T)^p\right)^{1/p} < \infty.$$

If this does not lead to a confusion, we write $T \in \boldsymbol{S}_p$ instead of $T \in \boldsymbol{S}_p(\mathcal{H},\mathcal{K})$. If $1 \le p < \infty$, the space $\boldsymbol{S}_p = \boldsymbol{S}_p(\mathcal{H},\mathcal{K})$ is a Banach space with norm $\|\cdot\|_{\boldsymbol{S}_p}$. We refer the reader to Gohberg and Krein [2] or Simon [2] for the proof of the triangle inequality in $\boldsymbol{S}_p$. Together with $\boldsymbol{C}(\mathcal{H},\mathcal{K})$ we use the notation $\boldsymbol{S}_\infty(\mathcal{H},\mathcal{K})$ for the space of compact operators from $\mathcal{H}$ to $\mathcal{K}$.

If $p < 1$, the space $\boldsymbol{S}_p$ is not a Banach space. However, the following triangle inequality is very useful for $p < 1$.

Theorem A1.1. *Let $0 < p < 1$ and A, $B \in \boldsymbol{S}_p$. Then*

$$\|A + B\|^p_{\boldsymbol{S}_p} \le \|A\|^p_{\boldsymbol{S}_p} + \|B\|^p_{\boldsymbol{S}_p}. \tag{A1.1}$$

Moreover, $\|A + B\|^p_{\boldsymbol{S}_p} = \|A\|^p_{\boldsymbol{S}_p} + \|B\|^p_{\boldsymbol{S}_p}$ if and only if $A^*B = AB^* = \mathbb{O}$.

Since it is not easy to find the proof of this theorem in monographs, we give a proof here that is due to A.B. Aleksandrov (private communication); apparently, it is published here for the first time. We refer the reader to Rotfel'd [1] and C.A. McCarthy [1] for other proofs.

Suppose that $\lambda = \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix}$ and $s = \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}$ are distinct vectors. Suppose also that $|\lambda_0| \ge |\lambda_1|$ and $|s_0| \ge |s_1|$. Consider the function f on $(0,\infty)$ defined by

$$f(q) = \|\lambda\|_{\ell^q} - \|s\|_{\ell^q}. \tag{A1.2}$$

We need the following fact whose proof is an easy exercise.

Lemma A1.2. *The function f can have at most one zero on $(0,\infty)$. If it has a zero, it changes sign at the zero.*

Lemma A1.3. *Let $0 < p < 1$ and let A and B be rank one operators. Suppose that $\|A\| \ge \|B\|$ and $\|A+B\|^p_{\boldsymbol{S}_p} \ge \|A\|^p_{\boldsymbol{S}_p} + \|B\|^p_{\boldsymbol{S}_p}$. Then $s_0(A+B) = \|A\|$ and $s_1(A+B) = \|B\|$ and, in particular, $\|A+B\|^p_{\boldsymbol{S}_p} = \|A\|^p_{\boldsymbol{S}_p} + \|B\|^p_{\boldsymbol{S}_p}$.*

Proof. Assume that $\|A\| \ge \|B\|$. Let $A = \lambda_0(\cdot, x_0)y_0$, $B = \lambda_1(\cdot, x_1)y_1$, where x_0, $x_1 \in \mathcal{H}$, y_0, $y_1 \in \mathcal{K}$, $\|x_0\| = \|x_1\| = \|y_0\| = \|y_1\| = 1$, $\lambda_0 = \|A\|$, $\lambda_1 = \|B\|$. Let $s_0 = s_0(A+B)$ and $s_1 = s_1(A+B)$. Put

$$\lambda = \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix}, \quad s = \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}.$$

By the assumptions, $\|s\|_{\ell^p} \ge \|\lambda\|_{\ell^p}$. By the triangle inequality in $\boldsymbol{S}_1$, $\|s\|_{\ell^1} \le \|\lambda\|_{\ell^1}$. Assume now that $\lambda \ne s$. It follows now from Lemma A1.2 that the function f defined by (A1.2) has one zero on $[p,1]$. Clearly, $f(q) < 0$ for any $q > 1$. It follows that $\lambda_0 > s_0$. Since $\operatorname{rank} A = 1$, it follows that $s_1 \le \lambda_1$. We have obtained a contradiction. ■

Corollary A1.4. *If A and B are rank one operators, then*

$$\|A+B\|^p_{\boldsymbol{S}_p} \le \|A\|^p_{\boldsymbol{S}_p} + \|B\|^p_{\boldsymbol{S}_p}.$$

Lemma A1.5. *Suppose that A and B are rank one operators such that $\|A\| \ge \|B\|$. The following are equivalent:*
(i) *$s_0(A+B) = \|A\|$ and $s_1(A+B) = \|B\|$;*
(ii) *$A^*B = \mathbb{O}$ and $AB^* = \mathbb{O}$.*

Proof. It is sufficient to prove that (i) implies (ii). Again, let $Ax = \lambda_0(x, x_0)y_0$, $Bx = \lambda_1(x, x_1)y_1$, $x \in \mathcal{H}$, where x_0, $x_1 \in \mathcal{H}$, y_0, $y_1 \in \mathcal{K}$, $\|x_0\| = \|x_1\| = \|y_0\| = \|y_1\| = 1$, $\lambda_0 = \|A\|$, $\lambda_1 = \|B\|$. Clearly, we may assume that $\lambda_1 > 0$. We have

$$s_1(A+B) \le \lambda_1 \sup_{x \perp x_0} \frac{|(x, x_1)|}{\|x\|} = \lambda_1 \big(1 - |(x_0, x_1)|^2\big)^{1/2}.$$

It follows that $(x_0, x_1) = 0$.

To prove that $(y_0, y_1) = 0$, we can apply the above reasoning to the operators A^* and B^*. ■

Corollary A1.6. *Let A and B be rank one operators. Then*

$$\|A+B\|^p_{\boldsymbol{S}_p} = \|A\|^p_{\boldsymbol{S}_p} + \|B\|^p_{\boldsymbol{S}_p}$$

if and only if $A^*B = \mathbb{O}$ and $AB^* = \mathbb{O}$.

Lemma A1.7. *Let $A : \mathcal{H} \to \mathcal{K}$ be an operator of rank at most N and $0 < p < 1$. Then*

$$\|A\|^p_{\boldsymbol{S}_p} = \inf \left\{ \sum_{j=1}^N \|A_j\|^p_{\boldsymbol{S}_p} : \ A = \sum_{j=1}^N A_j; \ \operatorname{rank} A_j \le 1 \right\}. \tag{A1.3}$$

Proof. Clearly, it is sufficient to consider on the right-hand side of (A1.3) only those A_j that satisfy $\operatorname{Ker} A \subset \operatorname{Ker} A_j$ and $\operatorname{Range} A_j \subset \operatorname{Range} A$. The set of such operators is finite-dimensional, and so the infimum on the right-hand side is attained.

Let $A = \sum\limits_{j=1}^{N} A_j$ be a representation that minimizes the infimum in (A1.3). Clearly, it is sufficient to show that $A_j^* A_k = \mathbb{O}$ and $A_j A_k^* = \mathbb{O}$. Suppose that $A_j^* A_k \neq \mathbb{O}$ or $A_j A_k^* \neq \mathbb{O}$ for some j and k. By Lemma A1.5, one can represent $A_j + A_k$ as the sum of rank one operators B_1 and B_2 such that

$$\|B_1\|_{\boldsymbol{S}_p}^p + \|B_2\|_{\boldsymbol{S}_p}^p < \|A_1\|_{\boldsymbol{S}_p}^p + \|A_2\|_{\boldsymbol{S}_p}^p$$

which contradicts the fact that the representation $A = \sum\limits_{j=1}^{N} A_j$ realizes the infimum in (A1.3). ■

Proof of Theorem A1.1. Let us prove (A1.1). Clearly, we can assume that A and B are finite rank operators. Let

$$A = \sum_{j=1}^{N} A_j, \quad B = \sum_{j=k}^{M} B_k,$$

where the A_j and B_k are of rank one. Then

$$A + B = \sum_{j=1}^{N} A_j + \sum_{k=1}^{M} B_k$$

and by Corollary A1.4,

$$\|A + B\|_{\boldsymbol{S}_p}^p \le \sum_{j=1}^{N} \|A_j\|_{\boldsymbol{S}_p}^p + \sum_{k=1}^{M} \|B_k\|_{\boldsymbol{S}_p}^p = \|A\|_{\boldsymbol{S}_p}^p + \|B\|_{\boldsymbol{S}_p}^p.$$

It is obvious that the conditions $A^*B = \mathbb{O}$ and $AB^* = \mathbb{O}$ imply that $\|A + B\|_{\boldsymbol{S}_p}^p = \|A\|_{\boldsymbol{S}_p}^p + \|B\|_{\boldsymbol{S}_p}^p$.

Suppose now that $\|A + B\|_{\boldsymbol{S}_p}^p = \|A\|_{\boldsymbol{S}_p}^p + \|B\|_{\boldsymbol{S}_p}^p$. Consider the Schmidt expansions of A and B:

$$A = \sum_j \lambda_j (\cdot, x_j) y_j, \quad B = \sum_k \mu_k (\cdot, u_k) v_k.$$

Then

$$\|A + B\|_{\boldsymbol{S}_p}^p = \sum_j |\lambda_j|^p + \sum_k |\mu_k|^p.$$

It is sufficient to show $x_j \perp u_k$ and $y_j \perp v_k$ for all j and k. If this is not true for some j_0 and k_0, then by Lemma A1.5, we can represent the operator $\lambda_j(\cdot, x_{j_0}) y_{j_0} + \mu_k(\cdot, u_{k_0}) v_{k_0}$ as the sum of A_1 and B_1 such that

$$\|\lambda_j(\cdot, x_{j_0}) y_{j_0} + \mu_k(\cdot, u_{k_0}) v_{k_0}\|_{\boldsymbol{S}_p}^p = \|A_1 + B_1\|_{\boldsymbol{S}_p}^p < |\lambda_{j_0}|^p + |\mu_{k_0}|^p.$$

Hence,

$$\begin{aligned}\|A+B\|^p_{\boldsymbol{S}_p} &\le \|A_1+B_1\|^p_{\boldsymbol{S}_p} + \sum_{j\ne j_0}|\lambda_j|^p + \sum_{k\ne k_0}|\mu_j|^p \\ &< |\lambda_{j_0}|^p + |\mu_{k_0}|^p + \sum_{j\ne j_0}|\lambda_j|^p + \sum_{k\ne k_0}|\mu_j|^p \\ &= \|A\|^p_{\boldsymbol{S}_p} + \|B\|^p_{\boldsymbol{S}_p},\end{aligned}$$

which contradicts the assumption. ■

Operators of class $\boldsymbol{S}_1$ are also called *nuclear operators* or *operators of trace class*. The last term reflects the fact that on the space $\boldsymbol{S}_1$ of operators on a Hilbert space $\mathcal{H}$ we can introduce the important functional trace. Suppose that $\{e_j\}_{j\ge 0}$ is an orthonormal basis in $\mathcal{H}$. Put

$$\operatorname{trace} T = \sum_{j\ge 0}(Te_j, e_j), \quad T \in \boldsymbol{S}_1.$$

Then trace is a linear functional on $\boldsymbol{S}_1$ and $|\operatorname{trace} T| \le \|T\|_{\boldsymbol{S}_1}$. Moreover, $\operatorname{trace} T$ does not depend on the choice of the orthonormal basis $\{e_j\}_{j\ge 0}$.

If $1 < p < \infty$, the dual space $\boldsymbol{S}_p^* = \boldsymbol{S}_p(\mathcal{H},\mathcal{K})^*$ can be identified with the space $\boldsymbol{S}_{p'} = \boldsymbol{S}_{p'}(\mathcal{H},\mathcal{K})$ with respect to the pairing

$$\langle T, R\rangle = \operatorname{trace} TR^*, \quad T \in \boldsymbol{S}_p, \quad R \in \boldsymbol{S}_{p'}. \tag{A1.4}$$

With respect to the same pairing we can identify $\boldsymbol{S}_1^*$ with the space $\mathcal{B}$ of bounded linear operators and we can identify the dual space $\boldsymbol{C}^*$ to the space of compact operators with the space $\boldsymbol{S}_1$.

The space $\boldsymbol{S}_2$ is also called the *Hilbert–Schmidt class*. It is a Hilbert space with respect to the inner product (A1.4). If $\mathcal{H} = L^2(\mathcal{X},\mu)$ and $\mathcal{K} = L^2(\mathcal{Y},\nu)$, an operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ belongs to $\boldsymbol{S}_2$ if and only if there exists a function $k \in L^2(\mathcal{X}\times\mathcal{Y}, \mu\times\nu)$ such that

$$(Tf)(y) = \int_{\mathcal{X}} k(x,y)f(x)d\mu(x).$$

Consider now the ideals $\boldsymbol{S}_{q,\omega}$, $1 \le q < \infty$, of operators T such that

$$\|T\|_{\boldsymbol{S}_{q,\omega}} \stackrel{\text{def}}{=} \left(\sum_{n\ge 0}\frac{(s_n(T))^q}{1+n}\right)^{1/q} < \infty.$$

It is easy to see that $\boldsymbol{S}_p \subset \boldsymbol{S}_{q,\omega}$ for any $p < \infty$ and any $q \in [1,\infty)$. The space $\boldsymbol{S}_{q,\omega}$ is a Banach space with norm $\|\cdot\|_{\boldsymbol{S}_{q,\omega}}$. This follows from the fact that the sequence space $d(q,w)$ that consists of sequences $\{a_n\}_{n\ge 0}$ such that

$$\|\{a_n\}_{n\ge 0}\|_{d(q,w)} \stackrel{\text{def}}{=} \left(\sum_{n\ge 0}\frac{(a_n^*)^q}{1+n}\right)^{1/q}$$

is a Banach space with norm $\|\cdot\|_{d(q,w)}$ (see Lindenstrauss and Tzafriri [1], Ch. 4) and the results of Ch. III, §3, of Gohberg and Krein [2]. Here we denote by $\{a_n^*\}_{n\ge 0}$ the nonincreasing rearrangement of $\{|a_n|\}_{n\ge 0}$.

Note also that the dual space $(\boldsymbol{S}_{q,\omega})^*$ with respect to the pairing (A1.4) consists of operators T such that the sequence $\{s_n(T)\}_{n\ge 0}$ belongs to the sequence space $(d(q,w))^*$, which is by definition the dual space to $d(q,w)$ with respect to the pairing

$$(a,b)=\sum_{n\ge 0}a_n\bar{b}_n,\quad a=\{a_n\}_{n\ge 0}\in d(q,w),\quad b=\{b_n\}_{n\ge 0}\in (d(q,w))^*.$$

The space $\boldsymbol{S}_\omega \stackrel{\text{def}}{=} \boldsymbol{S}_{1,\omega}$ is called the *Matsaev ideal.*

For $0<p<\infty$ and $0<q\le\infty$ we define the *Schatten–Lorentz class* $\boldsymbol{S}_{p,q}$ as the class of operators T on Hilbert space such that

$$\|T\|_{\boldsymbol{S}_{p,q}} \stackrel{\text{def}}{=} \left(\sum_{n\ge 0}(s_n(T))^q(1+n)^{q/p-1}\right)^{1/q}<\infty,\quad q<\infty,$$

$$\|T\|_{\boldsymbol{S}_{p,\infty}} \stackrel{\text{def}}{=} \sup_{n\in\mathbb{Z}_+}(1+n)^{1/p}s_n(T)<\infty,\quad q=\infty.$$

It can be easily verified that $\boldsymbol{S}_{p,p}=\boldsymbol{S}_p$, $\boldsymbol{S}_{p_1,q_1}\subset\boldsymbol{S}_{p_2,q_2}$, if $p_1<p_2$ and $\boldsymbol{S}_{p,q_1}\subset\boldsymbol{S}_{p,q_2}$, if $q_1<q_2$.

We need the following fact.

Theorem A1.8. *Suppose that* $1<q\le\infty$. *Then the space* $\boldsymbol{S}_{1,q}$ *is not normable.*

This fact is well-known. However, it is difficult to find a reference. Thus we sketch the proof.

Sketch of the proof. Suppose that $\{e_n\}_{n\ge 0}$ is an orthonormal basis in a Hilbert space $\mathcal{H}$. Then the subspace of $\boldsymbol{S}_{1,q}$ of operators T of the form $Tx=\sum\limits_{n\ge 0}\lambda_n(x,e_n)e_n$, $x\in\mathcal{H}$, is isomorphic to the Lorentz sequence space $\ell^{1,q}$. Thus it suffices to show that $\ell^{1,q}$ is not normable. Suppose that $\|\cdot\|$ is a norm on $\ell^{1,q}$ that is equivalent to the standard seminorm on $\ell^{1,q}$. We define the operators T_N on $\ell^{1,q}$ by

$$T_N(x_0,x_1,\cdots)=\left(\underbrace{\frac{x_0+\cdots+x_{N-1}}{N},\cdots,\frac{x_0+\cdots+x_{N-1}}{N}}_{N},0,0,\cdots\right).$$

It is easy to see that the operators T_N are uniformly bounded on $\ell^{1,q}$.

Suppose that $x=\{x_n\}_{n\ge 0}\in\ell^{1,q}$, $x_n\ge 0$ and $x=\{x_n\}_{n\ge 0}\notin\ell^1$. It is easy to verify that

$$\|T_Nx\|\ge\text{const}\sum_{n=0}^{N-1}x_n\to\infty\quad\text{as}\quad N\to\infty.\quad\blacksquare$$

We refer the reader to Gohberg and Krein [2] and Simon [2] for more detailed information on singular values and operator ideals.

2. Fredholm Operators and the Calkin Algebra

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$, and $\boldsymbol{\mathcal{C}}(\mathcal{H})$ the ideal of compact operators on $\mathcal{H}$. The quotient algebra

$$\mathcal{B}(\mathcal{H})/\boldsymbol{\mathcal{C}}(\mathcal{H}) = \mathcal{B}/\boldsymbol{\mathcal{C}}$$

is called the *Calkin algebra*. An operator T in $\mathcal{B}(\mathcal{H})$ is called *Fredholm* if its image in the Calkin algebra is invertible. In other words, T is Fredholm if there exists $R \in \mathcal{B}(\mathcal{H})$ such that the operators $RT - I$ and $TR - I$ are compact. It is well known that T is Fredholm if and only if $\operatorname{Range} T$ is closed in $\mathcal{H}$, $\dim \operatorname{Ker} T < \infty$, and $\dim \operatorname{Ker} T^* < \infty$. For a Fredholm operator T the *index* $\operatorname{ind} T$ is defined by

$$\operatorname{ind} T = \dim \operatorname{Ker} T - \dim \operatorname{Ker} T^*.$$

If T_1 and T_2 are Fredholm operators, then T_1T_2 is also Fredholm and

$$\operatorname{ind} T_1T_2 = \operatorname{ind} T_1 + \operatorname{ind} T_2.$$

If T is Fredholm, K is compact, then $T + K$ is Fredholm and $\operatorname{ind}(T + K) = \operatorname{ind} T$. If T is Fredholm, then there exists $\varepsilon > 0$ such that any perturbation $T - K$ of T of norm less than ε is also Fredholm and $\operatorname{ind}(T - K) = \operatorname{ind} T$.

The essential spectrum $\sigma_{\mathrm{e}}(T)$ of $T \in \mathcal{B}(\mathcal{H})$ is by definition the spectrum of the image of T in the Calkin algebra, i.e.,

$$\sigma_{\mathrm{e}}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \ \text{ is not Fredholm}\}.$$

Clearly, $\sigma_{\mathrm{e}}(T)$ is a nonempty compact subset of $\mathbb{C}$ that is contained in $\sigma(T)$.

We refer the reader to Douglas [2] for proofs and more detailed information.

3. The Gelfand–Naimark Theorem

Let A be an algebra with an involution, i.e., A is equipped with a map $x \mapsto x^*$, $x \in A$, of A into itself that satisfies the following properties:

$$\begin{aligned}
(x + y)^* &= x^* + y^*, \quad x,\, y \in A,\\
(\lambda x)^* &= \bar{\lambda} x, \quad x \in A,\ \lambda \in \mathbb{C},\\
(xy)^* &= y^* x^*, \quad x,\, y \in A,\\
x^{**} &= x, \quad x \in A.
\end{aligned}$$

A is called a C^**-algebra* if in addition to that A is a Banach algebra and the involution satisfies

$$\|x^* x\| = \|x\|^2, \quad x \in A.$$

The algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ is a C^*-algebra with involution $T \mapsto T^*$, $T \in \mathcal{B}(\mathcal{H})$. Clearly, any closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ is also a C^*-algebra. The following fundamental result due to Gelfand and Naimark asserts that an arbitrary C^*-algebra is $*$-isometrically isomorphic to such a subalgebra.

Theorem A1.9. *Let A be a C^*-algebra with unit. Then there exist a Hilbert space $\mathcal{H}$ and an isometric homomorphism $\mathcal{U}$ of A into $\mathcal{B}(\mathcal{H})$ such that $\mathcal{U}x^* = (\mathcal{U}x)^*$, $x \in A$.*

Note here that if A is an arbitrary C^*-algebra, the Hilbert space $\mathcal{H}$ does not have to be separable.

Consider now the Calkin algebra $\mathcal{B}/\boldsymbol{\mathcal{C}}$. It is a C^*-algebra with involution $T \mapsto T^*$, $T \in \mathcal{B}/\boldsymbol{\mathcal{C}}$. It follows from Theorem A1.9 that $\mathcal{B}/\boldsymbol{\mathcal{C}}$ is $*$-isometrically isomorphic to a self-adjoint subalgebra of the algebra of bounded linear operators on Hilbert space. We refer the reader to Arveson [4] for the proof of the Gelfand–Naimark theorem and for more information on C^*-algebras.

4. The von Neumann Integral

Let μ be a finite positive Borel measure on $\mathbb{R}$ and let $\{\mathcal{H}(t)\}_{t\in\mathbb{R}}$ be a measurable family of Hilbert spaces. That means that we are given an at most countable set Ω of functions f such that $f(t) \in \mathcal{H}(t)$, μ-a.e.,

$$\operatorname{span}\{f(t) : f \in \Omega\} = \mathcal{H}(t) \text{ for } \mu\text{-almost all } t$$

and the function

$$t \mapsto (f_1(t), f_2(t))_{\mathcal{H}(t)}$$

is μ-measurable for any $f_1, f_2 \in \Omega$. A function g with values $g(t)$ in $\mathcal{H}(t)$ is called measurable if the scalar-valued function $t \mapsto (f(t), g(t))_{\mathcal{H}(t)}$ is measurable for any $f \in \Omega$.

The von Neumann integral (direct integral) $\int \oplus \mathcal{H}(t) d\mu(t)$ consists of measurable functions f, $f(t) \in \mathcal{H}(t)$, such that

$$\|f\| = \left(\int \|f(t)\|^2_{\mathcal{H}(t)} d\mu(t) \right)^{1/2} < \infty.$$

If $f, g \in \int \oplus \mathcal{H}(t) d\mu(t)$, then their inner product is defined by

$$(f, g) = \int (f(t), g(t)) d\mu(t).$$

By von Neumann's theorem, each self-adjoint operator on a separable Hilbert space is unitarily equivalent to multiplication by the independent variable on a direct integral $\int \oplus \mathcal{H}(t) d\mu(t)$:

$$(Af)(t) = tf(t), \quad f \in \int \oplus \mathcal{H}(t) d\mu(t). \tag{A1.5}$$

Without loss of generality we can assume that $\mathcal{H}(t) \neq 0$, μ-almost everywhere. In this case μ is called a *scalar spectral measure* of A. The *spectral multiplicity function* ν_A of the operator A is defined μ-almost everywhere by

$$\nu_A(t) = \dim \mathcal{H}(t).$$

It is well known that self-adjoint operators A_1 and A_2 are unitarily equivalent if and only if their scalar spectral measures are mutually absolutely continuous and $\nu_{A_1} = \nu_{A_2}$ almost everywhere.

If A is a self-adjoint operator with scalar spectral measure μ and spectral multiplicity function ν_A, then A is unitarily equivalent to multiplication by the independent variable on

$$\int \oplus \mathcal{H}(t) d\mu(t),$$

where μ is a scalar spectral measure of A, the $\mathcal{H}(t)$ are embedded in a Hilbert space E with basis $\{e_j\}_{j\geq 1}$, and

$$\mathcal{H}(t) = \operatorname{span}\{e_j : 1 \leq j < \nu_A(t) + 1\}.$$

In this case a function g with values $g(t) \in \mathcal{H}(t)$ is measurable if and only if the scalar function $t \mapsto (g(t), e_j)$ is measurable for any $j \geq 1$.

If Ψ is a bounded operator-valued function defined μ-almost everywhere and such that $\Psi(t) \in \mathcal{B}(\mathcal{H}(t))$, μ-a.e., we say that Ψ is measurable if Ψf is measurable for any function $f \in \int \oplus \mathcal{H}(t) d\mu(t)$.

The following result describes the bounded linear operators that commute with the operator (A1.5).

Theorem A1.10. *Let T be a bounded linear operator on $\int \oplus \mathcal{H}(t) d\mu(t)$ that commutes with the operator* (A1.5). *Then there exists a bounded measurable operator function Ψ with values $\Psi(t) \in \mathcal{B}(\mathcal{H}(t))$, μ-a.e., such that*

$$(Tf)(t) = \Psi(t) f(t), \quad \mu\text{-}a.e.$$

Note that similar results also hold for unitary operators and normal operators.

We refer the reader to Birman and Solomyak [1], Ch. 7, for proofs and for more detailed information.

5. Unitary Dilations and Commutant Lifting

Let $\mathcal{H}$ be a Hilbert space and let T be a *contraction* on $\mathcal{H}$, i.e., $\|T\| \leq 1$. A remarkable theorem by Sz.-Nagy (see Sz.-Nagy–Foias [1], Ch. I, §4) asserts that there exists a Hilbert space $\mathcal{K}$ and a unitary operator U on $\mathcal{K}$ such that $\mathcal{H} \subset \mathcal{K}$ and

$$T^n = P_{\mathcal{H}} U^n \big| \mathcal{H}, \quad n \geq 0, \tag{A1.6}$$

where $P_{\mathcal{H}}$ is the orthogonal projection onto $\mathcal{H}$. A unitary operator satisfying (A1.6) is called a *unitary dilation* of T. Note that earlier Halmos [1] proved the existence of a unitary operator U satisfying (A1.6) with $n = 1$. U is called a *minimal unitary dilation* if

$$\operatorname{span}\{U^n x : x \in \mathcal{H}, n \in \mathbb{Z}\} = \mathcal{K}.$$

If we consider the subspace $\mathcal{K}^+$ of $\mathcal{K}$ defined by

$$\mathcal{K}^+ = \operatorname{span}\{U^n x : x \in \mathcal{H}, n \in \mathbb{Z}_+\},$$

it is easy to see that $U\mathcal{K}^+ \subset \mathcal{K}^+$ and $V \stackrel{\text{def}}{=} U\big|\mathcal{K}^+$ is an isometry. It is also easy to see that

$$T^n = P_{\mathcal{H}} V^n \big| \mathcal{H}, \quad n \ge 0,$$

i.e., V is an isometric dilation of T. In fact, V is a minimal isometric dilation of T in the natural sense.

It is very easy to construct an isometric dilation. Indeed, consider the natural imbedding of $\mathcal{H}$ in $\ell^2(\mathcal{H})$: $x \mapsto (x, \mathbb{O}, \mathbb{O}, \cdots)$, $x \in \mathcal{H}$. Define the operator V on $\ell^2(\mathcal{H})$ by

$$V(x_0, x_1, x_2, \cdots) = \Big(Tx_0, (I - T^*T)^{1/2}x_0, x_1, x_2, \cdots\Big).$$

It is easy to verify that V is an isometric dilation of T. To construct a unitary dilation of T, it suffices to construct a unitary extension of an isometric operator, which can be done using the so-called Kolmogorov–Wold decomposition of an isometry (see Sz.-Nagy–Foias [1], Ch. I, §1).

We are going to state a very important result, the commutant lifting theorem due to Sz.-Nagy and Foias.

Theorem A1.11. *Let T_1 and T_2 be contractions on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and let $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $XT_1 = T_2X$. Suppose that U_1 and U_2 are minimal unitary dilations of T_1 and T_2 on Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ and let $\mathcal{K}_1^+ \subset \mathcal{K}_1$ and $\mathcal{K}_2^+ \subset \mathcal{K}_2$ be the corresponding subspaces of minimal isometric dilations of T_1 and T_2. Then there exists an operator $Y \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ such that*

$$YU_1 = U_2Y, \quad X = P_{\mathcal{H}_2} Y \big| \mathcal{H}_1, \quad Y\mathcal{K}_1^+ \subset \mathcal{K}_2^+, \quad \text{and} \quad \|Y\| = \|X\|.$$

It is easy to see that the requirement of the minimality of unitary dilations is not important and the case of arbitrary unitary dilations can easily be reduced to the case of minimal unitary dilations.

We refer the reader to Sz.-Nagy and Foias [1], Ch. II, §2. We also recommend Foias and Frazho [1] for comprehensive information on commutant lifting. See also Sarason [7], where an approach due to Arocena [1] is presented that is based on ideas of Adamyan, Arov, and Krein.

6. Sz.-Nagy–Foias Functional Model

Let T be a *completely nonunitary contraction* on a Hilbert space $\mathcal{K}$, i.e., T is a contraction and T has no nonzero invariant subspace $\mathcal{H} \subset \mathcal{K}$ such that $T\big|\mathcal{H}$ is a unitary operator on $\mathcal{H}$. The *defect operators* D_T and D_{T^*} are defined by

$$D_T = (I - T^*T)^{1/2} \quad \text{and} \quad D_{T^*} = (I - TT^*)^{1/2}.$$

Consider the defect subspaces

$$\mathfrak{D}_T \stackrel{\text{def}}{=} \operatorname{clos} D_T\mathcal{K} \quad \text{and} \quad \mathfrak{D}_{T^*} \stackrel{\text{def}}{=} \operatorname{clos} D_{T^*}\mathcal{K}.$$

Since $T(I - T^*T) = (I - TT^*)T$, it follows that $TD_T = D_{T^*}T$, and so $T\mathfrak{D}_T \subset \mathfrak{D}_{T^*}$.

The *characteristic function* Θ_T of T is the analytic $\mathcal{B}(\mathfrak{D}_T, \mathfrak{D}_{T^*})$-valued operator function defined by

$$\Theta_T(z) = \big(-T + zD_{T^*}(I - zT^*)^{-1}D_T\big)\big|\mathfrak{D}_T.$$

Then Θ_T is a *purely contractive* function, i.e.,

$$\|\Theta_T(\zeta)x\|_{\mathfrak{D}_{T^*}} < \|x\|_{\mathfrak{D}_T} \quad \text{for any} \quad \zeta \in \mathbb{D} \quad \text{and any nonzero} \quad x \in \mathfrak{D}_T.$$

Suppose now that $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are Hilbert spaces and Θ is a purely contractive functions with values in $\mathcal{B}(\mathfrak{D}_1, \mathfrak{D}_2)$. Consider the Hilbert space

$$\mathcal{L} \stackrel{\text{def}}{=} H^2(\mathfrak{D}_2) \oplus \operatorname{clos} \operatorname{Range} \Delta,$$

where Δ is multiplication by $(I - \Theta^*\Theta)^{1/2}$ on $L^2(\mathfrak{D}_1)$ (see Appendix 2.3 for the definition of the Hardy space $H^2(\mathcal{H})$ of $\mathcal{H}$-valued functions). We define the operator T on the Hilbert space

$$\mathcal{K} \stackrel{\text{def}}{=} \mathcal{L} \ominus \{\Theta f \oplus \Delta f : \ f \in H^2(\mathfrak{D}_1)\}$$

by

$$T(f_1 \oplus f_2) = P_{\mathcal{K}}(zf_1 \oplus zf_2), \quad f_1 \oplus f_2 \in \mathcal{K}, \tag{A1.7}$$

where $P_{\mathcal{K}}$ is the orthogonal projection onto $\mathcal{K}$. Note that the operator $f \mapsto \Theta f \oplus \Delta f$ on $H^2(\mathfrak{D}_1)$ is an isometric imbedding of $H^2(\mathfrak{D}_1)$ in $\mathcal{L}$, and so $\{\Theta f \oplus \Delta f \ : \ f \in H^2(\mathfrak{D}_1)\}$ is a closed subspace of $\mathcal{L}$. Then T is a completely nonunitary contraction. The operator V on $\mathcal{L}$ defined by

$$V(f_1 \oplus f_2) = zf_1 \oplus zf_2.$$

is a minimal isometric dilation of T.

By the Sz.-Nagy–Foias theorem, the characteristic function of the contraction T defined by (A1.7) coincides with Θ modulo constant unitary factors. The representation of a contraction T in the form (A1.7) is called the *Sz.-Nagy–Foias functional model.*

If Θ is an inner operator function (see Appendix 2.3), the functional model looks simpler. Indeed, in this case $\Delta = \mathbb{O}$, $\mathcal{L} = H^2(\mathfrak{D}_2)$,

$$\mathcal{K} = K_\Theta \stackrel{\text{def}}{=} H^2(\mathfrak{D}_2) \ominus \Theta(\mathfrak{D}_1),$$

and

$$Tf = P_{K_\Theta} zf, \quad f \in K_\Theta.$$

A completely nonunitary contraction T on a Hilbert space $\mathcal{H}$ has inner characteristic function if and only if

$$\lim_{n\to\infty} \|T^n x\| = 0 \quad \text{for any} \quad x \in \mathcal{H}.$$

Such contractions are called C_{0*}-contractions. Note also that the characteristic function Θ_T is unitary-valued (i.e., Θ is inner and $\Theta(\zeta)\Theta^*(\zeta) = I$ for almost all $\zeta \in \mathbb{T}$) if and only if T is a C_{00}-contraction, i.e., both T and T^* are C_{0*}-contractions.

For more information on functional models see Sz.-Nagy and Foias [1], and Nikol'skii and Vasyunin [1].

7. The Heinz Inequality

Let A and B be self-adjoint operators on a Hilbert space $\mathcal{H}$ such that $\mathbb{O} \le A \le B$, i.e., $0 \le (Ax, x) \le (Bx, x)$ for any $x \in \mathcal{H}$. The Heinz inequality asserts that for any $\alpha \in (0, 1)$

$$A^\alpha \le B^\alpha.$$

We refer the reader to Birman and Solomyak [1], Ch. 10, §4, for the proof.

Appendix 2
Summary of Function Spaces

In this appendix we gather necessary information on function classes and give references for proofs and more details.

1. Hardy Classes

***H^p* spaces.** Let $0 < p < \infty$. The *Hardy class* H^p consists of functions f analytic in $\mathbb{D}$ and such that

$$\|f\|_p \stackrel{\text{def}}{=} \sup_{0<r<1} \left(\int_{\mathbb{T}} |f(r\zeta)|^p d\boldsymbol{m}(\zeta) \right)^{1/p} < \infty. \tag{A2.1}$$

It is equipped with the norm $\|\cdot\|_p$ (quasinorm, if $p < 1$). The Hardy class H^∞ is the space of bounded analytic functions in $\mathbb{D}$ with norm

$$\|f\|_\infty \stackrel{\text{def}}{=} \sup_{\zeta \in \mathbb{D}} |f(\zeta)|.$$

If $f \in H^p$, then for almost all $\zeta \in \mathbb{T}$ the function f has nontangential boundary values at ζ, i.e., for any $\delta \in (0,1)$ and for almost all $\zeta \in \mathbb{T}$ there exists a limit

$$f(\zeta) \stackrel{\text{def}}{=} \lim_{\lambda \in \Omega_\zeta^\delta,\, \lambda \to \zeta} f(\lambda),$$

where Ω_ζ^δ is the convex hull of $\{\lambda \in \mathbb{C} : |\lambda| \le \delta\} \cup \{\zeta\}$. Thus we can identify a function $f \in H^p$ with the boundary-value function in $L^p = L^p(\mathbb{T})$, which we also denote by f. Note that $\|f\|_{L^p(\mathbb{T})}$ is equal to the supremum in (A2.1). Under this identification H^p is a subspace of L^p. If $1 \le p \le \infty$, the subspace

H^p of L^p admits the following description:

$$H^p = \{f \in L^p : \hat{f}(n) = 0 \text{ for } n < 0\}$$

(as usual, $\hat{f}(n)$ is the nth Fourier coefficient of f).

If $0 < p < \infty$, then

$$\lim_{r \to 1} \int_{\mathbb{T}} |f(\zeta) - f(r\zeta)|^p d\boldsymbol{m}(\zeta) = 0.$$

If $f \in H^p$, then $\log|f| \in L^1$, and so if an H^p function vanishes on a set $E \subset \mathbb{T}$ of nonzero Lebesgue measure, then $f = \mathbb{O}$ (the brothers Riesz theorem).

Note that the Hardy class H^2 is a Hilbert space with respect to the inner product induced from L^2.

The disk algebra C_A is by definition the subspace of H^∞ that consists of functions analytic in $\mathbb{D}$ that can be extended to continuous functions on the closed unit disk. Using the above identification, we can write

$$C_A = \{f \in C(\mathbb{T}) : \hat{f}(n) = 0 \text{ for } n < 0\}.$$

Poisson kernel. Consider now the *Poisson kernel* P_λ, $\lambda \in \mathbb{D}$, defined by

$$P_\lambda(\zeta) = \operatorname{Re} \frac{\zeta + \lambda}{\zeta - \lambda} = \frac{1 - |\lambda|^2}{|1 - \bar{\zeta}\lambda|^2}.$$

If μ is a complex Borel measure on $\mathbb{T}$, then its *Poisson integral*

$$u(z) \stackrel{\text{def}}{=} \int_{\mathbb{T}} P_z(\zeta) d\mu(\zeta) = \sum_{n \ge 0} \hat{\mu}(n) z^n + \sum_{n<0} \hat{\mu}(n) \bar{z}^{-n}$$

is harmonic in $\mathbb{D}$, i.e.,

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0, \quad |x|^2 + |y|^2 < 1.$$

If μ is differentiable at a point $\zeta \in \mathbb{T}$ with respect to normalized Lebesgue measure on $\mathbb{T}$, then

$$\lim_{\lambda \in \Omega_\zeta^\delta,\, \lambda \to \zeta} u(\lambda) = \mu'(\zeta)$$

for any $\delta \in (0,1)$. In particular, if $f \in L^1$ and u is the Poisson integral of f, i.e.,

$$u(z) = \int_{\mathbb{T}} P_z(\zeta) f(\zeta) d\boldsymbol{m}(\zeta) = \sum_{n \ge 0} \hat{f}(n) z^n + \sum_{n<0} \hat{f}(n) \bar{z}^{-n},$$

then for almost all $\zeta \in \mathbb{T}$

$$\lim_{\lambda \in \Omega_\zeta^\delta,\, \lambda \to \zeta} u(\lambda) = f(\zeta).$$

Note that if $f \in L^1$, u is its Poisson integral, and $0 < r < 1$, then $u(rz)$ is the convolution of f and the *Poisson kernel* P_r,

$$P_r(\zeta) = \frac{1 - r^2}{1 - 2r \operatorname{Re} \zeta + r^2}, \quad \zeta \in \mathbb{T}.$$

The harmonic conjugate $\tilde{u}$ of the Poisson integral of f is given by

$$\tilde{u}(rz) = (f * Q_r)(z),$$

where Q_r is the *conjugate Poisson kernel*,

$$Q_r(\zeta) \stackrel{\text{def}}{=} \operatorname{Im} \frac{1 + r\zeta}{1 - r\zeta} = \frac{2r \operatorname{Im} \zeta}{1 - 2r \operatorname{Re} \zeta + r^2}.$$

Suppose now that u is a harmonic function in $\mathbb{D}$ and $1 < p \le \infty$. Then

$$\sup_{0<r<1} \int_{\mathbb{T}} |u(r\zeta)|^p d\boldsymbol{m}(\zeta) < \infty$$

if and only if u is the Poisson integral of a function in L^p. Next,

$$\sup_{0<r<1} \int_{\mathbb{T}} |u(r\zeta)| d\boldsymbol{m}(\zeta) < \infty \tag{A2.2}$$

if and only if u is the Poisson integral of a complex Borel measure on $\mathbb{T}$. Note also that if u is a positive harmonic function in $\mathbb{D}$, then (A2.2) holds and u is the Poisson integral of a finite positive Borel measure on $\mathbb{T}$.

If $1 \le p \le \infty$ and $f \in H^p$, then f is the Poisson integral of its boundary-value function, i.e.,

$$f(\lambda) = \int_{\mathbb{T}} P_\lambda(\zeta) f(\zeta) d\boldsymbol{m}(\zeta), \quad \lambda \in \mathbb{D}.$$

For a function φ analytic in $\mathbb{D}$, consider the *radial maximal function* $\varphi^{(*)}$ on $\mathbb{T}$ defined by

$$\varphi^{(*)}(\zeta) = \sup_{0<r<1} |\varphi(r\zeta)|, \quad \zeta \in \mathbb{T}.$$

If $\varphi \in H^p$, $0 < p \le \infty$, then $\varphi^{(*)} \in L^p$ and $\|\varphi^{(*)}\|_{L^p} \le \text{const}\, \|\varphi\|_{H^p}$ (see Zygmund [1]. Ch. 7, §7).

Let us now introduce the *Fejér kernels* K_n, $n \in \mathbb{Z}_+$:

$$K_n(z) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) z^j.$$

For $f \in L^1$ the convolution $f * K_n$ is the Césaro mean of the partial sums

$$\sum_{k=-j}^{j} \hat{f}(k) z^k, \quad 0 \le j \le n,$$

of the Fourier series of f. It can be easily verified that K_n is a nonnegative function on $\mathbb{T}$, which implies that $\|K_n\|_{L^1} = 1$, and so $\|f * K_n\|_p \le \|f\|_p$ for $f \in L^p$ and $1 \le p \le \infty$. If $f \in L^p$, $1 \le p < \infty$, then $\lim_{n \to \infty} \|f - f * K_n\|_p = 0$.

If $f \in L^1$, then $\lim_{n\to\infty}(f * K_n)(\zeta) = f(\zeta)$ for almost all $\zeta \in \mathbb{T}$. If $f \in C(\mathbb{T})$, the sequence $\{f * K_n\}_{n\ge 0}$ converges to f uniformly on $\mathbb{T}$.

On the space L^2 define the *Riesz projections* $\mathbb{P}_+$ and $\mathbb{P}_-$ onto the subspaces H^2 and $H^2_- \stackrel{\text{def}}{=} L^2 \ominus H^2$ by

$$\mathbb{P}_+ f = \sum_{n\ge 0} \hat{f}(n) z^n, \quad \mathbb{P}_- f = \sum_{n<0} \hat{f}(n) z^n. \tag{A2.3}$$

Clearly, $\mathbb{P}_+$ and $\mathbb{P}_-$ are orthogonal projections on L^2. By the M. Riesz theorem, for $1 < p < \infty$ the projections $\mathbb{P}_+$ and $\mathbb{P}_-$ defined on the set of trigonometric polynomials $\boldsymbol{\mathcal{P}}$ extend to bounded projections from L^p onto H^p and

$$H^p_- \stackrel{\text{def}}{=} \{f \in L^p : \ \hat{f}(n) = 0 \ \text{ for } \ n \ge 0\}.$$

However, the operators $\mathbb{P}_+$ and $\mathbb{P}_-$ are unbounded on L^1 and L^∞.

Consider now the operator $\mathbb{P}_+$ on L^1. Let $f \in L^1$ and let $\mathbb{P}_+ f$ be the function defined by (A2.3). Then for almost all $\zeta \in \mathbb{T}$, the function $\mathbb{P}_+ f$ has nontangential boundary values and we use the same notation $\mathbb{P}_+ f$ for the corresponding boundary-value function. By Kolmogorov's theorem, $\mathbb{P}_+$ has weak type $(1,1)$. i.e., if $f \in L^1$, then $\mathbb{P}_+ f$ belongs to the space

$$L^{1,\infty} \stackrel{\text{def}}{=} \left\{ f : \ \boldsymbol{m}\{\zeta \in \mathbb{T} : \ |(\mathbb{P}_+ f)(\zeta)| \ge \delta\} \le \frac{\text{const}}{\delta}, \ \delta > 0 \right\}.$$

For $f \in L^1$ we can define $\mathbb{P}_- f = f - \mathbb{P}_+ f$ as a function on $\mathbb{T}$.

If f is a real function in L^1, consider the function φ in $\mathbb{D}$ defined by

$$\varphi(\lambda) = \int_{\mathbb{T}} \frac{\zeta + \lambda}{\zeta - \lambda} f(\zeta) d\boldsymbol{m}(\zeta).$$

Then φ is analytic in $\mathbb{D}$, φ has nontangential boundary values almost everywhere and the boundary-value function (which we always denote by φ) satisfies $\operatorname{Re}\varphi(\zeta) = f(\zeta)$, for almost all ζ in $\mathbb{T}$. The function $\tilde{f} \stackrel{\text{def}}{=} \operatorname{Im}\varphi$ is called the *harmonic conjugate* of f. By linearity, the operator of harmonic conjugation extends to the complex functions in L^1. It is easy to see that

$$\tilde{f} = -\mathrm{i}\big(\mathbb{P}_+ f - \mathbb{P}_- f - \hat{f}(0)\big), \quad f \in L^1.$$

Let us mention here Zygmund's theorem according to which if ξ is a bounded real function such that $\|\xi\|_\infty < \pi/(2p)$, then $e^{\tilde{\xi}} \in L^p$. It follows easily from Zygmund's theorem that if ξ is a real continuous function on $\mathbb{T}$, then $e^{\tilde{\xi}} \in L^p$ for any $p < \infty$.

Inner–outer factorizations. A function ϑ analytic in $\mathbb{D}$ is called an *inner function* if $\vartheta \in H^\infty$ and

$$|\vartheta(\zeta)| = 1 \quad \text{for almost all} \quad \zeta \in \mathbb{T}.$$

A function h analytic in $\mathbb{D}$ is called *outer* if there exist a real function g in L^1 and a complex number c of modulus 1 such that

$$h(\lambda) = c \exp\left(\int_{\mathbb{T}} \frac{\zeta+\lambda}{\zeta-\lambda} g(\zeta) d\boldsymbol{m}(\zeta)\right), \quad \lambda \in \mathbb{D}.$$

Then the boundary values of h satisfy $|h(\zeta)| = e^{g(\zeta)}$ for almost all $\zeta \in \mathbb{T}$. If two outer functions have the same modulus on $\mathbb{T}$, then they differ from each other by a multiplicative unimodular (i.e., of modulus 1) constant.

If $f \in H^p$, $0 < p \le \infty$, and $\operatorname{Re} f \ge 0$, then f is outer.

If $0 < p \le \infty$, and $f \in H^p$, then f admits a representation $f = \vartheta h$, where ϑ is an inner function and h is an outer function. Moreover, such a representation is unique modulo a multiplicative unimodular constant. We denote by $f_{(\mathrm{i})}$ and $f_{(\mathrm{o})}$ *the inner and the outer factors* of f defined by

$$f_{(\mathrm{o})}(\lambda) = \exp\left(\int_{\mathbb{T}} \frac{\zeta+\lambda}{\zeta-\lambda} \log|f(\zeta)| d\boldsymbol{m}(\zeta)\right) \quad \text{and} \quad f_{(\mathrm{i})}(\lambda) = h^{-1}(\lambda) f(\lambda),$$

for $\lambda \in \mathbb{D}$. Clearly, $f = f_{(\mathrm{i})} f_{(\mathrm{o})}$.

Let us define a *Blaschke product*. For $\lambda \in \mathbb{D}$ we put

$$b_\lambda(z) = \frac{|\lambda|}{\lambda} \cdot \frac{\lambda - z}{\mathbf{1} - \bar{\lambda} z}, \quad \lambda \ne 0, \quad \text{and} \quad b_0(z) = z,$$

where $\mathbf{1}$ is the function identically equal to 1. If $\{\lambda_j\}_{j \ge 0}$ is a sequence in $\mathbb{D}$ and c is a complex number of modulus 1, then the product

$$B(\zeta) = c \prod_{j \ge 0} b_{\lambda_j}(\zeta)$$

converges for all $\zeta \in \mathbb{D}$ and is not identically equal to 0 if and only if the sequence $\{\lambda_j\}_{j \ge 0}$ satisfies the *Blaschke condition*

$$\sum_{j \ge 0} (1 - |\lambda_j|) < \infty. \tag{A2.4}$$

Such a function B is called a *Blaschke product*. If $0 < p \le \infty$, $f \in H^p$, and $\{\lambda_j\}_{j \ge 0}$ is the sequence of zeros of f in $\mathbb{D}$ (finite or infinite) counted with multiplicities, then $\{\lambda_j\}_{j \ge 0}$ satisfies the Blaschke condition (A2.4), and if B is the Blaschke product defined by (A2.4), then the function $g = B^{-1} f$ belongs to H^p, has no zeros in $\mathbb{D}$, and satisfies $|g(\zeta)| = |f(\zeta)|$ for almost all $\zeta \in \mathbb{T}$.

If ϑ is an inner function and $\{\lambda_j\}_{j \ge 0}$ is the sequence of zeros of ϑ in $\mathbb{D}$ counted with multiplicities, then the function $\omega = B^{-1} \vartheta$ is an inner function that has no zeros in $\mathbb{D}$. Such functions are called *singular inner functions*. They admit the following description. A function ω is a singular inner function if and only if it admits a representation

$$\omega(\lambda) = c \exp\left(\int_{\mathbb{T}} \frac{\zeta+\lambda}{\zeta-\lambda} d\mu(\zeta)\right), \quad \lambda \in \mathbb{D}, \tag{A2.5}$$

where c is a unimodular constant and μ is a positive Borel measure that is singular with respect to Lebesgue measure.

Thus each function in H^p can be represented as a product of a Blaschke product, a singular inner function, and an outer function. Moreover, such a representation is unique modulo multiplicative unimodular constants.

If ϑ is an inner function, then for any $\lambda \in \mathbb{D}$ the function ϑ_λ,

$$\vartheta_\lambda(\zeta) \stackrel{\text{def}}{=} \frac{\vartheta(\zeta) - \lambda}{1 - \bar{\lambda}\vartheta(\zeta)}$$

is also an inner function. By Frostman's theorem, ϑ_λ is a Blaschke product for almost all $\lambda \in \mathbb{D}$, and since obviously,

$$\lim_{\lambda \to 0} \|\vartheta - \vartheta_\lambda\|_\infty = 0,$$

it follows that each inner function is a uniform limit of a sequence of Blaschke products.

If ϑ is an inner function, it admits a representation $\vartheta = B\omega$, where B is a Blaschke product and ω is a singular inner function given by (A2.5).

The *spectrum* $\sigma(\vartheta)$ *of an inner function* ϑ is by definition the set of points $\lambda \in \operatorname{clos}\mathbb{D}$ such that the function $1/\vartheta$ does not extend analytically to a neighborhood of λ. It can be shown that

$$\sigma(\vartheta) = \left(\operatorname{clos} B^{-1}(0)\right) \bigcup \operatorname{supp} \mu.$$

If I is an open subarc of $\mathbb{T}$, then ϑ extends analytically across I if $I \cap \sigma(\vartheta) = \varnothing$. Note that if $\mathbb{T} \not\subset \sigma(\vartheta)$, ϑ extends analytically to $\hat{\mathbb{C}} \setminus \{\zeta \in \hat{\mathbb{C}} : 1/\bar{\zeta} \in \sigma(\vartheta)\}$ by the formula

$$\vartheta(\zeta) = \left(\overline{\vartheta(1/\bar{\zeta})}\right)^{-1}.$$

Here we use the notation $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

If ϑ is an inner function, we define its *degree* $\deg \vartheta$ by

$$\deg \vartheta = \dim(H^2 \ominus \vartheta H^2).$$

The function ϑ has a finite degree if and only if it is a finite Blaschke product in which case ϑ is a rational function and its degree is the degree of the rational function ϑ.

With the help of inner–outer factorizations it is easy to show that if $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and f is a function in H^p with $\|f\|_p = 1$, then there exist functions $g_1 \in H^q$ and $g_2 \in H^r$ such that $f = g_1 g_1$ and $\|g_1\|_q = \|g_2\|_r = 1$.

Similarly, one can consider the Hardy spaces $H^p(\mathbb{C}_+)$ of functions in the upper half-plane $\mathbb{C}_+$, which consists of functions f analytic in $\mathbb{C}_+$ and such that

$$\sup_{y>0} \int_{\mathbb{R}} |f(x + \mathrm{i}y)|^p dx < \infty.$$

As in the case of the unit disk functions in $H^p(\mathbb{C}_+)$ have nontangential boundary values almost everywhere on $\mathbb{R}$ and $H^p(\mathbb{C}_+)$ can be identified with a subspace of $L^p(\mathbb{R})$ (with respect to Lebesgue measure on $\mathbb{R}$). The Paley–Wiener theorem characterizes $H^2(\mathbb{C}_+)$ as the subspace of $L^2(\mathbb{R})$: if

$f \in L^2(\mathbb{R})$, then $f \in H^2(\mathbb{C}_+)$ if and only if the Fourier transform $\mathcal{F}f$ vanishes on $(-\infty, 0)$.

It is well known that if f is an arbitrary function in $H^1(\mathbb{C}_+)$, then

$$\int_{\mathbb{R}} f(t)dt = 0.$$

Consider the conformal map ω from $\mathbb{D}$ onto $\mathbb{C}_+$ defined by

$$\omega(\zeta) = \mathrm{i}\frac{1+\zeta}{1-\zeta}, \quad \zeta \in \mathbb{D}, \tag{A2.6}$$

and consider the operator $\mathcal{U}$ on L^2 defined by

$$(\mathcal{U}f)(t) = \pi^{-1/2}\frac{1}{t+\mathrm{i}}\big(f \circ \omega^{-1}\big)(t) = \pi^{-1/2}\frac{1}{t+\mathrm{i}} f\left(\frac{t-\mathrm{i}}{t+\mathrm{i}}\right), \quad t \in \mathbb{R}.$$

Then $\mathcal{U}$ is a unitary operator of L^2 onto $L^2(\mathbb{R})$ that maps H^2 onto $H^2(\mathbb{C}_+)$.

The Nevanlinna class. The *Nevanlinna class* N consists of meromorphic functions f in $\mathbb{D}$ of the form

$$f = \frac{g_1}{g_2}, \quad g_1 \in H^\infty, \quad g_2 \in H^\infty.$$

It is easy to see that if $0 < p \le \infty$ and $0 < q \le \infty$, then

$$N = \left\{\frac{g_1}{g_2} : \; g_1 \in H^p, \; g_2 \in H^q\right\}.$$

Carleson measures. Let μ be a positive measure in $\mathbb{D}$. It is called a *Carleson measure* if the identical imbedding operator $I_\mu : H^2 \to L^2(\mu)$ is bounded, i.e.,

$$\left(\int_{\mathbb{D}} |f(\zeta)|^2 d\mu(\zeta)\right)^{1/2} \le \text{const}\, \|f\|_2, \quad f \in H^2.$$

By the Carleson imbedding theorem, μ is a Carleson measure if and only if

$$\sup_I \frac{\mu(R_I)}{|I|} < \infty,$$

where the supremum is taken over all subarcs I of $\mathbb{T}$ and

$$R_I = \left\{\zeta \in \mathbb{D} : \; \frac{\zeta}{|\zeta|} \in I \quad \text{and} \quad 1 - |\zeta| \le |I|\right\}.$$

It is easy to deduce from Carleson's theorem that the identical imbedding operator from H^2 to $L^2(\mu)$ is compact if and only if

$$\lim_{|I| \to 0} \frac{\mu(R_I)}{|I|} = 0.$$

Such measures are called *vanishing Carleson measures.*

A complex measure μ on $\mathbb{D}$ is called a Carleson measure (a vanishing Carleson measure) if the total variation $|\mu|$ is a Carleson measure (a vanishing Carleson measure).

Interpolating sequences. A sequence $\{\lambda_j\}_{j\ge0}$ is called *interpolating* if

$$\bigl\{\{f(\lambda_j)\}_{j\ge0}:\ f\in H^\infty\bigr\}=\ell^\infty.$$

It is easy to see that if $\{\lambda_j\}_{j\ge0}$ is an interpolating sequence, then $\lambda_{j_1}\ne\lambda_{j_2}$ for $j_1\ne j_2$ and it satisfies the Blaschke condition (A2.4). Let B be the Blaschke products with zeros λ_j, $j\ge0$. A Blaschke product whose zeros form an interpolating sequence is called an *interpolating Blaschke product.* By a theorem of Carleson, $\{\lambda_j\}_{j\ge0}$ is an interpolating sequence if and only if

$$\inf_{j\ge0}|B_{\lambda_j}(\lambda_j)|>0, \tag{A2.7}$$

where B_{λ_j} is the Blaschke product with zeros λ_k, $k\ne j$, i.e., $B_{\lambda_j}\stackrel{\text{def}}{=}Bb_{\lambda_j}^{-1}$.

A sequence $\{\lambda_j\}_{j\ge0}$ satisfies (A2.7) if and only if it satisfies the separation condition

$$\inf_{j\ne k}\left|\frac{\lambda_j-\lambda_k}{1-\bar\lambda_j\lambda_k}\right|>0$$

and the measure μ defined by

$$\mu=\sum_{j\ge0}(1-|\lambda|_j)\delta_{\lambda_j}$$

is a Carleson measure. Here we denote by δ_{λ_j} the unit point mass at λ_j.

Corona theorem. Consider the Banach algebra H^∞. All point evaluations $\lambda\mapsto f(\lambda)$, $\lambda\in\mathbb{C}$, are multiplicative linear functionals on H^∞. They can be identified with the maximal ideals $\{f\in H^\infty:\ f(\lambda)=0\}$. The Carleson corona theorem says that such maximal ideals are dense in the space of maximal ideals of H^∞. It can easily be reformulated in the following form. Let $f_1,f_2,\cdots,f_n$ be functions in H^∞ such that

$$\sum_{j=1}^n|f_j(\zeta)|^2\ge\delta>0,\quad \zeta\in\mathbb{D}.$$

Then there exists functions $g_1,g_2,\cdots,g_n$ in H^∞ such that

$$\sum_{j=1}^n f_j(\zeta)g_j(\zeta)=1,\quad \zeta\in\mathbb{D}.$$

Note that using T. Wolff's method, Tolokonnikov and Rosenblum generalized this result to the case of infinitely many functions $f_1,f_2,\cdots$ (see N.K. Nikol'skii [2], Appendix 3).

We refer the reader to Duren [1], Garnett [1], Hoffman [1], Koosis [1], and N.K. Nikol'skii [2], [4] for more information on Hardy classes.

2. Invariant Subspaces

Bilateral shift $\mathcal{S}$ is by definition multiplication by z on L^2. Its restriction S to H^2 is called *unilateral shift*. The terminology becomes clear if we consider the action of $\mathcal{S}$ and S on the Fourier coefficients:

$$\widehat{(\mathcal{S}f)}(n) = \hat{f}(n-1), \quad n \in \mathbb{Z},$$

and

$$\widehat{(Sf)}(n) = \hat{f}(n-1), \quad n \geq 1, \quad \text{and} \quad \widehat{(Sf)}(0) = 0.$$

The fundamental Beurling–Helson theorem describes all invariant subspaces of $\mathcal{S}$ and S.

Theorem A2.1. *Let $\mathcal{L}$ be a subspace of L^2 such that $\mathcal{S}\mathcal{L} \subset \mathcal{L}$. Then either $\mathcal{S}\mathcal{L} = \mathcal{L}$ and there exists a measurable subset Δ of $\mathbb{T}$ such that $\mathcal{L} = \chi_\Delta L^2$, or $\mathcal{S}\mathcal{L} \neq \mathcal{L}$ and there exists a unimodular function u on $\mathbb{T}$ such that $\mathcal{L} = uH^2$.*

Here χ_Δ is the characteristic function of Δ. Recall that u is called a unimodular function if $|u(\zeta)| = 1$ for almost all $\zeta \in \mathbb{T}$.

It follows from Theorem A2.1 that if $f \in L^2$, then the invariant subspace $\mathcal{L} = \operatorname{span}\{z^n f : n \in \mathbb{Z}_+\}$ of $\mathcal{S}$ satisfies $\mathcal{S}\mathcal{L} \neq \mathcal{L}$ if and only if $\log|f| \in L^1$.

Beurling's theorem, which describes the invariant subspaces of S, follows easily from Theorem A2.1.

Corollary A2.2. *Let $\mathcal{L}$ be a nonzero subspace of H^2 such that $S\mathcal{L} \subset \mathcal{L}$. Then there exists an inner function ϑ such that $\mathcal{L} = \vartheta H^2$.*

It follows immediately from Corollary A2.2 that a function $f \in H^2$ is outer if and only if f is a cyclic vector of S, i.e., the linear combinations of the functions $z^n f$, $n \in \mathbb{Z}_+$, are dense in H^2.

Similar results also hold for the corresponding operators on L^p and H^p, $1 \leq p < \infty$. In particular, if $\mathcal{L}$ is a nonzero subspace of H^p invariant under multiplication by z, then $\mathcal{L} = \vartheta H^p$ for an inner function ϑ.

Consider now the operator S^* of backward shift, $S^* f = (f - f(0))/z$, $f \in H^2$. By Corollary A2.2, all proper invariant subspaces of S^* have the form $K_\vartheta \stackrel{\text{def}}{=} H^2 \ominus \vartheta H^2$, where ϑ is an inner function. It is easy to see that $K_\vartheta = \vartheta H^2_- \cap H^2$. Thus if $\mathbb{T} \not\subset \sigma(\vartheta)$, each function in K_ϑ extends analytically to the set $\hat{\mathbb{C}} \setminus \{\zeta \in \hat{\mathbb{C}} : 1/\bar{\zeta} \in \sigma(\vartheta)\}$.

The space K_ϑ is finite-dimensional if and only if ϑ is a finite Blaschke product in which case

$$\dim K_\vartheta = \deg \vartheta.$$

The adjoint $(S\big|K_\vartheta)^* \stackrel{\text{def}}{=} S_{[\vartheta]}$ of $S\big|K_\vartheta$ is the so-called compressed shift. Clearly, $S_{[\vartheta]} f = P_\vartheta z f$, where P_ϑ is the orthogonal projection onto K_ϑ. Note that $S_{[\vartheta]}$ is the model operator in the Sz.-Nagy–Foias functional model.

We proceed now to the notion of pseudocontinuation. Let $f \in H^2$. We say that a meromorphic function g of the Nevanlinna class in $\hat{\mathbb{C}} \setminus \operatorname{clos}\mathbb{D}$ is called a *pseudocontinuation* of f if the boundary values on $\mathbb{T}$ of f and g coincide almost everywhere. The Nevanlinna class of functions meromorphic in

$\hat{\mathbb{C}} \setminus \operatorname{clos}\mathbb{D}$ can be defined by analogy with the Nevanlinna class of meromorphic functions in $\mathbb{D}$ defined above.

A function f in H^2 has a pseudocontinuation if and only if f belongs to K_ϑ for some inner function ϑ. Note also that if $f \in H^2$, f has a pseudocontinuation, and f extends analytically across an arc $I \subset \mathbb{T}$, then its analytic extension coincides with its pseudocontinuation.

We refer the reader to N.K. Nikol'skii [2] and Helson [1] for more detailed information.

3. Invariant Subspaces of Multiple Shift and Inner–Outer Factorization of Operator Functions

Let $\mathcal{H}$ be a Hilbert space (recall that we consider here only separable Hilbert spaces). Consider the operator $S_\mathcal{H}$ of multiple unilateral shift on the Hardy space $H^2(\mathcal{H})$ of $\mathcal{H}$-valued functions: $S_\mathcal{H} f = zf$, $f \in H^2(\mathcal{H})$. The Hardy space $H^2(\mathcal{H})$ consists of $\mathcal{H}$-valued functions $f \in L^2(\mathcal{H})$ with Fourier coefficients satisfying $\hat{f}(n) = \mathbb{O}$ for $n < 0$. As in the scalar case, a function $f \in H^2(\mathcal{H})$ can be identified with the $\mathcal{H}$-valued analytic in $\mathbb{D}$ function

$$\sum_{n \ge 0} \hat{f}(n) z^n.$$

To state the description of the invariant subspaces of $S_\mathcal{H}$, we need the notion of an inner operator function.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. We denote by $L^\infty(\mathcal{B}(\mathcal{K},\mathcal{H}))$ the space of bounded weakly measurable functions on $\mathbb{T}$ that take values in the space $\mathcal{B}(\mathcal{K},\mathcal{H})$ of bounded linear operators from $\mathcal{K}$ to $\mathcal{H}$. By $H^\infty(\mathcal{B}(\mathcal{K},\mathcal{H}))$ we denote the subspace of $L^\infty(\mathcal{B}(\mathcal{K},\mathcal{H}))$ that consists of functions $\Phi \in L^\infty(\mathcal{B}(\mathcal{K},\mathcal{H}))$ whose Fourier coefficients $\hat{\Phi}(n)$ satisfy $\hat{\Phi}(n) = 0$, $n < 0$, and such a function Φ can be identified with the corresponding operator function

$$\sum_{n \ge 0} \hat{\Phi}(n) z^n.$$

A function $\Theta \in H^\infty(\mathcal{B}(\mathcal{K},\mathcal{H}))$ is called *inner* if $\Theta^*(\zeta)\Theta(\zeta) = I$ for almost all $\zeta \in \mathbb{T}$. In other words, a function $\Theta \in H^\infty(\mathcal{B}(\mathcal{K},\mathcal{H}))$ is inner if $\Theta(\zeta)$ is an isometry for almost all $\zeta \in \mathbb{T}$.

The invariant subspaces of $S_\mathcal{H}$ are described by the Beurling–Lax–Halmos theorem.

Theorem A2.3. *Let $\mathcal{L}$ be a subspace of $H^2(\mathcal{H})$. Then $S_\mathcal{H}\mathcal{L} \subset \mathcal{L}$ if and only if there exists a Hilbert space $\mathcal{K}$ and an inner operator function $\Theta \in H^\infty(\mathcal{B}(\mathcal{K},\mathcal{H}))$ such that $\mathcal{L} = \Theta H^2(\mathcal{K})$. If $\Theta_1 \in H^\infty(\mathcal{B}(\mathcal{K}_1,\mathcal{H}))$ and $\Theta_2 \in H^\infty(\mathcal{B}(\mathcal{K}_2,\mathcal{H}))$ are inner operator functions such that $\Theta_1 H^2(\mathcal{K}_1) = \Theta_2 H^2(\mathcal{K}_2)$, then there exixts a unitary operator V from $\mathcal{K}_1$ onto $\mathcal{K}_2$ such that $\Theta_1 V = \Theta_2$.*

If $\mathcal{L} = \{\mathbb{O}\}$, we can take the zero-dimensional space $\mathcal{K}$. Note also that if $\Theta_1 \in H^\infty(\mathcal{B}(\mathcal{K}_1, \mathcal{H}))$ and $\Theta_2 \in H^\infty(\mathcal{B}(\mathcal{K}_2, \mathcal{H}))$ are inner operator functions such that $\Theta_2 H^2(\mathcal{K}_1) \subset \Theta_1 H^2(\mathcal{K}_2)$, then there exixts an inner operator function $\Upsilon \in H^\infty(\mathcal{B}(\mathcal{K}_2, \mathcal{K}_1))$ such that $\Theta_2 = \Theta_1 \Upsilon$.

We proceed now to inner–outer factorizations for operator functions. Consider the class $L^2_{\mathrm{s}}(\mathcal{B}(\mathcal{K}, \mathcal{H}))$ of weakly measurable $\mathcal{B}(\mathcal{K}, \mathcal{H})$-valued functions Φ on $\mathbb{T}$ such that

$$\int_{\mathbb{T}} \|\Phi(\zeta)x\|^2_{\mathcal{H}} d\boldsymbol{m}(\zeta) < \infty \quad \text{for every} \quad x \in \mathcal{K}.$$

Clearly, $L^\infty(\mathcal{B}(\mathcal{K}, \mathcal{H})) \subset L^2_{\mathrm{s}}(\mathcal{B}(\mathcal{K}, \mathcal{H}))$. For $\Phi \in L^2_{\mathrm{s}}(\mathcal{B}(\mathcal{K}, \mathcal{H}))$ the Fourier coefficients are defined by

$$\hat{\Phi}(n)x = \int_{\mathbb{T}} \bar{\zeta}^n \Phi(\zeta) x d\boldsymbol{m}(\zeta), \quad x \in \mathcal{K}, \quad n \in \mathbb{Z}.$$

We say that a function Φ in $L^2_{\mathrm{s}}(\mathcal{B}(\mathcal{K}, \mathcal{H}))$ belongs to the Hardy class $H^2_{\mathrm{s}}(\mathcal{B}(\mathcal{K}, \mathcal{H}))$ if $\hat{\Phi}(n) = \mathbb{O}$ for $n < 0$.

A function $F \in H^2_{\mathrm{s}}(\mathcal{B}(\mathcal{K}, \mathcal{H}))$ is called *outer* if the functions of the form $q\Phi x$, where q ranges over the analytic polynomials $\boldsymbol{\mathcal{P}}_+$ and $x \in \mathcal{K}$, are dense in $H^2(\mathcal{H})$.

It follows from Theorem A2.3 that if $\Phi \in H^2_{\mathrm{s}}(\mathcal{B}(\mathcal{K}, \mathcal{H}))$, then there exist a Hilbert space $\mathcal{M}$ an inner function $\Theta \in H^\infty(\mathcal{B}(\mathcal{M}, \mathcal{H}))$ and an outer function $F \in H^2_{\mathrm{s}}(\mathcal{B}(\mathcal{K}, \mathcal{M}))$ such that

$$\Phi = \Theta F.$$

This factorization is called the inner–outer factorization of Φ.

We say that a function $G \in H^2_{\mathrm{s}}(\mathcal{B}(\mathcal{K}, \mathcal{H}))$ is *co-outer* if the function $F(z) = G^*(\bar{z})$ is an outer function in $H^2_{\mathrm{s}}(\mathcal{B}(\mathcal{H}, \mathcal{K}))$. Note that if $\mathcal{K}$ and $\mathcal{H}$ are finite-dimensional spaces and we identify them with $\mathbb{C}^n$ and $\mathbb{C}^m$ and we identify the space $\mathcal{B}(\mathcal{K}, \mathcal{H})$ with the space $\mathbb{M}_{m,n}$ of $m \times n$ matrices, then G is co-outer if and only if the transposed matrix function G^{t} is outer.

We mention the following useful fact.

Theorem A2.4. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space and let $F \in H^2_{\mathrm{s}}(\mathcal{B}(\mathcal{H}, \mathcal{H}))$. Then F is outer if and only if the determinant* $\det F$ *is a scalar outer function.*

Consider now the invariant subspace of multiple bilateral shift. For a Hilbert space $\mathcal{H}$ we denote by $\mathcal{S}_{\mathcal{H}}$ multiplication by z on $L^2(\mathcal{H})$. We are not going to state here the description of all invariant subspaces of $\mathcal{S}_{\mathcal{H}}$ and refer the reader to N.K. Nikol'skii [2], Lect. I, §5. We state here only the description of *completely nonreducing noninvariant subspaces*, i.e., such invariant subspaces that do not contain nontrivial reducing subspaces of $\mathcal{S}_{\mathcal{H}}$. (A reducing subspace of $\mathcal{S}_{\mathcal{H}}$ is a subspace invariant under both $\mathcal{S}_{\mathcal{H}}$ and $\mathcal{S}^*_{\mathcal{H}}$.)

Theorem A2.5. *Let $\mathcal{L}$ be a nonzero subspace of $L^2(\mathcal{H})$. Then $\mathcal{L}$ is a completely nonreducing invariant subspace of $\mathcal{S}_{\mathcal{H}}$ if and only if there exist* *a Hilbert space* $\mathcal{K}$ *and an isometric-valued function* $U \in L^\infty(\mathcal{K}, \mathcal{H})$ *such that* $\mathcal{L} = UH^2(\mathcal{K})$. *The operator function* U *is uniquely determined by* $\mathcal{L}$ *modulo a left constant unitary factor.*

We refer the reader to N.K. Nikol'skii [2] for more information.

4. Weighted L^2 and H^2 Spaces

Let μ be a finite positive measure on $\mathbb{T}$. Consider the weighted space $L^2(\mu)$ and denote by $H^2(\mu)$ the closure in $L^2(\mu)$ of the set of analytic polynomials $\boldsymbol{\mathcal{P}}_+$. We denote also by $H^2_-(\mu)$ the closed linear span of z^n, $n < 0$, in $L^2(\mu)$. Let $\mu = \mu_{\rm a} + \mu_{\rm s}$ be the Lebesgue decomposition of μ, where $\mu_{\rm a}$ is absolutely continuous and $\mu_{\rm s}$ is singular with respect to Lebesgue measure. We need the following Szegö–Kolmogorov alternative.

Theorem A2.6. *The space* $H^2(\mu)$ *admits the decomposition*

$$H^2(\mu) = H^2(\mu_{\rm a}) \oplus L^2(\mu_{\rm s}).$$

If

$$\int_{\mathbb{T}} \log \frac{d\mu_{\rm a}}{d\boldsymbol{m}} = -\infty,$$

then $H^2(\mu) = L^2(\mu)$. *However, if*

$$\int_{\mathbb{T}} \log \frac{d\mu_{\rm a}}{d\boldsymbol{m}} > -\infty,$$

then $H^2(\mu) \neq L^2(\mu)$ *and* $\bigcap_{n \geq 0} z^n H^2(\mu) = L^2(\mu_{\rm s})$.

We refer the reader to Hoffman [1] and Dym and McKean [1] for the proof.

5. The Spaces BMO and VMO

Let f be a function in L^1 on the unit circle and let I be a subarc of $\mathbb{T}$. Put

$$f_I \stackrel{\text{def}}{=} \frac{1}{\boldsymbol{m}(I)} \int_I f d\boldsymbol{m},$$

the mean value of f over I. The space BMO of functions of *bounded mean oscillation* consist of functions $f \in L^1$ such that

$$\sup_I \frac{1}{\boldsymbol{m}(I)} \int_I |f - f_I| d\boldsymbol{m} < \infty.$$

The space VMO of functions of *vanishing mean oscillation* consists of functions $f \in BMO$ for which

$$\lim_{\boldsymbol{m}(I) \to 0} \frac{1}{\boldsymbol{m}(I)} \int_I |f - f_I| d\boldsymbol{m} = 0.$$

There are many natural norms on BMO which make BMO a Banach space. One can consider, for example, the following one:

$$\|f\| = \sup_I \frac{1}{\boldsymbol{m}(I)} \int_I |f - f_I| d\boldsymbol{m} + |\hat{f}(0)|. \tag{A2.8}$$

VMO is the closure of the set $\boldsymbol{\mathcal{P}}$ of trigonometric polynomials in BMO, and so it is a closed subspace of BMO.

Clearly, $L^\infty \subset BMO$. However, the space BMO contains unbounded functions. A more nontrivial inclusion is $BMO \subset L^p$ for every $p < \infty$. The space BMO was introduced by John and Nirenberg [1]. It was a remarkable result by Fefferman [1] that demonstrated an exclusive importance of this space. Let us state it.

Theorem A2.7.

$$BMO = \{\xi + \tilde{\eta} : \ \xi, \ \eta \in L^\infty\}.$$

A similar result for VMO is due to Sarason [4]:

Theorem A2.8.

$$VMO = \{\xi + \tilde{\eta} : \ \xi, \ \eta \in C(\mathbb{T})\}.$$

(Recall that $C(\mathbb{T})$ is the space of continuous functions on $\mathbb{T}$.)

It follows from Theorems A2.7 and A2.8 that

$$\mathbb{P}_+ BMO = \mathbb{P}_+ L^\infty = BMO \cap H^2$$

and

$$\mathbb{P}_+ VMO = \mathbb{P}_+ C(\mathbb{T}) = VMO \cap H^2.$$

We put $BMOA \stackrel{\text{def}}{=} BMO \cap H^2$ for the space of analytic functions in BMO and $VMOA \stackrel{\text{def}}{=} VMO \cap H^2$ for the space of analytic functions in VMO.

The space QC of *quasicontinuous functions* is defined by

$$QC = (H^\infty + C) \cap \overline{(H^\infty + C)}.$$

It follows easily from Theorem A2.8 that $QC = VMO \cap L^\infty$.

Consider now the Garsia norm (seminorm, to be more precise) on BMO. For $\varphi \in L^2$ we consider its Poisson integral

$$\varphi_\#(\zeta) \stackrel{\text{def}}{=} \int_{\mathbb{T}} \varphi(\tau) P_\zeta(\tau) d\boldsymbol{m}(\tau), \quad \zeta \in \mathbb{D},$$

and we define the Garsia norm $\|\varphi\|_G$ of φ by

$$\|\varphi\|_{\mathrm{G}} = \left(\sup_{\zeta \in \mathbb{D}} \int_{\mathbb{T}} |\varphi(\tau) - \varphi_\#(\zeta)|^2 P_\zeta(\tau) d\boldsymbol{m}(\tau) \right)^{1/2}.$$

Theorem A2.9. *Let $\varphi \in L^2$. Then $\varphi \in BMO$ if and only if $\|\varphi\|_{\mathrm{G}} < \infty$. Moreover, $\|\cdot\|_{\mathrm{G}}$ is equivalent to the norm* (A2.8) *on BMO modulo the constants.*

Theorem A2.10. *Let $\varphi \in L^2$. Then $\varphi \in VMO$ if and only if*

$$\lim_{|\zeta|\to 1} \int_{\mathbb{T}} |\psi(\tau) - \psi_{\#}(\zeta)|^2 P_\zeta(\tau) d\boldsymbol{m}(\tau) = 0.$$

The space BMO admits the following characterization in terms of Carleson measures.

Theorem A2.11. *Let $f \in L^1$ and let u be the harmonic extension of f. Then $f \in BMO$ if and only if*

$$|(\nabla u)(\zeta)|^2 (1 - |\zeta|^2) d\boldsymbol{m}_2(\zeta)$$

is a Carleson measure on $\mathbb{D}$.

In particular, if $f \in H^1$, then $f \in BMOA$ if and only if

$$|f'(\zeta)|^2 (1 - |\zeta|^2) d\boldsymbol{m}_2(\zeta)$$

is a Carleson measure on $\mathbb{D}$.

Let us now describe BMO and VMO in terms of the Poisson balayage of Carleson (vanishing Carleson) measures.

Theorem A2.12. *Let $\varphi \in L^1$. Then $\varphi \in BMO$ if and only if there exists a complex Carleson measure μ on $\mathbb{D}$ whose Poisson balayage coincides with φ, i.e.,*

$$\varphi(\zeta) = \int_{\mathbb{D}} P_\lambda(\zeta) d\mu(\lambda) \quad \text{a.e. on } \mathbb{T}.$$

Similarly, a function $\varphi \in L^1$ belongs to VMO if and only if it is a Poisson balayage of a complex vanishing Carleson measure.

We can also consider the space $BMO(\mathbb{R})$ of locally integrable functions f on $\mathbb{R}$ satisfying

$$\sup_I \frac{1}{|I|} \int_I |f(x) - f_I| dx < \infty,$$

where the supremum is taken over intervals I of $\mathbb{R}$ and $|I|$ is the length of I. The space $VMO(\mathbb{R})$ consists of functions $f \in BMO(\mathbb{R})$ such that

$$\lim_I \frac{1}{|I|} \int_I |f(x) - f_I| dx = 0$$

as $|I| \to 0$, $|I| \to \infty$, or the center of I goes to ∞ or $-\infty$. The following analogs of Theorems A2.7 and A2.8 hold for $BMO(\mathbb{R})$ and $VMO(\mathbb{R})$:

$$BMO(\mathbb{R}) = \{\xi + \boldsymbol{P}^+\eta : \ \xi, \eta \in L^\infty(\mathbb{R})\},$$

$$VMO(\mathbb{R}) = \{\xi + \boldsymbol{P}^+\eta : \ \xi, \eta \in C(\hat{\mathbb{R}})\},$$

where $\boldsymbol{P}^+\eta \stackrel{\text{def}}{=} (\mathbb{P}_+(\eta \circ \omega)) \circ \omega^{-1}$, ω is the conformal map defined by (A2.6), and $C(\hat{\mathbb{R}})$ is the space of continuous functions on $\mathbb{R}$ that have equal finite limits at ∞ and $-\infty$. It follows that $\varphi \in BMO$ if and only if $\varphi \circ \omega \in BMO(\mathbb{R})$, and $\varphi \in VMO$ if and only if $\varphi \circ \omega \in VMO(\mathbb{R})$, where ω is the conformal map defined by (A2.6).

We refer the reader to Garnett [1] and Koosis [1] for more information about BMO and VMO. The results on $VMO(\mathbb{R})$ can be found in Neri [1] (note that in that paper the author uses the notation CMO for $VMO(\mathbb{R})$).

6. Spaces of Smooth Functions. Besov Spaces

Recall that a distribution on the unit circle $\mathbb{T}$ is a continuous linear functional on the space $C^\infty(\mathbb{T})$ of infinitely differentiable functions. The space L^1 is naturally imbedded in the space of distributions. If $\varphi \in L^1$, we identify φ with the following continuous linear functional on $C^\infty(\mathbb{T})$:

$$f \mapsto (f, \varphi) \stackrel{\text{def}}{=} \int_{\mathbb{T}} f(\zeta)\varphi(\zeta)d\boldsymbol{m}(\zeta), \quad f \in C^\infty(\mathbb{T}).$$

If φ is a distribution, its Fourier coefficients are defined by

$$\hat{\varphi}(j) = (z^j, \varphi), \quad j \in \mathbb{Z}.$$

If φ and ψ are distributions, then the convolution $\varphi * \psi$ is the distribution with Fourier coefficients

$$\widehat{\varphi * \psi}(j) = \hat{\varphi}(j)\hat{\psi}(j), \quad j \in \mathbb{Z}.$$

For a positive integer n we consider the polynomial W_n with Fourier coefficients satisfying $\hat{W}_n(2^n) = 1$, $\hat{W}_n(j) = 0$ for $j \notin \left(2^{n-1}, 2^{n+1}\right)$, and $\hat{W}_n$ is a linear function on $\left[2^{n-1}, 2^n\right]$ and $\left[2^n, 2^{n+1}\right]$. If n is a negative integer, we put $W_n = \overline{W}_{-n}$. Finally, $W_0 \stackrel{\text{def}}{=} \bar{z} + \mathbf{1} + z$.

The *Besov class* B^s_{pq}, $1 \le p \le \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$, consists of distributions f on $\mathbb{T}$ such that

$$\left\{2^{|n|s}\|W_n * f\|_p\right\}_{n\in\mathbb{Z}} \in \ell^q(\mathbb{Z}). \tag{A2.9}$$

If $s > 0$, then $B^s_{pq} \subset L^p$. We use the notation B^s_p for B^s_{pp}.

The spaces $\Lambda_s \stackrel{\text{def}}{=} B^s_\infty$ are called the *Hölder–Zygmund spaces*. For $s \in (0,1)$ they admit the following description:

$$f \in \Lambda_s, \quad 0 < s < 1, \quad \Longleftrightarrow \quad |f(\zeta_1) - f(\zeta_2)| \le \text{const}\,|\zeta_1 - \zeta_2|^s, \quad \zeta_1, \zeta_2 \in \mathbb{T}.$$

The class Λ_1 is the *Zygmund class*, which can be characterized as follows:

$$f \in \Lambda_1 \quad \Longleftrightarrow \quad |f(\zeta\tau) - 2f(\zeta) + f(\zeta\bar{\tau})| \le \text{const}\,|1-\tau|, \quad \zeta, \tau \in \mathbb{T}.$$

To state a similar description for arbitrary Besov spaces B^s_{pq} with $s > 0$, we introduce the difference operators Δ^n_τ, $\tau \in \mathbb{T}$. We define the operator Δ_τ by

$$(\Delta_\tau f)(\zeta) = f(\tau\zeta) - f(\zeta), \quad \zeta \in \mathbb{T},$$

and for a positive integer n the operator Δ^n_τ is the nth power of Δ_τ. Now let $s > 0$, $1 \le p \le \infty$, and $1 \le q \le \infty$. The following description of B^s_{pq} holds:

$$B^s_{pq} = \left\{f \in L^p : \int_{\mathbb{T}} \frac{\|\Delta^n_\tau f\|^q_p}{|1-\tau|^{1+sq}} d\boldsymbol{m}(\tau) < \infty\right\}, \quad q < \infty, \tag{A2.10}$$

and

$$B^s_{p\infty} = \left\{ f \in L^p : \sup_{\tau \neq 1} \frac{\|\Delta^n_\tau f\|_p}{|1-\tau|^s} < \infty \right\},$$

where n is an integer such that $n > s$, the choice of n does not make any difference.

We denote the closure of the trigonometric polynomials in $B^s_{p\infty}$ by $b^s_{p\infty}$. We also use the notation λ_s for b^s_∞.

We need the following consequence of (A2.9). Let $1 \le p,\ q \le \infty$, $s > 0$, and $f \in H^p$. Then

$$f \in \left(B^s_{pq}\right)_+ \quad \Longleftrightarrow \quad \left\{2^{ns} \operatorname{dist}_{L^p}\left(\left(S^*\right)^{2^n} f, H^p_-\right)\right\}_{n \ge 0} \in \ell^q. \tag{A2.11}$$

Let us prove (A2.11) in the case of the space $\Lambda_s = B^s_\infty$. For all other spaces the proof is similar. Suppose that $\operatorname{dist}_{L^\infty}\left((S^*)^{2^n} f, H^\infty_-\right) \le \operatorname{const} 2^{-ns}$. We have $f * W_n = \left(f - z^{2^{n-1}} g\right) * W_n$ for any $g \in H^\infty_-$. Hence,

$$\|f * W_n\|_\infty \le \operatorname{const} \cdot \operatorname{dist}_{L^\infty}\left((S^*)^{2^{n-1}} f, H^\infty_-\right) \le \operatorname{const} 2^{-ns}.$$

Conversely, if $f \in \Lambda_s$, then

$$\operatorname{dist}_{L^\infty}\left((S^*)^{2^{n-1}} f, H^\infty_-\right) \le \sum_{k \ge n} \|f * W_k\|_\infty \le \operatorname{const} 2^{-ns}. \quad \blacksquare$$

In a similar way it can be shown that if $f \in \left(B^{1/p}_p\right)_+$, $1 \le p < \infty$, and $s > 0$, then

$$f \in \left(B^{s+1/p}_p\right)_+ \quad \Longleftrightarrow \quad \left\|(S^*)^{2^n} f\right\|_{B^{1/p}_p} \le \operatorname{const} 2^{-ns},\ n \in \mathbb{Z}_+. \tag{A2.12}$$

The proof of (A2.12) can be found in Peller and Khrushchëv [1], Lemma 4.4.

Let us now describe the dual spaces to Besov spaces. Suppose that $q < \infty$. Then

$$(B^s_{pq})^* = B^{-s}_{p'q'}, \quad 1 \le p \le \infty,\ 1 \le q < \infty,\ s \in \mathbb{R}, \quad p' = \frac{p}{1-p},\ q' = \frac{q}{1-q},$$

with respect to the pairing

$$(f, \varphi) = \sum_{j \in \mathbb{Z}} \hat{f}(j)\overline{\hat{\varphi}(j)}, \quad f \in \boldsymbol{\mathcal{P}},\ \varphi \in B^{-s}_{p'q'}, \tag{A2.13}$$

i.e., the left-hand side is defined at least on the set of trigonometric polynomials and extends to the whole space B^s_{pq} by continuity. Note that we take the complex conjugate of $\hat{\varphi}(j)$ for convenience; we could certainly write $\hat{\varphi}(j)$ instead of $\overline{\hat{\varphi}(j)}$ in (A2.13). We also note that

$$(b^s_{p\infty})^* = B^{-s}_{p'1}, \quad 1 \le p \le \infty,\ s \in \mathbb{R},$$

with respect to the same pairing (A2.13). In particular,

$$(\lambda_s)^* = B^{-s}_1, \quad s \in \mathbb{R}.$$

The Riesz projections $\mathbb{P}_+$ and $\mathbb{P}_-$ defined by (A2.3) can also be defined in the same way on the space of distributions. It is easy to see from the definition (A2.9) that $\mathbb{P}_+$ is a bounded linear projection on B^s_{pq}, $1 \le p \le \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$, onto the subspace $(B^s_{pq})_+$ of analytic functions in B^s_{pq}. This subspace admits the following description. Let f be a function analytic in $\mathbb{D}$. Then $f \in (B^s_{pq})_+$ if and only if

$$\int_0^1 (1-r)^{q(n-s)-1}\|f_r^{(n)}\|_p^q\,dt < \infty, \quad q < \infty, \tag{A2.14}$$

$$\sup_{0<r<1}(1-r)^{n-s}\|f_r^{(n)}\|_p < \infty, \quad q = \infty,$$

where n is a nonnegative integer satisfying $n > s$ (note that n can be equal to 0 if $s < 0$). Again, the choice of n does not make any difference.

The space $\mathfrak{B} \stackrel{\text{def}}{=} (B^0_{\infty})_+$ is called the *Bloch space*. It consists of functions φ analytic in $\mathbb{D}$ such that

$$\sup_{\zeta\in\mathbb{D}}|\varphi'(\zeta)|(1-|\zeta|) < \infty.$$

We define on the space of distributions the operators I_α, $\alpha \in \mathbb{R}$, of fractional integration by

$$I_\alpha f = \sum_{j\in\mathbb{Z}}(1+|j|)^{-\alpha}\hat{f}(j)z^j.$$

The following relations hold:

$$I_\alpha B^s_{pq} = B^{s+\alpha}_{pq}, \quad 1 \le p,\, q \le \infty, \quad \alpha,\, s \in \mathbb{R}. \tag{A2.15}$$

Sometimes if we deal with analytic functions, instead of the operators I_α it is more convenient to deal with the operators $\tilde{I}_\alpha$ of fractional integration, which are defined by

$$\tilde{I}_\alpha f = \begin{cases} \sum\limits_{j\ge0} D_j^{(\alpha)}\hat{f}(j)z^j, & \alpha > 0, \\ \sum\limits_{j\ge0} \frac{\hat{f}(j)}{D_j^{(-\alpha)}}z^j, & \alpha < 0, \\ f, & \alpha = 0. \end{cases}$$

Here $f = \sum\limits_{j\ge0}\hat{f}(j)z^j$ and

$$D_j^{(\alpha)} \stackrel{\text{def}}{=} \begin{pmatrix} j+\alpha \\ j \end{pmatrix} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+j)}{j!}.$$

It is well known that

$$\left|D_j^{(\alpha)} - \frac{j^\alpha}{\Gamma(\alpha+1)}\right| \le \text{const}(1+j)^{-1}, \quad \alpha > -1, \quad j \in \mathbb{Z}_+,$$

where Γ is the Γ function; see Zygmund [1], Ch. 3, §1.

The last inequality together with (A2.15) implies that

$$\tilde{I}_\alpha(B^s_{pq})_+ = (B^{s+\alpha}_{pq})_+, \quad 1 \le p,\, q \le \infty, \quad \alpha,\, s \in \mathbb{R}.$$

The Besov classes B^s_{pq}, $s > 0$, admit an interesting description due to Dyn'kin in terms of *pseudoanalytic continuation*. A function f belongs to B^s_{pq} if and only if there exists a function ψ on the annulus $\{\zeta \in \mathbb{C} : 1/2 < |\zeta| < 2\}$ such that

$$\int_{1/2}^{2} \left(\int_{\mathbb{T}} |1 - r|^{q(1-s)-1} |\psi(r\zeta)|^p d\boldsymbol{m}(\zeta) \right)^{q/p} dr < \infty, \quad q < \infty, \tag{A2.16}$$

$$\sup_{1/2<r<2} (1-r)^{1-s} \left(\int_{\mathbb{T}} |\psi(r\zeta)|^p d\boldsymbol{m}(\zeta) \right)^{1/p} < \infty, \quad q = \infty, \tag{A2.17}$$

and

$$f(\zeta) = (G\psi)(\zeta) \stackrel{\text{def}}{=} -\frac{1}{\pi} \int\limits_{1/2<|\zeta|<2} \frac{\psi(w)}{w - \zeta} d\boldsymbol{m}_2(w) \tag{A2.18}$$

for almost all $\zeta \in \mathbb{T}$.

This definition is equivalent to the following one. A function $f \in L^p$ belongs to B^s_{pq} if and only if there exists a function $\mathfrak{f}$ in $\{\zeta : 1/2 < |\zeta| < 2\}$ such that the restrictions of $\mathfrak{f}$ to $\{\zeta : 1/2 < |\zeta| < 1\}$ and $\{\zeta : 1 < |\zeta| < 2\}$ belong to the Sobolev space W_1^ρ of functions in these domains for some $\rho > 1$ (i.e., the function itself and its first order partial derivatives in the distributional sense belong to the space L^ρ with respect to planar Lebesgue measure),

$$\lim_{r \to 1} \mathfrak{f}(r\zeta) = f(\zeta) \quad \text{for almost all} \quad \zeta \in \mathbb{T},$$

and

$$\left(\int_{1/2}^{2} \left(\int_{\mathbb{T}} |1 - r|^{q(1-s)-1} \left| \frac{\partial \mathfrak{f}}{\partial \bar{z}}(\zeta) \right|^p d\boldsymbol{m}(\zeta) \right)^{q/p} dr \right)^{1/q} < \infty, \quad q < \infty, \tag{A2.19}$$

$$\sup_{1/2<r<2} (1-r)^{1-s} \left(\int_{\mathbb{T}} \left| \frac{\partial \mathfrak{f}}{\partial \bar{z}}(\zeta) \right|^p d\boldsymbol{m}(\zeta) \right)^{1/p} < \infty, \quad q = \infty, \tag{A2.20}$$

where

$$\frac{\partial \mathfrak{f}}{\partial \bar{z}} \stackrel{\text{def}}{=} \frac{1}{2} \left(\frac{\partial \mathfrak{f}}{\partial x} + \frac{\partial \mathfrak{f}}{\partial y} \right).$$

Such a function $\mathfrak{f}$ is called a *pseudoanalytic continuation* of f. Note also that the norm of f in B^s_{pq} modulo the constants is equivalent to the infimum of the left hand-side of (A2.19) (or (A2.20), if $q = \infty$) over all pseudoanalytic continuations $\mathfrak{f}$.

Note that the class $(B^s_{pq})_+$ of analytic functions in B^s_{pq} consists of functions f of the form

$$f(\zeta) = (G\psi)(\zeta) = -\frac{1}{\pi}\int\limits_{1<|\zeta|<2} \frac{\psi(w)}{w-\zeta}\,d\boldsymbol{m}_2(w), \quad \zeta \in \mathbb{D},$$

where ψ satisfies (A2.16), if $q < \infty$, and (A2.17), if $q = \infty$, and ψ is zero on the annulus $\{\zeta : \ 1/2 < |\zeta| < 1\}$.

A pseudoanalytic continuation can be obtained with the help of a linear operator. If $f \in (B^s_{pq})_+$, we define the pseudoanalytic continuation $\mathfrak{f}$ by

$$\mathfrak{f}(\zeta) = \begin{cases} \left(\sum\limits_{k=0}^{m-1} f^{(k)}\left(1/\bar{\zeta}\right) \frac{\left(\zeta - 1/\bar{\zeta}\right)^k}{k!}\right) F(|\zeta|), & 1 < |\zeta| < 2, \\ 0, & 1/2 < |\zeta| < 1, \end{cases} \tag{A2.21}$$

where m is an integer greater than s, F is an infinitely differentiable function on $[1,\infty)$ such that $\operatorname{supp} F \subset [1,2]$, and $F\big|[1,3/2]$ is identically equal to 1. Then $\mathfrak{f}$ belongs to the Sobolev class W^ρ_1 for some $\rho > 1$, the boundary values of $\mathfrak{f}$ on $\mathbb{T}$ coincide with f almost everywhere, and $\mathfrak{f}$ satisfies (A2.19), if $q < \infty$, and (A2.20), if $q = \infty$. Moreover,

$$f(\zeta) = \left(G\left(\frac{\partial \mathfrak{f}}{\partial \bar{z}}\right)\right)(\zeta), \quad \zeta \in \mathbb{D}.$$

Similarly, one can define a linear operator of pseudoanalytic continuation of a function in $(B^s_{pq})_-$ and obtain a function vanishing on the annulus $\{\zeta : \ 1 < |\zeta| < 2\}$. To obtain a linear operator of analytic continuation on B^s_{pq} we can take $f \in B^s_{pq}$ and apply the above operators to $\mathbb{P}_+ f$ and $\mathbb{P}_- f$.

Using the above description of Besov classes in terms of pseudoanalytic continuation one can show that if $f \in B^s_{pq}$, $s > 0$, g is a function analytic in a neighborhood $\mathcal{O}$ of $f(\mathbb{T})$. Then $g \circ f \in B^s_{pq}$ and the norm of $g \circ f$ in B^s_{pq} modulo the constants can be estimated in terms of $\sup\limits_{\zeta \in \mathcal{O}} |g'(\zeta)|$ and the norm of f in B^s_{pq} modulo the constants.

It follows that if f_1 and f_2 are bounded functions in B^s_{pq}, then $f_1 f_2 \in B^s_{pq}$. Since $B^s_{pq} \subset L^\infty$ for $s > 1/p$ and $B^{1/p}_{p1} \subset L^\infty$, it follows that the spaces B^s_{pq} for $s > 1/p$ and $B^{1/p}_{p1}$ form algebras.

We proceed now to the Besov classes $B^s_p = B^s_{pp}$ for $0 < p < 1$ (we consider here for simplicity only the case $p = q$). Let v be an infinitely differentiable function on $\mathbb{R}$ such that $v \geq \mathbb{O}$, $\operatorname{supp} v = [1/2, 2]$, and

$$\sum_{n \geq 0} v\left(\frac{x}{2^n}\right) = 1, \quad x \geq 1.$$

It is very easy to construct such a function. We can take a nonnegative C^∞ function v on the interval $[1/2, 1]$ such that $v(1/2) = 0$, $v(1) = 1$, $v^{(k)}(1/2) = v^{(k)}(1) = 0$ for $k \geq 1$. Then we can put $v(x) = 1 - v(x/2)$ for $x \in [1,2]$ and $v(x) = 0$ for $x \notin [1/2, 1]$.

Consider now the trigonometric polynomials V_n defined by

$$V_n = \begin{cases} \sum\limits_{j\in\mathbb{Z}} v\left(\frac{j}{2^n}\right) z^j, & n \ge 1, \\ \overline{V}_{-n}, & n \le -1, \\ \bar{z} + \mathbf{1} + z, & n = 0. \end{cases}$$

The Besov class B_p^s, $0 < p \le \infty$, $s \in \mathbb{R}$, consists of distributions f such that

$$\sum_{n\in\mathbb{Z}} \left(2^{s|n|}\|f * V_n\|_p\right)^p < \infty.$$

As in the case $p \ge 1$, it follows easily from the definition that $\mathbb{P}_+ B_p^s \subset B_p^s$. As in (A2.15), we have

$$I_\alpha B_p^s = B_p^{s+\alpha}, \quad p > 0,\ s \in \mathbb{R}.$$

Under the condition $s > 1/p - 1$ the Besov classes B_p^s are contained in L^1 and they admit the description (A2.10) with $q = p$. Note also that the classes $(B_p^s)_+$ admit the description (A2.14) with $q = p$.

For $p < 1$ and $s > 1/p - 1$ the classes B_p^s also admit a description in terms of pseudoanalytic continuation. However, this description has to be modified slightly. A function f belongs to B_p^s if and only if it admits a representation (A2.18) for a function ψ satisfying

$$\int\limits_{\{1/2<|\zeta|<2\}} \big|1 - |\zeta|\big|^{p(1-s)-1} (\psi_*(\zeta))^p d\boldsymbol{m}_2(\zeta) < \infty,$$

where the maximal function ψ_* is defined by

$$\psi_*(\zeta) = \operatorname{ess\,sup}\left\{|\psi(\lambda)| :\ |\lambda - \zeta| < \frac{1}{2}\big|1 - |\zeta|\big|,\ \frac{1}{2} < |\lambda| < 2\right\}.$$

As in the case $p \ge 1$, under the condition $s > 1/p - 1$, a function $f \in L^p$ belongs to B_p^s if and only if it has a pseudoanalytic continuation $\mathfrak{f}$ in $\{\zeta :\ 1/2 < |\zeta| < 2\}$ such that the restrictions of $\mathfrak{f}$ to $\{\zeta :\ 1/2 < |\zeta| < 1\}$ and $\{\zeta :\ 1 < |\zeta| < 2\}$ belong to the Sobolev space W_1^ρ of functions in these domains for some $\rho > 1$, the boundary values of $\mathfrak{f}$ on $\mathbb{T}$ coincide with f almost everywhere, and

$$\int\limits_{\{1/2<|\zeta|<2\}} \big|1 - |\zeta|\big|^{p(1-s)-1} \left(\left(\frac{\partial \mathfrak{f}}{\partial \bar{z}}\right)_* (\zeta)\right)^p d\boldsymbol{m}_2(\zeta) < \infty.$$

A pseudoanalytic continuation $\mathfrak{f}$ can be obtained with the help of the same linear operator (A2.21) as in the case $p > 1$.

Again, as in the case $p > 1$, the description of B_p^s in terms of pseudoanalytic continuation shows that if $s > 1/p - 1$, $f \in B_p^s$, and φ is a function analytic in a neighborhood of $f(\mathbb{T})$, then $\varphi \circ f \in B_p^s$ and the seminorm of $\varphi \circ f$ in B_p^s modulo the constants admits the same estimate as in the

Banach case. Thus if $f_1,\ f_2 \in B_p^s \cap L^\infty$, then $f_1 f_2 \in B_p^s$. In particular, for $s \ge 1/p$ the space B_p^s is an algebra with respect to pointwise multiplication.

A similar description in terms of pseudoanalytic continuation can be applied for Besov classes $B_p^s(\mathbb{R})$ of functions on $\mathbb{R}$, $s > 1/p - 1$. We deal in this book only with classes $B_p^{1/p}(\mathbb{R})$, which are defined in §6.7. It follows from this description that if $g \in BMO(\mathbb{R})$, then $\boldsymbol{P}^- g \in B_p^{1/p}(\mathbb{R})$ if and only if $\mathbb{P}_-(\psi \circ \omega) \in B_p^{1/p}$, where

$$\omega(\zeta) = \mathrm{i}\frac{1+\zeta}{1-\zeta}, \quad \zeta \in \mathbb{D},$$

and $\boldsymbol{P}^- g = (\mathbb{P}_-(g \circ \omega)) \circ \omega^{-1}$ (see Dyn'kin [2]).

We refer the reader for more information on Besov classes to Peetre [1], Triebel [1], [2], Bergh and Löfström [1], S.M. Nikol'skii [1], Oswald [1], and for the theory of pseudoanalytic continuation to Dyn'kin [1] and [2].

The spaces $\mathcal{L}_p^s$ of *Bessel potentials*, $1 < p < \infty$, $s \in \mathbb{R}$, are defined by

$$\mathcal{L}_p^s \stackrel{\text{def}}{=} I_s(L^p).$$

If s is a positive integer, $\mathcal{L}_p^s$ is a classical Sobolev space. It is well known that $B_p^s \subset \mathcal{L}_p^s$ for $1 \le p \le 2$ and $\mathcal{L}_p^s \subset B_p^s$ for $2 \le p < \infty$. The spaces $\mathcal{L}_p^s$ for $s > 0$ can also be described in terms of pseudoanalytic continuation. We do not state this description and refer the reader to Dyn'kin [1]. Using that description, one can show that if f_1 and f_2 are bounded functions in $\mathcal{L}_p^s$, $s > 0$, then $f_1 f_2 \in \mathcal{L}_p^s$. In particular, $\mathcal{L}_p^s$ is a Banach algebra with respect to pointwise multiplication if $s > 1/p$.

We refer the reader to Triebel [1], S.M. Nikol'skii [1], and Dyn'kin [1] for more information.

We need the following description of the spaces

$$I_{-\alpha} H^p = \{f :\ I_\alpha f \in H^p\}, \quad \alpha > 0, \quad 1 \le p < \infty,$$

of analytic functions:

$$\varphi \in I_{-\alpha} H^p \quad \Longleftrightarrow \quad \int_{\mathbb{T}} \left(\int_0^1 |\varphi(r\zeta)|^2 (1-r)^{2\alpha-1} dr \right)^{p/2} d\boldsymbol{m}(\zeta) < \infty$$

(see Zygmund [1] and Triebel [2]).

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